Neutralino Annihilation into Fermion Pairs

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Chapter 1

Dark Matters

1.1 Composition of the Universe

One of the most basic questions humanity has had about the world surrounding us is the composition of the universe, but has been proven one to be one of the most difficult to answer. Our knowledge of matter and energy has progressed astonishingly over the last few centuries, leading to a detailed understanding of nearly all of the materials and phenomena around us. We have even begun to understand the principles that have shaped the evolution of our universe itself over the past few billion years.

One of the most shocking discoveries of recent science, however, is that we still have much to learn. Only a tiny fraction of the universe seems to be composed of the protons, neutrons, and electrons we have laboured so long to understand. The identity of the vast majority of the cosmos remains a mystery.

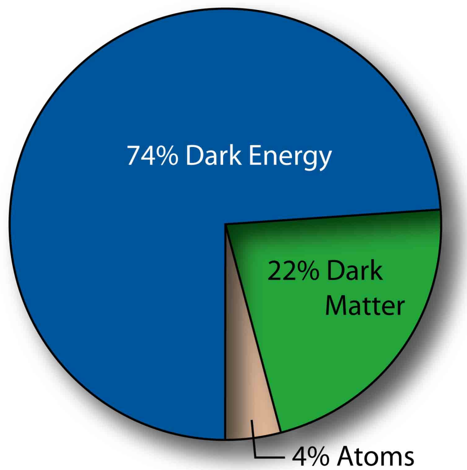
Our current understanding of the composition of the universe is roughly as follows:

- Baryonic Matter: Approximately a 4% of the mass in the universe. This is ordinary matter composed of protons, neutrons, and electrons. It comprises gas, dust, stars, planets, people, etc.

- An approximately 22% of mass of the universe is the so-called *dark matter* of the universe. It comprises the halos that surround galaxies and galaxy clusters, and aids in the formation of structure in the universe.

- Through observations of distant supernovae, two research groups have independently discovered that the expansion of the universe appears to be getting faster with time. This seems to require some kind of *antigravity* effect which we do not understand. Cosmologists believe that the acceleration may be caused by some kind of new energy field that permeates the universe, perhaps even the cosmological constant that Einstein imagined almost a century ago. Whatever the source of this phenomenon turns out to be, cosmologists refer to it generically as *dark energy*. This last contribution is about the 74% of the composition of the universe.

Figure 1.1: Diagram showing the proportion of components of the universe.



Here, we only speak about the dark matter and now we are going to shed some light on the hints and evidences of the existence of this so-called dark matter.

1.2 Some Evidences of Dark Matter

In spite of not being directly detected, the evidences of dark matter arise in several astrophysical studies. Here we are going to address very few of them just to give a taste and to convince ourselves that there are strong foundations to consider it. Far more evidences can be found with a very easy searching.

1.2.1 Coma Cluster

The mass of Coma was first estimated by Zwicky [29, 30] to be $M > 5 \cdot 10^{14} M_{\odot}$, using the virial theorem. This estimation was based on a value [31] of 1200 km/s for the radial velocity dispersion of the cluster galaxies, σ_v.

The corresponding mass-to-light ratio was large, $M/L > \dfrac{50M_{\odot}}{L_{\odot}}$, and a form of invisible matter seemed needed.

1.2.2 Galactic Rotation Velocity Curves

At large distances from the galactic centre the gravitational potential should be that produced by a central point mass and, in the absence of forces other than gravitation, it should be expected that

$$\frac{Gm}{r^2} = \frac{\omega^2}{r}, \tag{1.1}$$

where G is the universal gravitation constant, m is the galactic mass, r is the radius of the galaxy centre and ω is the rotation velocity; therefore $\omega \propto r^{-1/2}$, which is called, the Keplerian rotation curve. This Keplerian decline was not observed, but rather, flat rotation curves with $\omega = cte$ were obtained.

Apparently, this has the direct implication that $m \propto r$, thus depending on the quality of the telescope used. The dark matter hypothesis interprets this result in the sense that the Keplerian regime holds at much greater distances than those at which we obtain observations. There should be great quantities of dark matter extending far beyond the visible matter in a more or less spherically symmetric dark matter halo. If its distribution is spherically symmetric, the mass interior to a sphere of radius r would be $m(r) \propto r$, so that we obtain a first rough model of dark matter density distribution:

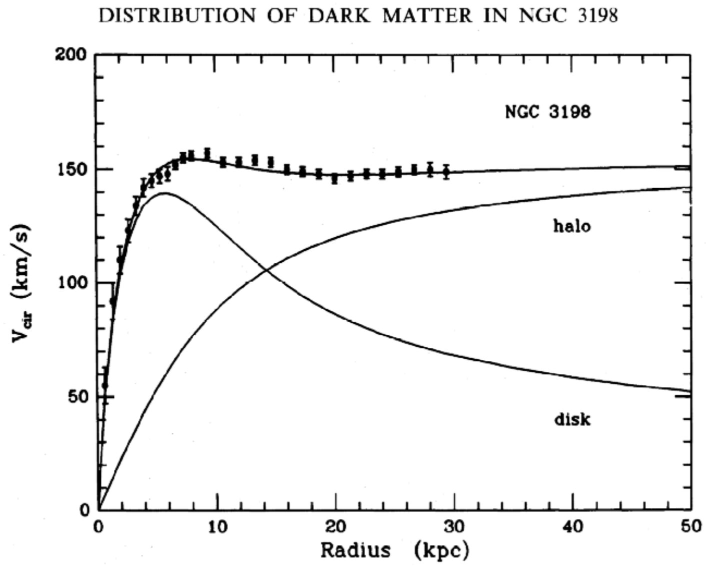
$$\rho = \frac{1}{4}\pi r^2 \frac{\mathrm{d}m}{\mathrm{d}r} = \frac{\omega}{4\pi G r^2}, \tag{1.2}$$

i.e. $\rho \propto R^{-2}$, for distances far beyond the visible radius. This model is over simplified, as we will see, but it coincides with the so called *nonsingular isothermal* profile

$$\rho = \frac{\rho_0}{1 + \left(\frac{r}{r_0}\right)^2}, \tag{1.3}$$

(with ρ_0 and r_0) one of the most frequently types of halos.

The interpretation of rotation curves of spiral galaxies as evidence of dark matter halos was probably first proposed by Freeman in 1970 [8] who noticed that the expected Keplerian decline was not present in NGC 300 and M33, and considered an undetected mass, with a different distribution for the visible mass. The observation of flat rotation curves was later confirmed and the dark matter hypothesis reinforced by successive studies. Rubin, Ford and Thonnard (1980) [24] and Bosma (1978, 1981a, b) [3–5] carried out an extensive study, after which the existence of dark matter in spiral galaxies was widely accepted. Van Albada et al. (1985) [25] analysed the rotation of NGC 3198 and the distribution of its hypothetical dark matter, concluding that this galaxy has a dark halo.



1.2.3 Cosmic Microwave Background

Angular fluctuations in the cosmic microwave background (CMB) spectrum provide evidence for dark matter as well. Since the 1964 discovery and confirmation of the CMB radiation, [19] many measurements of the CMB have supported and constrained this theory. The NASA Cosmic Background Explorer (COBE) found that the CMB spectrum a blackbody spectrum with a temperature of 2.726 K. In 1992, COBE detected fluctuations (anisotropies) in the CMB spectrum, at a level of about one part in 105 . [2] In the following decade, CMB anisotropies were further investigated by a large number of ground-based and balloon experiments. The primary goal of those was to measure the angular scale of the first acoustic peak of the power spectrum of the anisotropies, for which COBE did not have sufficient resolution. Between 2000 and 2001, several experiments, most notably BOOMERanG [16] found the universe to be almost spatially flat by measuring the typical angular size of the anisotropies. During the 1990s, the first peak was measured with increasing sensitivity and by 2000 the BOOMERanG experiment reported that the highest power fluctuations occur at scales of approximately one degree. These measurements were able to rule out cosmic strings as the leading theory of cosmic structure formation, and suggested cosmic inflation was the correct theory.

A number of ground-based interferometers provided measurements of the fluctuations with higher accuracy over the next three years, including the Very Small Array, the Degree Angular Scale Interferometer (DASI) and the Cosmic Background Imager (CBI). DASI made the first detection of the polarization of the CMB, [14,15] and the CBI provided the first E-mode polarization spectrum with compelling evidence that it is out of phase with the T-mode spectrum. [21] COBE's successor, the Wilkinson Microwave Anisotropy Probe (WMAP) has provided the most detailed measurements of (large-scale) anisotropies in the CMB as of 2009 [9] and ESA's Planck spacecraft returning more detailed results in 2012-2014. [1] WMAP's measurements played the key role in establishing the current Standard Model of Cosmology, namely the Lambda-CDM model, a flat universe dominated by dark energy, supplemented by dark matter and atoms with density fluctuations seeded by a Gaussian, adiabatic, nearly scale invariant process. The basic properties of this universe are determined by five numbers: the density of matter, the density of atoms, the age of the universe (or equivalently, the Hubble constant today), the amplitude of the initial fluctuations, and their scale dependence.

