On tensor invariants and entanglement

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From perturbative to non-perturbative QFT

Muenster – 16/06/23

Partially based on results with Benoit Collins & Razvan Gurau

1 – Tensors and multipartite quantum systems

2 – Local unitaries and tensor invariants

3 – Randomized measurements and tensor HCIZ

4 – An ensemble of density matrices to work with

5 – Results for randomized measurements for this ensemble

1 – Tensors and multipartite quantum sysytems

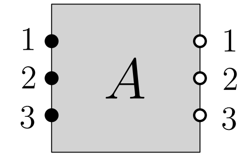
Tensors

Tensors

$$A = \{A_{i_1,\ldots,i_D\,;\,j_1,\ldots,j_D}\}$$

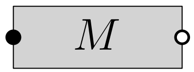
$1 \leq c \leq D$: "color"

$1 \leq i_c\,, j_c \leq N$



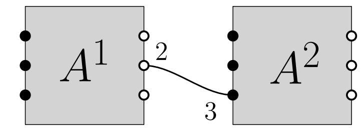
$D = 3$

Matrix D=1



Index summation / contraction

$$\sum_{j=1}^{N} A^1_{i_1,i_2,i_3\,;\,j_1,\boxed{j},j_3} A^2_{i'_1,i'_2,\boxed{j}\,;\,j'_1,j'_2,j'_D}$$



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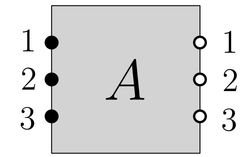

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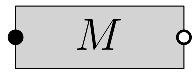

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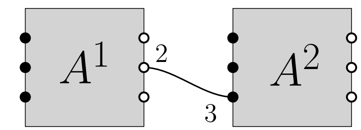

Why this definition?

Density matrices or observables on a D-partite quantum system $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_D$
are tensors of this form (= matrices with subdivided index-set)

- For states : seen as a matrix it is Hermitian, positive semi-definite, and of trace 1
- For operators : Hermitian

Size of the index set = Local dimension $\dim(\mathcal{H}_c) = N_c$

Tensors

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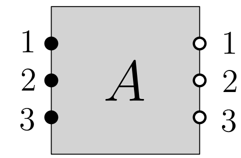
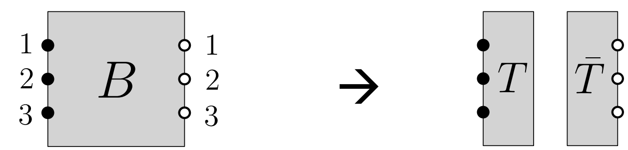


$D = 3$

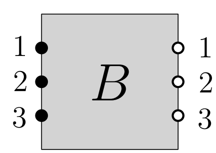
The kind of tensors that we have seen this week are pure…

Pure tensor $B = |T\rangle\langle T| = T \otimes \bar{T}$



- For states: pure state (not mixed)
- For observables: projection

Mixed tensor ⇔ not pure



Mixed state…

Normalized identity is maximally mixed…

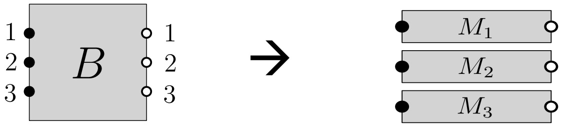
Tensors

$$\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_D$$

1 ≤ c ≤ D : “color”

Factorized tensor

$$B = M_1 \otimes \cdots \otimes M_D$$



- For states: factorized state (may be pure or mixed but it has no entanglement)
- For observables: local observable (applied independently in each subsystem)

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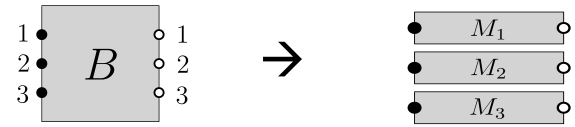


- For states: factorized state (may be pure or mixed but it has no entanglement)
- For observables: local observable (applied independently in each subsystem)

Entanglement: how far is the density matrix (a tensor) from a convex combination of factorized states. Quantum correlations between subsystems (⇔ colors).

$$\rho_{\mathrm{sep}} = \sum_{k=1}^{K} p_k \, \rho_1^{(k)} \otimes \cdots \otimes \rho_D^{(k)} \qquad \sum_{i=1}^{K} p_i = 1 \qquad \rho_c^{(i)} \in \mathcal{M}_{N_c}(\mathbb{C})$$

density matrices

→ The key resource exploited by quantum technologies (computers, communications, teleportation…)

→ Fundamental in the study of quantum black holes, holography, …

D=2 : bipartite entanglement
D > 2 multipartite entanglement

Grouping subsystems / colors is equivalent to ``multiplying’’ the index sets

Some important questions

Given an unknown quantum state (that is, an unknown tensor),

1. How efficiently can we reconstruct the full tensor using some measurements? $\langle \mathcal{O} \rangle_\rho = \mathrm{Tr}(\mathcal{O}\rho)$
   (tomography$\ldots$ exponential in the size of the system$\ldots$)

2. How can we recover (theoretically / experimentally) only the information needed to characterize the amount of entanglement between the different parts$\ldots$?

