Chapter 5
Sine-Gordon Equation

The sine-Gordon equation is a nonlinear hyperbolic partial differential equation involving the d'Alembert operator and the sine of the unknown function. The equation, as well as several solution techniques, were known in the nineteenth century in the course of study of various problems of differential geometry. The equation grew greatly in importance in the 1970s, when it was realized that it led to *solitons* (so-called ”kink“ and ”antikink“). The sine-Gordon equation appears in a number of physical applications [14, 7, 27], including applications in relativistic field theory, Josephson junctions [21] or mechanical trasmission lines [24, 21].
The equation reads

$$u_{tt} - u_{xx} + \sin u = 0\,, \tag{5.1}$$

where $u = u(x,t)$. In the case of mechanical trasmission line, $u(x,t)$ describes an angle of rotation of the pendulums. Note that in the low-amplitude case ($\sin u \approx u$) Eq. (5.1) reduces to the Klein-Gordon equation

$$u_{tt} - u_{xx} + u = 0\,,$$

admiting solutions in the form

$$u(x,t) = u_0 \cos(kx - \omega t)\,, \qquad \omega = \sqrt{1+k^2}\,.$$

Here we are interested in large amplitude solutions of Eq. (5.1).

5.1 Kink and antikink solitons

Let us look for travelling wave solutions of the sine-Gordon equation (5.1) of the form

$$u(\xi) := u(x - ct)\,,$$

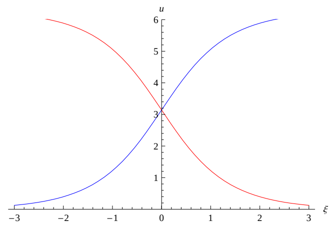

Fig. 5.1 Representation of the kink (blue) and antikink (red) solutions (5.4)

where c is an arbitrary velocity of propagation and $u \to 0$, $u_\xi \to 0$, when $\xi \to \pm\infty$ [21, 27]. In the co-moving frame Eq. (5.1) reads

$$(1-c^2)\,u_{\xi\xi} = \sin u\,.$$

Multiplying both sides of the last equation by u_ξ and integrating yields

$$\frac{1}{2}\,u_\xi^2\,(1-c^2) = -\cos u + c_1\,, \tag{5.2}$$

where c_1 is an arbitrary constant of integration. Notice that we look for solutions for which $u \to 0$ and $u_\xi \to 0$ when $\xi \to \pm\infty$, so $c_1 = 1$. Now we can rewrite the last equation as

$$\frac{d\,u}{\sin\frac{u}{2}} = \pm\frac{2}{\sqrt{1-c^2}}\,d\,\xi\,. \tag{5.3}$$

Integrating Eq. (5.3) yields

$$\pm\frac{2}{\sqrt{1-c^2}}\,(\xi-\xi_0) = 2\,\ln\left(\tan\frac{u}{4}\right),$$

or

$$u(\xi) = 4\,\arctan\left(\exp\left(\pm\frac{\xi-\xi_0}{\sqrt{1-c^2}}\right)\right).$$

That is, the solution of Eq. (5.1) becomes

$$u(x,\,t) = 4\,\arctan\left(\exp\left(\pm\frac{x-x_0-c\,t}{\sqrt{1-c^2}}\right)\right). \tag{5.4}$$

Equation (5.4) represents a localized solitary wave, travelling at any velocity $|c| < 1$. The $\pm$ signs correspond to localized solutions which are called *kink* and *antikink*, respectively. For the mechanical transmission line, when c increases from $-\infty$ to $+\infty$ the pendlums rotate from 0 to $2\,\pi$ for the kink and from 0 to $-2\,\pi$ for the antikink. (see Fig. 5.1)

One can show [14, 21], that Eq. (5.1) admits more solutions of the form

$$u(x,\,t) = 4\,\arctan\left(\frac{F(x)}{G(t)}\right).$$

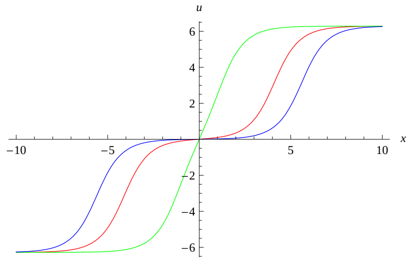

Fig. 5.2 The kink-kink collision, calculated at three different times: At $t = -7$ (red curve) both kinks propagate with opposite velocities $c = \pm 0.5$; At $t = 0$ they collide at the origin (green curve); At $t = 10$ (blue curve) they move away from the origin with velocities $c = \mp 0.5$.

