HIGHER DIMENSIONAL LINKS ARE SINGULAR SLICE

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ABSTRACT. We show that for $n \geq 2$ all links of embedded n-spheres in S^{n+2} are singular slice, i.e. bound pairwise disjoint (but not embedded) $n+1$-disks in D^{n+3}. The proof relies on a careful analysis of immersions in codimension two, that allows us to work in a nilpotent setting.

1. INTRODUCTION

An n-dimensional *link* $\mathcal{L}$ is a smooth embedding $S^n \amalg \cdots \amalg S^n \hookrightarrow S^{n+2}$. $\mathcal{L}$ is said to be slice if there are *slice disks* for $\mathcal{L}$, namely an embedding $f : D^{n+1} \amalg \cdots \amalg D^{n+1} \hookrightarrow D^{n+3}$ that extends $\mathcal{L}$. In the classical dimension (n=1) the linking number detects examples of non-slice links. For example the Hopf link is not slice. The linking number obstructs even more: there are no singular slice disks for the Hopf link. A link $\mathcal{L}$ is called *singular slice* if there are *singular slice disks* for $\mathcal{L}$, namely a link map $f : D^{n+1} \amalg \cdots \amalg D^{n+1} \to D^{n+3}$ extending $\mathcal{L}$. (A *link map* is a map that keeps different components disjoint in the image.) More examples of such links are detected by Milnor's μ-invariants (with non-repeating indices), see [13]. For example the Borromean rings have non-vanishing $\mu(1,2,3)$ and are thus not singular slice.

In [5], Cochran shows that certain proposed generalizations of the μ-invariants to higher dimensional (embedded) links vanish. He used a result of Bousfield and Kan on the homology of nilpotent quotients of the free group (compare 2.3). This is also an important ingredient in the proof of our result:

Theorem 1.1. *All links of dimension $n \geq 2$ are singular slice.*

A *link homotopy* is a motion that keeps different components disjoint, i.e. a homotopy through link maps. Link homotopy was introduced by Milnor in [13] to study classical links. He constructed a certain nilpotent quotient of the fundamental group of the link complement, later known as the *Milnor group*, which is invariant under link homotopy. A classical link is homotopically trivial if and only if its Milnor group is isomorphic to the Milnor group of the trivial link. Another way of formulating this result is to consider μ-invariants (with non-repeating indices). Then a classical link is homotopic to the trivial link if and only if all of these invariants vanish. Note that homotopically trivial links are singular slice. In fact the two notions are equivalent: singular slice links are also homotopically trivial. For $n \geq 2$ this result is due to Teichner [17]. For $n = 2$ a proof of his result can be found in [1], but the general case is yet unpublished. [1] also contains our theorem for $n = 2$. The following consequence of Teichner's result together with Theorem 1.1 is in contrast to the classical situation.

Corollary 1.2. *All links of dimension $n \geq 2$ are homotopically trivial.*

It should be noted that link homotopy makes sense for link maps $S^n \amalg \cdots \amalg S^n \to S^{n+2}$ that are not necessary embeddings. The first example of a link map $S^2 \amalg S^2 \to S^4$ that is homotopically essential was constructed by Fenn and Rolfsen in [8]. Link homotopy is not restricted to codimension two. A generalisation of μ-invariants to link maps $S^{p_1} \amalg \cdots \amalg S^{p_r} \to S^m$ is due to Koschorke ([11]). The vanishing of these invariants on the links studied here has been conjectured by Kaiser.

It is known that all even dimensional knots are slice (see [9]), but there are non-slice knots in all odd dimensions (see [12]). In [3], boundary links are studied, leading to examples of links in odd dimensions that are not slice, even though all their components are slice as knots. The question of wether all even dimensional links are slice is still open. An approach to this question is to use surgery to build a slice complement, for example see [4]. For a link $\mathcal{L}$ let $X_{\mathcal{L}}$ be obtained from S^{n+2} by surgery on all components of $\mathcal{L}$. Then $\mathcal{L}$ is slice if and only if $X_{\mathcal{L}}$ bounds a manifold (namely the slice complement) satisfying certain conditions. The main problem here is to find both a suitable model space and a map from the link complement to the model that controls the surgeries. For example, the canonical slice complement of the trivial link (W_0 from Section 2) is homotopy equivalent to a wedge of circles, and constructing a suitable map boils down to group theory. In this way, one can prove that boundary links of even dimensions are slice (compare also [3]). An obvious consequence of 1.2 is the following: invariants that could detect non-slice links cannot be invariant under link homotopy.

The proof of Theorem 1.1 uses the technique sketched above for the slice problem: we construct the link map f by building its complement in D^{n+3}. To find a suitable model the following statement about the trivial link $\mathcal{L}_0$ is essential.

Proposition 1.3. *There are immersed singular slice disks $f_1 : D^{n+1} \amalg \cdots \amalg D^{n+1} \looparrowright D^{n+3}$ for $\mathcal{L}_0$ such that their complement has a nilpotent fundamental group and nilpotent homotopy groups (over the fundamental group) in dimensions $\leq n/2$.*

In fact, the nilpotent fundamental group will be MF, the Milnor group of the trivial link. Assuming Proposition 1.3 we construct our *nilpotent* model MOD and maps into it in Section 2. In particular, we find maps comparing MOD with the Eilenberg-MacLane spaces K_r for nilpotent quotients of the free group. In order to construct maps into our nilpotent model we have to control only obstructions in cohomology with untwisted coefficients. To control these obstructions we will use the consequence of Bousfield's and Kan's result that was obtained in [5]. For a given link $\mathcal{L}$ we construct in Section 3 a potential boundary of a singular slice complement, a closed manifold $Y_{\mathcal{L}}$. Recall that $X_{\mathcal{L}}$ is obtained from the link complement by adding $\amalg^{\nu} D^{n+1} \times S^1$. To construct $Y_{\mathcal{L}}$ we replace $\amalg^{\nu} D^{n+1} \times S^1$ by a more complicated manifold ΣF, reflecting the presence of selfintersections. In fact, the choice of ΣF fixes the structure of selfintersections of the singular slice disks we are looking for. The next step is to construct a manifold A with boundary $Y_{\mathcal{L}}$ and a suitable map $A \to MOD$. If n is even we can then obtain the desired complement of singular slice disks by surgery on A. In the case of odd dimensional links, a surgery obstruction (echoing the existence of non-slice links) complicates the situation. An additional geometric construction (*symmetric surgery* from [18]) is needed in Section 4 to finish the proof for this case. In [18] it was used to show that all boundary link maps are homotopically trivial. This construction introduces further selfintersections into the singular slice disks.

The remaining sections of the paper contain the proof of the above proposition. In Sections 5 and 6 we study immersions $M \looparrowright N$, describing in particular how certain moves can be used to change the selfintersections and simplify the homotopy type of the complement $N - M$ (6.9). Here we use the language of stratified handles from [17]. Finally Section 7 provides the necessary algebra to finish the proof of 1.3. We construct nilpotent quotients of modules over the Milnor group MF. These quotients will be realized using the moves from Section 6 to obtain nilpotent homotopy groups in the construction of f_1. We work in the smooth category.

Our proof generalizes the argument from [1] where the moves involved are only *finger moves* and where the nilpotent model is the classifying space for the Milnor group MF.

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2. The model

Let $n \geq 2$ and $\mathcal{L}_0 : S^n \amalg \cdots \amalg S^n \hookrightarrow S^{n+2}$ be the trivial link with ν components. Let $f_0 : D^{n+1} \amalg \cdots \amalg D^{n+1} \hookrightarrow D^{n+3}$ be standard slice disks for $\mathcal{L}_0$ and let f_1 be the singular slice disks for $\mathcal{L}_0$ from 1.3. We denote by F the free group on ν generators and by F_r the r-th term of its lower central series (compare Section 7). Let $\widetilde{f}_0 : \amalg^\nu D^{n+1} \times D^2 \hookrightarrow D^{n+3}$ and $\widetilde{f}_1 : \amalg^\nu D^{n+1} \times D^2 \looparrowright D^{n+3}$ be thickenings of f_0 and f_1 which agree on $\amalg^\nu S^n \times D^2$. Denote by $int(D^2)$ the interior of D^2. We will need the following manifolds:

$$\begin{array}{rcl} W_0^{n+3} & := & D^{n+3} - \widetilde{f}_0(\amalg^\nu D^{n+1} \times int(D^2)) \\ W_1^{n+3} & := & D^{n+3} - \widetilde{f}_1(\amalg^\nu D^{n+1} \times int(D^2)) \\ W^{n+3} & := & W_0 \cup_{S^{n+2} - \widetilde{f}_0(\amalg^\nu S^n \times int(D^2))} W_1 \\ \Sigma H^{n+3} & := & \widetilde{f}_1(\amalg^\nu D^{n+1} \times D^2) \\ \Sigma F^{n+2} & := & \partial \Sigma H - \widetilde{f}_0(\amalg^\nu S^n \times int(D^2)). \end{array}$$

These manifolds come with some corners, but all of them can be smoothed, essentially by some application of Figure 5. We will ignore this matter for now. Note that W_0, W_1 and ΣH inherit a framing (of stable tangent bundles) as codimension 0 submanifolds of D^{n+3}. These framings induce framings of ΣF and W. There is a unique framing of $D^{n+1} \times S^1$ that extends to a framing of $D^{n+1} \times D^2$. It is this framing that the components of $\amalg^\nu D^{n+1} \times S^1$ inherit as part of the boundary of W. The following space will be used to

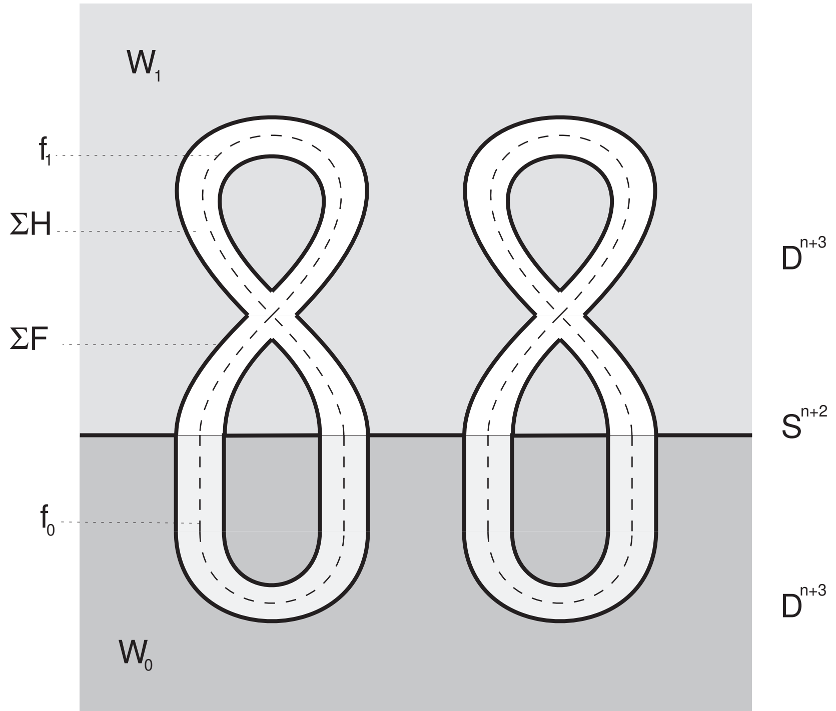

FIGURE 1. A schematic picture of W.

model the complement of singular slice disks for an arbitrary link: let MOD be obtained from W_1 by attaching cells of dimension $\geq n/2+2$ such that $\pi_k(MOD) = 0$ for $k \geq n/2+1$.

Proposition 2.1. *There is a sequence of spaces and maps*

$$MOD \simeq Z_m \to Z_{m-1} \to \cdots \to Z_1 = K(\pi_1(MOD), 1)$$

such that the $Z_{j+1} \to Z_j$ are fibrations with fiber $K(G_j, l_j)$ where $\pi_1(Z_j) = \pi_1(MOD)$ acts trivially on G_j. Here $l_j \geq 2$. In particular, to lift a map $X \to K(\pi_1(MOD), 1)$ to a map $X \to MOD$ only obstructions in ordinary cohomology as opposed to cohomology with twisted coefficients have to be considered.