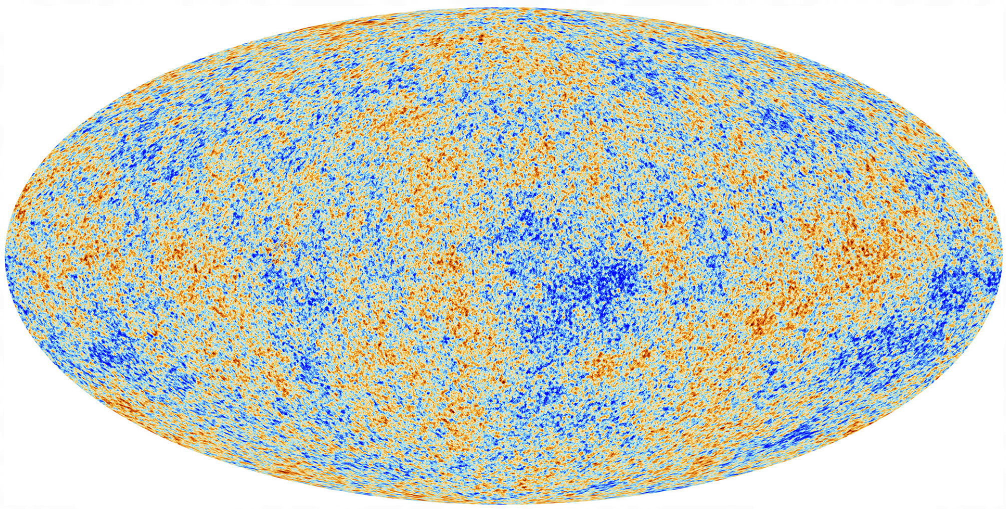
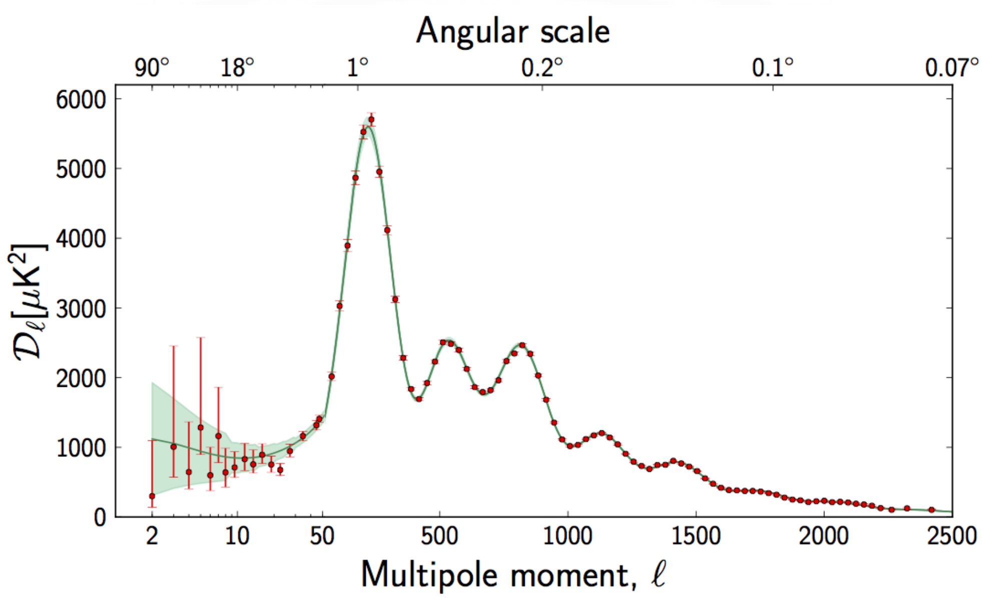


Figure 1.2: Cosmic microwave background as seen by Planck mission.

A successful Big Bang cosmology theory must fit with all available astronomical observations, including the CMB. In cosmology, the CMB is explained as relic radiation from shortly after the big bang. The anisotropies in the CMB are explained as the result of acoustic oscillations in the photon-baryon plasma (prior to the emission of the CMB after the photons decouple from the baryons 379,000 years after the Big Bang) whose restoring force is gravity. [13] Ordinary (baryonic) matter interacts strongly by way of radiation whereas dark matter particles, such as WIMPs for example, do not; both affect the oscillations by way of their gravity, so the two forms of matter will have different effects. The typical angular scales of the oscillations in the CMB, measured as the power spectrum of the CMB anisotropies, thus reveal the different effects of baryonic matter and dark matter. The CMB power spectrum shows a large first peak and smaller successive peaks, with three peaks resolved as of 2009. [9] The first peak tells mostly about the density of baryonic matter and the third peak mostly about the density of dark matter, measuring the density of matter and the density of atoms in the universe.

Figure 1.3: Planck Cosmic Microwave Background temperature angular power spectrum.



1.3 Dark Matter Candidates

1.3.1 Weakly Interacting Massive Particles (WIMPs)

According to the observations dark matter should be fairly *cold*, that is, its particles are relatively slow-moving. If dark matter were composed of relativistic particles (e.g. neutrinos), it would not clump as well under its own gravity and would wash out certain smaller structures of galaxies in the universe, which is not what we observe.

At the same time, it should represent around the 20% of the total mass density of the universe. This can be compared against predictions from particular models for dark matter.

As we pointed before one of the main candidate for a cold dark matter particle is a Weakly Interacting Massive Particle (WIMP). That is a collective name describing neutral stable particles with masses of around hundreds of GeV which are predicted to have electro-weak scale cross sections for interactions with matter. It is a remarkable coincidence that such relic particles have just the relevant abundance and decoupling temperatures to be a cold dark matter. One of the most popular WIMP candidates which is considered in the community is provided by the Supersymmetry (SUSY) theory [17]. That is a lightest supersymmetric particle (LSP), namely neutralino (χ).

At this point, it shall be mentioned that neutralinos are not the only considered candidates for the dark matter particle. For example, neutrinos were considered a good dark matter particle candidate since it has been proven that they have mass. As neutrinos are abundant in the Universe there would be no need to invent new type of particle in order to have a candidate. However, neutrinos were excluded from being a dark matter because they were relativistic at the time of the Universe formation and would prevent structure formation.

1.3.2 Massive Compact Halo Objects (MACHOs)

There is another possibility instead of considering scenarios where new particles arise. Massive Compact Halo Objects are non-luminous objects that make up the halos around galaxies. Machos are thought to be primarily brown dwarf stars and black holes. Like many astronomical objects, their existence had been predicted by theory long before there was any proof. The existence of brown dwarfs was predicted by theories that describe star formation.

Brown dwarfs are star but different from normal ones because of their relatively low mass, brown dwarfs do not have enough gravity to ignite when they form. Thus, a brown dwarf is not a *real* star; it is an accumulation of hydrogen gas held together by gravity. Brown dwarfs give off some heat and a small amount of light.

On the other hand black holes, unlike brown dwarfs, have an over-abundance of matter. All that matter collapses under its own enormous gravity into a relatively small area. The black hole is so dense that anything that comes too close to it, even light, cannot escape the pull of its gravitational field. Stars at safe distance will orbit around the black hole.

Detecting MACHOs is quite a challenge and so far they have been sought directly with the Hubble Space Telescope and indirectly by gravitational effect of nearby objects and gravitational lensing. Since the MACHOs are very compact objects we can expect a great bending from the light of distant stars in the background.

1.3.3 Axions

Axions, introduced in an attempt to solve the problem of $\mathcal{CP}$ violation in particle physics, have been also often discussed as candidates well motivated by theory. Axions are expected to interact extremely weakly with ordinary particles, which implies that they were not in thermal equilibrium in the early universe. Moreover, if they are forming a relic remnant they have to be very abundant as their expected mass is in the range $10^{-6} - 3 \cdot 10^{-2}$ eV. The searches for these particles are ongoing.

1.3.4 Other Exotic candidates

There are also other exotic dark matter candidates like WIMPziallas, Kaluza-Klein Dark Matter, Mirror Dark Matter, or self Interacting dark matters. Of course this is beyond any consideration here and they are referred only for the sake of the completeness.

From all these possibilities for dark matter,here we focus in the neutralino, the particle given by the Supersymmetric Model. Let us go deeper with this particle.

Chapter 2

Neutralino, a Supersymmetric Dark Matter Candidate

2.1 Supersymmetry in a Nutshell

Supersymmetry is widely considered extension to the Standard Model of particle physics.

Table 2.1: Standard Model particles and their correspondent supersymmetric partners.

Standard Model particles		SUSY partners	
Symbol	Name	Symbol	Name
u, c, t	up-quarks	$\tilde{q}_u^1, \ldots, \tilde{q}_u^6$	up squarks
u, c, t	down-quarks	$\tilde{q}_d^1, \ldots, \tilde{q}_d^6$	down squarks
e, μ, τ	leptons	$\tilde{l}^1, \ldots, \tilde{l}^6$	sleptons
ν_e, ν_μ, ν_τ	neutrinos	$\tilde{\nu}_1, \ldots, \tilde{\nu}_3$	sneutrinos
g	gluon	$\tilde{g}$	sgluinos
$W^\pm$	W bosons		
W^3	W^3-field (Z^0)	$\tilde{W}^3$	wino
B	B-field (photon)	$\tilde{B}^0$	bino
H_1^0, H_2^0, H_3^0	Higgs bosons	$\tilde{H}_1^0, \tilde{H}_2^0, \tilde{H}_3^0$	higgsinos
$H^\pm$	charged Higgs bosons	$\tilde{\chi}_1^\pm, \tilde{\chi}_2^\pm$	charginos

SUSY assumes that for each known particle a supersymmetric partner particle exists, referred to as sparticle. Sparticles would have spin different than particles by 1/2. Therefore, sfermions would be bosons and sbosons would be fermions. The spectrum of supersymmetric particles which is introduced in one of the simplest SUSY models, Minimal Supersymmetric Standard Model (MSSM), is shown in Table 2.1. The MSSM contains two Higgs doublets needed to give mass to up and down squarks. There two squarks for each quark. The superpartners of the W and charged Higgs bosons, the charged higgsino and gaugino, mix after electro-weak symmetry breaking and create two mass eigenstates which are called charginos ($\chi^\pm$). The same thing happens for the superpartners of the photon, Z boson, and two neutral Higgs bosons. The following fields are produced: B (photino), $\tilde{W}^3$ (wino) and $\tilde{H}_1^0$, $\tilde{H}_2^0$ (higgsinos). These fields create 4 mass eigenstates called neutralinos:

$$\tilde{\chi} = a_1\tilde{B} + a_2\tilde{W}^3 + a_3\tilde{H}_1^0 + a_4\tilde{H}_2^0 \tag{2.1}$$

The lightest neutralino is most likely the LSP and is considered the best motivated WIMP candidate. The LSP is stable in the MSSM because theory contains a multiplicatively conserved quantum number called $\mathcal{R}$-parity (which serves to prevent rapid proton-decay). The value of R is -1 for sparticles and 1 for particles. The LSP has to be stable since it can no longer decay into lighter SUSY particles and neither it can decay into particles from to the Standard Model only (due to $\mathcal{R}$-parity conservation).

The limits on the neutralino mass (m_χ) depend on the model assumed for the SUSY breaking, constraints from searches for sparticles at LEP e^+e^- and Tevatron hadron collliders, and are based on the cosmological considerations. The lower limit for m_χ is around 7-10 GeV and can be derived from searches for supersymmetric particles at accelerators. The upper limit on neutralino mass, from cosmology, is approximately 7 TeV. These limits should be considered as a general indication of scale of WIMP masses as variation of the SUSY and Universe evolution models can yield slightly different constraints. Neutralinos are expected to have only the electroweak scale interactions with oridinary matter. One could distinguish two types of their interactions: spin-dependent (SD) and spin-independent (SI) [11]. In the first case χ couple to the spin of the target nucleus (axial vector interactions). That is mainly the case for neutralino interactions with nuclei which have odd number of nucleons (spin is unpaired). In case of spin-independent reactions WIMPs couple to the mass of the target nucleus (scalar interactions).

