
Problem Sheet 3: **To hand in until: 29.05.2017**

Problem 1: Transport Equations

a) Write the equations

$$\partial_t \rho = -\boldsymbol{\nabla} \cdot (\rho \mathbf{v}) \tag{1}$$

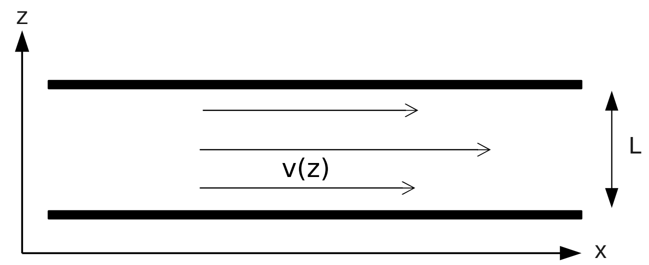
$$\partial_t (\rho \mathbf{v}) = -\boldsymbol{\nabla} \cdot (\rho \mathbf{v}\mathbf{v} + \underline{P}) + \mathbf{f} \tag{2}$$

in index notation and simplify Eq. (2) by inserting Eq. (1). Note that the tensor $\mathbf{vv}$ in index notation is $v_i v_j$.

b) Take the resulting equation and specify $\underline{P} = p\underline{\mathbf{1}} - \eta\left(\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^T\right)$. Simplify and obtain the Navier-Stokes (NS) equation

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\right] = -\boldsymbol{\nabla} p + \eta\,\Delta\mathbf{v} + \mathbf{f}\,. \tag{3}$$

Problem 2: Chanel Flow



The NS Eq. (3) (for $\mathbf{f} = 0$) can be solved in constrained geometries under certain assumptions. Here we consider a two dimensional flow between two plates, and assume that the velocity field has only a component in x-direction that is independent of x (see also figure above). Simplify the Navier-Stokes equation for this geometry and for a constant pressure gradient in x-direction and solve for steady velocity profiles in the following cases:

a) No-slip boundary conditions at the plates ($v(0) = v(L) = 0$).

b) Navier-slip boundary conditions ($v(0) = b\ \partial_z v(0),\ v(L) = -b\ \partial_z v(L)$, where b is the slip-length).

c) Navier-slip boundary conditions for $b \gg L$.

Problem 3: **Nondimensionalisation of the NS equation with boundary conditions**

Consider the incompressible 2D NS Eq. (3) and the boundary conditions for a free surface including Marangoni and capillary forces. When $\mathbf{n}$ and $\mathbf{t}$ are the normal and tangential vectors at the surface, the continuity of the forces at the surface yields:

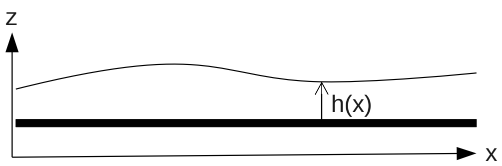
$$\text{normal force} \qquad \mathbf{n}\cdot(\boldsymbol{\tau_1}-\boldsymbol{\tau_2})\cdot\mathbf{n} = \gamma\kappa \tag{4}$$

$$\text{tangential force} \qquad \mathbf{t}\cdot(\boldsymbol{\tau_1}-\boldsymbol{\tau_2})\cdot\mathbf{n} = \frac{\partial\gamma}{\partial s}. \tag{5}$$

Nondimensionalise the NS equation and the boundary conditions. To do so, write the boundary conditions componentwise using $\boldsymbol{\tau} = \eta\left(\boldsymbol{\nabla} v + (\boldsymbol{\nabla} v)^T\right)$. Write $\gamma(s)$ as $\gamma(s) = \gamma_0(1+\frac{\gamma_\Gamma}{\gamma_0}\xi(s))$, where ξ is dimensionless. Introduce general scales l_0, v_0, $t_0 = \frac{l_0}{v_0}$, $p_0 = \rho v_0^2$ for length, velocity, time and pressure, respectively. Scale the curvature κ with $\frac{1}{l_0}$ and define the following dimensionless parameters:

$$\text{Reynolds number}\ \ \mathrm{Re} = \frac{\rho v_0 l_0}{\eta};\ \ \text{Marangoni number}\ \ \mathrm{Ma} = \frac{\gamma_\Gamma}{v_0\eta};\ \ \text{Capillary number}\ \ \mathrm{Ca} = \frac{v_0\eta}{\gamma_0}.$$

Problem 4: **Long wave expansion of the NS equation (thin film equation)**



We now introduce two different spatial scales to describe a thin film on a substrate, where typical length scales in x-direction (along the substrate) are much larger than those in z-direction: $z = l_0 z'$, $x = L_0 x' = \frac{l_0}{\epsilon}x$, $\epsilon \ll 1$. Furthermore, we consider a free surface between the liquid and a stress-free gas phase ($\boldsymbol{\tau_2} \approx \mathbf{0}$). The free surface is assumed to be without overhangs and can therefore be described by a height profile $h(x)$, yielding the tangential and normal vectors and the curvature

$$\mathbf{n} = \frac{(-\partial_x h, 1)}{(1+(\partial_x h)^2)^{1/2}},\quad \mathbf{t} = \frac{(1,\partial_x h)}{(1+(\partial_x h)^2)^{1/2}},\quad \kappa = \frac{-\partial_{xx}h}{(1+(\partial_x h)^2)^{3/2}}. \tag{6}$$

In order to approximately solve the Navier-Stokes equation for $\mathbf{v} = (u, w)$ combined with the continuity equation $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$, the following boundary conditions are needed additionally to the stress continuity at the free surface:

$$\text{No-slip and no-penetration at the solid boundary:} \qquad \mathbf{v}(z=0) = \mathbf{0} \tag{7}$$

$$\text{Kinematic boundary condition at the free surface:} \qquad w = \partial_t h + u\partial_x h. \tag{8}$$

a) Rescale all boundary conditions and the bulk equations similarily to the scaling in the previous exercise but with the two different spatial scales as outlined above. Omit Marangoni forces. As the rescaled continuity equation should not contain the smallness parameter ϵ, one can deduce which scaling of the two velocities u, w one has to choose.

b) Show that in lowest order in ϵ you arrive at the following set of equations (for the rescaled variables):

$$\text{bulk equations:}\quad u_{zz} = p_x;\quad p_z = 0;\quad u_x + w_z = 0 \tag{9}$$

$$\text{boundary conditions:}\quad w = \partial_t h + u\partial_x h;\quad u_z = 0;\quad p = -\frac{h_{xx}}{\widetilde{\mathrm{Ca}}} \qquad \text{at}\ \ z = h(x) \tag{10}$$

$$u = w = 0 \qquad \text{at}\ \ z = 0. \tag{11}$$

The term in the boundary condition due to capillarity is not of lowest order but essential for further derivations. The capillary number Ca defined in the previous exercise is therefore rescaled by a certain power of ϵ to obtain $\widetilde{\mathrm{Ca}}$.

c) The kinematic boundary condition (first part of Eq. (10)) combined with the continuity equation yields the following conservation law for $h(x,t)$:

$$\partial_t h = -\partial_x \left(\int_0^h u(x,z)\ dz \right) . \tag{12}$$

Solve equation (9) for $u(x,z)$ with the boundary conditions (10) and (11). Insert the result into equation (12) and show that the resulting equation is

$$\partial_t h = -\partial_x \left[\frac{h^3}{3} \partial_x \left(\partial_{xx} h \right) \right] .$$