Proof. From 1.3 and the construction of MOD we know that only a finite number of its homotopy groups are nonzero and that they all are nilpotent as modules over the (nilpotent) fundamental group. Thus, MOD is a *nilpotent space* and its Postnikov tower has a refinement as stated, see [2, II.4.7]. $\square$

Lemma 2.2. *The inclusion map $W_1 \to MOD$ extends to a map $W \to MOD$.*

Proof. Note that W_0 is simply the boundary connected sum of ν copies of $S^1 \times D^{n+2}$. Up to homotopy equivalence, we can obtain W from W_1 by adding cells of dimensions $n+2$ and $n+3$. But by construction, $\pi_{n+2}(MOD)$ and $\pi_{n+3}(MOD)$ are trivial; hence we can extend our map over the additional cells. $\square$

We will need a consequence of a result of Bousfield and Kan [2, p.123] stating that the tower

$$\cdots \to H_k(F/F_r) \to H_k(F/F_{r-1}) \to \cdots \to H_k(F/F_2)$$

is *protrivial*. Let K_r be an Eilenberg-MacLane space for F/F_r that is constructed by adding cells of dimension ≥ 2 to W_0. This form of K_r will be used later to extend maps from W_0 to K_r, see 2.4 below. Let J_r be obtained from K_r by attaching 2-cells to the ν meridians in W_0. Thus, the J_r are 1-connected and satisfy $H_k(J_r) \cong H_k(F/F_r)$ for $k \geq 2$. For $r' > r$, the maps $K_{r'} \to K_r$ induced by the projections $F/F_{r'} \to F/F_r$ can be extended to maps $J_{r'} \to J_r$. The following result is now a consequence of the *eventual Hurewicz theorem* from [5]. (Compare also [6].)

Theorem 2.3. *For any r and k there is an integer $r' > r$ such that the map from the k-skeleton of $J_{r'}$ to J_r is null-homotopic.*

Note that 2.2 gives a map $W_0 \to MOD$.

Lemma 2.4. *For sufficiently large r the map $W_0 \to MOD$ can be extended to a map $K_r \to MOD$.*

Proof. By 1.3 $\pi_1(MOD)$ is nilpotent. Hence there is an l such that the map induced by $W_0 \to MOD$ on fundamental groups factors as

$$F \to F/F_{r'} \to F/F_r \to \pi_1(MOD)$$

for all $r' > r > l$. This gives a commutative diagram:

$$\begin{tikzcd}
W_0 \arrow[rr] \arrow[d] & & MOD \arrow[d] \\
K_{r'} \arrow[r] & K_r \arrow[r] & K(\pi_1(MOD),1)
\end{tikzcd}$$

We will now use the notation of 2.1. From the exact sequences of the pairs (K_r, W_0) and $(K_{r'}, W_0)$, we see that

$$H^{l_1+1}(K_r, W_0; G_1) \cong H^{l_1+1}(K_r; G_1)\ ,\ H^{l_1+1}(K_{r'}, W_0; G_1) \cong H^{l_1+1}(K_{r'}; G_1).$$

Let $u \in H^{l_1+1}(K_r, W_0; G_1)$ be the obstruction to lift $K_r \to K(\pi_1(MOD), 1)$ to a map $K_r \to Z_2$. By 2.3, we can choose r' sufficiently large such that u pulls back to $0 \in H^{l_j+1}(K_{r'}, W_0)$. Thus, there is a lift $K_{r'} \to Z_2$. Repeating this process, we work our way up the tower of 2.1 and find that there is an extension of $W_0 \to MOD$ to $K_r \to MOD$ for sufficiently large r. $\square$

There are ν projections $F \to \mathbb{Z}$ coming from the generators. They induce projections $F/F_r \to \mathbb{Z}$ and give maps

$$\alpha_j : K_r \to S^1$$

for $j = 1, \ldots, \nu$.

Let Ω_* be a generalized homology theory and denote the corresponding reduced theory by $\widetilde{\Omega}_*$.

Proposition 2.5. *Let $r \geq 2$. For any k there is a short exact sequence*

$$0 \to \bigoplus^{\nu} \widetilde{\Omega}_k(S^1) \to \widetilde{\Omega}_k(K_r) \to \widetilde{\Omega}_k(J_r) \to 0.$$

A splitting of this sequence is given by

$$(\alpha_{1*}, \ldots, \alpha_{\nu*}) : \widetilde{\Omega}_k(K_r) \to \bigoplus^{\nu} \widetilde{\Omega}_k(S^1).$$

Proof. Consider the long exact sequence of the pair (J_r, K_r). Observe that

$$\Omega_k(J_r, K_r) \cong \bigoplus^{\nu} \Omega_k(D^2, S^1) \cong \bigoplus^{\nu} \widetilde{\Omega}_{k-1}(S^1).$$

This provides the long exact sequence

$$\cdots \to \bigoplus^{\nu} \widetilde{\Omega}_k(S^1) \to \widetilde{\Omega}_k(K_r) \to \widetilde{\Omega}_k(J_r) \to \bigoplus^{\nu} \widetilde{\Omega}_{k-1}(S^1) \to \ldots .$$

Now $(\alpha_{1*}, \ldots, \alpha_{\nu*})$ splits this into short exact sequences as claimed. □

3. Even dimensional links

We will continue to use the notation of section 2. Let $n \geq 2$ and $\mathcal{L} : S^n \amalg \cdots \amalg S^n \hookrightarrow S^{n+2}$ be a link with ν components. We can add locally to each component of $\mathcal{L}$ its respective mirror image, the inverse in the knot concordance group (see [9]). Since knot maps are null homotopic, this addition can be achieved by a link homotopy taking place in small disjoint $(n+2)$-disks, one for each component (compare [18]). We will from now on always assume that the components of $\mathcal{L}$ are slice as knots. Note that it is sufficient to prove Theorem 1.1 for such links, since we can always add a link homotopy to singular slice disks. This assumption implies in particular that we can extend $\mathcal{L}$ to an immersion

$$f : D^{n+1} \amalg \cdots \amalg D^{n+1} \looparrowright D^{n+3}.$$

We can construct f using general position slice disks for the components of $\mathcal{L}$ such that the restriction of f to each $(n+1)$-disk gives an embedding. Let $\widetilde{f} : \amalg^{\nu} D^{n+1} \times D^2 \looparrowright D^{n+3}$ be a thickening of f. Surgery on $\mathcal{L}$ produces the manifold

$$X_{\mathcal{L}}^{n+2} := S^{n+2} - \widetilde{f}(\amalg^{\nu} S^n \times int(D^2)) \cup_{\amalg^{\nu} S^n \times S^1} \amalg^{\nu} D^{n+1} \times S^1.$$

Note that $X_{\mathcal{L}}$ bounds the manifold

$$V_{\mathcal{L}}^{n+3} := D^{n+3} \cup_{\amalg^{\nu} S^n \times D^2} \amalg^{\nu} D^{n+1} \times D^2.$$

The immersion $\widetilde{f}$ can be used to define a framing of $\amalg^{\nu} D^{n+1} \times D^2$. This fits with the standard framing of D^{n+3} and gives a framing of $V_{\mathcal{L}}$. As its boundary, $X_{\mathcal{L}}$ inherits an induced framing. Restricted to $\amalg^{\nu} D^{n+1} \times S^1$, this gives again the unique framing that extends over $\amalg^\nu D^{n+1} \times D^2$. From $\amalg^\nu D^{n+1} \times S^1 \subset W_0$ we have inclusions $\amalg^\nu D^{n+1} \times S^1 \to K_r$.

Lemma 3.1. *For any r there is a map $\varphi_r : X_{\mathcal{L}} \to K_r$ extending the inclusion $\amalg^\nu D^{n+1} \times S^1 \to K_r$. For $r' > r$, this gives a commutative triangle:*

$$\begin{tikzcd} X_{\mathcal{L}} \arrow[d] \arrow[dr] & \\ K_{r'} \arrow[r] & K_r \end{tikzcd}$$

Proof. We find $X_{\mathcal{L}} \to K_1$ since K_1 is contractible. The fibration $K_{r+1} \to K_r$ has $K(F_r/F_{r+1}, 1)$ as its fiber, and $\pi_1(K_r) = F/F_r$ acts trivially on F_r/F_{r+1}. Thus, the obstructions to lift our maps lie in $H^2(X_{\mathcal{L}}, \amalg^\nu D^{n+1} \times S^1)$ with appropriate untwisted coefficients. Now

$$\begin{aligned} H^2(X_{\mathcal{L}}, \amalg^\nu D^{n+1} \times S^1) &\cong H^2(S^{n+2}, \widetilde{f}(\amalg^\nu S^n \times D^2)) \\ &\cong H^1(\amalg^\nu S^n \times D^2) \\ &= 0, \end{aligned}$$

and all obstructions vanish. □

Denote by Ω_*^{fr} the generalized homology theory given by framed bordism. Using the maps from 3.1 and our framing of $X_{\mathcal{L}}$, we have elements $[X_{\mathcal{L}}, \varphi_r] \in \Omega_{n+2}^{fr}(K_r)$.

Proposition 3.2. $[X_{\mathcal{L}}, \varphi_r] = 0$ *for all r.*

Proof. Recall that $X_{\mathcal{L}} = \partial V_{\mathcal{L}}$ and hence $[X_{\mathcal{L}}, \varphi_r] \in \widetilde{\Omega}_{n+2}^{fr}(K_r)$. Let $\mathcal{K}_i$ be the i-th component of $\mathcal{L}$ and denote by $\mathcal{L}_i$ the link obtained by deleting $\mathcal{K}_i$ from $\mathcal{L}$. Then a framed manifold $V_{\mathcal{L}_i}$ can be constructed analogously to $V_{\mathcal{L}}$ by adding $\nu - 1$ handles to D^{n+3}. Let $\widetilde{g} : D^{n+1} \times D^2 \hookrightarrow D^{n+3}$ be the restriction of $\widetilde{f}$ to the i-th component. Now the framed manifold $U^{n+3} := V_{\mathcal{L}_i} - \widetilde{g}(D^{n+1} \times int(D^2))$ bounds $X_{\mathcal{L}}$. Moreover, the meridian to $\mathcal{K}_i$ still gives a homology class in $H_1(U)$. Hence $\alpha_i \circ \varphi_r : X_{\mathcal{L}} \to S^1$ can be extended to a map $U \to S^1$. This proves

$$[X_{\mathcal{L}}, \varphi_r] \in ker(\alpha_1, \ldots, \alpha_\nu),$$

where $(\alpha_1, \ldots, \alpha_\nu)$ is the splitting from 2.5. Let ι denote the inclusion $K_r \to J_r$. We then have

$$[X_{\mathcal{L}}, \iota \circ \varphi_r] \in im(\widetilde{\Omega}_{n+2}^{fr}(J_{r'}) \to \widetilde{\Omega}_{n+2}^{fr}(J_r))$$

for all $r' > r$. Let J_r^{n+3} denote the $(n+3)$-skeleton of J_r. A consequence of the Atiyah-Hirzebruch spectral sequence is that $\widetilde{\Omega}_{n+2}^{fr}(J_r) \cong \widetilde{\Omega}_{n+2}^{fr}(J_r^{n+3})$. Hence $[X_{\mathcal{L}}, \iota \circ \varphi_r] = 0$ by 2.3. The splitting of 2.5 implies then $[X_{\mathcal{L}}, \varphi_r] = 0$. □

Let

$$Y_{\mathcal{L}}^{n+2} := S^{n+2} - \widetilde{f}(\amalg^\nu S^n \times int(D^2)) \cup_{\amalg^\nu S^n \times S^1} \Sigma F.$$

Recall that we have $\Sigma F \hookrightarrow W_1 \to MOD$ from Section 2.

Proposition 3.3. *$Y_{\mathcal{L}}$ bounds a framed manifold A^{n+3} such that there is a map $A \to MOD$ which makes the following diagram commutative:*

$$\begin{tikzcd}
 & Y_{\mathcal{L}} \arrow[r] & A \arrow[d] \\
\Sigma F \arrow[ur] \arrow[r] & W_1 \arrow[r] & MOD
\end{tikzcd}$$

Moreover, the framings which ΣF inherits from A and from W_1 (and ΣH) coincide.