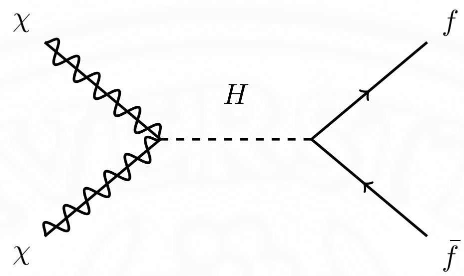
2.2 Neutralino Annihilation

WIMPs can annihilate into numerous final states. The most studied ones are those referring to the neutralino. This is the only particle we are going to consider. The annihilation cross section is a relevant quantity related to the relic abundance of cosmological dark matter. So it is useful to know the procedure to compute this cross section.

The dominant annihilation processes are those at the lowest order in perturbation theory, with two vertices (the so-called *tree level*). We are going to analyse the annihilation of two neutralinos to a fermion-antifermion pair in the s-channel.

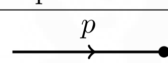
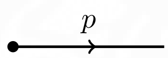
In this process, an exchange of a Higgs scalar or pseudoscalar bosons may occur so we will consider both of them separately. The reason we only consider these kind of exchange and not others is quite simple. The calculations will be easier since we only have scalar bosons. More realistic calculations would include more possibilities and they may be found in several other works.

As we use Feynman diagrams to compute the cross section it is important to give here a very brief sum up of the rules for these diagrams in the MSSM.

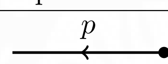
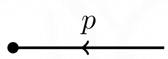


2.3 Feynman Rules

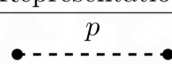
- External fermions:

Direction	Representation	Expression
Incoming	 p	$\ldots u^{(s)}(p)$
Outgoing	 p	$\bar{u}^{(s)}(p)\ldots$

- External antifermions:

Direction	Representation	Expression
Incoming	 p	$\bar{v}^{(s)}(p)\ldots$
Outgoing	 p	$\ldots v^{(s)}(p)$

- Internal 0-spin bosons:

Representation	Expression
 p	$\dfrac{i}{p^2 - m^2}$

Here $u^{(s)}(p)$ and $v^{(s)}(p)$ are the Dirac spinors.

Chapter 3

Helicity Amplitudes in Spherical Basis

We now consider the relativistic partial wave formalism [10, 27] to calculate S-matrix elements of a general process $a + b \to c + d$. Our goal will be to get an expression for the S-matrix in a basis, where the incoming (non-relativistic) state is an eigenstate of the total angular momentum J and its three-component J_z, the total orbital angular momentum L and the total spin S. The outgoing (in general relativistic) state is also an eigenstate of J and J_z, but has definite values for the helicities, of the constituting particles. To obtain these matrix elements, which are also called *helicity amplitudes in spherical basis*, we first have to define plane wave helicity states and to construct eigenstates of total angular momentum out of them. Afterwards we have to find a relation between these helicity states and the spectroscopic states with definite j, m, l and s before we can finally construct the desired matrix element and the cross section. In the end of this Chapter, we present the explicit spherical helicity amplitudes the neutralino annihilation process.

3.1 Angular Momentum States

3.1.1 Euler Angles

Before considering any angular momentum arises the necessity of defining the Euler angles. Consider a right-handed system of coordinates Σ_1 with the standard x, y, z axis. The orientation of any other system of coordinates Σ_4 is identical to the orientation of Σ_1 after three elementary rotations. Using the convention of rotations around the axis z-y-z, these transformations are:

1. A rotation of angle $\alpha \in [0, 2\pi)$ around the z-axis, which yields a frame Σ_2.
2. A rotation of angle $\beta \in [0, 2\pi)$ around the y-axis, which yields a frame Σ_3.
3. A rotation of angle $\gamma \in [0, 2\pi)$ around the z-axis, which yields a frame Σ_4.

3.1.2 Rotation Operator

As the components $J_{x,y,z}$ of the angular momentum are the generators of the rotation group, the rotation operator $\mathcal{R}_{\alpha,\beta,\gamma}$ expressed in terms of the Euler angles is

$$\mathcal{R}_{\alpha,\beta,\gamma} \equiv e^{-i\gamma J_{z,3}} e^{-i\beta J_{y,2}} e^{-i\alpha J_{z,1}} = R_{z,3}(\gamma) R_{y,2}(\beta) R_{z,1}(\alpha), \tag{3.1}$$

where the indices refer to the explained frames. As the obtained expression is very unhandy for calculations because the elementary rotations are defined in different coordinate systems. To write them again in terms containing only the coordinates of Σ_1, we have to regard $R_{y,2}$ and $R_{z,3}$ as rotated operators itself, i.e. display them as the results of unitary transformations induced by $R_{z,1}(\alpha)$ and $[R_{y,2}(\beta)R_{z,1}(\alpha)]$:

$$R_{y,2} = R_{z,1}(\alpha) R_{y,1}(\beta) R_{z,1}(\alpha)^\dagger \tag{3.2}$$

$$\begin{aligned} R_{z,3}(\beta) &= [R_{y,2}(\beta) R_{z,1}(\alpha)] R_{z,1}(\gamma) [R_{y,2}(\beta) R_{z,1}(\alpha)^\dagger] \\ &= R_{z,1}(\alpha) R_{y,1}(\beta) R_{z,1}(\gamma) R_{y,1}(\beta)^\dagger R_{z,1}(\alpha)^\dagger, \end{aligned} \tag{3.3}$$

the last step following after using the first equation. Substituting these expressions into eq. (3.1) yields the more useful form:

$$\mathcal{R}_{\alpha,\beta,\gamma} = e^{-i\alpha J_z} e^{-\beta J_y} e^{-i\gamma J_z} \tag{3.4}$$

3.1.3 The Wigner Functions

The *Wigner D-function* gives the matrix elements of the rotation operator R in the jm-representation. For the Euler angles α, β, γ, the D-function is defined as:

$$\begin{aligned} \mathcal{R}_{\alpha,\beta,\gamma} |jm\rangle &= \sum_{j'=0}^{\infty} \sum_{m'=-j'}^{j'} |j'm'\rangle \langle j'm'|\mathcal{R}_{\alpha,\beta,\gamma}|jm\rangle \\ &= \sum_{m'=-j}^{j} |jm'\rangle \langle jm'|\mathcal{R}_{\alpha,\beta,\gamma}|jm\rangle \\ &\equiv \sum_{m'=-j}^{j} |jm'\rangle D^j_{m'm}(\alpha,\beta,\gamma). \end{aligned} \tag{3.5}$$

In the second step of (3.5) we have used the fact that J^2 commutes pairwise with all of its components and therefore also with $\mathcal{R}_{\alpha,\beta,\gamma}$. Now, using the expression (3.6) and the fact that $e^{-i\gamma J_z} |jm\rangle = e^{-im\gamma} |jm\rangle$, we can see that the Wigner D-function can always be written using the Wigner d-function as:

$$\begin{aligned} D^j_{m'm}(\alpha,\beta,\gamma) &= \langle j,m|\mathcal{R}_{\alpha,\beta,\gamma}|jm'\rangle \\ &= \langle jm|e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}|jm'\rangle \\ &= e^{-im\alpha} \langle jm|e^{-i\beta J_y}|jm'\rangle e^{-im'\gamma} \\ &\equiv e^{-im\alpha} d^j_{m,m'}(\beta) e^{-im'\gamma}, \end{aligned} \tag{3.6}$$

where

$$d^j_{mm'}(\beta) = \langle jm|e^{-i\beta J_y}|jm'\rangle \tag{3.7}$$

The last step in (3.6) is the definition of another kind of Wigner functions, namely the *Wigner d-Functions* or *reduced rotation matrices*. Explicit general expressions of these functions will not be derived, but they can be found in [18]:

$$\begin{aligned} d^j_{m'm}(\beta) = \sum_n & \frac{(-1)^n\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{(j-m'-n)!(j+m-n)!n!(n+m'-m)!} \\ & \times \cos^{2j+m-m'-2n}\frac{\beta}{2}\sin^{2n+m'-m}\frac{\beta}{2}, \end{aligned} \tag{3.8}$$

the sum running over all n that yield positive factorial terms. With our z-y-z-convention for the Euler angles the $d^j_{m'm}(\beta)$ are always real valued, in other conventions half of the elements can also be purely imaginary. Although eq. 3.8 is somewhat complicated, the $d_{m'm}(\beta)$ has many simple properties [22]. Clearly these functions are real and from eq. 3.8,

$$d_{m'm}(-\beta) = (-1)^{m'-m}d_{m'm}(\beta). \tag{3.9}$$

Also, $R^\dagger = R^{-1}$ implies $\langle jm|R|jm'\rangle = \langle jm'|R^{-1}|jm\rangle^*$ so that

$$D^j_{mm'}(0,\beta,0) = \langle jm|e^{-i\beta J_y}|jm'\rangle = \langle jm'|e^{i\beta J_y}|jm\rangle = D^j_{mm'}(0,-\beta,0), \tag{3.10}$$

where we have used the reality of the $d^j_{m'm}(\beta)$ functions. Thus

$$d^j_{m'm}(-\beta) = d^j_{mm'}(\beta). \tag{3.11}$$

Using (3.9) and (3.11), we find that

$$d^j_{m'm}(\beta) = (-1)^{m'-m}d^j_{mm'}(\beta) \tag{3.12}$$

and from (3.11) we have

$$D^j_{mm'}(\alpha,\beta,\gamma) = e^{-im\gamma}d_{mm'}(-\beta)^j e^{-im'\alpha} = D^j_{mm'}(\gamma,-\beta,\alpha). \tag{3.13}$$

From eq. (3.8) we can calculate $d^j_{m'm}(\beta)$ for $\beta = \pi, 2\pi$:

$$d^j_{m'm}(\beta=\pi) = (-1)^{j-m}\delta_{m',-m}, \tag{3.14}$$

$$d^j_{m'm}(\beta=2\pi) = (-1)^{2j}d^j_{m'm}(0) = (-1)^{2j}\delta_{m'm}. \tag{3.15}$$

Finally, there is the extremely useful orthogonality relation:

$$\int_0^{2\pi}\mathrm{d}\alpha\int_0^{2\pi}\mathrm{d}\gamma\int_0^{\pi}\mathrm{d}\beta\ \sin\beta D^{j*}_{mn}(\alpha,\beta,\gamma)D^{j'}_{m'n'}(\alpha,\beta,\gamma) = \frac{8\pi^2}{2j+1}\delta_{mm'}\delta_{nn'}\delta_{jj'} \tag{3.16}$$

From now expressions containing $\mathcal{R}$ without explicit argument are understood to be evaluated at (α,β,γ), and respectively D or d matrices evaluated at β.

