Nachtrag:

t-Kanal 

$Q^2 = -p^2 > 0$, i.e. logarithm is real.

Define effective coupling constant:

$$\alpha_{eff.} = \frac{e_{eff}^2}{4\pi} = \frac{e_R^2}{4\pi}\left[1 + \frac{e_R^2}{4\pi}\ln\frac{Q^2}{m^2}\right] = \frac{1}{137}\left[1 + 0.00077\ln\frac{Q^2}{m^2}\right]$$

Renormalization group equation: (RGE) "β-function"

$$\boxed{Q\frac{de_{eff}}{dQ} =: \beta(e) = \frac{e_{eff}^3}{12\pi^2}}$$

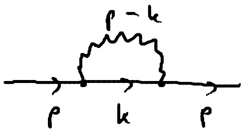
Resummation of all orders:

$$\boxed{\alpha_{eff} = \frac{\alpha_R}{1 - \frac{\alpha_R}{3\pi}\ln\frac{Q^2}{m^2}}}$$

Electron propagator at tree-level:

$$iG_0(\not p) = \frac{i}{\not p - m + i\epsilon}$$

One-loop-correction:

$$iG_2(\not p) =$$ 

$$= iG_0(\not p)\left[i\Sigma_2(\not p)\right] iG_0(\not p)$$

Feynman gauge:

$$i\Sigma_2(\not p) = (-ie)^2\int\frac{d^4k}{(2\pi)^4}\gamma^\mu\frac{i(\not k + m)}{k^2 - m^2 + i\epsilon}\gamma_\mu\frac{-i}{(p-k)^2 + i\epsilon}$$

Feynman parameter: (Complete square in denominator, $k \to k + px$, drop linear term in k)

$$i\Sigma_2(\not p) = 2e^2\mu^\epsilon\int_0^1 dx\int\frac{d^{\cancel{4}\,d}k}{(2\pi)^{\cancel{4}\,d}}\frac{x\not p - 2m}{(k^2 - \Delta + i\epsilon)^2}, \quad \text{where } \Delta = (1-x)(m^2 - p^2x)$$

Dimensional regularization: $\tilde\mu^2 = 4\pi e^{-\gamma_E}\mu^2$

$$\Sigma_2(\not p) = \frac{\alpha}{4\pi}\int_0^1 dx\left[(2-\epsilon)x\not p - (4-\epsilon)m\right]\left[\frac{2}{\epsilon} + \ln\frac{\tilde\mu^2}{(1-x)(m^2 - p^2x)}\right]$$

Divergence only:

$$\boxed{\Sigma_2(\not p) = \frac{\alpha}{\pi}\left(\frac{\not p - 4m}{2\epsilon} + \text{finite}\right).} \qquad (*)$$

Renormalization:

Bare Lagrangian:

$$\mathcal{L} = i\bar\psi_0\not\partial\psi_0 - m\bar\psi_0\psi_0 - e_0\bar\psi_0\not A_0\psi_0 - \frac14 F_{\mu\nu,0}^2$$

Renormalizable parameters: ψ_0 (remember $\sum\psi\bar\psi = \not p + m$), m_0, e_0 (sec. 2.3.1), A_0 (see below)

Renormalizable fermion wave function:

$$\varphi_R(x) =: \frac{1}{\sqrt{Z_2}}\varphi_0 = \frac{1}{\sqrt{1+\delta_2}}\varphi_0$$

"Counterterme"

Renormalized mass:

$$m_0 = Z_m m_R = (1+\delta_m) m_R$$

Renormalized propagator:

$$iG_R(p) = \frac{1}{Z_2}\frac{i}{\not p - m_0} + \text{loops} =$$

$$= \frac{1}{(1+\delta_2)}\left(\frac{i}{\not p - m_R - \delta_m m_R}\right) + \text{loops}$$

$$= \frac{i}{\not p - m_R} + \frac{i}{\not p - m_R}\, i\left[\delta_2 \not p - (\delta_2+\delta_m)m_R + \Sigma_2(\not p)\right]\frac{i}{\not p - m_R}$$