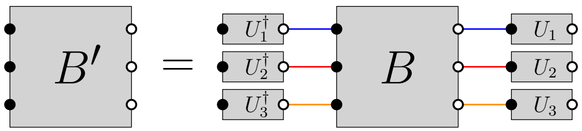
$\ldots$

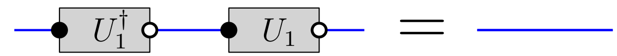
2 – Local unitaries and tensor invariants

Local unitaries

Local unitaries (LU)

LU transformation $$B' = (U_1^\dagger \otimes \ldots \otimes U_D^\dagger)\, B\, (U_1 \otimes \ldots \otimes U_D)$$





LU equivalence $$B' \sim_{\mathrm{LU}} B \quad \Leftrightarrow \quad \exists U = U_1 \otimes \cdots \otimes U_D, \quad B' = U^\dagger B U$$

LU invariance

Function: $f(\rho) = f(U\rho U^\dagger)$

Distribution: $d\mu(\rho) = d\mu(U\rho U^\dagger)$

$$U = U_1 \otimes \cdots \otimes U_D$$

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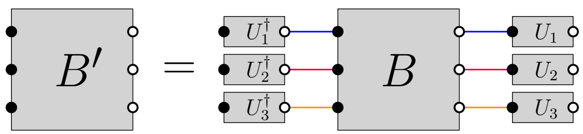

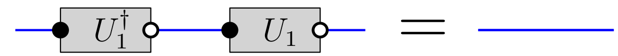

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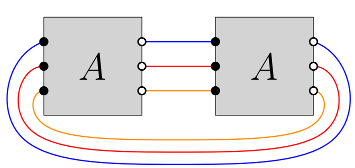
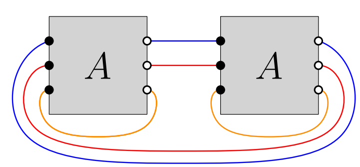
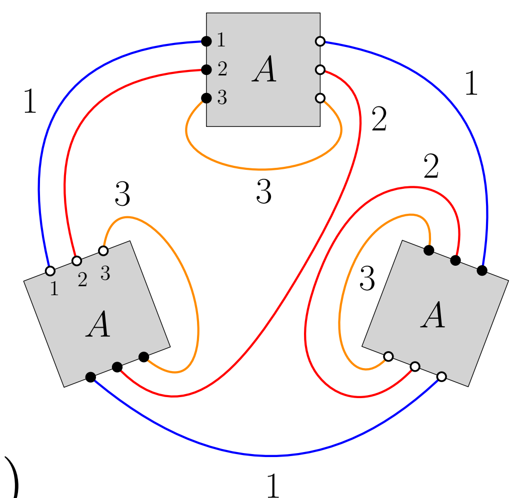
"LU invariants" : LU-invariant **polynomial** encoded by colorwise summation of indices:



Tr A^2

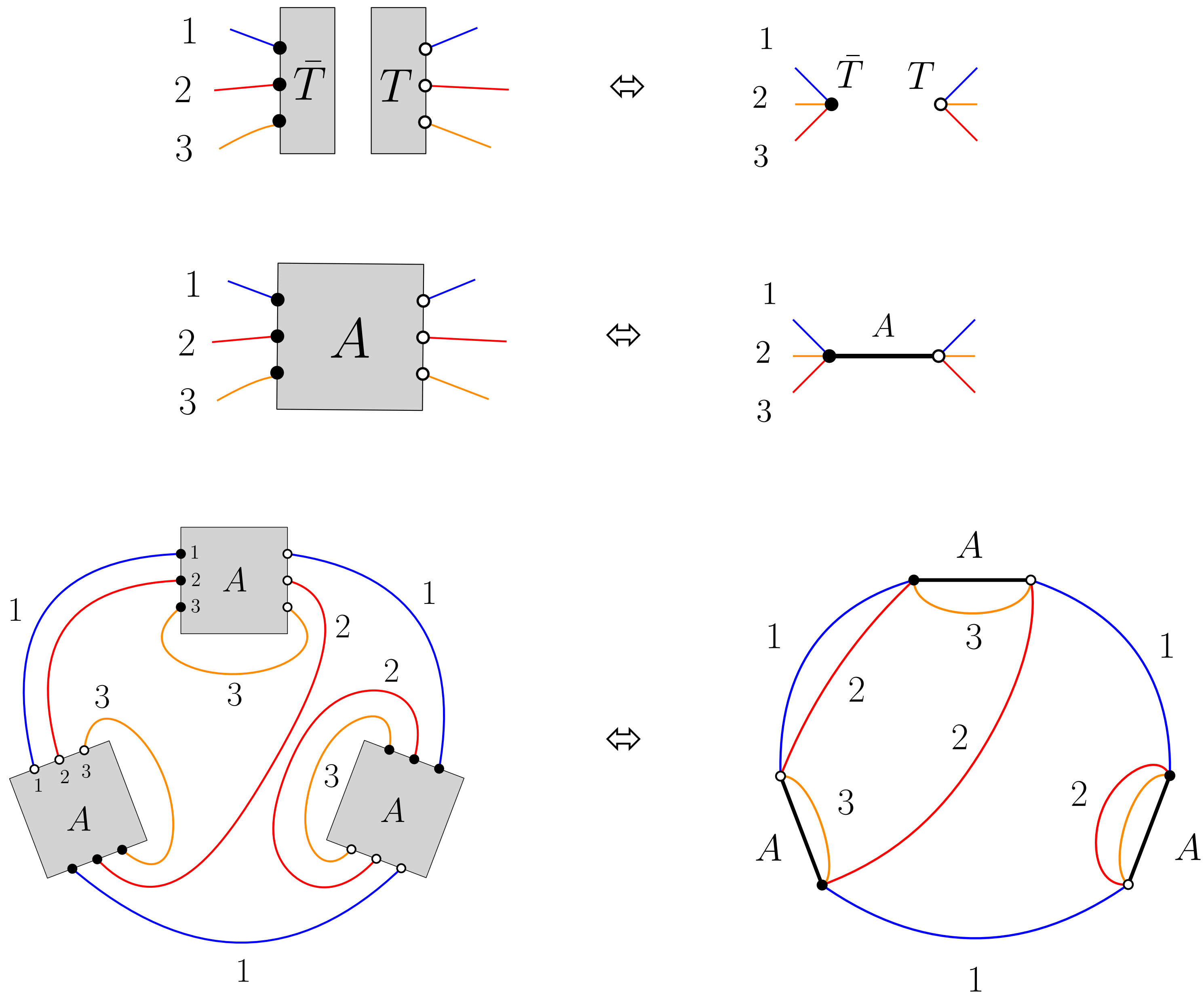


Tr_{12} [$(\mathrm{Tr}_3\ A)^2$] = Tr $A_{12}{}^2$



$\mathrm{Tr}_G(A)$

LU-invariants ⇔ bubbles



Local unitaries (LU) : why do we care?

→ Two density matrices share the same *theoretical* entanglement properties IFF they are LU-equivalent

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“theoretical” because there are some ``operational’’ notions of entanglement equivalence classes that are more adapted to the use of entanglement in quantum operations (LOCC…).

Two density matrices in the same LU-entanglement class also have the same ``operational’’ entanglement properties, but density matrices in different LU-entanglement classes may still have the same ``operational’’ entanglement…

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Anyways:

- All entanglement measures $f : \rho \to f(\rho) \in \mathbb{R}$ are LU-invariant *functions*.
- LU-invariant *distributions* allow studying "typical properties" of LU-entanglement classes

Random quantum states: Page, Hayden, Leung, Winter, Collins, Nechita, Zyckowski, Aubrun, Majumdar…

LU invariant distributions: two examples

- **EX 1:** The perturbed Gaussians we are used to are LU-invariant distributions

$$\mathbb{E}\left[f(T,\bar{T})\right] = \int dT d\bar{T}\, e^{-N^{D-1}\left[T\cdot\bar{T}+V(T,\bar{T})\right]} f(T,\bar{T}),$$

$$V(T,\bar{T})[\lambda] = \sum_{n\geq 2}\lambda^n \sum_{\substack{G \text{ colored graph} \\ \text{with } n \text{ vertices}}} N^{-\zeta(G)} z_G \mathrm{Tr}_G(T\otimes\bar{T})$$

If normalized, they provide distributions over pure states inside an LU-equivalence class

The Gaussian distribution is often used for random pure states

(⇔ big Haar unitary on a fixed state $U^\dagger|0\rangle\langle 0|U$ … Page curve for instance)

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- **EX 2:** Fix *ρ* and consider $(U_1^\dagger \otimes \ldots \otimes U_D^\dagger)\, \rho\, (U_1 \otimes \ldots \otimes U_D)$ with U_c Haar distributed

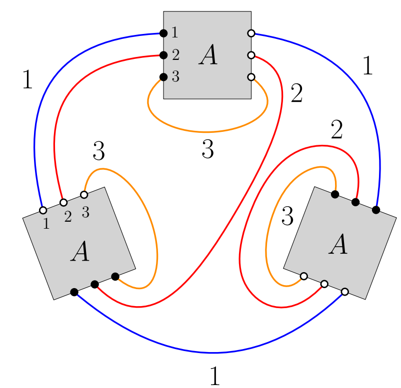
→ Average over properties that don't matter when studying entanglement.

For instance: recall that $\langle \mathcal{O} \rangle_\rho = \mathrm{Tr}(\mathcal{O}\rho)$ is the average of observable *O* for state *ρ.*

Study $\langle \mathcal{O} \rangle_{U^\dagger \rho U} = \mathrm{Tr}\Big(\mathcal{O}\, U^\dagger \rho\, U\Big)$ $\qquad U = U_1 \otimes \cdots \otimes U_D$

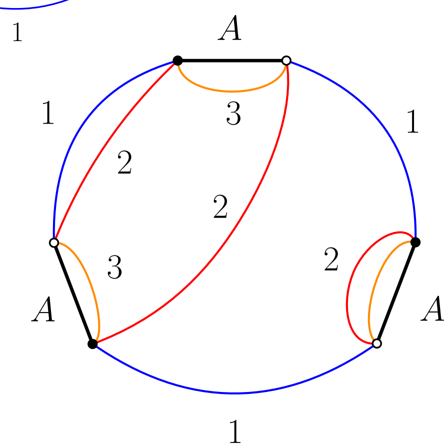
More on LU-invariant polynomials = tensor invariants

Importance of LU-invariants (the polynomials)



→ They separate the LU-entanglement classes
Contain all information on LU-entanglement

→ Basis for LU-invariant functions in the limit $N \to \infty$



Think of : − products of traces of power of a matrix for unitary invariance
− products of power sums for symmetric functions