3.1.4 The Clebsch-Gordan Coefficients

Let us now consider the action of the rotation operator on a direct product state of two irreducible state vectors according to the rotation group:

$$\mathcal{R}(|j_1 m_1\rangle \otimes |j_2 m_2\rangle) = \mathcal{R}\,|j_1 j_2 m_1 m_2\rangle = \sum_{m_1'=j_1}^{j_1} \sum_{m_2'=j_2}^{j_2} D^{j_2}_{m_1' m_1} D^{j_2}_{m_2' m_2} \,|j_1 j_2 m_1' m_2'\rangle \,. \tag{3.17}$$

From [28] we can see the properties of group theory and it turns out that the direct product of states as well as operators acting on these states can be decomposed into a direct sum of quantities of the respective kind existing in an irreducible representation. For the states we get the well known expansion

$$\begin{aligned} |j_1 m_1\rangle \otimes |j_2 m_2\rangle &= \sum_{j_1=|j_1-j_2|}^{j_1+j_2} \langle j m j_1 j_2 | j_1 j_2 m_1 m_2\rangle \,|j m j_1 j_2\rangle \\ &\equiv \sum_{j_1=|j_1-j_2|}^{j_1+j_2} \mathscr{C}(j_1 j_2 j; m_1 m_2)\,|j m j_1 j_2\rangle \,, \end{aligned} \tag{3.18}$$

where $\mathscr{C}$ is the so called Clebsch-Gordan Coefficients (CGC) [26, 28]. The reduction of the group yields symbolically:

$$D^{j_1} \otimes D^{j_2} = \bigoplus_j D^j, \tag{3.19}$$

while for a fixed j the D_j act on all elements (indicated by m) of the coupled state obtained above[1] To get an explicit expression one can use the desired behaviour for the rotation of the coupled states

$$\mathcal{R}\,|j m j_1 j_2\rangle = \sum_{m'} |j m j_1 j_2\rangle\, D^j_{m' m} \tag{3.20}$$

and the orthogonality relations of the CGC to obtain the important coupling rule:

$$D^{j_1}_{m_1' m_1} D^{j_2}_{m_2' m_2} = \sum_{j=|j_1-j_2|}^{j_1+j_2} \mathscr{C}(j_1 j_2 j; m_1' m_2') \mathscr{C}(j_1 j_2 j; m_1 m_2) D^j_{m_1'+m_2', m_1+m_2} \tag{3.21}$$

Both (3.18) and (3.21) are known as the *Clebsch-Gordan Series* [28]. It is remarkable that (3.21) still holds if we replace the D-functions with the d-functions.

[1] This is according to the definition of *irreducible*, i.e. the action of the group operator in the irreducible representation leaves no invariant subspaces of the Hilbert space.

3.1.5 Irreducible Spherical Tensors

As a generalization of angular momentum states we may use their rotation property to define arbitrary tensorial quantities as irreducible [28] according to the rotation group if they transform in exactly the same way as the eigenstates of angular momentum do. Therefore we assign the component m of an irreducible spherical tensor of rank j with T_{jm} and request all of the $2j+1$ components to transform under rotations in the following way:

$$\mathcal{R}T_{jm} = \sum_{m'=-j}^{j} T_{jm'} D^j_{m'm}. \tag{3.22}$$

For example the angular momentum states $|jm\rangle$ itself are the components indicated by m of an irreducible spherical tensor T_{jm}.

We can also treat three-vectors as irreducible spherical tensors when choosing a basis for them where they transform in the representation of D^1. This basis is found by diagonalizing the 3×3 matrix of J_z in the Cartesian basis. The result is the following transformation rule for a four-vector A^μ,

$$A^\mu \to A^a = A_{m,n} = \begin{pmatrix} A^\diamond \\ A^1 \\ A^0 \\ A^{-1} \end{pmatrix} = A_{m,n} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} A_t \\ A_x \\ A_y \\ A_z \end{pmatrix} \tag{3.23}$$

We denote spherical vectors with roman indices $a, b, c, \ldots$ covering a time component index " $\diamond$ " and the numbers -1, 0, 1. To indicate only the spatial part we will use the letters l, m, n. Note that the index 0 now labels the actual z-component of this vector and not the time component, which stays the same. The complex conjugation of intrinsically complex vectors should be done with care, since

$$(A^*)^m = (-1)^m (A^{-m})^*. \tag{3.24}$$

Therefore, the contraction of two vectors in this basis is written as:

$$A \cdot B = A^\diamond B^\diamond - \sum_{m=-1}^{1} (-1)^m A^m B^m. \tag{3.25}$$

In fact may we interpret all angular dependent quantities that appear in a Feynman amplitude as spherical tensors in the according representation. If they are not coupled already this will be the $\frac{1}{2}$- (for spinors) or 1- (for vectors) representation . We will refer to this fact when defining the explicit wave functions for fermions and bosons in the helicity basis.

3.1.6 Plane Wave Helicity States

One Particle

In order to expand the S-matrix, that is usually defined in a linear momentum basis, we first need to construct plane wave states which have quantum numbers that are invariant under rotations. [22] That is we cannot use the spin three-component, but we can use the quantity defined as the projection of the three-component spin in the direction of motion,

$$\lambda \equiv \frac{\mathbf{S}\cdot\mathbf{p}}{|\mathbf{p}|} = \mathbf{S}\cdot\hat{\mathbf{p}}, \tag{3.26}$$

called *helicity*. The normalization is to preserve partially the invariance under Lorentz- boosts, unless the boost changes the direction of $\mathbf{p}$.

We begin with the rest state $|\mathbf{p}=0,s,\lambda\rangle$, which has spin s and spin projection λ along the z-axis. In the rest frame the spin projection and the helicity are equivalent. But when this state is rotated only the helicity remains invariant, and we will use it to label the state. Physically, the invariance is due to the fact that the quantisation of the axis $\hat{\mathbf{p}}$ rotates along with the spin $\mathbf{S}$ of the system.

To obtain the state $|\mathbf{p},s,\lambda\rangle$, we first consider a rest frame with $|\mathbf{p}|=0$. In this rest frame the helicity equals the spin component in its quantisation axis that we choose to be the z-axis and hence also equals the z-component of the total angular momentum. Such a state is labelled by $||\mathbf{p}|=0,s,\lambda=s_z\rangle$, with s being the quantum number of the squared spin; λ can therefore receive $2s+1$ different values $\lambda\in\{-s,-s+1,\ldots,s-1,s\}$. To get a general state we have to rotate it into θ-ϕ-direction using the D-function and Lorentz boost it with the operator $\mathcal{B}(\mathbf{p})$ along that direction until it has the momentum $\mathbf{p}$:

$$\mathcal{R}_{\phi,\theta,-\phi}\,|\mathbf{p}=0,s,\lambda\rangle = \mathcal{B}\sum_{s_z=-s}^{s} D^s_{s_z,\lambda}(\phi,\theta,-\phi)\,|0,s,s_z\rangle\,. \tag{3.27}$$

The reason to choose $\gamma=-\phi$ is explained in the first of the three following comments about the phases [10]:

1. The angle γ is arbitrary. $\gamma=-\phi$ is convenient to omit an additional phase factor in the case $\theta=0\neq\phi$:

$$\begin{aligned}\mathcal{R}_{\phi,\theta=0,-\phi}\,|\mathbf{p}=0,s,\lambda\rangle &= \sum_{m'} D^j_{m'\lambda}(\phi,0,-\phi)\,|0,s,m'\rangle = \sum_{m'} e^{im'\phi}\delta_{m'\lambda}e^{i\lambda\phi}\,|0,s,m'\rangle\\ &= |0,s,m'\rangle\,.\end{aligned} \tag{3.28}$$

2. The description of the state propagating in negative z-direction leaves another freedom in choosing the angle ϕ. Here we have to impose the boundary condition:

$$\lim_{-\mathbf{p}_z\to 0}|-\mathbf{p}_z,s,\lambda\rangle = \lim_{\mathbf{p}_z\to 0}|\mathbf{p}_z,s,\lambda\rangle\,, \tag{3.29}$$

using eq. (3.14) yields

$$
\begin{aligned}
e^{-i\pi J_y}\left|\mathbf{p}=0,s,\lambda\right\rangle &= \sum_{\lambda'} D^s_{\lambda\lambda'}(0,\pi,0)\left|\mathbf{p}=\mathbf{0},\mathbf{s},\lambda'\right\rangle \\
&= \sum_{\lambda'}(-1)^{s-\lambda}\delta_{\lambda',-\lambda}\left|\mathbf{p}=0,s,\lambda'\right\rangle = (-1)^{s-\lambda}\left|\mathbf{p}=0,s,-\lambda\right\rangle,
\end{aligned}
\tag{3.30}
$$

we have

$$
(-1)^{s-\lambda}e^{-i\pi J_y}\left|\mathbf{p}=0,s,\lambda\right\rangle = \left|\mathbf{p}=0,s,-\lambda\right\rangle, \tag{3.31}
$$

comparing with (3.29)

$$
\left|-\mathbf{p}_z,s,\lambda\right\rangle = (-1)^{s-\lambda}e^{-i\pi J_y}\left|\mathbf{p}_z,s,\lambda\right\rangle. \tag{3.32}
$$

3. The phases relative to different values of λ can be fixed by the requirement that the ladder operators of the total angular momentum should act like the corresponding spin matrices in the correct representation at $p=0$:

$$
(J_x \pm iJ_y)\left|\mathbf{p}=0,\lambda\right\rangle = (S_x \pm iS_y)\left|p=0,\lambda\right\rangle = \sqrt{(s\mp\lambda)(s\pm\lambda+1)}\left|p=0,\lambda+1\right\rangle \tag{3.33}
$$

The only difference when describing a massless state is, that it is not possible to transform into its rest frame and that there are *per se* two independent solutions according to the helicities $\lambda=\pm s$ that do not have to be boosted, only rotated. To retain consistency the first and second statements above stay conventionally true but the third one has to be replaced by what follows from the next construction.