Minimal subtraction scheme (MS): Only poles absorbed.

$$\boxed{\delta_2 = -\frac{\alpha}{4\pi}\frac{2}{\varepsilon}\;, \qquad \delta_m = -\frac{3\alpha}{4\pi}\frac{2}{\varepsilon}}$$

"$\underline{\underline{MS}}$"

Modified Minimal Subtraction ($\overline{MS}$) scheme: $\boxed{\frac{2}{\varepsilon} \to \frac{2}{\tilde\varepsilon} = \frac{2}{\varepsilon} - \gamma_E + \ln(4\pi)}$

(or equivalently $\mu \to \tilde\mu$).

Sum of one-particle irreducible (1PI) diagrams:

$$iG^{\text{bare}}(\not p) =$$  $$+ \;\ldots$$

$$= \frac{i}{\not p - m_0 + \Sigma(\not p)}$$

Renormalized Green's function:

$$iG_R(\not p) = \frac{i}{\not p - m_R + \Sigma_R(\not p)}\;, \text{ where } \Sigma_R(\not p) = \Sigma_2(\not p) + \delta_2(\not p) - (\delta_2+\delta_m)m_R + O(e^4)$$

Physical electron mass:

Location of the pole $\boxed{(\not p = m_p)}$ with residue i → defines pole mass (which is scheme-ind.)

On-shell subtraction scheme:

Defined by $m_R = m_P$, so

$$\Sigma_R(m_p) = m_R - m_P$$

Residue i implies:

$$i = \lim_{\not p \to m_p} (\not p - m_p)\frac{i}{\not p - m_R + \Sigma_R(\not p)} = \lim_{\not p\to m_p} \frac{i}{1 + \frac{d}{d\not p}\Sigma_R(\not p)}$$

using L'Hôpital's rule. Therefore:

$$\boxed{\frac{d}{d\not p}\Sigma_R(\not p)\Big|_{\not p = m_p} = 0}$$

Using $\Sigma_R(\not p) = \Sigma_2(\not p) + \delta_2 \not p - (\delta_2+\delta_m)m_R + O(e^4)$, this condition and $\Sigma_R(m_p) = 0$ imply:

$$\boxed{\delta_2 = -\frac{d}{d\not p}\Sigma_2(\not p)\Big|_{\not p = m_p} \quad , \qquad \delta_m m_p = \Sigma_2(m_p)}$$ "On-shell"

Back to fermion selfenergy in QED : ®

In dim. reg. we found:

"mass regularization"

$$\Sigma_2(\not p) = \frac{\alpha}{4\pi}\int_0^1 dx\left[(2-\varepsilon)x\not p - (4-\varepsilon)m\right]\left[\frac{2}{\varepsilon} + \ln\frac{\tilde\mu^2}{(1-x)(m^2-p^2x) + x\,m_\gamma^2}\right]$$

Wave function renormalization constant: (On-shell scheme)

$$\boxed{\delta_2 = -\Sigma_2'(m_p) = -\frac{\alpha}{2\pi}\left(\frac{1}{\varepsilon} + \frac{1}{2}\ln\frac{\tilde\mu^2}{m_p^2} + 2 + \ln\frac{m_\gamma^2}{m_p^2}\right)}$$ "on-shell"

Mass ren. constant:

$$\boxed{\delta_m = \frac{1}{m_p}\Sigma_2(m_p) = \frac{\alpha}{2\pi}\left(-\frac{3}{\varepsilon} - \frac{3}{2}\ln\frac{\tilde\mu^2}{m_p^2} - 2\right)}$$ "on-shell"

Relation of pole mass to $\overline{MS}$ mass:

$$\overline{m}_R = m_p\left[1 - \frac{\alpha}{4\pi}\left(4 + 3\ln\frac{\tilde\mu^2}{m_p^2}\right) + O(\alpha^2)\right]$$

Modified LSZ reduction:

$$\langle f|S|i\rangle \sim (\not p_f - m_p)\cdots(\not p - m_p)\langle \varphi_f^R \cdots \varphi_i^R\rangle$$

not quite correct, since we assumed asymptotic states to be non-interacting.

