Annihilation of Cohomology over Gorenstein Rings

A conference in memoriam Ragnar-Olaf Buchweitz

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Introduction

MAXIMAL COHEN-MACAULAY MODULES AND TATE-COHOMOLOGY

OVER

GORENSTEIN RINGS

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(Hannover)

The main theme of this article is :

Why should one consider Maximal Cohen-Macaulay Modules ?

Although there has been a lot of work and success lately in the theory of such modules, of which this conference witnessed, it has remained mysterious - at least to the present author - why these modules provide such a powerful tool in studying the algebra and geometry of singula-

The Singularity Category of A Regular Ring is Trivial

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Lemma 7.7.4.: Let S be a commutative Gorenstein ring of finite Krull dimension. Then for a prime p in $\mathrm{Spec}(S)$ the groups $\mathrm{Ext}_S^{\cdot}(X,Y)_p$ are zero for all (complexes of) S-modules Y (in $D^-(\mathrm{Mod}\text{-}S)$) and X (in $D^b(S)$) if and only if S_p is a regular local ring.

Coming back to the promised application of the Duality Theorem, let us recall that $t = \dim T$ and $s = \mathrm{vdim}\, S = t - d$ by (7.6.3.(iii)) .
Then we have :

Theorem 7.7.5. (The Duality Theorem for isolated singularities) :
Assume again given a homomorphism of rings $f : T \longrightarrow S$ which satisfies (7.6.2.), ω_T and ω_S dualizing modules over T and S respectively as in (7.7.1.), M a complex of S-modules in $D^b(S)$ and N a S-bimodule which is MCM on both sides.
(i) If M and the underlying right S-module of N are stably trans-

Question.

For a commutative Gorenstein ring R, describe the ideal

$$\bigcap_{M \in \mathrm{MCM}(R)} \underline{\mathrm{Hom}}_R(M, N).$$

The Cohomology Annihilator Ideal - S.Iyengar and R.Takahashi

- The Jacobian ideal of a commutative ring and annihilators of cohomology (1610:02599),
- Annihilation of cohomology and decompositions of derived categories (1405:5299),
- Annihilation of cohomology and strong generation of module categories (1404:1476)

The Jacobian Ideal Annihilates All Tate Cohomology Groups

trivially valued) field k in n variables.

Assume given a sequence $\underline{f} = (f_1,\ldots,f_m)$ of elements in the unique maximal ideal $\mathfrak{m} = (x_1,\ldots,x_n)$ and let $J(\underline{f})$ denote the corresponding <u>Jacobian ideal</u> generated by all maximal minors of the Jacobian matrix of $\underline{f}$ with respect to the chosen coordinate-functions $(x_1,\ldots,x_n)$, whose entries are hence given by the partial derivatives $\partial f_j/\partial x_i$.

With these notations we have :

<u>Corollary 7.8.7.</u>: If $\underline{f} = (f_1,\ldots,f_m)$ constitutes a <u>regular sequence</u> in $\underline{m} = (x_1,\ldots,x_n)$, the quotient ring $R = P/\underline{f}P$ is a complete intersection ring, hence Gorenstein, of dimension $n - m$.

The Jacobian ideal $J(\underline{f})$ then annihilates all Tate-cohomology groups over R which become consequently modules over the <u>Jacobian ring</u> of R, $\bar{R} = R/J(\underline{f})R$ in a natural way.

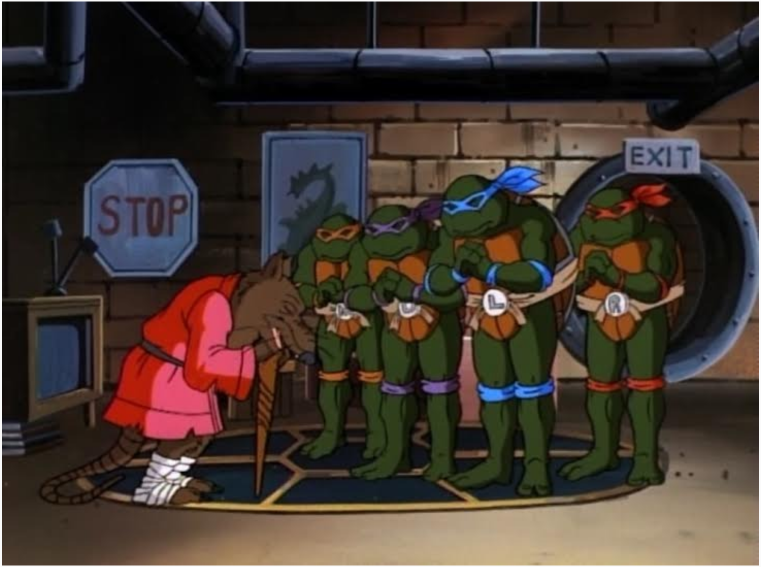
<u>For the proof</u> it needs only to be remarked that a composition

$$k\langle x_{i_1},\ldots,x_{i_{n-m}}\rangle \xrightarrow{\text{incl.}} P \xrightarrow{\text{proj.}} R$$

yields a finite flat homomorphism of rings if the corresponding minor

The Problem

The Group in Toronto



My Claim and Ragnar's Response

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I claimed (silently): I think you are wrong Ragnar.

Theorem

Theorem

Over a 1-dimensional complete reduced Gorenstein local ring, the cohomology annihilator ideal coincides with the conductor ideal.

An Application

Theorem

Let $f(x,y) \subseteq \mathbb{C}[\![x,y]\!]$ *be a reduced plane curve singularity. Consider*

$$R = \mathbb{C}[\![x,y,z_1,\ldots,z_n]\!]/(f(x,y)+z_1^2+\ldots+z_n^2).$$

Then,

$$\dim_{\mathbb{C}} R/J(R) = 2\dim_{\mathbb{C}} R/\mathrm{ca}(R) - r + 1$$

where r *is the number of branches of the curve* f *at its singular point.*

Orders

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Examples

Examples of orders where this works:

- If R is of finite maximal Cohen-Macaulay type of dimension $0, 1$ or 2, then its Auslander algebra,
- If R is the generic determinantal hypersurface singularity, then the NCCR described by Buchweitz-Leuschke-van den Bergh,
- If R is a one dimensional torus invariant ring, then the NCCR described by van den Bergh.

At least we have this

Theorem

Let R be a Gorenstein ring. If Λ is an R-order of finite global dimension containing R as a direct summand, then

$$(\operatorname{ann}_R \underline{\operatorname{End}}_R(\Lambda))^{1+\operatorname{gldim}\Lambda} \subseteq \operatorname{ca}(R) \subseteq \operatorname{ann}_R \underline{\operatorname{End}}_R(\Lambda).$$

Danke Schön!