Let $\mathcal{P}$ be the well known parity operator acting like $x,y,z\to -x,-y,-z$. Then a reflection in the xy plane can be expressed with the operator $Y=e^{-i\pi J_y}\mathcal{P}$:

$$
Y\left|\mathbf{p}_z,s,\lambda=s\right\rangle = \eta\left|\mathbf{p}_z,s,\lambda=-s\right\rangle, \tag{3.34}
$$

That is the desired relation between the two states. The intrinsic parity η, with $|\eta|=1$, depends on the explicit construction of the helicity states, i.e. the polarization vectors. Finally, we choose the convenient Lorentz invariant normalization:

$$
\langle\mathbf{p}',s',\lambda|\mathbf{p},s,\lambda\rangle = (2\pi)^3 2E\delta^{(3)}(\mathbf{p}'-\mathbf{p})\delta_{s's}\delta_{\lambda'\lambda}. \tag{3.35}
$$

Two Particles

We now construct states that represent two particles that are in plane wave states with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$. They are simply the direct product states

$$
\left|\mathbf{p}_1,\lambda_1\mathbf{p}_2,\lambda_2\right\rangle \equiv \left|\mathbf{p}_1,\lambda_1\right\rangle\otimes\left|\mathbf{p}_2,\lambda_2\right\rangle. \tag{3.36}
$$

We now take the center of mass (CM) frame, so that $\mathbf{p_1} = -\mathbf{p_2}$. Therefore we write again this state in terms of the common absolute value of the linear three-momentum p, the angles θ, ϕ denoting the particle 1 direction and the helicities λ_1, λ_2. Furthermore is it useful for later calculations to factor out the summed linear momentum part:

$$|\mathbf{p_1}, \lambda_1, -\mathbf{p_1}, \lambda_\mathbf{2}\rangle \equiv |p, \theta\phi\lambda_1\lambda_2\rangle = (2\pi)^3 \sqrt{\frac{4E_{\mathrm{CM}}}{p}} \, |\theta\phi\lambda_1\lambda_2\rangle \, |P^\mu\rangle \, , \tag{3.37}$$

with summed four-momentum $P^\mu = (E_{\mathrm{CM}}, 0, 0, 0)$ and CM energy $E_{\mathrm{CM}} = \sqrt{p^2 + m_1^2} + \sqrt{p + m_2^2}$. Last factorization is chosen such that the following normalizations hold:

$$\langle p'; \theta'\phi'\lambda_1'\lambda_2' | p'; \theta\phi\lambda_1\lambda_2\rangle = (2\pi)^6 \frac{4\sqrt{s}}{p} \delta^{(4)}(P^\mu - P'^\mu)\delta(\cos\theta - \cos\theta')\delta(\phi - \phi')\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'}. \tag{3.38}$$

$$\langle P^\mu | P'^\mu\rangle = \delta^{(4)}(P^\mu - P'^\mu). \tag{3.39}$$

Note, that the state (3.37) can also be written as:

$$|p; \theta, \phi\lambda_1\lambda_2\rangle = \mathcal{R}_{\phi,\theta,-\phi} \, |p; 0, 0\lambda_1\lambda_2\rangle \, . \tag{3.40}$$

3.1.7 Spherical Waves Helicity States

Next task here is to transform the above states into a spherical basis keeping all of the helicity properties (since they are invariant under rotation). The canonical quantities according to a spherical basis are the total angular momentum j and its three-component m. Therefore we can assign the new state as

$$|p; jm\lambda_1\lambda_2\rangle \, . \tag{3.41}$$

Being also a state in the irreducible representation of R, it has the same transformation properties as displayed in (3.5). It is important to note that p only assigns the absolute value of the linear momentum but does not refer to a defined direction because neither of the particles is in an eigenstate of $\mathbf{p}$.

We may start by expanding the plane wave state as shown in (3.40), afterwards, we insert a complete set of states (3.41) and after using (3.5) finally identify the resulting terms (p, λ_i and the angles will be suppressed since they stay unchanged):

$$\begin{aligned}
\mathcal{R}_{\phi,\theta,-\phi} \, |p, \theta = 0, \phi = 0, \lambda_1\lambda_2\rangle &= \sum_{j,m} \langle jm|\mathcal{R}|\theta = 0, \phi = 0\rangle \, |jm\rangle \\
&= \sum_{j,m,j',m'} \langle jm|\mathcal{R}|j'm'\rangle \, \langle j'm'|\theta = 0, \phi = 0\rangle \, |jm\rangle \\
&= \sum_{j,m} N_j D^j_{m\lambda_1-\lambda_2}(\phi, \theta, -\phi) \, |p; jm\lambda_1\lambda_2\rangle \, .
\end{aligned} \tag{3.42}$$

The normalization constant $N_j \equiv \langle j, m = \lambda | 0, 0 \rangle$ with $\lambda \equiv \lambda_1 - \lambda_2$ is obtained, because $braket j', m' | 0, 0$ does not depend on m', since it is a projection matrix element on the state $|p; \theta = 0, \phi = 0 \lambda_1 \lambda_2\rangle$ that has no orbital angular momentum in z-direction and consequently a fixed $m = \lambda_1 - \lambda_2$, leading to a $\delta_{m' \lambda_1 - \lambda_2}$. We may determine the value of N_j by claiming the normalisation:

$$\langle j', m', \lambda_1' \lambda_2' | j, m, \lambda_1 \lambda_2 \rangle = \delta_{jj'} \delta_{mm'} \delta_{\lambda_1 \lambda_1'} \delta_{\lambda_2 \lambda_2'} \tag{3.43}$$

taking the states, where the linear momentum part is factorised out in exactly the same (3.43)

$$N_j = \sqrt{\frac{2j+1}{4\pi}}. \tag{3.44}$$

The final matrix element

$$\langle jm\lambda_1\lambda_2 | \theta\phi\lambda_1'\lambda_2' \rangle = \sqrt{\frac{2j+1}{4\pi}} D^j_{m\lambda}(\phi, \theta, -\phi) \delta_{\lambda_1 \lambda_1'} \delta_{\lambda_2 \lambda_2'} \tag{3.45}$$

way also be used to express spherical basis states in terms of plane waves in the following way:

$$\begin{aligned} |p; jm\lambda_1\lambda_2\rangle &= N_j \int \mathrm{d}\Omega \; D^{j*}_{m,\lambda_1-\lambda_2}(\phi, \theta, -\phi) \mathcal{R}_{\phi,\theta,\phi} \, |p; 0, 0, \lambda_1, \lambda_2\rangle \\ &= \int \mathrm{d}\Omega \; D^{j*}_{m,\lambda_1-\lambda_2}(\phi, \theta, -\phi) \, |p; \theta, \phi, \lambda_1, \lambda_2\rangle \end{aligned} \tag{3.46}$$

with $\mathrm{d}\Omega = \sin\theta d\theta d\phi$

3.1.8 Expansion of the S-Matrix in Spherical Basis

It is now straightforward to expand the S-matrix for a process

$$a + b \to c + d \tag{3.47}$$

in terms of spherical basis states. The initial states are chosen to propagate parallel to the z-axis with common absolute three-momentum p_i and $\lambda_i \equiv \lambda_a - \lambda_b$ and the final state along a direction θ, ϕ with p_f and $\lambda_f \equiv \lambda_c - \lambda_d$. After splitting the S-matrix in such a way that

$$S = 1 + i(2\pi)^4 \delta^{(4)}(P_a^\mu + P_b^\mu - P_c^\mu - P_d^\mu) T \tag{3.48}$$

we can write down the part T of the S-matrix as [20]

$$T_{fi} = (2\pi)^2 \frac{E_{\mathrm{CM}}}{\sqrt{p_i p_f}} \langle \theta, \phi, \lambda_c \lambda_d | T(E_{\mathrm{CM}}) | 0, 0, \lambda_a \lambda_b \rangle \tag{3.49}$$

and make use of the conservation of angular momentum on it by inserting complete sets of spherical helicity states using the matrix element (3.45)

$$T_{fi} = (2\pi)^2 \frac{E_{\mathrm{CM}}}{\sqrt{p_i p_f}} \sum_{j,m} \sum_{j',m'} \sqrt{\frac{2j+1}{4\pi}} \sqrt{\frac{2j'+1}{4\pi}} D^j_{m\lambda_f}(\phi,\theta,-\phi)\delta_{mm'}\delta_{jj'} \\ \cdot \langle jm\lambda_c\lambda_d|T^j(s)|j'm'\lambda_a\lambda_b\rangle\, D_{m'\lambda_i}(0,0,0), \tag{3.50}$$

where $\lambda_i \equiv \lambda_a - \lambda_b$ and $\lambda_f = \lambda_c - \lambda_d$. Now let

$$4\pi \frac{E_{\mathrm{CM}}}{\sqrt{p_i p_f}} (2j+1) \langle jm\lambda_c\lambda_d|T^j(s)|j'm'\lambda_a\lambda_b\rangle \equiv T^j_{\lambda_a\lambda_b;\lambda_c\lambda_d}. \tag{3.51}$$

Then, our final formula is:

$$T_{fi} = \sum_j T^j_{\lambda_a\lambda_b;\lambda_c\lambda_d} d^j_{\lambda_i\lambda_f} e^{i(\lambda_i-\lambda_f)\phi} \tag{3.52}$$

3.1.9 Cross Section for a Non-Relativistic Initial State

Our final goal is to calculate the cross section of the neutralino annihilation in a non-relativistic limit, as pointed out before. Therefore is it not necessary to calculate the full relativistic Feynman diagrams but only some selected parts of it that can contribute to the process. For the treatment of partial waves in the non-relativistic limit it is more convenient to use another basis, and since we are interested in the total cross section, so do not need to know the dependence of the polarizations of the initial particles, we may choose the the orbital angular momentum l and the total spin s instead of helicity quantum numbers. That has the following advantages.

Firstly, an expansion of the initial plane wave at the same time in l and v yields coefficients that behave like $c_l v^l + c_{l+2} v^{l+2} + \ldots$, what reduces the initial states to account for to S-($l = 0$) and P-($l = 1$) wave states since the cross section goes with the squared amplitude. One can prove this statement by analyzing the representation of a plane wave propagating with absolute momentum p in θ direction in terms of Legendre polynomials P_l and spherical Bessel functions j_l:

$$\langle p, \theta|\mathbf{x}\rangle = \sum_{l=0}^{\infty} i^l (2l+1) j_l(p\,|\mathbf{x}|) P_l(\cos\theta). \tag{3.53}$$

The argument of the j_l is proportional to v in the non-relativistic limit and the j_l itself have an expansion around zero exactly like shown above.