Proof. Let r be such that $W_0 \to MOD$ extends to a map $\psi : K_r \to MOD$ (see 2.4). By 3.2 we see, $[X_{\mathcal{L}}, \psi \circ \varphi_r] = 0 \in \Omega^{fr}_{n+2}(MOD)$. Therefore, $X_{\mathcal{L}}$ is the boundary of a framed manifold C^{n+3} over MOD. Together with maps and spaces from Section 2, we have the following commutative diagram:

$$\begin{tikzcd}
\amalg^{\nu} D^{n+1} \times S^1 \arrow[r] \arrow[d] & W_0 \arrow[r] \arrow[d] & W \arrow[d] & \Sigma F \arrow[l] \arrow[d] \\
X_{\mathcal{L}} \arrow[r] \arrow[d] & K_r \arrow[r] & MOD & W_1 \arrow[l] \\
C \arrow[urr, bend right] & & &
\end{tikzcd}$$

Let

$$A := C \cup_{\amalg^{\nu} D^{n+1} \times S^1} W.$$

The framing of C induces the unique framing on $\amalg^{\nu} D^{n+1} \times S^1$ that extends over $\amalg^{\nu} D^{n+1} \times D^2$. Recall from Section 2 that there is a framing of W with the same property. This gives the framing of A. The maps from C and W to MOD can be combined to a map $A \to MOD$. $\square$

In the following theorem, a link map that is also an immersion is called a *link immersion.*

Theorem 3.4.

(i) *If $n = 2k$ is even, then $\mathcal{L}$ is singular slice via a link immersion*

$$f : D^{n+1} \amalg \cdots \amalg D^{n+1} \looparrowright D^{n+3}.$$

(ii) *If $n = 2k-1$ is odd, then a framed k-connected manifold B^{n+3} exists with boundary $\partial B = S^{n+2}$ and a link immersion*

$$f : D^{n+1} \amalg \cdots \amalg D^{n+1} \looparrowright B,$$

extending $\mathcal{L}$. Moreover, $\pi_{k+1}(B) \cong H_{k+1}(B)$ has a basis $e_1, \ldots, e_r, e'_1, \ldots, e'_r$ satisfying the following:

(a) *The intersection form* λ *is given by*
$$\lambda(e_i, e_j) = 0,\ \lambda(e'_i, e'_j) = 0,\ \lambda(e_i, e'_j) = \delta_{ij}.$$
(So the e_i, e'_i *form a* hyperbolic *basis.)*

(b) *Every embedding* $S^{k+1} \hookrightarrow B$ *representing one of the* e_i *has trivial normal bundle.*

(c) *All the* e_i, e'_i *can be represented by immersions* $S^{k+1} \looparrowright B$ *missing the image of* f.

Proof. For $A \to MOD$ as in 3.3 set
$$B^{n+3}(A) := A \cup_{\Sigma F} \Sigma H.$$
Then $\partial B(A) = S^{n+2}$, and the map $f_1 : \amalg^\nu D^{n+1} \looparrowright \Sigma H$ gives a link immersion $f : \amalg^\nu D^{n+1} \looparrowright B(A)$ extending $\mathcal{L}$. 3.3 provides a framing for $B(A)$. Recall that MOD is obtained from W_1 by adding cells of dimension $\geq k+2$ and that $W_1 \cup_{\Sigma F} \Sigma H = D^{n+3}$. Hence
$$MOD \cup_{\Sigma F} \Sigma H = D^{n+3} \cup (\text{ cells of dimension } \geq k+2) =: D^+.$$
After surgery on classes of dimension $\leq k$, we may assume that $A \to MOD$ is a $(k+1)$-equivalence. Then we can compare the push-out diagrams
$$\begin{tikzcd} \Sigma F \arrow[r] \arrow[d] & \Sigma H \arrow[d] \\ A \arrow[r] & B(A) \end{tikzcd} \qquad \begin{tikzcd} \Sigma F \arrow[r] \arrow[d] & \Sigma H \arrow[d] \\ MOD \arrow[r] & D^+. \end{tikzcd}$$
Repeated applications of van Kampen's Theorem prove that
$$\pi_1(B(A)) = \pi_1(D^+) = 1.$$
Comparing the Mayer-Vietoris sequences of the diagrams, we see that $H_j(B(A)) = 0$ for $j = 1, \ldots, k$. Therefore, $B(A)$ is k-connected. Note that $\pi_{k+1}(A)$ maps onto the kernel of $H_{k+1}(A) \to H_{k+1}(MOD)$ via the Hurewicz homomorphism.
$$\begin{tikzcd} & \pi_{k+1}(A) \arrow[r] \arrow[d] & \pi_{k+1}(B(A)) \arrow[d, "\cong"] & \\ H_{k+1}(\Sigma F) \arrow[r] \arrow[d, "="] & H_{k+1}(\Sigma H) \oplus H_{k+1}(A) \arrow[r] \arrow[d] & H_{k+1}(B(A)) \arrow[r, "0"] \arrow[d] & H_k(\Sigma F) \arrow[d, "="] \\ H_{k+1}(\Sigma F) \arrow[r] & H_{k+1}(\Sigma H) \oplus H_{k+1}(MOD) \arrow[r] & H_{k+1}(D^+) = 0 \arrow[r] & H_k(\Sigma F) \end{tikzcd}$$
From the above diagram, we see that $\pi_{k+1}(A) \to \pi_{k+1}(B(A))$ is surjective.

Now let $n = 2k$. It is a classical result of Milnor and Kervaire in [10] that $B(A)$ can be changed to a contractible manifold by a sequence of surgeries on classes in $\pi_{k+1}(B(A))$. We just saw that $\pi_{k+1}(A)$ maps onto $\pi_{k+1}(B(A))$, so we can represent these classes by $(k+1)$-spheres in A (and by general position, embedding these spheres comes for free).

Thus, we can do surgery to A to obtain A' such that $B(A')$ is contractable. We still have a link immersion $f : \amalg^\nu D^{n+1} \looparrowright B(A')$ extending $\mathcal{L} : \amalg^\nu S^n \hookrightarrow S^{n+2} = \partial B(A')$. The h-cobordism theorem implies $B(A') \cong D^{n+3}$, but the induced diffeomorphism $\alpha : S^{n+2} = \partial B(A') \cong S^{n+2}$ can be different from $Id|_{S^{n+2}}$. However, $Id|_{S^{n+2}}$ can be extended to a homeomorphism $\beta : B(A') \cong D^{n+3}$, that is differentiable in the complement of a single point in $B(A') - f(\amalg^\nu D^{n+1})$. Now $\beta \circ f$ is the desired link immersion.

Consider $n = 2k - 1$. Form the connected sum

$$A' := A\# - (B(A) \cup_{S^{n+2}} D^{n+3})$$

such that

$$B(A') \cong B(A)\# - (B(A) \cup_{S^{n+2}} D^{n+3})$$

has vanishing signature if $k + 1$ is even and vanishing Arf invariant if $k + 1$ is odd. Again, (a) and (b) follow from [10]. As before, $\pi_{k+1}(A') \to \pi_{k+1}(B(A'))$ is surjective. Using immersion theory, we can represent all elements of $H_{k+1}(B(A'))$ by immersions $S^{k+1} \looparrowright A' \subset \mathcal{S}_0(f)$. This impies (c) and finishes the proof of (ii). $\square$

In the second case general position does not give embeddings $S^{k+1} \hookrightarrow A'$ and we have to do more work in the next section.

4. Odd dimensional links

Let $n = 2k - 1 > 1$ and $\mathcal{L} : S^n \amalg \cdots \amalg S^n \hookrightarrow S^{n+2}$ be a ν-component link. Again we assume that all its components are slice as knots. By 3.4(ii) there is a k-connected manifold B^{n+3} with boundary $\partial B = S^{n+2}$ admitting a link immersion

$$f : D^{n+1} \amalg \cdots \amalg D^{n+1} \looparrowright B$$

extending $\mathcal{L}$. Moreover, a hyperbolic basis e_i, e_i' of $\pi_{k+1}B$ can be represented by immersions

$$\alpha_i, \alpha_i' : S^{k+1} \looparrowright B - f(\amalg^\nu D^{n+1}).$$

We want to replace these immersions by embeddings

$$\beta_i, \beta_i' : S^{k+1} \hookrightarrow B$$

which realize the algebraic intersections from 3.4(ii)(a) as geometric intersections. Thus, we want $\beta_i \cap \beta_i'$ to consist of exactly one point for every i, and these points should be the only intersections among the β_i, β_i'. The standard procedure for achieving this is the Whitney trick (see [14] and [15]). The algebraic intersection property 3.4(ii)(a) implies the following: after possibly introducing further self-intersections to the α_i, α_i', we may assume that there are framed Whitney disks W_j such that the Whitney moves along the W_j lead from the α_i, α_i' to the embeddings β_i, β_i'. Surgery on the β_i gives a $(k + 1)$-connected manifold (by 3.4(ii)(b) the β_i have trivial normal bundles). By Poincaré duality and the h-cobordism theorem this manifold is D^{n+3}, see [10]. But the W_j may intersect our link immersion f and so f does not survive the surgeries. This

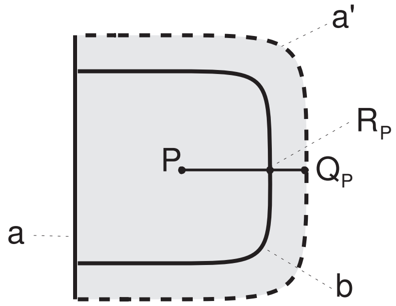

FIGURE 2. The Whitney disk W.

very failure is measured by an obstruction in $\Gamma_{n+3}(\mathbb{Z}[G] \to \mathbb{Z})$. Similar obstructions appear in the concordance classification of boundary links in [3] and detect non-slice links. However, boundary links are still link homotopically trivial (by [18]). The proof makes use of a procedure called *symmetric surgery*, and this will be useful in the present situation. This produces additional self-intersections in our link immersion and kills the above obstruction. We will use symmetric surgery in disguise of the following result.

Theorem 4.1. *For $1 \leq j \leq \nu$ let V_j^{2k} be the connected sum of D^{2k} with a finite number of copies of $S^k \times S^k$. Suppose that $\mathcal{L}$ extends to a link immersion*

$$g : V_1 \amalg \cdots \amalg V_\nu \looparrowright D^{n+3}.$$

Let $a_i, a_i' : S^k \hookrightarrow V_1 \amalg \cdots \amalg V_\nu$ be representing the union of the standard hyperbolic bases for the $H_k(V_j)$. If there are immersions $D^{k+1} \looparrowright D^{n+3}$ extending the $g \circ a_i, g \circ a_i'$ and mapping the interior of D^{k+1} disjoint from g, then there is a link immersion

$$f : D^{n+1} \amalg \cdots \amalg D^{n+1} \looparrowright D^{n+3}$$

extending $\mathcal{L}$.

Proof. The arguments of [18, section 3] imply this result as a special case: One starts with a hyperbolic basis a_i, a_i' for V_1 and does symmetric surgery on this first basis using the corresponding disks. Then one clears the *contraction* which is a $2k$-disk from all intersections with the remaining $(k+1)$-disks. One repeats this procedure dealing with one hyperbolic basis at a time. For more details see [18]. $\square$

Theorem 4.2. *There is a link immersion*

$$D^{n+1} \amalg \cdots \amalg D^{n+1} \looparrowright D^{n+3}$$

extending $\mathcal{L}$.