→ Also the correlation functions for LU-invariant random tensors

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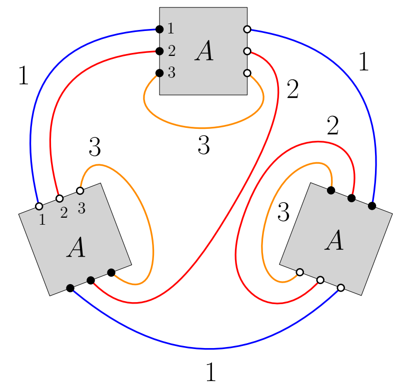

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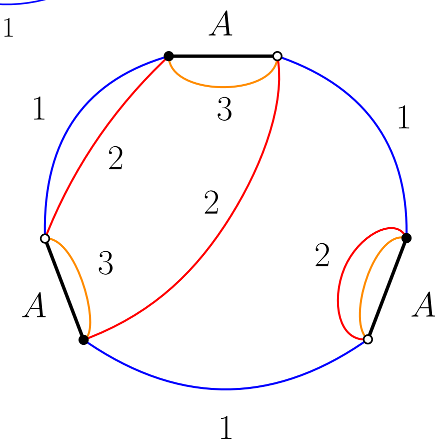

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- So we study these polynomials for ensembles of density matrices or random tensors, & work with expansions of other quantities on this family.

- Growing interest of LU-invariants for D > 2 in characterizing the multipartite entanglement structure, for instance in holography.

Some important questions (BIS)

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   (tomography… exponential in the size of the system…)

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3. Identify the LU-entanglement class of the tensor
   ⇔ Reconstruct only the information up to LU transformations.
   ⇔ Compute a certain number f(N) of LU-invariants (all of them for N → ∞)…

   It's ``a lot less'' than the first point but ``a lot more'' than what's needed for the second point (still a lot…)

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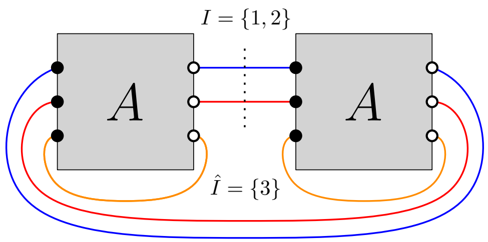
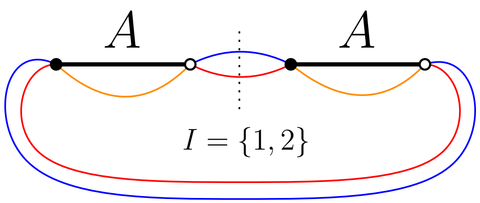
4. Compute SOME of the LU-invariants of your tensor… e.g. Rényi-n entropies!
   → What info on multipartite entanglement do they contain???

- Purities / Rényi entropies (bipartite, classical)

$$S_n^I(\rho) = \frac{1}{1-n} \log \mathrm{Tr}(\rho_I^n)$$

(measures how mixed ρ_I is)





Cyclic melonic & necklace

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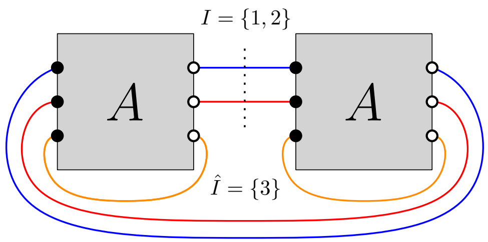
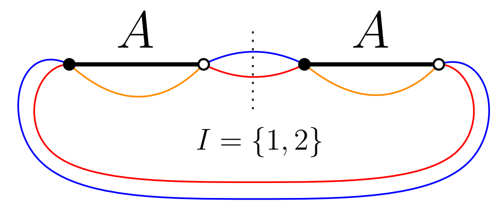


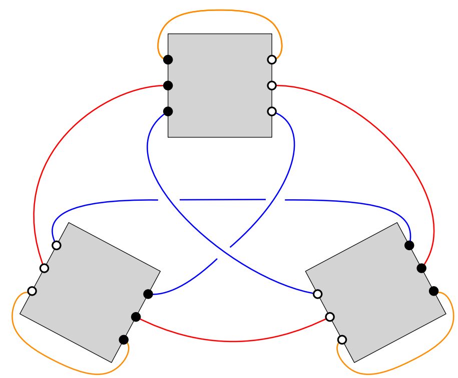


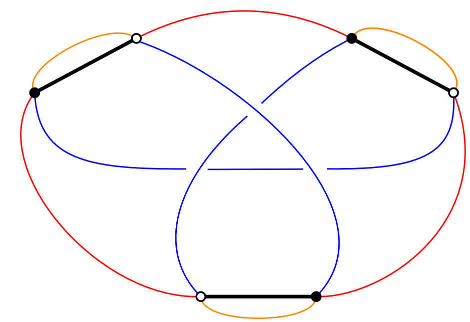
Cyclic melonic & necklace

- Moments of the partial transpose

$$\mathrm{Tr}_{12}\left(\mathrm{Tr}_3(\rho^{T_2})^n\right)$$

…
Calabrese, Cardy 12; Tamaoka 18 ;
Dong, Qi, Walter 21 ;
Kudler-Flam, Narovlansky, Ryu 21