Secondly, the number of selection rules is increased since any eigenstate of the total orbital angular momentum l is a definite state of parity $\mathcal{P}$:

$$\mathcal{P} = \eta_1 \cdots \eta_n (-1)^l\,|l\rangle \tag{3.54}$$

for a coupled system of n particles owing the intrinsic parities η_i with $|\eta_i| = 1$. The phases are only fixed relative to each other, for example a product of fermionic and antifermionic intrinsic parities has to be odd. In our case of two Majorana fermions we also have to take Fermi statistics into account, which fixes the symmetric S-wave to an antisymmetric spin state $s = 0$ (and consequently $j = 0$) and the P -wave to a symmetric $s = 1$ state. Combining the parity with the charge conjugation (which is even for two equal Majorana fermions) we may further state that the S-wave must be $\mathcal{CP}$-odd and the P-wave $\mathcal{CP}$-even. This gives a restriction to the neutral Higgs particles that can be exchanged in the s-channel.

Altogether, we find that only four amplitudes arising from the states ${}^1S_0$, ${}^3P_0$, ${}^3P_1$ and ${}^3P_2$ have to be included. Before we can construct the cross section out of them, we need to know the projection matrix [10] between the actual spherical helicity basis and the l, s basis. For that we first use (3.46) to express the helicity states in terms of z-components m_1 , m_2 of the individual spins s_1, s_2 ($\lambda \equiv \lambda_1 - \lambda_2$):

$$|p; jm\lambda_1\lambda_2\rangle = N_j \sum_{m_1,m_2} \int \mathrm{d}\Omega\; D^{j*}_{m\lambda} D^{s_1}_{m_1\lambda_1} D^{s_2}_{m_2,-\lambda_2}. \tag{3.55}$$

The product of the D-functions may be simplified using the coupling rule (3.21) and the symmetry properties of the D-functions and the CGC (A.2) [7]:

$$\begin{aligned} D^{j*}_{m\lambda} D^{s_1}_{m_1\lambda_1} D^{s_2}_{m_2,-\lambda_2} = \sum_{l=|l-s|}^{j+s} \sum_{s=s_1-s_2}^{s_1+s_2} \frac{2l+1}{2j+1} \mathscr{C}(lsj;0,\lambda)\mathscr{C}(s_1s_2s;\lambda_1,-\lambda_2) \\ \cdot \mathscr{C}(lsj;l_z,m_1+m_2)\mathscr{C}(s_1s_2sm_1m_2) D^{l*}_{m_l0}. \end{aligned} \tag{3.56}$$

After the decomposition of the coupled state yields

$$|p; jmls\rangle = \sum_{m_1,m_2} \mathscr{C}(lsj;l_z,m_1+m_2)\mathscr{C}(s_1s_2s;m_1m_2)\,|p; ll_z; m_1m_2\rangle \tag{3.57}$$

and the expansion of the last uncoupled state in a manner of (3.46)

$$|p; ll_z; m_1m_2\rangle = \sqrt{\frac{2l+1}{4\pi}} \int \mathrm{d}\Omega\; D^{l*}_{l_z0}\,|p;\theta\phi;m_1m_2\rangle\,, \tag{3.58}$$

so finally we obtain

$$\begin{aligned} |p; jm\lambda_1\lambda_2\rangle &= \sum_{l,s} \sqrt{\frac{2l+1}{2j+1}} \mathscr{C}(lsj;l_z,0,\lambda)\mathscr{C}(s_1s_2s;\lambda_1,-\lambda_2)\,|p; jmls\rangle \\ &\equiv \sum_{l,s} \sqrt{\frac{2l+1}{2j+1}} \mathscr{P}_{\lambda_1\lambda_2}({}^{2s+1}l_j)\,|p; jmls\rangle \end{aligned} \tag{3.59}$$

Some of the projectors $\mathscr{P}$ have the following explicit expressions:

$$\mathscr{P}_{\lambda_1\lambda_2}({}^1S_0) = \frac{1}{\sqrt{2}}(-1)^{\lambda_2-\frac{1}{2}}\delta_{\lambda_1\lambda_2}, \tag{3.60}$$

$$\mathscr{P}_{\lambda_1\lambda_2}({}^3P_0) = -\frac{1}{\sqrt{6}}\delta_{\lambda_1\lambda_2}. \tag{3.61}$$

These are the states that we will use for the calculations in next section. Expanding the above amplitude $T^j_{\lambda_a\lambda_b;\lambda_c\lambda_d}$ in the l, s initial states defines the helicity amplitudes $A(^{2s+1}l_j)$ and the full amplitude in partial wave expansion is written as:

$$T_{fi} = \sum_{j,l,s} A(^{2s+1}l_j)\mathscr{P}_{\lambda_a\lambda_b}(^{2s+1}l_j)d^j_{\lambda_i\lambda_f}(\theta)e^{i(\lambda_i-\lambda_f)}. \tag{3.62}$$

whereas the $A(^{2s+1}l_j)$ can be obtained out of the $T^j_{\lambda_a\lambda_b;\lambda_c\lambda_d}$ by taking use of the completeness property of the projectors:

$$A(^{2s+1}l_j) = \sum_{\lambda_a,\lambda_b} T^j_{\lambda_a\lambda_b;\lambda_c\lambda_d}\mathscr{P}_{\lambda_a\lambda_b}(^{2s+1}l_j). \tag{3.63}$$

The advantage of the spherical basis shows up when computing the unpolarised cross section. Since the angular part factorises in terms of the D we can easily integrate it out using the orthogonality relation (A.6) and furthermore simplify the cross section with CGC orthogonalities. The following formulae already regard the initial state particles to have the same mass and therefore same energy $E_{\mathrm{CM}}/2$ and v is again the relative velocity. Moreover, we set $\phi = 0$ for all calculations from now on since that does not change the final results. We obtain:

$$\begin{aligned}
\sigma(\chi\chi \to f\bar{f}) &= \frac{1}{E_{\mathrm{CM}}v}\int \frac{\mathrm{d}^3p_c}{(2\pi)^3 2E_c}\frac{\mathrm{d}^3p_d}{(2\pi)^3 2E_d}(2\pi)^4\delta^{(4)}(P^\mu_a + P^\mu_b - P^\mu_c - P^\mu_d)\frac{1}{4}\sum_{\substack{\lambda_a,\lambda_b,\\ \lambda_c,\lambda_d}} |T_{fi}|^2 \\
&= \frac{p_f}{4\pi E^3_{\mathrm{CM}}v}\int \frac{\mathrm{d}\Omega}{4\pi}\frac{1}{4}\sum_{\substack{\lambda_a,\lambda_b,\\ \lambda_c,\lambda_d}}\left|\sum_{j,l,s} A(^{2s+1}l_j)\mathscr{P}_{\lambda_a\lambda_b}(^{2s+1}l_j)d^j_{\lambda_i\lambda_f}(\theta)\right|^2 \\
&= \frac{p_f}{4\pi E^3_{\mathrm{CM}}v}\sum_{\substack{\lambda_c,\lambda_d,\\ j,l,s}}\frac{1}{2l+1}\left|A(^{2s+1}l_j)\right|^2.
\end{aligned} \tag{3.64}$$

With the kinematical factor $\bar{\beta}_f = \frac{2p_f}{\sqrt{s}}$ the important part of our cross section according to the expansion up to order $\mathcal{O}(v^2)$ is given by

$$\sigma(\chi\chi \to f\bar{f})v = \frac{\bar{\beta}_f}{32\pi E^2_{\mathrm{CM}}Q}\sum_{\lambda_c,\lambda_d}\left[\left|A(^1S_0)\right|^2 + \frac{1}{3}\left(\left|A(^3P_0)\right|^2 + \left|A(^3P_1)\right|^2 + \left|A(^3P_2)\right|^2\right)\right], \tag{3.65}$$

with Q being a symmetry factor that is 2 for identical final particles and 1 otherwise.

3.2 Calculation of the Diagrams

3.2.1 Construction of the External State Wave Functions

In order to calculate Feynman diagrams in the helicity formalism we have to refine the usual four-spinors of fermions into helicity eigenstates and moreover express them in terms of d-functions since we later want to expand combined expressions in terms of these. We will use exactly the approach that has been explained in Sec. 3.1.6, i.e. we will take the corresponding solutions of the equations of motion in the rest frame, then boost and rotate them, ending up with expressions describing particles with arbitrary energy E propagating into an arbitrary direction θ. For reasons of simplicity we fix the azimuth angle to $\phi = 0$ which of course does not change the final results of the S-matrix. We further work only in the CM frame of a two particle system, therefore we additionally define a corresponding particle 2 moving with the same absolute value p of three-momentum in the opposite direction $-\theta$. To distinct between both representations, we label them with the labels (1) and (2) and denote the momentum of particle two with $\tilde{p}$.