Proof. We have to study the Whitney moves from above in more detail. Let W be a 2-disk with three arcs a, a', and b as given in Figure 2. Let $U := W \times \mathbb{R}^k \times \mathbb{R}^k \subset B$ be

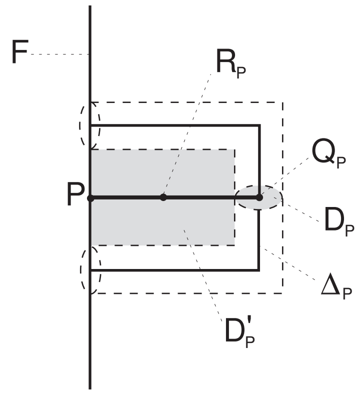

FIGURE 3. The disks D_P and D'_P.

such that

$$\begin{array}{rclcccccl} U \cap \text{ image } \alpha_i, \alpha'_i & = & & a & \times & \mathbb{R}^k & \times & \{0\} & \text{(the } a\text{-sheet)} \\ & & \cup & b & \times & \{0\} & \times & \mathbb{R}^k & \text{(the } b\text{-sheet).} \end{array}$$

We may assume that $F := f(D^{n+1} \amalg \cdots \amalg D^{n+1})$ intersects U in $S \times \mathbb{R}^k \times \mathbb{R}^k$ where S is a finite set of points in W. Note that we can assume that no point of S belongs to the selfintersections of f by general position. There are no points of S between a' and b if we choose a' close to b. Now the Whitney move replaces the a-sheet by the a'-sheet consisting of

$$a \times (\mathbb{R}^k - D^k) \times \{0\} \cup W \times S^{k-1} \times \{0\} \cup a' \times D^k \times \{0\}.$$

(Note that a'-sheet $\cap$ b-sheet $= \emptyset$.) However, the a'-sheet intersects F in $S \times S^{k-1} \times \{0\}$. For every $P \in S$ pick an embedded arc γ_P in W connecting P with a point Q_P on a'. We can assume that the γ_P are disjoint and meet b in single points R_P. Then

$$\Delta_P := \gamma_P \times S^{k-1} \times \{0\} \cup \{Q_P\} \times D^k \times \{0\}$$

is a k-disk inside the a'-sheet bounding $\{P\} \times S^{k-1} \times \{0\}$. Now thicken Δ_P normal to the a'-sheet to obtain $\Delta_P \times D^{k+1} \subset B$. Then we can do ambient surgery on $\{P\} \times S^{k-1} \times \{0\} \subset F$ and replace $\partial\Delta_P \times D^{k+1}$ by $\Delta_P \times S^k$. This changes F to the connected sum $F \# S^k \times S^k$. Note that there are $(k+1)$-disks D_P and D'_P bounding the standard hyperbolic basis of $S^k \times S^k$ such that $D_P \cap a'$-sheet $= \{Q_P\}$ and $D'_P \cap b$-sheet $= \{R_P\}$. Moreover, except for Q_P and R_P the interior of the two disks miss F and the β_i, β'_i. Here $D_P = Q_P \times D^{k+1} \subset \Delta_P \times D^{k+1}$, and D'_P is constructed as a subdisk of $\gamma_P \times D^k \times \{0\} \subset U$, see Figure 3. Applying this procedure to all our Whitney disks W_j and all the intersections of them with F, we obtain a new link immersion

$$g : V_1 \amalg \cdots \amalg V_\nu \looparrowright B$$

where the V_i are connected sums of D^{n+1} with copies of $S^k \times S^k$. We can now do the Whitney moves to get from the α_i, α_i' to the β_i, β_i'. Then g misses the β_i, β_i', and we have embedded $(k+1)$-disks D_j bounding the standard hyperbolic basis of $\oplus \pi_k V_r$. The interior of the D_j miss g and intersect the β_i, β_i' in points. Recall that β_i and β_i' meet exactly in one point. Thus, for every intersection point T of one of the D_j with a β_i we can add a push-off of the dual sphere β_i' to D_j joined by a tube around an arc in β_i from T to $\beta_i \cap \beta_i'$. (This is a standard trick in four dimensional topology, see [7].) Therefore, we find new immersed disks E_j for the hyperbolic basis that miss the β_i. (Of course, we produce a lot of intersections among the E_j.) Finally, we can do surgery on the β_i and change B to a contractable manifold B'. Since the β_i are disjoint from g, we still have

$$g : V_1 \amalg \cdots \amalg V_\nu \looparrowright B'.$$

Moreover, we find the E_j again in B'. Using the argument that finished the proof of 3.4(i) we can arrange $B' = D^{n+3}$. The statement follows now from 4.1. $\square$

5. Stratified triads

In this section we will set up some notation that will be used to describe *stratified handles*. Since a product formalism is used to define these handles, we will discuss to some extent how to smooth corners in this stratified setting. However, we will allow some corners because it will make the smoothing simpler in our context.

A smooth n-manifold with *corners* in codimension k is a Hausdorff space X^n that is locally modeled on

$$\mathbb{R}^n_k := (\mathbb{R}_+)^k \times \mathbb{R}^{n-k},$$

where $\mathbb{R}_+ = [0, \infty)$. In other words, there are charts $\psi_i : U_i \to \mathbb{R}^n_k$ defined on an open cover $\{U_i\}$ of X such that the compositions

$$\psi_i \circ \psi_j^{-1} : \psi_j(U_i \cap U_j) \to \psi_i(U_i \cap U_j)$$

are smooth. (A map $f : V \to \mathbb{R}^m$ for $V \subset \mathbb{R}^n_k$ is called smooth if it has a smooth extension to an open neighborhood of V in $\mathbb{R}^n$.) The product of manifolds with corners in codimension k and k' inherits the structure of a manifold with corners in codimension $k + k'$. There is a more refined notion of manifolds with corners to the effect that boundaries stay in the same category, but we will not need it here. A manifold with corners has still a tangent bundle, and so there are notions of embeddings and immersions.

A *triad* of dimension n is a n-manifold X^n with corners in codimension 2 and subspaces $\partial_0 X$ and $\partial_1 X$ of X satisfying the following condition: there are charts $\psi_i : U_i \to \mathbb{R}^n_2$ for an open cover $\{U_i\}$ of X such that

$$\begin{aligned} \psi_i^{-1}(\mathbb{R}_+ \times \{0\} \times \mathbb{R}^{n-2}) &= \partial_0 X \cap U, \\ \psi_i^{-1}(\{0\} \times \mathbb{R}_+ \times \mathbb{R}^{n-2}) &= \partial_1 X \cap U. \end{aligned}$$

So $\partial_0 X$ and $\partial_1 X$ are $(n-1)$-manifolds with common boundaries. We will frequently write $(X, \partial_0 X, \partial_1 X)$ for a triad. In this notation, the first example of a triad is $(\mathbb{R}^n_2, \mathbb{R}_+ \times \{0\} \times$

$\mathbb{R}^{n-2}, \{0\} \times \mathbb{R}_+ \times \mathbb{R}^{n-2})$. Unfortunately, so far we have defined only triads of dimension ≥ 2. A 0-dimensional triad X is by definition just a collection of points with $\partial_0 X = \partial_1 X = \emptyset$. A 1-dimensional triad is a 1-manifold X with boundary $\partial X = \partial_0 X \cup \partial_1 X$, where the union is disjoint. Note that, in particular, every manifold M can be viewed as a triad by setting $\partial_0 M := \partial M$ and $\partial_1 M := \emptyset$ or vice versa. So everything discussed below will also apply to manifolds.

Recall that a collection of subspaces V_i of a vector space W is said to be in *general position* if the diagonal map $W \to \bigoplus W/V_i$ is surjective. An immersion $f : X \looparrowright Y$ of triads is said to be *generic* if for every $y \in Y$ the following holds: let $x_1, \ldots, x_p$ be the preimages of y under f; then the vector spaces $df(T_{x_i})$ are in general position in $T_y Y$. Of course, embeddings are generic. A generic immersion f is called *proper* if the following is satisfied: for $y \in \partial_j Y$ the preimages $x_1, \ldots, x_p$ lie in $\partial_j X$, and the subspaces $df(T_{x_1} X), \ldots, df(T_{x_p} X), T_y(\partial_j Y)$ are in general position in $T_y Y$ (here $j = 0, 1$). If $y \in \partial_0 Y \cap \partial_1 Y$, then $df(T_{x_1} X), \ldots, df(T_{x_p} X), T_y(\partial_0 Y)$, and $T_y(\partial_1 Y)$ have to be in general position in $T_y Y$.

A *subtriad* S of a triad X is a triad $S \subset X$ such that the inclusion is a proper embedding. A collection $\{\mathcal{S}_r X | r = 0, \ldots, l\}$ of subtriads of a triad X is called a *stratification* if the following hold:

(i) X is the disjoint union of the $\mathcal{S}_r X$,
(ii) $\mathcal{S}^r X := \mathcal{S}_0 X \cup \cdots \cup \mathcal{S}_r X$ is open in X for all $r = 0, \ldots, l$.

$\mathcal{S}_r X$ is called the *stratum* of *depth* r. We will call $X = (X, \{S_r X\})$ a *stratified triad* or an *s-triad*. Note that this also stratifies $\partial_0 X$ and $\partial_1 X$. There are more sophisticated definitions of stratifications, but this one will serve our purposes. An embedding $f : X \hookrightarrow Y$ of s-triads is called *stratified* or an *s-embedding* if $\mathcal{S}_r X = f^{-1}(\mathcal{S}_r Y)$ for all r.

Let $(X, \{\mathcal{S}_r X\})$ and $(Y, \{\mathcal{S}_r Y\})$ be s-triads. We want to give $X \times Y$ the structure of an s-triad. As spaces we define

$$\begin{array}{rcll} \partial_j(X \times Y) & := & \partial_j X \times Y \cup X \times \partial_j Y & \text{for } j = 0, 1, \\ \mathcal{S}_r(X \times Y) & := & \bigcup_{i+j=r} \mathcal{S}_i X \times \mathcal{S}_j Y & \text{for } r \geq 0. \end{array}$$

Note that this comes from the usual definition of the product of pairs. In particular, we have

$$(X \times Y, \partial_j(X \times Y)) = (X, \partial_j X) \times (Y, \partial_j Y) \quad \text{for } j = 0, 1.$$

The product of the smooth structures on X and Y gives corners in codimension 4. So we have to smooth some of them. A way to do this is to compose the product charts with a fixed homeomorphism between $\mathbb{R}^n_4$ and $\mathbb{R}^n_2$. We will specify a homeomorphism in the proof of 5.1 and consider products always with this smooth structure. If $f : X \looparrowright X'$ and $g : Y \looparrowright Y'$ are proper generic immersions, then $f \times g : X \times Y \looparrowright X' \times Y'$ is also a proper generic immersion. However, if f or g fail to be proper, the situation is more complicated. Note that, even before smoothing, $O(X \times Y) := X \times Y - (\partial_0 X \times \partial_0 Y \cup \partial_1 X \times \partial_1 Y)$ is an s-triad, see Figure 4.

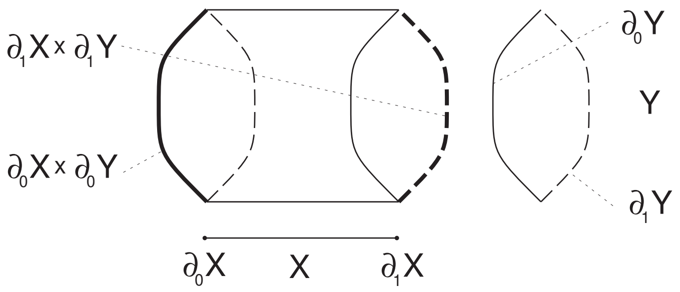

FIGURE 4. A product of triads.

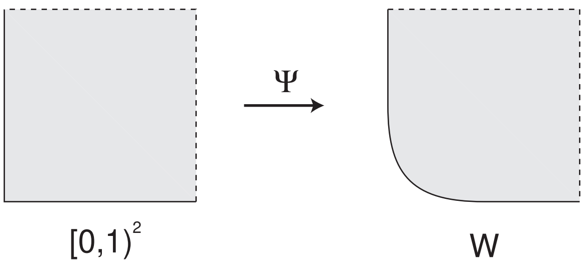

FIGURE 5. Smoothing corners.