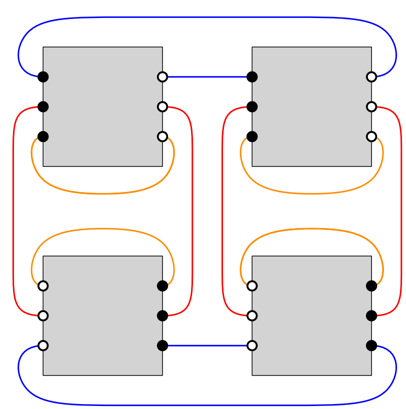


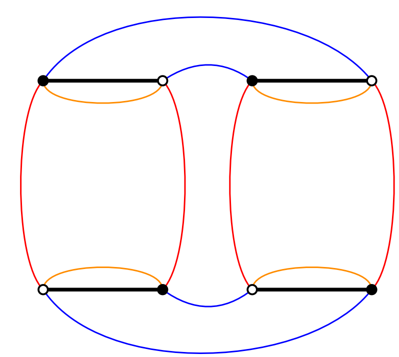


K33 if pure…

- Other LU-invariants

…
Dutta, Faulkner 19
Akers, Faulkner, Lin, Rath 21 & 22
Akers, Rath 20
Pennington, Walter, Witteveen 22
Gadde, Krishna, Sharma 22 & 23





Cube (octahedron) if pure…

3 – Randomized measurements
& the tensor HCIZ integral

Randomized measurements

Study of the real random variable

$$\mathrm{Tr}(U^\dagger A U \rho) \qquad U = U_1 \otimes \cdots \otimes U_D$$

$U_c \in U(N)$ Haar distributed

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Study of the real random variable

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Seeing *A* as an observable

$$\mathrm{Tr}(U^\dagger A U \rho) \begin{cases} \langle A \rangle_{U\rho U^\dagger} \\ \langle U^\dagger A U \rangle_\rho \end{cases}$$

$\langle A \rangle_{U\rho U^\dagger}$: Observation of *A* on a random state in the LU-entanglement class of *ρ*

$\langle U^\dagger A U \rangle_\rho$: Locally randomly rotated observation on a fixed density matrix. "Randomized measurements".

Access the properties of an unknown density matrix using a locally randomly rotated A as a **probe**

This is one example of **LU invariant distribution**

Some important questions (BIS-BIS)

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5. Study the distribution of local randomized measurements performed on your unknown system for the your favorite observable…

$$\langle \mathcal{O} \rangle_{U^\dagger \rho U} = \mathrm{Tr}\Big(\mathcal{O}\, U^\dagger \rho\, U\Big) \qquad U = U_1 \otimes \cdots \otimes U_D$$

Recover **some** of the LU-invariant information (related to **dominant** LU-invariants)

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Recover **some** of the LU-invariant information (related to **dominant** LU-invariants)

…What about the choice of observables… Are they all as good in the role of ``probe''?

Randomized measurements

→ For small values of *N,D*: good detection of entanglement.

→ Also: - testing outcomes of distant experiments,
- computing measures of chaos and thermalization,
- identification of topological phases
- …

→ Solves some experimental issues,
Allows for post-treatment of information on classical computers (``classical shadows''),
tools from AI

→ All of this also tested **experimentally** on quantum plateforms…

See e.g. 2203.11374 for a review

With Collins, Gurau, Hu, we work on computing the distribution of the random variable $\mathrm{Tr}(U^\dagger A U \rho)$ for *D*-partite quantum states, for finite *D* and in the limit $N \to \infty$.

The tensor HCIZ integral

$$\mathrm{Tr}(U^\dagger A U \rho) \qquad U = U_1 \otimes \cdots \otimes U_D$$

$$U_c \in U(N) \quad \text{Haar distributed}$$

Characterize the random variable by computing its moments (correlations)

$$\left\langle \left(\mathrm{Tr}(U^\dagger A U \rho)\right)^n \right\rangle = \int_{U(N)^D} \mathrm{d}U_1 \cdots \mathrm{d}U_D \left(\mathrm{Tr}(U^\dagger A U \rho)\right)^n$$

Generating function of moments **: tensor HCIZ integral**

$$I_D(A, \rho\,; z) = \int_{U(N)^D} \mathrm{d}U_1 \cdots \mathrm{d}U_D \, e^{z\,\mathrm{Tr}(U^\dagger A U \rho)}$$

If D=1 (matrix case), usual HCIZ integral. Analyticity properties [Goulden, Guay-Paquet, Novak 11,12 ; Novak 20, 22]

Equivalently, characterize the random variable by its cumulants (*connected* correlations):

$$\log I_D(A, \rho\,; z) = \sum_{n \geq 1} \frac{z^n}{n!} \left\langle \left(\mathrm{Tr}(U^\dagger A U \rho)\right)^n \right\rangle_c$$

More meaningful in limit of infinite N

The tensor HCIZ integral

$$\mathrm{Tr}(U^\dagger A U \rho) \qquad U = U_1 \otimes \cdots \otimes U_D$$

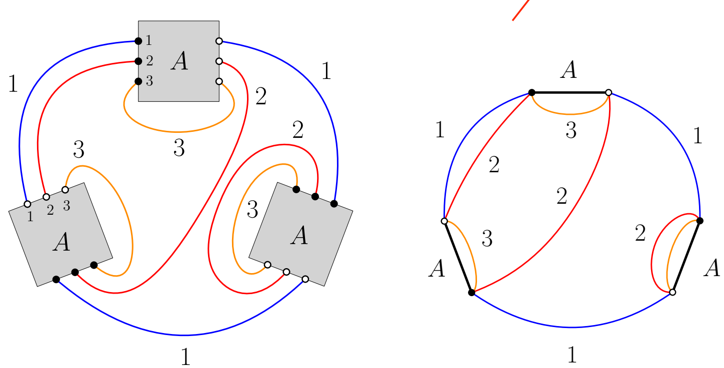
$$U_c \in U(N) \quad \text{Haar distributed}$$