We will start with the Dirac equation [20] in momentum space with zero three-momentum ($E = m, p = 0$):

$$(\gamma_\mu p^\mu - m)u(p=0) = 0 \rightarrow (m\gamma^0 - m)u(0) = m \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u(0) = 0. \tag{3.66}$$

Its solution for the two values $h = \pm 1/2$[2] of the helicity is given as the direct product of two spinors being labelled here as the *Weyl spinor*, containing only the angular dependence, and the *Lorentz spinor*, describing the dependence of energy, mass and momentum:

$$u_h(0)\chi_h(\theta = 0) \otimes \xi_h(p = 0), \tag{3.67}$$

with

$$\chi_+(\theta = 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_-(\theta = 0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.68}$$

and

$$\xi_h(p = 0) = \sqrt{m} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.69}$$

the normalisation being in accordance with (3.40). The following boost with *rapidity*, η, in z-direction, realised by the operator [20]

$$e^{-\frac{\eta}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}} \tag{3.70}$$

of the Lorentz group, affects only the Lorentz part of the spinor, while rotations done by the d-matrix in $j = 1$ representation only affect the Weyl part. We end up with the spinors

$$\chi_h(\theta) = \sqrt{m} \begin{pmatrix} d^{\frac{1}{2}}_{\frac{1}{2},h}(\theta) \\ d^{\frac{1}{2}}_{-\frac{1}{2},h}(\theta) \end{pmatrix}, \quad \xi_h(p) = \sqrt{m} \begin{pmatrix} e^{-h\eta} \\ e^{h\eta} \end{pmatrix} \tag{3.71}$$

[2] From now we will denote it $\pm$

where the exponential $e^{\pm h\eta} = \gamma(1 \pm 2h\beta)$ is a function of the kinematical quantities $\gamma = \frac{E}{m}$ and $\beta = \frac{p}{E}$. The corresponding particle 2 with helicity h can be obtained by boosting $u_{-h}(0)$ along the negative z-axis and then rotating about the same angle θ. The antiparticle wave functions are defined through $v = \mathcal{C}\bar{u}^T$. In the chiral representation the charge conjugation operator is $\mathcal{C} = i\sigma^2 \otimes \sigma^3$. After introducing another spinor

$$\zeta_h(p) = \sqrt{m}\begin{pmatrix} (-1)^{h-\frac{1}{2}} e^{-h\eta} \\ (-1)^{h+\frac{1}{2}} e^{h\eta} \end{pmatrix} \tag{3.72}$$

we can summarise our results which are a complete set of spinors that can be used to calculate Feynman diagrams and have the appealing property that the angular and energy dependences are separated:

$$u_h^{(1)}(p,\theta) = \chi_h(\theta) \otimes \xi_h(p), \tag{3.73}$$

$$u_h^{(2)}(\tilde{p},\theta) = \chi_{-h}(\theta) \otimes \xi_h(p), \tag{3.74}$$

$$v_h^{(1)}(p,\theta) = -\chi_{-h}(\theta) \otimes \zeta_h(p), \tag{3.75}$$

$$v_h^{(2)}(\tilde{p},\theta) = \chi_h(\theta) \otimes \zeta_h(p). \tag{3.76}$$

It is important to note that only the χ and ζ spinors depend on the representation of the Clifford algebra.

3.2.2 Definition of the kinematical quantities

Our following discussions will refer to annihilation processes of the type

$$\chi\left[(E_0,p_0),h\right] + \chi\left[(E_0,p_0),h\right] \to f\left[(E_0,p_0),\lambda\right] + \bar{f}\left[(E_0,p_0),\bar{\lambda}\right] \tag{3.77}$$

calculated in the CM frame. Hence, the initial state three-momenta ($\pm p_0$) are parallel to the z-axis and the final state three-momenta ($p_f = -p_{\bar{f}}$) are parallel to the θ-direction. This frame is called the initial frame. It is of course possible and sometimes even easier to do calculations in the final frame where the final states propagate along the z-axis and the initial states along the $(-\theta)$-direction. In the present case this would yield to trivial expressions for the final momenta, spinors and polarization states since the according d- functions reduce to Delta functions at $\theta = 0$, but nontrivial expressions for the according initial momenta, spinors and polarisation states.

Since we expand all functions of the kinematical variables in the relative velocity v of the neutralinos we can express them in terms of the mass ratios $R \equiv m_f/m_\chi = m_{\bar{f}}/m_\chi$[3] only. The conservation of energy and momentum then dictates the final state energies and momenta to be

$$E_f = E_{\bar{f}} = E_0, \tag{3.78}$$

[3]Since the mass of a particle and its antiparticle is the same. From now we will denote it as m

$$p_f = p_{\bar{f}} = \bar{\beta}_f E_0 = \bar{\beta}_f m_\chi \left[1 + \frac{v^2}{8} + \mathcal{O}(v^4)\right], \tag{3.79}$$

while the kinematical factor $\bar{\beta}_f$ is

$$\bar{\beta}_f \equiv \sqrt{1 - 4\frac{m}{E_{\text{CM}}}} \tag{3.80}$$

with $E_{\text{CM}} = E_0$. In our following calculations we will expand the function $\bar{\beta}_f$ and the s-channel propagator $S(q_s^2)$:

$$\bar{\beta}_f(v) = \bar{\beta}_f(v=0) + \frac{v^2 R^2}{8\bar{\beta}_f(0)} + \mathcal{O}(v^4), \tag{3.81}$$

$$S(q_s^2) = \frac{1}{m_\chi(4 - R_s^2)}\left(1 - \frac{v^2}{m_\chi(4 - R_s^2)}\right) + \mathcal{O}(v^4), \tag{3.82}$$

where $R_s \equiv m_s/m_\chi$ is the ratio of the propagating particle.

3.2.3 Expansion of the Amplitudes in J

The main task for calculating a $2 \to 2$ tree-level processes in the presented formalism is to expand the combined functions of different $d^{1/2}$, d^1 in terms of $d^j_{\lambda_i,\lambda_f}$ since the final result should have a structure like (3.62), where the coefficients are Taylor expansions up to $\mathcal{O}(v^2)$. The basic step to combine d-functions is to use the coupling rule (3.21) in such a way that one finally ends up with d-functions with the indices λ_i, λ_f while all other index contributions should vanish. Due to the type of process they can only exist in two frames denoted by the angles 0 and θ. In the first case all related d-functions reduce to Kronecker Delta between the corresponding component of the quantity (spinor or vector respectively) and the helicity. In the second case we get non-trivial d-functions that are already constructed in such a way that a contraction between two of them yields a summation in the indices according to the coupling rule (3.21), ending up with contributions only from the helicities of the according quantities. For vectors we can directly see that from the contraction rule (3.24)

$$\sum_m (-1)^m d_{m,\lambda_1} d^1_{-m,\lambda_2} = \sum_{j=0}^{1} d^j_{0,\lambda_1+\lambda_2} = \sum_m (-1)^m \mathscr{C}(11j; m, -m)\mathscr{C}(11j; \lambda_1\lambda_2). \tag{3.83}$$

The spinors $\chi_h = (\chi_{h,\frac{1}{2}}, \chi_{h,-\frac{1}{2}})$ have to be taken in the right representation for the corresponding particle using symmetry properties of the d-function:

$$\chi_{h,l} = \begin{cases} d^{\frac{1}{2}}_{l,h} & \text{for a particle in the } \theta\text{-direction.} \\ (-1)^{h-l} d^{\frac{1}{2}}_{-l,-h} & \text{for an antiparticle in the } (-\theta)\text{-direction.} \end{cases} \tag{3.84}$$

$$\chi_{-h,l} = \begin{cases} d^{\frac{1}{2}}_{l,-h} & \text{for a particle in the } (-\theta)\text{-direction.} \\ (-1)^{h-l} d^{\frac{1}{2}}_{-l,h} & \text{for an antiparticle in the } \theta\text{-direction.} \end{cases} \tag{3.85}$$

If we consider the mixed case we automatically get the right index structure so that particle 1 states contribute with their positive and particle 2 states with their negative helicity on the appropriate side. This can be seen from the contraction rule for vectors or the construction of the spinors respectively. It originates from the group property of rotations, namely the fact that successive rotations behave additively in the angle. Assuming in particular the initial angle to be θ_2, leading to general helicity states proportional to $d^1_{\lambda_2,-m}(\theta_2)$, it then follows for a vector contraction:

$$\sum_m (-1)^m d^1_{m,\lambda_2,m}(\theta) d^1_{\lambda_2,m}(\theta_2) = (-1)^{\lambda_2} d_{\lambda_2,\lambda_1}(\theta - \theta_2) \tag{3.86}$$

After discussing separate contractions of vectors and spinors we consider the general case. We will present three ways of computing the Feynman amplitude. At first, by expanding the appearing bilinears in the amplitude into some of the fundamental bilinear covariants. This can be done by first computing the elementary two-spinor bilinears for the ξ, ζ and χ spinors respectively and then combine them using the tensor product representation of the covariants. In chiral representation these are given by:

$$\gamma^i = -\sigma^i \otimes i\sigma_y, \tag{3.87}$$

$$\gamma^5 = -\mathbb{1} \otimes \sigma_z, \tag{3.88}$$

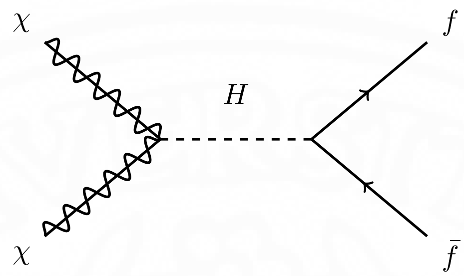
3.2.4 Expression of the Helicity Amplitudes

We will subsequently list the nonvanishing amplitudes for the two relevant spectroscopical initial states, namely the ${}^1S_0$ and ${}^3P_0$ states. The result will be given separately for different final state helicities starting with the conventional amplitude with definite initial and final helicities using the notation introduced in (3.77). The expression of the propagators are given by (3.82).

The c_i, c_{ij} are coupling coefficients (in some cases collected in the matrices $C = \text{diag}(c_1, c_2)$ and $C_i = \text{diag}(c_{i1}, ci2))$ that have to be replaced with the amplitudes for standard MSSM vertices. [6,23]

Since the neutralinos are Majorana fermions we have the freedom to chose whether to take the u- or $\bar{v}$-spinor for both initial states or the direction of the fermion line respectively. In our cases we use $u^{(1)}$ (or respectively $\bar{v}^{(1)}$) for the neutralino in positive (negative) z-direction. This fixes the fermion line in such a way that we do not have to take charge conjugated expressions for the vertices in each of other cases.

For the computation of the cross section, full left/right mixing is taken into account, but neglected generation mixing. From now, the spinor indicated by λ and $\bar{\lambda}$ refer to the fermion and antifermion respectively.



