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Algebraic Geometry
Exercise Sheet 8

To be hand in on 13.12.2018

Exercise 1:

Let k be a algebraically closed field. Consider the following morphism:

$$f : X := \operatorname{Spec} k[x] \to Y := \operatorname{Spec} k[y]$$

which is induced by $y \mapsto x^2$.

(i) Show that f is finite.

(ii) Let $a \in Y$ be a closed point, show that the fiber

$$X_a := X \times_Y \operatorname{Spec} k(a) = \begin{cases} \operatorname{Spec} k[x]/(x^2) & \operatorname{char}(k) = 2 \text{ or } a = (y) \\ \operatorname{Spec} k \times k & \text{otherwise} \end{cases}.$$

(iii) Let $\eta \in Y$ be the generic point, describe the generic fiber $X_\eta := X \times_Y \operatorname{Spec} k(\eta)$.

Exercise 2:

Let $f : X \to Y$ be a morphism of schemes.

(i) If f is an open immersion, show that f is locally of finite type. Moreover, if Y is locally noetherian, show that f is of finite type.

(ii) f is called *quasi-finite* if it is of finite type and $\forall y \in Y$, the fiber $X_y := X \times_Y \operatorname{Spec} k(y)$ is a finite set equipped with the discrete topology. Show that finite morphisms are quasi-finite.

(iii) Find an example that f is quasi-finite but not finite.

Exercise 3: Let n be a positive integer. Consider the functor $F : \text{Schemes} \to \text{Sets}$:

$$X \mapsto \left\{ (f_1, \dots, f_n) \in \Gamma(X, \mathcal{O}_X)^{\oplus n} \,\middle|\, X = \bigcup_{1 \le i \le n} X_{f_i} \right\}.$$

(i) If $n = 1$, show that F is the functor: $X \mapsto \Gamma(X, \mathcal{O}_X)^\times$.

(ii) Show that F is representable by $\mathbb{A}^n_{\mathbb{Z}} \backslash V(x_1, \dots, x_n)$.

(iii) For any ring R, show that:

$$F(\operatorname{Spec} R) = \{x \in R^{\oplus n} \backslash \{0\} \text{ such that } R^{\oplus n}/Rx \text{ is locally free}\}.$$

Exercise 4: Consider the following functor on Rings.

(i) Show that the functor $F(R) = \{A \in \operatorname{Mat}_{2\times 2}(R) | A^2 = 0\}$ is representable by a closed subscheme X of $\mathbb{A}^4_{\mathbb{Z}}$.

(ii) Show that the functor $G(R) = \{A \in \operatorname{Mat}_{2\times 2}(R) | \det(A) = \operatorname{trace}(A) = 0\}$ is representable by a closed subscheme Y of X.

(iii) Show that X is not reduced, but Y is a reduced closed subsheme of X with the same underlying topology.

Homepage: http://www.uni-muenster.de/Arithm/hellmann/veranstaltungen