Expansion of cumulants (connected correlations) of randomized measurements, on the family of LU-invariants [Collins, Gurau, L. 20]:

$$\left\langle \left(\mathrm{Tr}(U^\dagger A U \rho)\right)^n \right\rangle_c = \frac{1}{N^{nD}} \sum_{G_1, G_2 \in \mathbb{G}_n} \sum_{l \geq l_{\min}(G_1, G_2)} \left(-\frac{1}{N}\right)^l \mathrm{Tr}_{G_1}(A)\, \mathrm{Tr}_{G_2}(\rho)\, f[G_1, G_2; l]$$

Explicit

Combinatorial: a generalization of Hurwitz numbers, which counts certain branched coverings of nodal surfaces

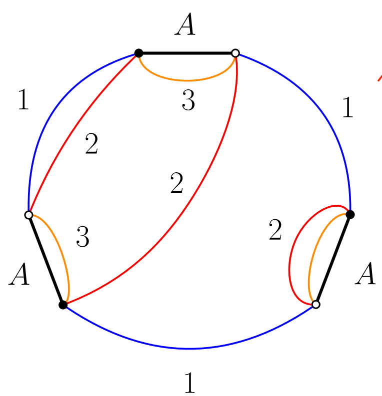


(not necessarily connected colored graphs)

4 – An ensemble of density matrices to work with

Remember that we have shown the following in full generality:

$$\left\langle \left(\mathrm{Tr}(U^\dagger A U \rho)\right)^n \right\rangle_c = \frac{1}{N^{nD}} \sum_{G_1, G_2 \in \mathbb{G}_n} \sum_{l \geq l_{\min}(G_1, G_2)} \left(-\frac{1}{N}\right)^l \mathrm{Tr}_{G_1}(A)\ \mathrm{Tr}_{G_2}(\rho)\, f[G_1, G_2; l]$$



(not necessarily connected colored graphs)

→ To take a large N limit, we need to know how the LU-invariants scale with N.

Only assumption needed.

Scaling ansatz for LU-invariants

We are going to derive some results for ensembles of density matrices satisfying that scale in the same following way:

$$\mathrm{Tr}_G(\rho) = N^{-s_G(\rho)}\,\mathrm{tr}_G(\rho)\,(1 + O(1/N))$$

With:

$$s_G(\rho) = \beta(\rho) s_G\left(\mathbb{1}/N^D\right) + \epsilon(\rho)\, s_G\Big(|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|\Big)$$

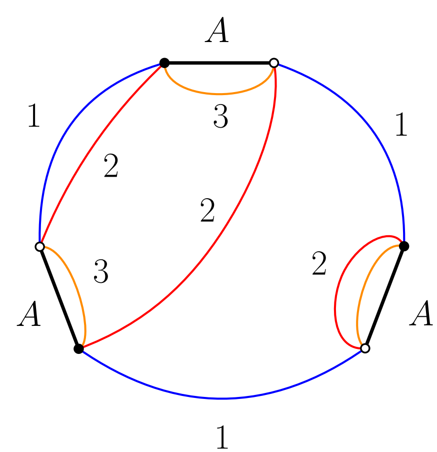
Where: $$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle \cdots |i\rangle$$

Multipartite generalization of Bell state (very entangled…``maximally'')

In tensor notation: $$|\mathrm{GHZ}\rangle_{i_1,\ldots,i_D} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \delta_{i_1,j} \cdots \delta_{i_D,j}$$

The scalings are given by: $$s_G\left(\mathbb{1}/N^D\right) = \sum_{c=1}^{D} (n - F_{Ac}(G))$$

$$s_G\big(|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|\big) = n - C_{\text{pure}}(G)$$



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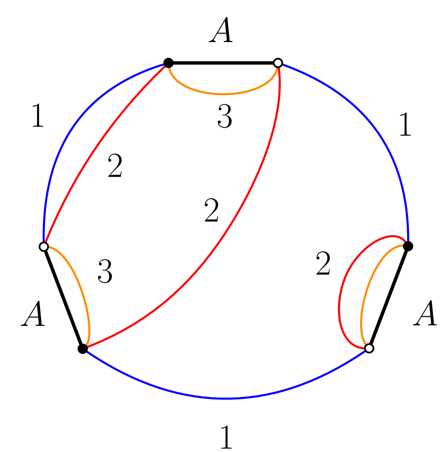

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Could also replace GHZ by Psi with :

$$s_G\big(|\mathrm{Psi}\rangle\langle\mathrm{Psi}|\big) = \frac{1}{D-1}\sum_{1\le c_1<c_2\le D}\left(n - F_{c_1c_2(G)}\right)$$

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Where: $$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} |i\rangle \cdots |i\rangle$$ Multipartite generalization of Bell state (very entangled)

