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Übungen im WWW: http://pauli.uni-muenster.de/tp/

Problem Sheet 1: **To hand in until: 01.05.2017**

Problem 1: Competitive Lotka-Volterra equations

Interspecific competition is a form of competition in which individuals of different species compete for the same resource in an ecosystem. The impacts of interspecific competition on two populations $p_1 = p_1(t)$ and $p_2 = p_2(t)$ can be described by the following dynamical system:

$$\dot{p}_1 = r_1\, p_1 \left(1 - \left(\frac{p_1 + \alpha_{12}\, p_2}{K_1}\right)\right) , \qquad (1)$$

$$\dot{p}_2 = r_2\, p_2 \left(1 - \left(\frac{p_2 + \alpha_{21}\, p_1}{K_2}\right)\right) . \qquad (2)$$

Here, r_i and K_i are the growth rate and the carrying capacity of the population p_i, whereas α_{ij} describe the effect that the species p_j has on the population of species p_i, i, $j = 1, 2$.

a) Find the fixed points (p_1^*, p_2^*).

b) Calculate the Jacobian matrix J of the system in question.

c) Find all eigenvalues λ of J and classify the fixed points (p_1^*, p_2^*) for $r_1 = 0.2$, $r_2 = 0.1$, $K_1 = 50.0$, $K_2 = 100.0$, $\alpha_{12} = 0.75$, $\alpha_{21} = 3.0$.

d) Sketch the neighboring to (p_1^*, p_2^*) trajectories, and try to fill the rest of the phase portrait. Which population shall survive?

Problem 2: Lorenz system

The Lorenz equations are given by

$$\begin{aligned} \dot{x} &= \sigma\,(y - x)\,, \\ \dot{y} &= r\,x - y - x\,z\,, \\ \dot{z} &= x\,y - b\,z\,. \end{aligned}$$

Here, $\sigma > 0$ is the Prandtl number, $b > 0$ is connected with the cell geometry and $r > 0$ is the relative Rayleigh number. In what follows, we will always use r as the control parameter.

a) Find the fixed points (x^*, y^*, z^*) of the system in question: Show that the origin is a fixed point of the system for any values of the parameters, whereas the other two fixed points C^+ and C^- exist if and only if $r > 1$.

b) Demonstrate that the Jacobian $\mathbf{J}$ of the system is given by

$$\mathbf{J} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z^* & -1 & -x^* \\ y^* & x^* & -b \end{pmatrix}$$

c) Find the eigenvalues at the origin and show that the origin is a stable node for $r < 1$. Further, show that for $r > 1$ the origin changes from a stable node to a saddle point.
Tip: Note that for $(x^*, y^*, z^*) = (0, 0, 0)$ the matrix $\mathbf{J}$ can be written as a 2×2 matrix, as the linearized equation for $z(t)$ is decoupled.

d) Now consider the case $r > 1$, so that both non-trivial fixed points C^+ and C^- exist. Demonstrate that the characteristic polynomial reads

$$\lambda^3 + (\sigma + b + 1)\,\lambda^2 + (r + \sigma)\,b\,\lambda + 2\,b\,\sigma\,(r - 1) = 0\,.$$

d) Show now that C^+ and C^- are stable for

$$1 < r < r_{\mathrm{H}}, \qquad r_{\mathrm{H}} = \frac{\sigma\,(\sigma + b + 3)}{\sigma - b - 1}, \quad \sigma > b + 1\,.$$

Tip: Calculate, under which condition the characteristic equation has one negative real root λ_1 and a pair of complex conjugated roots $\lambda_{2,3}$ that are purely imaginary. Use the ansatz $\lambda_{2,3} = \pm\mathrm{i}\,\omega$ and find ω.

e) Using the information about the fixed points of the system and the results from linear stability analysis discuss the possible types of bifurcations at $r = 1$ as well as at $r = r_{\mathrm{H}}$.