Proposition 5.1. *Let X and Y be compact s-triads. Then there is an s-embedding $\iota : X \times Y \hookrightarrow O(X \times Y)$. Here ι can be taken to be the identity away from an arbitrarily small neighborhood of $\partial_0 X \times \partial_0 Y \cup \partial_1 X \times \partial_1 Y$. If $f : X \hookrightarrow X'$ and $g : Y \hookrightarrow Y'$ are s-embeddings where $f^{-1}(\partial_i X') = g^{-1}(\partial_i Y') = \emptyset$ for $i = 0, 1$, then $(f \times g) \circ \iota : X \times Y \hookrightarrow X' \times Y'$ is an s-embedding.*

Proof. Let $W^2 \subset [0,1)^2 - (0,0)$ and $\Psi : [0,1)^2 \to W$ be a homeomorphism satisfying the following conditions:

(i) W is diffeomorphic to $[0,1) \times (0,1)$,
(ii) $\Psi|_{[0,1)^2 - (0,0)}$ is a diffeomorphism onto its image,
(iii) $\Psi \equiv id$ outside of $[0, 0.5)^2$.

For $j = 0, 1$ and $Z = X, Y$ let $\partial_j Z \times [0,1) \cong C_j^Z \subset Z$ be a collar of $\partial_j Z$ in Z such that

(i) $C_j^Z \cap \partial_{1-j} Z$ is collar of $\partial\partial_j Z$,
(ii) $C_j^Z \cap \mathcal{S}_r Z$ is a collar of $\mathcal{S}_r \partial_j Z \times [0,1)$.

Such collars can be constructed successively over the strata, starting at the deepest stratum. Here we use the compactness of X and Y. Let $O_0 := X \times Y - \partial_0 X \times \partial_0 Y$. In a first step, we construct a map $\iota_0 : X \times Y \hookrightarrow O_0$ and define ι_0 on $C_0^X \times C_0^Y$ by

$$\partial_0 X \times \partial_0 Y \times [0,1)^2 \xrightarrow{id \times \Psi} \partial_0 X \times \partial_0 Y \times W$$

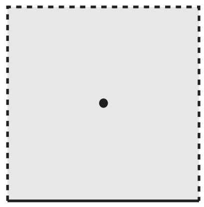

FIGURE 6. The stratified disk.

conjugated with $C_0^X \times C_0^Y \cong \partial_0 X \times \partial_0 Y \times [0,1)^2$. This can be extended to $X \times Y$ by the identity. Note that ι_0 respects the stratification. Now define $\iota_1 : O_0 \to O(X \times Y)$ as follows: On $(C_1^X \times C_1^Y) \cap O_0$ we take the restriction of

$$\partial_1 X \times \partial_1 Y \times [0,1)^2 \xrightarrow{id \times \Psi} \partial_1 X \times \partial_1 Y \times W$$

conjugated with $C_1^X \times C_1^Y \cong \partial_1 X \times \partial_1 Y \times [0,1)^2$. Again this can be extended by the identity. Now $\iota := \iota_1 \circ \iota_0$ is the desired map.

We can apply this construction to $\mathbb{R}^2_2 \times \mathbb{R}^2_2$ (where the collars exist even though $\mathbb{R}^2_2$ is not compact). Then $\iota(\mathbb{R}^2_2 \times \mathbb{R}^2_2) \cong \mathbb{R}^4_2$, which provides homeomorphisms $\mathbb{R}^n_4 \cong \mathbb{R}^n_2$. If we use this to construct smooth product structures, then $\iota : X \times Y \hookrightarrow O(X \times Y)$ is an s-embedding. $\square$

6. IMMERSIONS IN CODIMENSION TWO

Given an immersion $f : M^n \looparrowright N^{n+2}$, one may study its complement $N - f(M)$. In particular, we are interested in the homotopy groups of the complement and their behavior under certain moves of M in N. We will study these moves using stratified handles as defined in [17]. Parts of this section are taken from there. Our main contribution is 6.9.

Definition 6.1.

(i) *The stratified disk* is the triad

$$\mathcal{D}^2 := (D^1_c, \{-1\}, \{1\}) \times (D^1_t, \emptyset, \{-1, 1\}).$$

It is stratified by $\mathcal{S}_0(\mathcal{D}^2) := \mathcal{D}^2 - \{(0,0)\}$ and $\mathcal{S}_1(\mathcal{D}^2) := \{(0,0)\}$.

(ii) The *handle of index λ in dimension n* can be considered as the s-triad

$$H^n_\lambda := (D^\lambda_c, S^{\lambda-1}_c, \emptyset) \times (D^{n-\lambda}_t, \emptyset, S^{n-\lambda-1}_t)$$

with the trivial stratification.

(iii) The *stratified handle of index (r, λ) in dimension n* is the s-triad

$$\mathcal{H}^n_{r,\lambda} := (\mathcal{D}^2)^r \times H^{n-2r}_\lambda.$$

Here the index c stands for core, and the index t marks the directions in which we have to thicken the core to obtain the whole handle. We will use this structure in 6.7. One can define more general stratified handles, but in codimension 2 these are sufficient.

Let $f : X^n \looparrowright Y^{n+2}$ be a proper generic immersion. We can form then the *multiple point stratification* of Y by setting

$$\mathcal{S}_r(f) := \{y \in Y | y \text{ has precisely } r \text{ preimages under } f\}, r \geq 0.$$

We will write $MPS(f)$ for Y with this stratification. The points in $S_r(f)$ are called *r-fold* points of f. Given generic proper immersions $f_i : X_i^{n_i} \looparrowright Y_i^{n_i+2}$, we can form a $\star$-product

$$f_0 \star f_1 : X_0 \times Y_1 \amalg Y_0 \times X_1 \looparrowright Y_0 \times Y_1$$

where $f_0 \star f_1 = f_0 \times id_{Y_1} \amalg id_{Y_0} \times f_1$. From section 5 we know that $f_0 \star f_1$ is a proper generic immersion. Counting preimages, we see that

$$MPS(f_0 \star f_1) = MPS(f_0) \times MPS(f_1).$$

Let $incl : \{(0,0)\} \to D_c^1 \times D_t^1$ be the inclusion. Then $\mathcal{D}^2 = MPS(incl)$. Moreover, $H_\lambda^n = MPS(\emptyset \to H_\lambda^n)$, and hence using the $\star$-product all stratified handles are $MPS(f)$ for some proper generic immersion f.

Lemma 6.2. *Let $f : M^n \looparrowright N^{n+2}$ be a proper generic immersion of compact manifolds. If $y \in \mathcal{S}_r(f) - \partial N$, then y has a neighborhood $U \cong (\mathcal{D}^2)^r \times D^{n+2-2r}$. This diffeomorphism respects the stratifications (but there are no triad structures).*

Proof. All self-intersections of $f(M)$ are transverse. So y has a neighborhood $U \cong (\mathbb{R}^2)^r \times \mathbb{R}^{n+2-2r}$ such that

$$f(M) \cap U \cong \bigcup_{j=1}^{r} (\mathbb{R}^2)^{j-1} \times \{0\} \times (\mathbb{R}^2)^{r-j} \times \mathbb{R}^{n+2-2r}.$$

As $\mathcal{D}^2 \subset \mathbb{R}^2$, $D^{n+2-2r} \subset \mathbb{R}^{n+2-2r}$, the product $(\mathcal{D}^2)^r \times D^{n+2-2r}$ defines a smaller neighborhood of y. □

Lemma 6.3. *Let $X = (X, \partial_0 X, \partial_1 X)$ be any triad. Then*

$$\partial_0(\mathcal{D}^2 \times X) \cong \partial_1(\mathcal{D}^2 \times X),$$

and the diffeomorphism can be taken to be the identity on $\partial\partial_0(\mathcal{D}^2 \times X) = \partial\partial_1(\mathcal{D}^2 \times X)$. Here we ignore stratifications.

Proof. Let $Z := (D_t^1, \emptyset, \{-1,1\}) \times X$. Then $\mathcal{D}^2 \times X \cong (D_c^1, \{-1\}, \{1\}) \times Z$ and hence

$$\begin{aligned} \partial_0(\mathcal{D}^2 \times X) &\cong \{-1\} \times Z \cup D_c^1 \times \partial_0 Z, \\ \partial_1(\mathcal{D}^2 \times X) &\cong \{1\} \times Z \cup D_c^1 \times \partial_1 Z. \end{aligned}$$

From here the diffeomorphism is easily produced. □

Lemma 6.4. *Let $r \geq 2$. There are compact manifolds U^n and V^{n+2} and proper generic immersions $g_j : U \looparrowright V$ such that $\partial_j \mathcal{H}_{r,\lambda}^{n+3} \cong MPS(g_j)$ for $j = 0,1$. Moreover, $g_0|_{\partial U} = g_1|_{\partial U}$ and the diffeomorphism $\partial\partial_j \mathcal{H}_{r,\lambda}^{n+3} \cong MPS(g_j|_{\partial U})$ is independent of j.*

Proof. Let $f : Y \looparrowright \mathcal{H}^{n+1}_{r-1,\lambda}$ be a proper generic immersion such that $\mathcal{H}^{n+1}_{r-1,\lambda} = MPS(f)$. Hence $\mathcal{H}^{n+3}_{r,\lambda} = \mathcal{D}^2 \times \mathcal{H}^{n+1}_{r-1,\lambda} = MPS(incl \star f)$, where

$$incl \star f : \{(0,0)\} \times \mathcal{H}^{n+1}_{r-1,\lambda} \amalg \mathcal{D}^2 \times Y \looparrowright \mathcal{H}^{n+3}_{r,\lambda}.$$

For $j = 0, 1$ let

$$\begin{array}{rcl} U_j & := & \partial_j(\{(0,0)\} \times \mathcal{H}^{n+1}_{r-1,\lambda} \amalg \mathcal{D}^2 \times Y) \\ & \cong & \partial_j \mathcal{H}^{n+1}_{r-1,\lambda} \amalg \partial_j(\mathcal{D}^2 \times Y), \\ V_j & := & \partial_j \mathcal{H}^{n+3}_{r,\lambda}, \\ g_j & := & (incl \star f)|_{U_j} : U_j \looparrowright V_j. \end{array}$$

By Lemma 6.3, $U_0 \cong U_1, V_0 \cong V_1$ and $g_0|_{\partial U_j} = g_1|_{\partial U_j}$. □

Let M^{n-1} be a manifold and $\varphi : \partial_0 H^n_\lambda \hookrightarrow M$ an embedding. Recall that surgery then produces the manifold

$$M - \varphi(int(\partial_0 H^n_\lambda)) \cup_{\partial\partial_0 H^n_\lambda = \partial\partial_1 H^n_\lambda} \partial_1 H^n_\lambda.$$

We will study an analogous procedure in the stratified setting, using the stratified handles $\mathcal{H}^n_{r,\lambda}$. Now let M^{n-1} be a stratified manifold and

$$\varphi : \partial_0 \mathcal{H}^n_{r,\lambda} \hookrightarrow M$$

an s-embedding. We can then form

$$M^\varphi := M - int(\varphi(\partial_0 \mathcal{H}^n_{r,\lambda})) \cup_{\partial\partial_0 \mathcal{H}^n_{r,\lambda} = \partial\partial_1 \mathcal{H}^n_{r,\lambda}} \partial_1 \mathcal{H}^n_{r,\lambda},$$

and the stratifications fit together to form a stratification $\{\mathcal{S}^\varphi_M\}$ of M. (Here we have to give M^φ a suitable smooth structure, but this can easily be done using collars, which respect the stratification.)

Proposition 6.5. *Let $f_0 : M^n \looparrowright N^{n+2}$ be a proper generic immersion. Let $r \geq 2$ and $\varphi : \partial_0 \mathcal{H}^{n+3}_{r,\lambda} \hookrightarrow MPS(f_0)$ be a stratified embedding. Then a proper generic immersion $f_1 : M^n \looparrowright N^{n+2}$ exists such that*

$$MPS(f_0)^\varphi = MPS(f_1).$$

Proof. We use the notation from 6.4. Thus, φ gives an s-embedding $MPS(g_0) \hookrightarrow MPS(f_0)$. On the level of manifolds, we denote this map by $\psi : V \hookrightarrow N$. Now let $x \in U$ and $O_U \subset U$ be an open neighborhood of x such that $g_0|_{O_U}$ is an embedding. Then $\psi(g_0(O_U)) \subset N - \mathcal{S}_0(f_0)$. Moreover, a dense subset of $\psi(g_0(O_U))$ is contained in $\mathcal{S}_1(f_0)$. Hence there is a unique open set $O_M \subset M$ such that $\psi(g_0(O_U)) = f_0(O_M)$. This gives rise to an embedding $U \hookrightarrow M$ and the commutative diagram

$$\begin{tikzcd} U \arrow[r, "g_0"] \arrow[d] & V \arrow[d, "\psi"] \\ M \arrow[r, "f_0"] & N \end{tikzcd}.$$

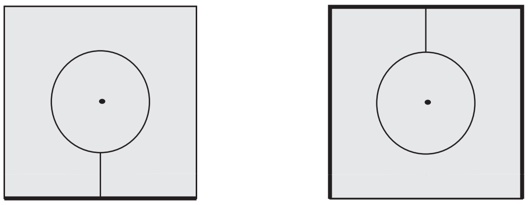

FIGURE 7. The images of σ and τ.