We will compute the expansions on LU-invariants of the cumulants of the distribution of randomized measurements $\mathrm{Tr}(U^\dagger A U \rho)$, in the limit of infinite local dimension N,

for ρ satisfying the assumption above,

For *A* a local observable: $A = A_1 \otimes \ldots \otimes A_D$

$$A_{i_1,\ldots,i_D\,;\,j_1,\ldots,j_D} = \prod_{c=1}^{D} (A_c)_{i_c,j_c}$$

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Interpretation: "entropies"

$$\begin{aligned} S_G(\rho) &= -\frac{1}{c_G} \log \mathrm{Tr}_G(\rho) \\ &= \frac{s_G(\rho)}{c_G} \log\big(N\big) - \frac{1}{c_G} \log\Big(\mathrm{tr}_G(\rho)\,(1 + o(1))\Big) \end{aligned}$$

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→ $s_G(\rho)$ is the dominant part of the "entropies" associated to the LU-invariants

→ We are assuming that the dominant parts of the entropies all interpolate between those of the maximally mixed, separable state and a maximally entangled, pure state

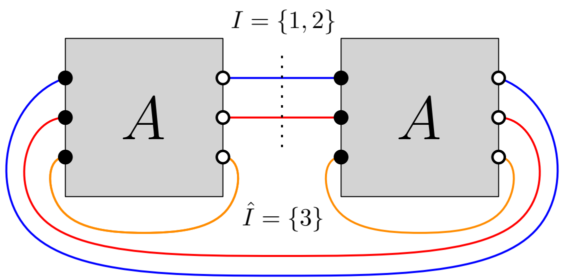
Meaning of the parameters ε and β

Meaning of the parameters ε and β

To relate these parameters to known properties of the state ρ, look at the Rényi entropies

$$S_n(\rho_I) = \frac{1}{1-n} \log \mathrm{Tr}(\rho_I^n)$$

(measures how mixed ρ_I is)



- $S_n(\rho) = \beta(\rho) D \log N + O(1)$ → β(ρ) > 0 informs on how mixed the state ρ is
- $S_n(\rho_I) = \big(\epsilon(\rho) + \beta(\rho)|I|\big) \ln N + O(1)$

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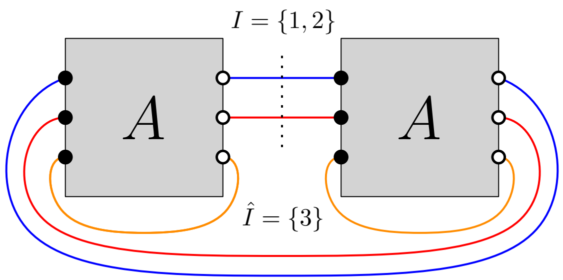

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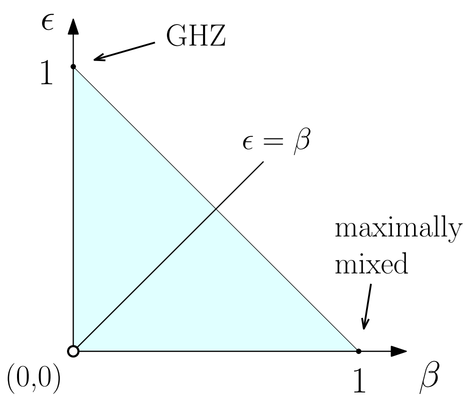
- $S_n(\rho_I) = \big(\epsilon(\rho) + \beta(\rho)|I|\big) \ln N + O(1)$

- Can see that $\epsilon(\rho) + \beta(\rho) \leq 1 + O(1/\ln(N))$



- Mutual information (Rényi)

$$\mathcal{I}_n[I](\rho) = S_n(\rho_I) + S_n(\rho_{\hat{I}}) - S_n(\rho) = 2\epsilon(\rho) \log N + O(1)$$

→ ε(ρ) > 0 informs on how entangled the state ρ is

Meaning of the parameters ε and β

We can make precise statements by looking at the **conditional n-inequalities**:

If $|I| > D/2$ $$\max\{S_n^I(\rho), S_n^{\hat{I}}(\rho)\} > S_n(\rho) \quad \Leftrightarrow \quad \epsilon > \beta(D - |I|)$$

→ If this inequality is satisfied, all bipartitions of the D-parts in two groups of size $|I|$ and $D - |I|$ are entangled (…) [Horodeckies 96]

In particular:

if $\epsilon(\rho) > \beta(\rho)(D-1)$

→ all bipartitions of the subsystems are entangled!

→ The state ρ is said to be ``*genuinely entangled*"

5 – Results on randomized measurements

(part of which with Collins and Gurau)

Moments

Moments of the distribution of randomized measurements

For tensors in the ensemble (and more) :

$$\lim_{N\to\infty}\left\langle \mathrm{Tr}(U^\dagger A U \rho)^n \right\rangle = \left(\mathrm{Tr}(A) N^{-D}\right)^n (1+o(1))$$

…The **moments** of the distribution of randomized measurements contain no information on ρ in this limit (at first order)…

→ Look at **connected correlations instead (cumulants**)

Results for observables of small rank

Results for an observable of small rank

$$\boxed{\mathrm{Tr}(U^\dagger A U \rho) \qquad\qquad A = \bigotimes_{c=1}^{D} |0\rangle\langle 0|}$$

$$U = U_1 \otimes \cdots \otimes U_D$$

$U_c \in U(N)$ Haar distributed

$$A_{i_1,\dots,i_D\,;\,j_1,\dots,j_D} = \prod_{c=1}^{D} \delta_{i_c,1}\delta_{j_c,1}$$

$$\left\langle \left(\mathrm{Tr}(U^\dagger A U \rho)\right)^n \right\rangle_c = \frac{1}{N^{nD}} \sum_{G_1,G_2\in\mathbb{G}_n} \; \sum_{l\geq l_{\min}(G_1,G_2)} \left(-\frac{1}{N}\right)^l \mathrm{Tr}_{G_1}(A)\,\mathrm{Tr}_{G_2}(\rho)\, f[G_1,G_2;l]$$