Annihilation via Higgs Pseudoscalar

In the s-channel:

$$\begin{aligned}
T_{fi} &= \bar{v}_{\bar{h}}^{(2)}(c_{11}P_L + c_{12}P_R)u_h^{(1)}S(q_s^2)\bar{u}_{\bar{\lambda}}^{(1)}c_2\gamma_5 v_{\bar{\lambda}}^{(2)} \\
&= c_2\delta_{\bar{h}h}\delta_{\bar{\lambda}\lambda}S(q_s^2)v_{\bar{h}}^{\dagger}\sigma_x \underbrace{(c_{11}P_L + c_{12}P_R)}_{C_1} u_h u_{\bar{\lambda}}^{\dagger}\sigma_x(-\sigma_z)v_{\bar{\lambda}} \\
&= c_2\delta_{\bar{h}h}\delta_{\bar{\lambda}\lambda}S(q_s^2)\zeta_{\bar{h}}^T\sigma_x C_1\xi_h\xi_{\lambda}^T\sigma_x(-\sigma_z)\zeta_{\bar{\lambda}} \\
&= c_2\delta_{\bar{h}h}\delta_{\bar{\lambda}\lambda}\frac{1}{(4-R_s)^2}\left(1-\frac{v^2}{m_\chi^2(4-R_s^2)}\right) \\
&\times\frac{1}{2}\left[(-1)^{\bar{h}+\frac{1}{2}}c_{11}e^{-\eta(\bar{h}+h)} + (-1)^{\bar{h}-\frac{1}{2}}c_{12}e^{\eta(\bar{h}+h)}\right]\left[(-1)^{\bar{\lambda}-\frac{1}{2}}e^{\eta(\bar{\lambda}+\lambda)} + (-1)^{\bar{\lambda}-\frac{1}{2}}e^{-\eta(\bar{\lambda}+\lambda)}\right]
\end{aligned} \tag{3.89}$$

At this point some things must be clarified. Firstly $h = \bar{h}$ and $\lambda = \bar{\lambda}$ and this must hold because o the two Kronecker delta.[4] The second is that applying the expansion given in 3.2.1 and doing a Taylor approximation until $\mathcal{O}(v^3)$ since the incoming particles are non relativistic then

$$\begin{aligned}
T_{fi} = c_2&\frac{1}{2(4-R_s^2)}\left(1-\frac{v^2}{m_\chi^2(4-R_s^2)}\right) \\
&\times\left(1+v+\frac{v^2}{2}\right)^2\left[(-1)^{h+\frac{1}{2}}c_{11} + (-1)^{h-\frac{1}{2}}c_{12}\right]\left[(-1)^{\lambda+\frac{1}{2}} + (-1)^{\lambda-\frac{1}{2}}\right]
\end{aligned} \tag{3.90}$$

And this result should yield from the contributions to the amplitude of two total spin states of $s = 0$ and $s = 1$ multiplied by the projectors $\mathscr{P}$. These are theoretically [12]

1. $A(^1S_0)$

$$(-1)^{\bar{\lambda}+\frac{1}{2}}\delta_{\lambda_f 0}\frac{2\sqrt{2}c_2(c_{11}-c_{12})}{R_s^2-4}\left(1+\frac{v^2}{4}\right) \tag{3.91}$$

[4] We are going to denote for simplicity for now $2h \equiv h + \bar{h}$ and $2\lambda \equiv \lambda + \bar{\lambda}$.

2. $A(^3P_0)$

$$v(-1)^{\bar{\lambda}+\frac{1}{2}}\delta_{\lambda_f 0}\frac{\sqrt{6}c_2(c_{11}+c_{12})}{R_s-4} \tag{3.92}$$

Annihilation via Higgs Scalar

In the s-channel:

$$\begin{aligned} T_{fi} &= \bar{v}_{\bar{h}}^{(2)}(c_{11}P_L + c_{12}P_R)u_h^{(1)}S(q_s^2)\bar{u}_{\bar{\lambda}}^{(1)}c_2 v_{\bar{\lambda}}^{(2)} \\ &= c_2\delta_{\bar{h}h}\delta_{\bar{\lambda}\lambda}S(q_s^2)v_{\bar{h}}^{\dagger}\sigma_x \underbrace{(c_{11}P_L + c_{12}P_R)}_{C_1} u_h u_{\bar{\lambda}}^{\dagger}\sigma_x v_{\bar{\lambda}} \\ &= c_2\delta_{\bar{h}h}\delta_{\bar{\lambda}\lambda}S(q_s^2)\zeta_{\bar{h}}^T\sigma_x C_1\xi_h\xi_{\lambda}^T\sigma_x\zeta_{\bar{\lambda}} \end{aligned} \tag{3.93}$$

And subsequently:

1. $A(^1S_0)$

$$\delta_{\lambda_f 0}\frac{2\sqrt{2}\bar{\beta}_f c_2(c_{11}-c_{12})}{R_s^2-4} - v^2\frac{c_2(c_{11}-c_{12})\left[(R_f^2-2)R_s+4R_f^2\right]}{2\sqrt{2}\bar{\beta}_f(R_s^2-4)^2} \tag{3.94}$$

2. $A(^3P_0)$

$$v\delta_{\lambda_f 0}\frac{\sqrt{6}\bar{\beta}_f c_2(c_{11}-c_{12})}{R_s^2-4} \tag{3.95}$$

Chapter 4

Conclusions

Once with the expressions of the amplitude we could compute the cross section of the neutralino annihilation into a fermion pair. As we pointed out before, this calculation is of great relevance, since it with it we can compute for example the LSP relic density. However, here we only address a computation with annihilation into fermion pairs. A more refined work would include more possible states with more difficult particles, but since this is only a bachelor thesis we only include one of these possibilities to have a brief and a enlightening idea about how does this work.

The obtained explicit expressions for the amplitudes T_{fi} include the theoretical results shown. In order to get them from the theoretical expression we can take (3.62) to verify them. This is not shown until the very last step in this thesis due to date problems. This document was crafted in a very short time taking into account the needed to take the calculations until the last stage.

Nevertheless, we can see that the explicit calculations are in the right way and they would eventually show up the results we addressed, and that is the important issue to take into account.

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Appendices

Appendix A

Properties of the Wigner functions

In this section we summarise useful relations needed for handling both, the D- and d-functions. More concretely, these are orthogonality and completeness as well as symmetry relations and simplifications for certain cases of the indices or arguments.

A.1 Orthogonality and Completeness Relations

$$\sum_{m''=-j} D^j_{m'm''}(\alpha,\beta,\gamma) D^{j*}_{mm''}(\alpha,\beta,\gamma) = \delta_{mm'} \tag{A.1}$$

$$\sum_{m''=-j}^{j} d_{m'm''j}(\beta) d^j_{mm''}(\beta) = \delta_{mm'} \tag{A.2}$$

$$\sum_{j=0}^{\infty} \sum_{m,m'=-j}^{j} \frac{2j+1}{8\pi^2} D^{j*}_{mm'}(\alpha_2,\beta_2,\gamma_2) = \delta(\alpha_1-\alpha_2)\delta(\cos\beta_1-\cos\beta_2)\delta(\gamma_1-\gamma_2) \tag{A.3}$$

$$\sum_{j=0}^{\infty} \frac{2j+1}{2} d^j_{mm'}(\beta_1) d^j_{mm'}(\beta_2) = \delta(\cos\beta_1-\cos\beta_2) \tag{A.4}$$

$$\int \mathrm{d}\alpha\ \sin\beta\,\mathrm{d}\beta\,\mathrm{d}\gamma\ D^{j_1}_{m_1m'_1}(\alpha,\beta,\gamma) = \frac{8\pi^2}{2j_1+1}\delta_{j_1j_2}\delta_{m_1m_2}\delta_{m'_1m'_2} \tag{A.5}$$

$$\int \sin\beta\,\mathrm{d}\beta\ d^{j_1}_{mm'}(\beta) d^{j_2}_{mm'}(\beta) = \frac{2}{2j_1+1}\delta_{j_1j_2} \tag{A.6}$$

A.2 Symmetry Relations

$$D^j_{-m,-m'}(\alpha,\beta,\gamma) = D^j_{mm'}(-\alpha,-\beta,-\gamma) = D^{j*}_{mm'}(\alpha,-\beta,\gamma) \tag{A.7}$$

$$D^j_{mm'}(-\alpha,-\beta,-\gamma) = D^j_{m'm}(\gamma,\beta,\alpha) = D^{j*}_{mm'}(-\gamma,-\beta,-\alpha) \tag{A.8}$$

$$D^j_{mm'}(\alpha, \beta, \gamma) = (-1)^{m-m'} D_{m'm}(\gamma, \beta, \alpha) \tag{A.9}$$

$$D^j_{m,m'}(\alpha, \pi - \beta, \gamma) = (-1)^{j+m} D_{m,-m'}(\alpha, \beta, -\gamma) \tag{A.10}$$

$$d^j_{-m,-m'}(\beta) = d^j_{mm'}(-\beta) = d^j_{m'm}(\beta) \tag{A.11}$$

$$d^j_{mm'}(\beta) = (-1)^{m-m'} d^j_{m'm}(\beta) \tag{A.12}$$

$$d^j_{mm'}(\pi - \beta) = (-1)^{j+m} d^j_{m,-m'}(\beta) \tag{A.13}$$

A.3 Special Cases

From now, the $Y_{lm}(\alpha, \beta)$ are the spherical harmonics and $P_l(\cos\beta)$ are the Legendre polynomials.

$$D^l_{m0}(\alpha, \beta, \gamma) = \sqrt{\frac{4\pi}{2l+1}} Y^*_{lm}(\alpha, \beta) \tag{A.14}$$

$$D^l_{00}(\alpha, \beta, \gamma) = d^l_{00}(\beta) = P_l(\cos\beta) \tag{A.15}$$

$$D^l_{mm'}(0, 0, 0) = d^j_{mm'}(0) = \delta_{mm'} \tag{A.16}$$

$$D^l_{m0}(0, \pi, 0) = d_{mm'}(0)^j = (-1^{j+m})\delta m, -m' \tag{A.17}$$

Appendix B

Pauli Matrices

These matrices are a set of three 2×2 complex matrices which are Hermitian and unitary. They are

$$\sigma_1 = \sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{B.1}$$

$$\sigma_2 = \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{B.2}$$

$$\sigma_3 = \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{B.3}$$

They verify:

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -i\sigma_1\sigma_2\sigma_3 = \mathbb{1}. \tag{B.4}$$

$$\det \sigma_i = 0. \tag{B.5}$$

$$\mathrm{Tr}\ \sigma_i = 0. \tag{B.6}$$

Appendix C

Gamma Matrices

The gamma-matrices satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{C.1}$$

Given the four-dimensional representation of the gamma-matrices introduced in the lecture and check explicitly that they satisfy:

$$\gamma^0 = (\gamma^0)^\dagger, \tag{C.2}$$

$$, \gamma^i = -(\gamma^i)^\dagger. \tag{C.3}$$

The matrix γ_5 is defined as $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. They satisfy that

$$\gamma^5 = (\gamma^5)^\dagger, \tag{C.4}$$

$$(\gamma^5)^2 = 1, \tag{C.5}$$

$$\{\gamma^5, \gamma^\nu\} = 0. \tag{C.6}$$