where F and G are arbitrary functions. Namely, one distinguishes the kink-kink and the kink-antikink collisions as well as the breather solution. The *kink-kink collision* solution reads

$$u(x,t) = 4 \arctan\left(\frac{c \sinh\left(\frac{x}{\sqrt{1-c^2}}\right)}{\cosh\left(\frac{ct}{\sqrt{1-c^2}}\right)}\right) \tag{5.5}$$

and describes the collision between two kinks with respective velocities c and $-c$ and approaching the origin from $t \to -\infty$ and moving away from it with velocities $\pm c$ for $t \to \infty$ (see Fig. 5.2). In a similar way, one can construct solution, corresponding to the *kink-antikink collision*. The solution has the form:

$$u(x,t) = 4 \arctan\left(\frac{\sinh\left(\frac{ct}{\sqrt{1-c^2}}\right)}{c \cdot \cosh\left(\frac{x}{\sqrt{1-c^2}}\right)}\right) \tag{5.6}$$

The *breather* soliton solution, which is also called a *breather mode* or *breather soliton* [21], is given by

$$u_B(x,t) = 4 \arctan\left(\frac{\sqrt{1-\omega^2}\sin(\omega t)}{\omega \cosh(\sqrt{1-\omega^2}x)}\right) \tag{5.7}$$

which is periodic for frequencies $\omega < 1$ and decays exponentially when moving away from $x = 0$. Now we are in the good position to look for numerical solutions

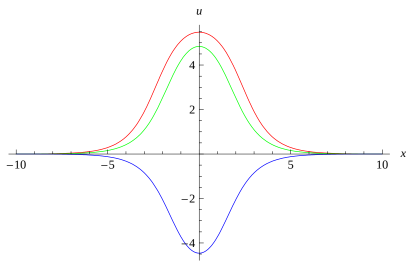

Fig. 5.3 The breather solution, oscillating with the frequency $\omega = 0.2$, calculated for three different times $t = 0$ (red curve), $t = 5$ (green curve) and $t = 10$ (blue curve).

of Eq. (5.1).

5.2 Numerical treatment

A numerical scheme

Consider an IVP for the sine-Gordon equation (5.1):

$$u_{tt} - u_{xx} + \sin(u) = 0$$

on the interval $x \in [a, b]$ with initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x)\,, \tag{5.8}$$

and with, e.g., no-flux boundary conditions

$$\left.\frac{\partial u}{\partial x}\right|_{x=a,b} = 0\,.$$

Let us try to apply a simple explicit scheme (4.9) to Eq. (5.1). The discretization scheme reads

$$u_i^{j+1} = -u_i^{j-1} + 2\,(1-\alpha^2)\,u_i^j + \alpha^2\,(u_{i+1}^j + u_{i-1}^j) - \triangle t^2 \sin(u_i^j) \tag{5.9}$$

with $\alpha = \triangle t/\triangle x$, $i = 0,\dots,M$ and $t = 0,\dots,T$. To the implementation of the second initial condition one needs again the virtual point u_i^{-1},

$$u_t(x_i,0) = g(x_i) = \frac{u_i^1 - u_i^{-1}}{2\triangle t} + \mathscr{O}(\triangle t^2)\,.$$

Hence, one can rewrite the last expression as

$$u_i^{-1} = u_i^1 - 2\triangle t g(x_i) + \mathscr{O}(\triangle t^2)\,,$$

and the second time row u_i^1 can be calculated as

$$u_i^1 = \triangle t\, g(x_i) + (1-\alpha^2)\, f(x_i) + \frac{1}{2}\,\alpha^2 (\, f(x_{i-1}) + f(x_{i+1})) - \frac{\triangle t^2}{2}\sin(\, f(x_i))\,. \tag{5.10}$$