By means of 6.4, replacing $\varphi(\partial_0\mathcal{H}^{n+3}_{r,\lambda})$ by $\partial_1\mathcal{H}^{n+3}_{r,\lambda}$ has the same effect as replacing g_0 by g_1 in the above diagram. This defines f_1. $\square$

Define $\sigma : (D^1, S^0) \to (\mathcal{S}_0\mathcal{D}^2, \mathcal{S}_0\partial_0\mathcal{D}^2)$ by

$$\sigma(t) := \begin{cases} (t,0) & : \quad -1 \leq t \leq -0.5 \\ (0.5\cos 2\pi t, 0.5\sin 2\pi t) & : \quad -0.5 \leq t \leq 0.5 \\ (-t,0) & : \quad 0.5 \leq t \leq 1 \end{cases}.$$

Observe that

$$\begin{array}{lclcccc} (D^{r+\lambda}, S^{r+\lambda-1}) & \cong & (D^1,S^0)^r & \times & (D^\lambda_c, S^{\lambda-1}_c) & \times & (\{0\},\emptyset), \\ (\mathcal{S}_0\mathcal{H}^n_{r,\lambda}, \mathcal{S}_0\partial_0\mathcal{H}^n_{r,\lambda}) & \cong & (\mathcal{S}_0\mathcal{D}^2, \mathcal{S}_0\partial_0\mathcal{D}^2)^r & \times & (D^\lambda_c, S^{\lambda-1}_c) & \times & (D^{n-\lambda-2r}_t, \emptyset). \end{array}$$

Define

$$\alpha_0 : (D^{r+\lambda}, S^{r+\lambda-1}) \to (\mathcal{S}_0\mathcal{H}^n_{r,\lambda}, \mathcal{S}_0\partial_0\mathcal{H}^n_{r,\lambda})$$

by $\sigma^r \times id_{D^\lambda_c} \times incl_{\{0\}}$, composed with the above diffeomorphisms. Let $\beta_0 := \alpha_0|_{S^{r+\lambda-1}}$. Similarly we can construct first $\tau : (D^1, S^0) \to (\mathcal{S}_0\mathcal{D}^2, \mathcal{S}_0\partial_1\mathcal{D}^2)$ and then

$$\alpha_1 : (D^{n-\lambda-r}, S^{n-\lambda-r-1}) \to (\mathcal{S}_0\mathcal{H}^n_{r,\lambda}, \mathcal{S}_0\partial_1\mathcal{H}^n_{r,\lambda}).$$

Again we set $\beta_1 := \alpha_1|_{S^{n-\lambda-r-1}}$.

Lemma 6.6. *There are homotopy equivalences of pairs*

$$\begin{array}{lcl} (\mathcal{S}_0\partial_0\mathcal{H}^n_{r,\lambda} \cup_{\beta_0} D^{r+\lambda}, \mathcal{S}_0\partial_0\mathcal{H}^n_{r,\lambda}) & \to & (\mathcal{S}_0\mathcal{H}^n_{r,\lambda}, \mathcal{S}_0\partial_0\mathcal{H}^n_{r,\lambda}) \\ (\mathcal{S}_0\partial_1\mathcal{H}^n_{r,\lambda} \cup_{\beta_1} D^{n-\lambda-r}, \mathcal{S}_0\partial_1\mathcal{H}^n_{r,\lambda}) & \to & (\mathcal{S}_0\mathcal{H}^n_{r,\lambda}, \mathcal{S}_0\partial_1\mathcal{H}^n_{r,\lambda}) \end{array}$$

induced by α_0 and α_1.

Proof. For $\mathcal{H}^n_{r,\lambda} = \mathcal{D}^2$ or H^n_λ the statement is clearly true. Note for the general case that

$$\begin{array}{cl} & (\mathcal{S}_0\partial_0\mathcal{H}^n_{r,\lambda} \cup_{\beta_0} D^{r+\lambda}, \mathcal{S}_0\partial_0\mathcal{H}^n_{r,\lambda}) \\ = & (\mathcal{S}_0\partial_0\mathcal{D}^2 \cup_{\sigma|_{D^0}} D^1, \mathcal{S}_0\partial_0\mathcal{D}^2)^r \times (\partial_0 H^n_\lambda \cup_{S^{\lambda-1}} D^\lambda, \partial_0 H^n_\lambda) \end{array}$$

is just a product of relative CW-complexes. The product of homotopy equivalences (of pairs) is a homotopy equivalence. This implies the first homotopy equivalence, and the second follows from the same formalism. $\square$

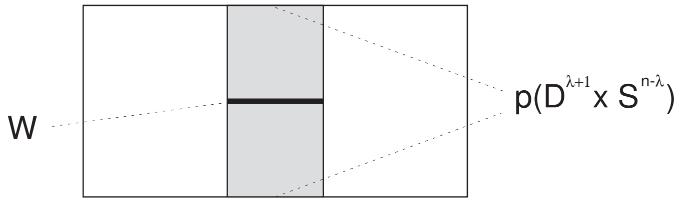

FIGURE 8. The disk W in $\mathcal{S}_0\partial_0\mathcal{H}^{n+3}_{2,\lambda}$

Pick $(-1,0) \in D^1_c \times D^1_t = \mathcal{D}^2$ as a basepoint. Then σ gives the meridian to $\mathcal{S}_1\mathcal{D}^2$ in $\mathcal{D}^2$, an element $[\sigma] \in \pi_1(\mathcal{S}_0\mathcal{D}^2)$. From $\mathcal{S}_0\mathcal{D}^2 \times \{(-1,0)\}$ and $\{(-1,0)\} \times \mathcal{S}_0\mathcal{D}^2$ in $\mathcal{S}_0(\mathcal{D}^2)^2$ we have meridians $[\sigma_1], [\sigma_2] \in \pi_1(\mathcal{S}_0(\mathcal{D}^2)^2)$. The map $\sigma \times \sigma : D^1 \times D^1 \to \mathcal{S}_0(\mathcal{D}^2)^2$ shows that $[\sigma_1]$ and $[\sigma_2]$ commute in $\pi_1(\mathcal{S}_0(\mathcal{D}^2)^2)$.

Let M^n and N^{n+2} be oriented manifolds and $f : M \looparrowright N$ a proper generic immersion. Let $U \cong \mathcal{D}^2 \times D^n$ be a neighborhood of some $y \in \mathcal{S}_1(f)$, compare 6.2. Then, up to an orientation, a meridian $m \in \pi_1(\mathcal{S}_0(f))$ to y is determined by $\sigma : D^1 \to \mathcal{D}^2$ and an arc connecting the basepoints. We can fix orientations of $\mathcal{D}^2$ and D^n to determine the orientation of m. A double point $y \in \mathcal{S}_2(f)$ determines two meridians $m_1, m_2 \in \pi_1(\mathcal{S}_0(f))$ via a neighborhood $U \cong \mathcal{D}^2 \times \mathcal{D}^2 \times D^{n-2}$ of y and an arc connecting the basepoints. Then we have

$$[m_1, m_2] = 1 \in \pi_1(\mathcal{S}_0(f)).$$

We will say that the double point y *represents* this relation. (Note that this relation holds for all choices of an arc if it holds for one.)

Pick a basepoint $x_0 \in \partial_0 H^{n-4}_\lambda$. The product of x_0 with our basepoint of $\mathcal{D}^2$ gives a basepoint of $\partial_0\mathcal{H}^n_{2,\lambda}$. Now, $\beta_0 := \alpha_0|_{S^{\lambda+1}}$ gives an element $[\beta_0] \in \pi_{\lambda+1}(\mathcal{S}_0\partial_0\mathcal{H}^n_{2,\lambda})$. We also find meridians $[\sigma_1], [\sigma_2] \in \pi_1(\mathcal{S}_0\partial_0\mathcal{H}^n_{2,\lambda})$. Note that we have $\beta_0 = [[\sigma_1],[\sigma_2]] \in \pi_1(\mathcal{S}_0\partial_0\mathcal{H}^n_{2,0})$ for the stratified handle of index $(2,0)$.

Lemma 6.7. *Let $\lambda \geq 1$. There exists an embedding $p : D^{\lambda+1} \times D^{n-\lambda+1} \hookrightarrow \mathcal{S}_0\partial_0\mathcal{H}^{n+3}_{2,\lambda}$ such that the following hold:*

(i)
$$\partial\partial_0\mathcal{H}^{n+3}_{2,\lambda} \cap p(D^{\lambda+1} \times D^{n-\lambda+1}) = p(D^{\lambda+1} \times S^{n-\lambda}).$$

(ii)
$$\begin{aligned}[\beta_0] &= (1 - [\sigma_1] + [\sigma_1][\sigma_2] - [\sigma_2])[W] \\ &\in \pi_{\lambda+1}(\mathcal{S}_0\partial_0\mathcal{H}^{n+3}_{2,\lambda}, \mathcal{S}_0\partial_0\mathcal{H}^{n+3}_{2,\lambda} - p(D^{\lambda+1} \times D^{n-\lambda+1}))\end{aligned}$$

where $[W]$ is the class represented by $W := p(D^{\lambda+1} \times \{0\})$, see Figure 8.

Proof. Let $l^1_c := ([-1,-0.6), \{-1\}, \emptyset) \subset (D^1_c, \{-1\}, \{1\})$. Thus,

$$X := (l^1_c \times (D^1_t, \emptyset, \{-1,1\}))^2 \times (D^\lambda_c, S^{\lambda-1}_c, \emptyset) \times (D^{n-\lambda-1}_t, \emptyset, S^{n-\lambda-2}_t)$$

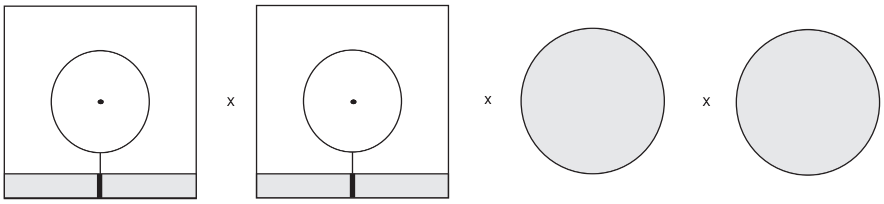

FIGURE 9. The subtriade X in $\mathcal{H}^{n+3}_{2,\lambda} = \mathcal{D}^2 \times \mathcal{D}^2 \times D^\lambda_c \times D^{n-\lambda-1}_t$

is a subtriad of $\mathcal{S}_0\mathcal{H}^{n+3}_{2,\lambda}$. Now $X \cong X^{\lambda+2}_c \times X^{n-\lambda+1}_t$ where

$$\begin{array}{lclcl} X^{\lambda+2}_c & := & (l^1_c)^2 & \times & (D^\lambda_c, S^{\lambda-1}_c, \emptyset), \\ X^{n-\lambda+1}_t & := & (D^1_t, \emptyset, \{-1,1\})^2 & \times & (D^{n-\lambda-1}_t, \emptyset, S^{n-\lambda-2}_t). \end{array}$$

Therefore,

$$\begin{array}{rcl} \partial_0 X & \cong & \partial_0 X^{\lambda+2}_c \times ((D^1_t)^2 \times D^{n-\lambda-1}_t) \\ & \cong & \partial_0 X^{\lambda+2}_c \times (D^{n-\lambda+1}_t). \end{array}$$

Let $W^{\lambda+1}$ be the $(\lambda+1)$-disk $[-1,-0.8] \times \{-1\} \times D^\lambda_c \subset \partial_0 X^{\lambda+2}_c$. The situation is sketched in Figure 9. The product of the shaded areas is X; the product of the thickened lines and the disk D^λ_c is $X^{\lambda+2}_c$. The disk W is given by the product of half of the first thickened line with D^λ_c.

