Appendix A
Tridiagonal matrix algorithm

The tridiagonal matrix algorithm (TDMA), also known als *Thomas algorithm*, is a simplified form of Gaussian elimination that can be used to solve tridiagonal system of equations

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = y_i, \qquad i = 1, \ldots n, \tag{A.1}$$

or, in matrix form ($a_1 = 0$, $c_n = 0$)

$$\begin{pmatrix} b_1 & c_1 & 0 & \ldots & \ldots & 0 \\ a_2 & b_2 & c_2 & \ldots & \ldots & 0 \\ 0 & a_3 & b_3 & c_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & c_{n-1} \\ 0 & \ldots & \ldots & 0 & a_n & b_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_n \end{pmatrix}$$

The TDMA is based on the Gaussian elimination procedure and consist of two parts: a forward elimination phase and a backward substitution phase [8]. Let us consider the system (A.1) for $i = 1 \ldots n$ and consider following modification of first two equations:

$$\mathrm{Eq}_{i=2} \cdot b_1 - \mathrm{Eq}_{i=1} \cdot a_2$$

which relults in

$$(b_1 b_2 - c_1 a_2) x_2 + c_2 b_1 x_3 = b_1 y_2 - a_2 y_1.$$

The effect is that x_1 has been eliminated from the second equation. In the same manner one can eliminate x_2, using the <u>modified</u> second equation and the third one (for $i = 3$):

$$(b_1 b_2 - c_1 a_2)\mathrm{Eq}_{i=3} - a_3(\text{mod. } \mathrm{Eq}_{i=2}),$$

which would give

$$(b_3(b_1 b_2 - c_1 a_2) - c_2 b_1 a_3) x_3 + c_3(b_1 b_2 - c_1 a_2) x_4 = y_3(b_1 b_2 - c_1 a_2) - (y_2 b_1 - y_1 a_2) a_3$$

If the procedure is repeated until the n'th equation, the last equation will involve the unknown function x_n only. This function can be then used to solve the modified equation for $i = n - 1$ and so on, until all unknown x_i are found (backward substitution phase). That is, we are looking for a backward ansatz of the form:

$$x_{i-1} = \gamma_i x_i + \beta_i. \tag{A.2}$$

If we put the last ansatz in Eq. (A.1) and solve the resulting equation with respect to x_i, the following relation can be obtained:

$$x_i = \frac{-c_i}{a_i\gamma_i + b_i} x_{i+1} + \frac{y_i - a_i\beta_i}{a_i\gamma_i + b_i} \tag{A.3}$$

This relation possesses the same form as Eq. (A.2) if we identify

$$\boxed{\gamma_{i+1} = \frac{-c_i}{a_i\gamma_i + b_i}, \qquad \beta_{i+1} = \frac{y_i - a_i\beta_i}{a_i\gamma_i + b_i}}. \tag{A.4}$$

Equation (A.4) involves the recursion formula for the coefficients γ_i and β_i for $i = 2, \ldots, n-1$. The missing values γ_1 and β_1 can be derived from the first ($i = 1$) equation (A.1):

$$x_1 = \frac{y_1}{b_1} - \frac{c_1}{b_1} x_2 \Rightarrow \gamma_2 = -\frac{c_1}{b_1}, \beta_2 = \frac{1}{b_1} \Rightarrow \boxed{\gamma_1 = \beta_1 = 0.}$$

The last what we need is the value of the function x_n for the first backward substitution. We can obtain if we put the ansatz

$$x_{n-1} = \gamma x_n + \beta_n$$

into the last ($i = n$) equation (A.1):

$$a_n(\gamma x_n + \beta_n) + b_n x_n = y_n,$$

yielding

$$x_n = \frac{y_n - a_n\beta_n}{a_n\gamma_n + b_n}.$$

One can get this value directly from Eq. (A.2), if one formal puts

$$x_{n+1} = 0.$$

Altogether, the TDMA can be written as:

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1.Set γ1 = β1 = 0;
2.Evaluate for i = 1,...,n−1
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$$\gamma_{i+1} = \frac{-c_i}{a_i\gamma_i + b_i}, \qquad \beta_{i+1} = \frac{y_i - a_i\beta_i}{a_i\gamma_i + b_i};$$

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3.Set x_{n+1} = 0;
4.Find for i = n+1,...,2
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$$x_{i-1} = \gamma_i x_i + \beta_i.$$

The algorithm admits $\mathscr{O}(n)$ operations instead of $\mathscr{O}(n^3)$ required by Gaussian elimination.

Limitation

The TDMA is only applicable to matrices that are diagonally dominant, i.e.,

$$|b_i| > |a_i| + |c_i|, \qquad i = 1,\dots,n.$$

Appendix B
The Method of Characteristics

The method of characteristics is a method which can be used to solve *an initial value problem* for general first order PDEs [2]. Let us consider a quasilinear equation of the form

$$A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial t} + Cu = 0, \qquad u(x,0) = u_0, \tag{B.1}$$

where $u = u(x,t)$, and A, B and C can be functions of independent variables and u. The idea of the method is to change coordinates from (x,t) to a new coordinate system (x_0,s), in which Eq. (B.1) becomes *an ordinary differential equation* along certain curves in the (x,t) plane. Such curves, $(x(s),t(s))$ along which the solution of (B.1) reduces to an ODE, are called the *characteristic curves*. The variable s can be varied, whereas x_0 changes along the line $t = 0$ on the plane (x,t) and remains constant along the characteristics. Now if we choose

$$\frac{dx}{ds} = A, \quad \text{and} \quad \frac{dt}{ds} = B, \tag{B.2}$$

then we have

$$\frac{du}{ds} = u_x\frac{dx}{ds} + u_t\frac{dt}{ds} = Au_x + Bu_t,$$

and Eq. (B.1) becomes the ordinary differential equation

$$\frac{du}{ds} + Cu = 0 \tag{B.3}$$

Equations (B.2) and (B.3) give the characteristics of (B.1).
That is, a general strategy to find out the characteristics of the system like (B.1) is as follows:

- Solve Eq. (B.2) with initial conditions $x(0) = x_0$, $t(0) = 0$. Solutions of (B.2) give the transformation $(x,t) \to (x_0,s)$;
- Solve Eq. (B.3) with initilal condition $s(0) = u_0(x_0)$ (where x_0 are the initial points on the characteristic curves along the $t = 0$ axis). So, we have a solution $u(x_0,s)$;

- Using the results of the first step find s and x_0 in terms of x and t and substitute these values in $u(x_0, s)$ to get the solution $u(x,t)$ of the original equation (B.1).