In addition, no-flux boundary conditions lead to the following expressions for two virtual space points u_{-1}^j and u_{M+1}^j:

$$\left.\frac{\partial u}{\partial x}\right|_{x=a} = 0 \Leftrightarrow \frac{u_1^j - u_{-1}^j}{2\triangle x} = 0 \Leftrightarrow u_{-1}^j = u_1^j\,,$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=b} = 0 \Leftrightarrow \frac{u_M^j - u_{M+1}^j}{2\triangle x} = 0 \Leftrightarrow u_{M+1}^j = u_M^j\,.$$

One can try to rewrite the differential scheme to more general matrix form. In matrix notation the second time-row is given by

$$\boxed{\mathbf{u}^1 = \triangle t \gamma_1 + A\mathbf{u}^0 - \frac{\triangle t^2}{2}\beta_1,} \tag{5.11}$$

where

$$\begin{aligned} \gamma_1 &= \big(g(a), g(x_1), g(x_2), \ldots, g(x_{M-1}), g(b)\big)^T \quad \text{and} \\ \beta_1 &= \big(\sin(u_0^0), \sin(u_1^0), \ldots, \sin(u_{M-1}^0), \sin(u_M^0)\big)^T \end{aligned}$$

are $M+1$-dimensional vectors and A is a tridiagonal square $M+1 \times M+1$ matrix of the form

$$A = \begin{pmatrix} 1-\alpha^2 & \boxed{\alpha^2} & 0 & \ldots 0 \\ \alpha^2/2 & 1-\alpha^2 & \alpha^2/2 & \ldots 0 \\ 0 & \alpha^2/2 & 1-\alpha^2 & \ldots 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \boxed{\alpha^2} & 1-\alpha^2 \end{pmatrix}$$

The boxed elements indicate the influence of boundary conditions. Other time rows can also be written in the matrix form as

$$\boxed{\mathbf{u}^{j+1} = -\mathbf{u}^{j-1} + B\mathbf{u}^j - \triangle t^2 \beta, \quad j = 1, \ldots, T-1} \tag{5.12}$$

Here

$$\beta = \big(\sin(u_0^j), \sin(u_1^j), \ldots, \sin(u_{M-1}^j), \sin(u_M^j)\big)^T$$

is a $M+1$-dimensional vector and B is a square matrix, defined by an equation

$$B = 2A.$$

Now we can apply the explicit scheme (5.9) described above to Eq. (5.1). Let us solve it on the interval $[-L, L]$ with no-flux boundary conditions using the following parameters set:

`Space interval`	L=20
`Space discretization step`	$\triangle x = 0.1$
`Time discretization step`	$\triangle t = 0.05$
`Amount of time steps`	$T = 1800$
`Velocity of the kink`	$c = 0.2$

We start with the numerical representation of kink and antikink solutions. The initial condition for the kink is

$$
\begin{aligned}
f(x) &= 4\arctan\left(\exp\left(\frac{x}{\sqrt{1-c^2}}\right)\right), \\
g(x) &= -2\frac{c}{\sqrt{1-c^2}}\,\mathrm{sech}\left(\frac{x}{\sqrt{1-c^2}}\right).
\end{aligned}
$$

Figure 5.4 (a) shows the space-time plot of the numerical kink solution. For the antikink the initial condition reads

$$
\begin{aligned}
f(x) &= 4\arctan\left(\exp\left(-\frac{x}{\sqrt{1-c^2}}\right)\right), \\
g(x) &= -2\frac{c}{\sqrt{1-c^2}}\,\mathrm{sech}\left(\frac{x}{\sqrt{1-c^2}}\right).
\end{aligned}
$$

Numerical solutions is shown on Fig. 5.4 (b).

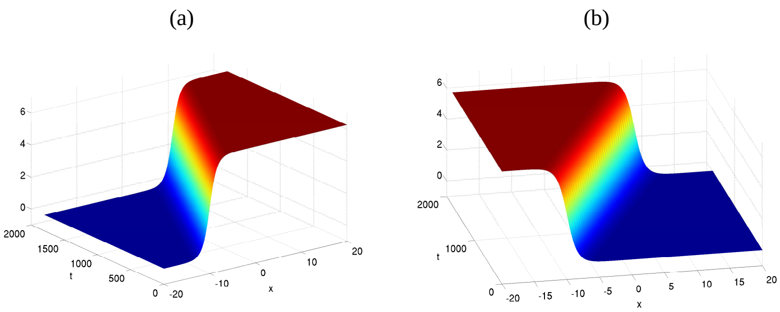

Fig. 5.4 Numerical solution of Eq. (5.1), calculated with the scheme (5.9) for the case of (a) the kink and (b) antikink solitons, moving with the velocity $c = 0.2$. Space-time information is shown.