Therefore:

$$\langle f|S|i\rangle \sim (\not p_f - m_p)\cdots(\not p_i - m_p)\langle \varphi_f^R \cdots \varphi_i^R\rangle_{\text{amputated}}$$

i.e. all external lines are chopped off until they begin interacting with other fields.

Self-energy diagrams must only be calculated for internal lines.

2.3.4 Renormalized perturbation theory (19)

Renormalized charge: $e_0 =: Z_e e_R = (1+\delta_e)e_R$

Photon field: $A_0^\mu =: \sqrt{Z_3}A_R^\mu = \sqrt{1+\delta_3}\,A_R^\mu$

Renormalized Lagrangian:

$$\mathcal{L} = i Z_2\bar\varphi_R\not\partial\varphi_R - Z_2 Z_m m\bar\varphi_R\varphi_R - e_R Z_e Z_2\sqrt{Z_3}\bar\varphi_R\not A_R\varphi_R - \frac{1}{4}Z_3(\partial^\mu A_R^\nu - \partial^\nu A_R^\mu)^2$$

It is customary to define $Z_1 := Z_e Z_2\sqrt{Z_3} = (1+\delta_1)$, so that $\boxed{\delta_e = \delta_1 - \delta_2 - \frac{1}{2}\delta_3 + O(e_R^4)}$

Expanded renormalized Lagrangian:

$$\mathcal{L} = i\bar\varphi_R\not\partial\varphi_R - m_R\bar\varphi_R\varphi_R - e_R\bar\varphi_R\not A_R\varphi_R - \frac{1}{4}F_R^{\mu\nu 2}$$

$$+\, i\delta_2\bar\varphi_R\not\partial\varphi_R - (\delta_2+\delta_m)m_R\bar\varphi_R\varphi_R - e_R\delta_1\bar\varphi_R\not A_R\varphi_R - \frac{1}{4}\delta_3 F_R^{\mu\nu 2}$$

Feynman rules for counterterms:

$$\text{—}\!\times\!\text{—} = i(\not p\,\delta_2 - (\delta_m + \delta_2)m_R)$$

$$\sim\!\times\!\sim = -i\delta_3(p^2 g^{\mu\nu} - p^\mu p^\nu)$$

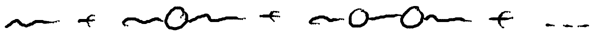
 $= -ie_R\delta_1\gamma^\mu$

Photon propagator:

$iG^{\mu\nu}(p) =$  $+ \dots$

$$= -i\,\frac{g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}}{p^2}\left[1-e_R^2\,\Pi_2(p^2)-\delta_3\right] + O(e_R^4)$$

Sum of all 1PI diagrams:

$iG^{\mu\nu}(p) =$  $+ \dots$

$$= -i\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\frac{1}{p^2(1+\Pi(p^2))+i\epsilon} \quad \text{with} \quad \Pi(p^2) = e_R\Pi_2(p^2)+\delta_3+\dots$$

On-shell subtraction:

Defined by $\Pi(0)=0$, so that photon remains massless. Consequently

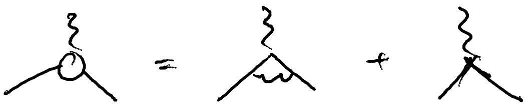
$$\boxed{\delta_3 = -e_R^2\,\Pi_2(0) = -\frac{e_R^2}{6\pi^2}\frac{1}{\varepsilon} - \frac{e_R^2}{12\pi^2}\ln\frac{\tilde{\mu}^2}{m_e^2}} \qquad \text{"on-shell"}$$

Renormalized propagator:

$$\Pi(p^2) = \frac{e_R^2}{2\pi^2}\int_0^1 dx\, x(1-x)\,\ln\left(\frac{m_e^2}{m_e^2-p^2x(1-x)}\right)$$

here obtained through wave fct. renormalization, not charge ren.

Vertex correction:



$\Rightarrow \boxed{\delta_1 \equiv \delta_2}$ True to all orders, consequence of gauge invariance?