The map p given by

$$D^{\lambda+1} \times D^{n-\lambda+1} \cong W \times D^{n-\lambda+1}_t \subset \partial_0 X \subset \partial_0 \mathcal{S}_0 \mathcal{H}^{n+3}_{2,\lambda}$$

satisfies (i). Observe that

$$(\sigma \times \sigma \times id_{D^\lambda_c})^{-1}(X^{\lambda+2}_c) \subset (D^1 \times D^1) \times D^\lambda_c \cong D^{\lambda+2}$$

consists of four disjoint copies of $X^{\lambda+2}_c$, each mapped homeomorphically onto $X^{\lambda+2}_c$ by $\sigma \times \sigma \times id_{D^\lambda_c}$. Hence $\beta_0^{-1}(W) \subset S^{\lambda+1}$ consists of four disks, each mapped homeomorphically onto W by β_0. In Figure 10, the product of the four shaded squares with D^λ_c represents the preimage of $X^{\lambda+2}_c$. The product of the four thickened vertical lines on the boundary of $D^1 \times D^1$ with D^λ_c gives the preimage of W under β_0. The boundary of $D^1 \times D^1$ maps to σ_1 and σ_2 as indicated. Thus we can understand the group elements picked up by arcs in $S^{\lambda+1}$ connecting the four disks to the basepoint. Now the formula in (ii) follows from the homotopy addition theorem, see [19, IV.6.1]. $\square$

Lemma 6.8. *Let $f : M^n \looparrowright N^{n+2}$ be a proper generic immersion such that the relation*

$$[m_1, m_2] = 1 \in \pi_1(\mathcal{S}_0(f))$$

is represented by a double point $y \in \mathcal{S}_2(f_0)$. If $\lambda \leq n-2$ and if $U \cong \mathcal{D}^2 \times \mathcal{D}^2 \times D^{n-2}$ is a neighborhood of y, then a stratified embedding $\varphi_0 : \partial_0 \mathcal{H}^{n+3}_{2,\lambda} \hookrightarrow U$ exists such that

$$(\varphi_{0*}(\sigma_j)) = m_j$$

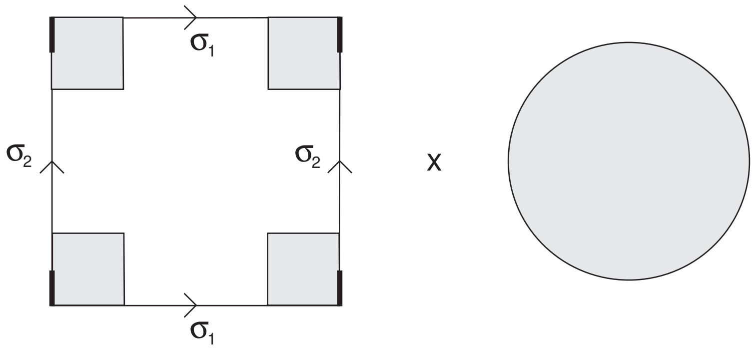

FIGURE 10. The preimage of $\sigma \times \sigma \times id_{D_c^\lambda}$ in $(D^1 \times D^1) \times D_c^\lambda$.

for $j = 0,1$.

Proof. Since $\lambda \leq n-2$, we have

$$\mathcal{H}_{2,\lambda}^{n+3} \cong \mathcal{H}_{2,\lambda}^{n+2} \times (D_t^1, \emptyset, \{-1,1\}).$$

This implies

$$\partial_0 \mathcal{H}_{2,\lambda}^{n+3} \cong \partial_0 \mathcal{H}_{2,\lambda}^{n+2} \times D_t^1.$$

Let $H_{2,\lambda}^{n-2} \hookrightarrow D^{n-2}$ be an embedding. Taking the product with $id_{\mathcal{D}^2}$ on the first two factors, we obtain an s-embedding

$$\psi : \mathcal{H}_{2,\lambda}^{n+2} \hookrightarrow U$$

(after smoothing by means of 5.1). Now shrink ψ slightly in a way that its image is in the interior of U, and extend $\psi|_{\partial_0 \mathcal{H}_{2,\lambda}^{n+2}}$ to an s-embedding

$$\varphi_0 : \partial_0 \mathcal{H}_{2,\lambda}^{n+2} \times D_t^1 \hookrightarrow U.$$

This can be done successively over the strata, starting at the stratum of depth 2. (From ψ we have inward directions at every point of $\partial_0 \mathcal{H}_{2,\lambda}^{n+2}$.) Again we have to smooth this map by 5.1. The last statement is clear from the construction. $\square$

Theorem 6.9. *Let* $n \geq 3$ *and* $f_0 : M^n \looparrowright N^{n+2}$ *be a proper generic immersion of oriented manifolds. Suppose that* $y \in \mathcal{S}_2(f_0)$ *is a double point representing the relation*

$$[m_1, m_2] = 1 \in \pi_1(\mathcal{S}_0(f_0)).$$

Choose an embedding $v : S^k \times D^{n+2-k} \hookrightarrow \mathcal{S}_0(f_0)$ *and let* $[v] \in \pi_k(\mathcal{S}_0(f_0))$ *be determined by* v *and a path connecting* v *to a basepoint. If* $2 \leq k < (n+1)/2$, *then a generic immersion* $f_1 : M^n \hookrightarrow N^{n+2}$ *exists such that* $\pi_j(\mathcal{S}_0(f_1)) \cong \pi_j(\mathcal{S}_0(f_0))$ *for* $j < k$ *and*

$$\pi_k(\mathcal{S}_0(f_1)) \cong \pi_k(\mathcal{S}_0(f_0))/(1 - m_1 + m_1 m_2 - m_2)[v].$$

Moreover, all relations in $\pi_1(\mathcal{S}_0(f_0))$ *represented by double points of* f_0 *are also represented by double points of* f_1.

Proof. Let $U \cong \mathcal{D}^2 \times \mathcal{D}^2 \times D^{n-2}$ be a neighborhood of y missing v. Set $\lambda := k-1$, and let $\varphi_0 : \partial_0 \mathcal{H}^{n+3}_{2,\lambda} \hookrightarrow U$ be an s-embedding as in 6.8. Let $p : D^{\lambda+1} \times D^{n-\lambda+1} \hookrightarrow \partial_0 \mathcal{H}^{n+3}_{2,\lambda}$ and $W, B \subset \partial_0 \mathcal{H}^{n+3}_{2,\lambda}$ be as in 6.7. By 6.7(i), any embedding $q : D^{\lambda+1} \times D^{n-\lambda+1} \hookrightarrow \mathcal{S}_0(f_0)$ that agrees with $\varphi_0 \circ p$ on a neighborhood of $S^\lambda \times D^{n-\lambda+1}$ and misses $\varphi_0(B)$ induces a new s-embedding $\varphi : \partial_0 \mathcal{H}^{n+3}_{2,\lambda} \hookrightarrow MPS(f_0)$ such that $\varphi \circ p = q$. The connected sum of $\varphi_0 \circ p$ with v inside $\mathcal{S}_0(f_0)$ along an arc gives the q we want. This implies that

$$(\varphi_*[W]) = [v] \in \pi_k(\mathcal{S}_0(f_0), \mathcal{S}_0 U).$$

(Here we choose the image of our basepoint in $\partial_0 \mathcal{H}^{n+3}_{2,\lambda}$ as the basepoint of $\mathcal{S}_0(f_0)$.) Hence by 6.7(ii) and 6.8 we have

$$(\varphi_*[\beta_0]) = (1 - m_1 + m_1 m_2 - m_2)[v] \in \pi_k(\mathcal{S}_0(f_0), \mathcal{S}_0 U).$$

As $\mathcal{S}_0 U \simeq S^1 \times S^1$ and hence $\pi_k(\mathcal{S}_0 U) = 0$, the same equation holds in $\pi_k(\mathcal{S}_0(f_0))$. By 6.5, there is a generic immersion $f_1 : M \looparrowright N$ such that $MPS(f_0)^\varphi \cong MPS(f_1)$. Now form

$$Z := \mathcal{S}_0(f_0) \cup_\varphi \mathcal{S}_0 \mathcal{H}^{n+3}_{2,\lambda}.$$

Then $\mathcal{S}_0(f_0)$ and $\mathcal{S}_0(f_1)$ are subspaces of Z. By 6.6, we have

$$\begin{aligned} (Z, \mathcal{S}_0(f_0)) &\simeq (\mathcal{S}_0(f_0) \cup_{\varphi \circ \beta_0} D^{\lambda+2}, \mathcal{S}_0(f_0)), \\ (Z, \mathcal{S}_0(f_1)) &\simeq (\mathcal{S}_0(f_1) \cup_{\varphi \circ \beta_1} D^{n-\lambda+1}, \mathcal{S}_0(f_1)). \end{aligned}$$

Since $k < (n+1)/2$, we have $n - \lambda + 1 > \lambda + 2 = k + 1$. So we find

$$\begin{aligned} \pi_j(\mathcal{S}_0(f_1)) &\cong \pi_j(Z) \cong \pi_j(\mathcal{S}_0(f_0)) \qquad \text{for} \quad j < k, \\ \pi_k(\mathcal{S}_0(f_1)) &\cong \pi_k(Z) \cong \pi_k(\mathcal{S}_0(f_0))/\varphi_*[\beta_0]. \end{aligned}$$

$\mathcal{S}_2(f_0)$ is of dimension $n - 2 \geq 1$. Thus, we can find double points representing relations in $\pi_1(\mathcal{S}_0(f_0))$ outside of U, and these double points are preserved in the construction. □

Proposition 6.10. *Let $n \geq 2$ and $f_0 : M^n \looparrowright N^{n+2}$ be a proper generic immersion of oriented manifolds. Pick a meridian $m \in \pi_1(\mathcal{S}_0(f_0))$ to a point $y \in \mathcal{S}_1(f_0)$ and choose $g \in \pi_1(\mathcal{S}_0(f_0))$ arbitrarily. Then there is a proper generic immersion $f_1 : M^n \looparrowright N^{n+2}$ such that*

$$\pi_1(\mathcal{S}_0(f_1)) \cong \pi_1(\mathcal{S}_0(f_0))/[m, m^g]$$

where the relation $[m, m^g]$ is represented by a double point $y \in \mathcal{S}_2(f_1)$. Moreover, the construction preserves double points of f_0.

Proof. We will use the stratified handle $\mathcal{H}^{n+3}_{2,0}$. Here

$$\begin{aligned} \partial_0 \mathcal{H}^{n+3}_{2,0} &= \partial_0(\mathcal{D}^2 \times \mathcal{D}^2) \times D^{n-1}_t \\ &= \{-1\} \times D^1_t \times \mathcal{D}^2 \times D^{n-1}_t \\ &\quad \cup \ \mathcal{D}^2 \times \{-1\} \times D^1_t \times D^{n-1}_t. \end{aligned}$$

Thus, $\partial_0 \mathcal{H}^{n+3}_{2,0}$ is just a boundary connected sum of two copies of $\mathcal{D}^2 \times D^n$. Now let $U \cong \mathcal{D}^2 \times D^n$ be a neighborhood of y. We can produce an s-embedding of $\partial_0 \mathcal{H}^{n+3}_{2,0}$ as follows : start with two disjoint s-embeddings $\mathcal{D}^2 \times D^n \hookrightarrow U$ and connect them by a tube to obtain an s-embedding

$$\varphi : \partial_0 \mathcal{H}^{n+3}_{2,0} \hookrightarrow MPS(f_0).$$

If our tube follows an arc corresponding to g, then we can arrange that

$$(\varphi_* \beta_0) = [m, m^g].$$

As in the proof of Theorem 6.9 we can now use 6.5 and 6.6 to produce a proper generic immersion $f_1 : M \looparrowright N$ such that $MPS(f_0)^\varphi \cong MPS(f_1)$ and

$$\pi_1(\mathcal{S}_0(f_1)) \cong \pi_1(\mathcal{S}_0(f_0))/[m, m^g].$$

Moreover, we see from $\mathcal{D}^2 \times \mathcal{D}^2 \times S^{n-2}_t \subset \partial_1 \mathcal{H}^{n+3}_{2,0}$ that the additional relation $[m, m^g]$ is represented by a double point of f_1. $\square$

Of course, the above is just a way of presenting finger moves using the stratified handle $\mathcal{H}^{n+3}_{2,0}$. This introduces new double points to an immersion. For $\lambda \geq 1$ we have used $\mathcal{H}^{n+3}_{2,\lambda}$ to change the double point set.