Now we are in position to find numerical solutions, corresponding to kink-kink and kink-antikink collisions. For the kink-kink collision we choose

$$
\begin{aligned}
f(x) &= 4\arctan\left(\exp\left(\frac{x+L/2}{\sqrt{1-c^2}}\right)\right) + 4\arctan\left(\exp\left(\frac{x-L/2}{\sqrt{1-c^2}}\right)\right), \\
g(x) &= -2\frac{c}{\sqrt{1-c^2}}\,\mathrm{sech}\left(\frac{x+L/2}{\sqrt{1-c^2}}\right) + 2\frac{c}{\sqrt{1-c^2}}\,\mathrm{sech}\left(\frac{x-L/2}{\sqrt{1-c^2}}\right),
\end{aligned}
$$

whereas for the kink-antikink collision the initial conditions are

$$
\begin{aligned}
f(x) &= 4\arctan\left(\exp\left(\frac{x+L/2}{\sqrt{1-c^2}}\right)\right) + 4\arctan\left(\exp\left(-\frac{x-L/2}{\sqrt{1-c^2}}\right)\right), \\
g(x) &= -2\frac{c}{\sqrt{1-c^2}}\,\mathrm{sech}\left(\frac{x+L/2}{\sqrt{1-c^2}}\right) - 2\frac{c}{\sqrt{1-c^2}}\,\mathrm{sech}\left(\frac{x-L/2}{\sqrt{1-c^2}}\right).
\end{aligned}
$$

Numerical solutions, corresponding to both cases is presented on Fig. 5.5 (a)-(b), respectively. Finally, for the case of breather we choose

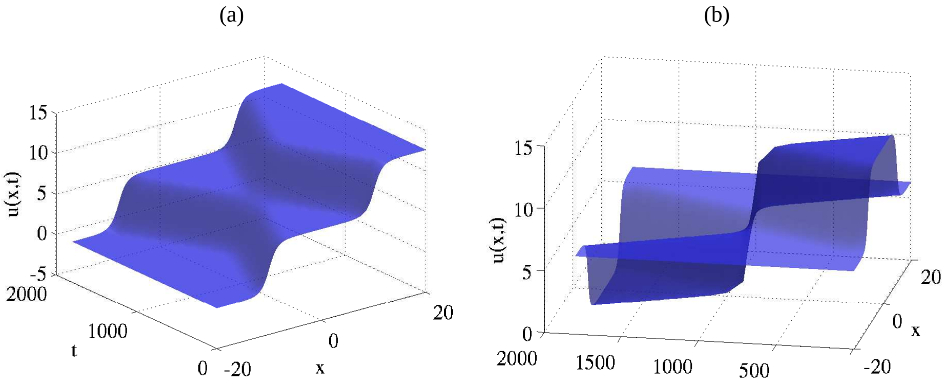

Fig. 5.5 Space-time representation of the numerical solution of Eq. (5.1) for (a) kink-kink collision and (b) kink-antikink collision.

$$f(x) = 0\,,$$
$$g(x) = 4\sqrt{1-c^2}\,\mathrm{sech}\left(x\sqrt{1-c^2}\right).$$

Corresponding numerical solution is presented on Fig. 5.6.

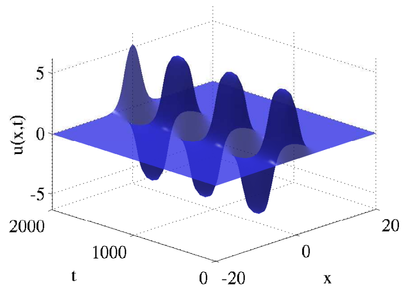

Fig. 5.6 Space-time plot of the numerical breather solution, oscillating with the frequency $\omega = 0.2$.