7. Nilpotent modules over the Milnor group

The *lower central series* of a group G is defined by $G_1 := G$ and $G_{k+1} := [G, G_k]$ for $k \geq 1$. A group G is said to be *nilpotent* of class $\leq k$ if $G_{k+1} = \{1\}$. Let $F(n) = F(x_1, \ldots, x_n)$ be the free group on n generators. Let $NF(n) \triangleleft F(n)$ be normally generated by all elements of the form $[x_i, x_i^g]$. Here $i = 1, \ldots, n$ and $g \in F(n)$. Then

$$MF(n) := F(n)/NF(n)$$

is the *free Milnor group* on n generators. A proof of the following result can be found in Milnor's paper on link homotopy [13] or in [1].

Proposition 7.1. *The free Milnor group $MF(n)$ is nilpotent of class $\leq n$. It is finitely generated by the x_i and also finitely presented, i.e. $FN(n)$ is normally generated by a finite number of commutators of the form $[x_i, x_i^g]$ where $1 \leq i \leq n$ and $g \in F(n)$.*

Let G be a group and V be a G-module. (We will only consider left actions of G.) Define $[G, V]$ to be the submodule of V generated by $(1 - G)V$. Note that $[G, V] \subset W$ whenever the action of G on V/W is trivial. Corresponding to the lower central series, we have then $V_1 := V$ and $V_{k+1} := [G, V_k]$ for $k \geq 1$. Again V is said to nilpotent of class $\leq k$ if $V_{k+1} = \{0\}$.

Let now V be an $MF(n)$-module. We can then construct a quotient of V that imitates the Milnor group. Let NV be the submodule generated by all elements of the form $(1 - x_i)(1 - x_i^g)v$. Here $g \in MF(n), v \in V$ and $1 \leq i \leq n$. Define

$$MV := V/NV.$$

The proof of the following statement is very similiar to the arguments in [13] and [1] that prove Proposition 7.1.

Proposition 7.2. *As an $MF(n)$-module, MV is nilpotent of class $\leq n+1$.*

Proof. First, suppose $n = 1$. Then $NV = V_3$, and $MV = V/V_3$ is nilpotent of class ≤ 2. The general case is done by induction on n. Let $A_j \leq MV$ be generated by $(1 - x_j)MV$, and let $B_j \leq V$ be generated by $(1 - x_j)V$. Then

$$MV/A_j \cong V/NV + B_j \cong M(V/B_j).$$

However, x_j acts trivially on V/B_j. Thus, $[MF(n), W] = [MF(n-1), W]$ for every submodule $W \leq V/B_j$. Here we use $MF(n-1) \cong MF(n)/ \ll x_j \gg$ and thus V/B_j is an $MF(n-1)$-module. By induction MV/A_j is nilpotent of class $\leq n$. Therefore, MV_{n+1} is contained in the intersection of the A_j. Now let $v \in MV_{n+1}$. For any j we can write $v = \sum_i g_i(1 - x_j)v_i \in A_j$ for some $g_i \in MF(n), v_i \in V$. Hence

$$\begin{aligned} (1 - x_j)v &= \textstyle\sum_i (1 - x_j)g_i(1 - x_j)v_i \\ &= \textstyle\sum_i (1 - x_j)(1 - x_j^{g_i})g_i v_i \\ &= 0 \in MV. \end{aligned}$$

This implies that the generators x_j of $MF(n)$ act trivially on MV_{n+1} and thus $MV_{n+2} = 0$. □

We will use the fact that the group ring of a finitely generated nilpotent group is Noetherian. A proof is indicated in [16, p.136].

Proposition 7.3. *There are a finite number of elements $g_j \in MF(n)$ such that NV is generated by a finite number of elements of the form $(1 - x_i)(1 - x_i^{g_j})v$ with $v \in V$ and $1 \leq i \leq n$ for every finitely generated $MF(n)$-module V.*

Proof. First, consider the Noetherian ring $\mathbb{Z}[MF(n)]$. The left ideal $N\mathbb{Z}[MF(n)]$ is generated by a finite number of elements

$$f_j = (1 - x_{i_j})(1 - x_{i_j}^{g_j})r_j$$

with $1 \leq i_j \leq n$, $g_j \in MF(n)$ and $r_j \in \mathbb{Z}[MF(n)]$. In the general case V is a Noetherian $\mathbb{Z}[MF(n)]$-module. Thus, NV is generated by a finite number of elements

$$(1 - x_{i_k})(1 - x_{i_k}^{h_k})v_k$$

with $1 \leq i_k \leq n$, $h_k \in MF(n)$ and $v_k \in V$. Now $(1 - x_{i_k})(1 - x_{i_k}^{h_k}) \in N\mathbb{Z}[MF(n)]$ and this implies that NV is generated by the elements $f_j v_k$. □

Let X be a space with Postnikov tower

$$\begin{tikzcd} & & & & X \arrow[dllll, bend right] \arrow[dl] \arrow[d] \\ \cdots \arrow[r] & X_n \arrow[r] & \cdots \arrow[r] & X_1 \arrow[r] & X_0. \end{tikzcd}$$

Here $\pi_k(X_n) = 0$ for $k > n$ and $\pi_k(X_n) \cong \pi_k(X)$ via $X \to X_n$ for $k \leq n$. Moreover, $X_n \to X_{n-1}$ is a fibration with fiber $K(\pi_n(X), n)$. Here $K(G, n)$ denotes an Eilenberg-MacLane space satisfying $\pi_k(K(G, n)) = 0$ for $k \neq n$ and $\pi_n(K(G, n)) = G$. Suppose that

$\pi_n(X)$ is nilpotent of class $\leq k$ as a $\pi_1(X)$-module. Then $X_n \to X_{n-1}$ has a refinement corresponding to the lower central series of $\pi_n(X)$

$$X_n = X_n^k \to X_n^{k-1} \to \cdots \to X_n^1 \to X_n^0 = X_{n-1}$$

where $X_n^j \to X_n^{j-1}$ is a fibration with fiber $K(V_j, n)$. Here $V_j = \pi_n(X)_j / \pi_n(X)_{j+1}$, and $\pi_1(X)$ acts trivially on V_j. There is a similiar refinement of $X_1 \to X_0$ if $\pi_1(X)$ is nilpotent.

Proposition 7.4. *Let X be a finite CW-complex such that $G = \pi_1(X)$ is nilpotent and $\pi_k(X)$ is nilpotent as a G-module for $k = 2, \ldots, r$. Then $\pi_{r+1}(X)$ is finitely generated as a G-module.*

Proof. We will use the above notation and denote universal covers by $\widetilde{X}_n$ and $\widetilde{X}$. The cellular chain complex of $\widetilde{X}$ consists of Noetherian G-modules since $\mathbb{Z}[G]$ is Noetherian. Thus, $H_*(\widetilde{X})$ is finitely generated as a G-module. Assume that $H_{n+2}(\widetilde{X}_n)$ is finitely generated over $\mathbb{Z}$. Considering $\widetilde{X} \to \widetilde{X}_n$ as an inclusion, it follows that $H_{n+2}(\widetilde{X}_n, \widetilde{X})$ is finitely generated over $\mathbb{Z}[G]$. From the relative Hurewicz theorem and the isomorphism $\pi_{n+2}(\widetilde{X}_n, \widetilde{X}) \cong \pi_{n+1}(\widetilde{X})$, we see that $\pi_{n+1}(X)$ is finitely generated as a G-module under the above assumption.

To finish the proof, it is now sufficient to show that $H_*(\widetilde{X}_n)$ is finitely generated over $\mathbb{Z}$ in every dimension for $n = 1, \ldots, r$. For $n = 1$, we have $\widetilde{H}_*(\widetilde{X}_1) = 0$. Assuming the statement for $n-1$, we see that $\pi_n(X)$ is finitely generated as a G-module. The universal covers of the above refinement of $X_n \to X_{n-1}$ give

$$\widetilde{X}_n = \widetilde{X}_n^k \to \widetilde{X}_n^{k-1} \to \cdots \to \widetilde{X}_n^1 \to \widetilde{X}_n^0 = \widetilde{X}_{n-1},$$

where $\widetilde{X}_n^j \to \widetilde{X}_n^{j-1}$ is a fibration with fiber $K(V_j, n)$. Now $V_j = \pi_n(X)_j / \pi_n(X)_{j+1}$ is finitely generated (as $\pi_n(X)$ is Noetherian). This implies that $K(V_j, n)$ has dimensionwise finitely generated homology groups ([19, XIII.7.12]). An easy application of the Leray-Serre spectral sequence is the following: if the fiber and the base of a fibration with simply connected base have finitely generated homology groups in each dimension, then this is also true for the total space of the fibration ([19, XIII.7.11]). Applying this argument k times to our refinement, we can conclude that $H_*(\widetilde{X}_n)$ is finitely generated in every dimension. □

Proof of Proposition 1.3. Start with the standard slice disks $f_0 : D^{n+1} \amalg \cdots \amalg D^{n+1}$ for the trivial link of ν components. Note that $\pi_1(\mathcal{S}_0(f_0)) \cong F(\nu)$. Now introduce self-intersections to obtain $\pi_1(\mathcal{S}_0(f)) = MF(\nu)$: we use 6.10 to introduce the relations of the form

$$[x_i, x_i^g] = 1$$

for $g \in F(\nu)$. Because of 7.1, we can do this in a finite number of steps. Using 6.10 again, we can introduce more double points to obtain $f : \amalg^\nu D^{n+1} \looparrowright D^{n+3}$ such that the relations $[x_i, x_i^g] = 1$ are represented by double points of f for all g of 7.3. We will now use 6.9 to introduce relations of the form

$$(1 - x_i)(1 - x_i^g)v = 0$$

to $\pi_k(\mathcal{S}_0(f))$. Here $2 \leq k < n/2 + 1$ and $v \in \pi_k(\mathcal{S}_0(f))$. (Note that dim $\mathcal{S}_0(f) = n + 3$ whereas in 6.9 the ambient dimension was $n + 2$.) We can represent v by an embedding $S^k \hookrightarrow \mathcal{S}_0(f)$ using general position. Let ξ be the normal bundle of this embedding and denote the trivial line bundle by ϵ. Then

$$\xi \oplus TS^k \oplus \epsilon \cong T\mathcal{S}_0(f)|_{S^k} \oplus \epsilon$$

is trivial because $\mathcal{S}_0(f) \subset D^{n+3}$ is parallelizable. However, $TS^k \oplus \epsilon$ is also trivial, and hence ξ is stably trivial. In fact, ξ is trivial since dim $\xi > k$.

Suppose that $\pi_2(\mathcal{S}_0(f)), \ldots, \pi_r(\mathcal{S}_0(f))$ are nilpotent as modules over $MF(\nu)$. Note that $\mathcal{S}_0(f)$ is homotopy equivalent to a compact manifold and hence to a finite CW-complex. Then it follows from 7.4 that $\pi_{r+1}(\mathcal{S}_0(f))$ is finitely generated over $MF(\nu)$. By virtue of 7.3 and 7.2, a finite number of applications of 6.9 give us a map $f_1 : \amalg^\nu D^{n+1} \looparrowright D^{n+3}$ such that $\pi_{k+1}(\mathcal{S}_0(f_1))$ is also nilpotent as an $MF(\nu)$-module. We can repeat this until $\pi_k(\mathcal{S}_0(f_1))$ is nilpotent for all $k < n/2 + 1$. Each application of 6.10 and 6.9 uses only one of the disks, and hence f_1 is still a link map. □

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