Plug-in the assumptions for *A* and *ρ*

Results for an observable of small rank

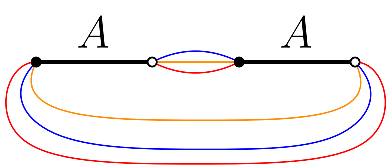
$$\mathrm{Tr}(U^\dagger A U \rho) \qquad A = \bigotimes_{c=1}^{D} |0\rangle\langle 0|$$

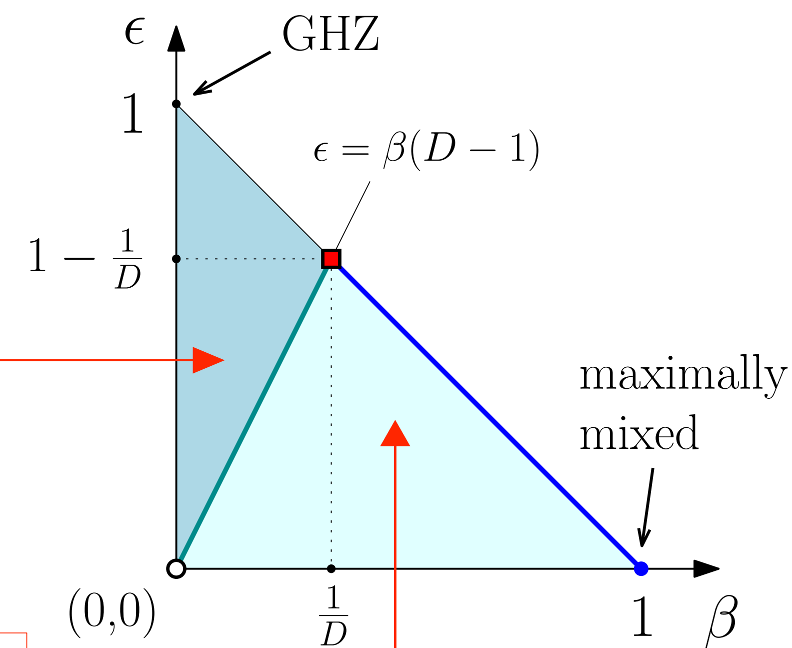
$$U = U_1 \otimes \cdots \otimes U_D$$

$U_c \in U(N)$ Haar distributed

$$\Big\langle \big(\mathrm{Tr}(U^\dagger A U \rho)\big)^n \Big\rangle_c \sim (n-1)!\,\mathrm{Tr}(\rho^n) N^{-D}$$

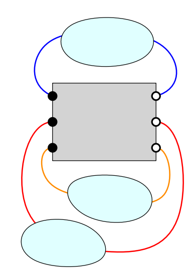
→ Extract the Rényi entropies (so also β(ρ))





$$\Big\langle \big(\mathrm{Tr}(U^\dagger A U \rho)\big)^n \Big\rangle_c \propto \text{“} \sum_{G \ \text{SYK-melonic}} \mathrm{Tr}_G(\rho) \text{”}$$

→ Extract info on other LU-invariants (extract β(ρ) + ε(ρ))



Results for an observable of small rank

$$\mathrm{Tr}(U^\dagger A U \rho) \qquad A = \bigotimes_{c=1}^{D} |0\rangle\langle 0|$$

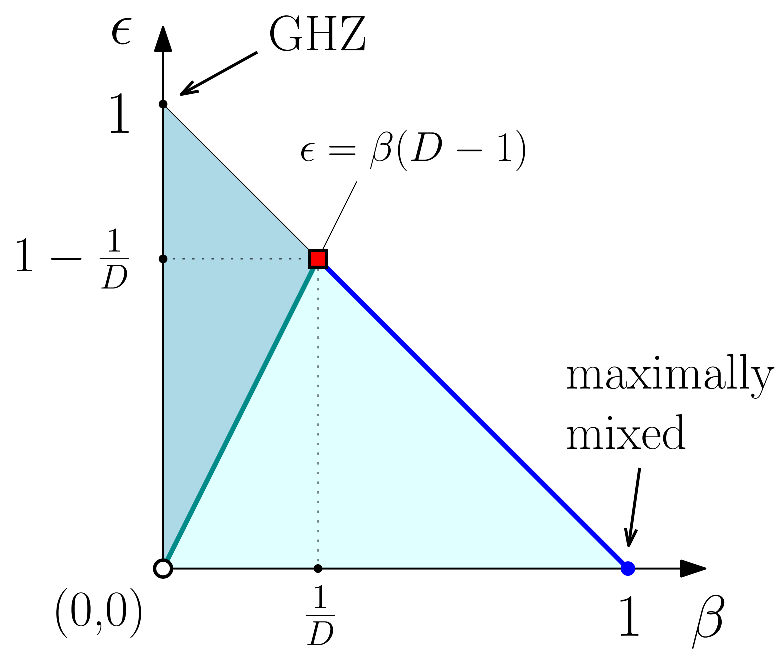
$$U = U_1 \otimes \cdots \otimes U_D$$

$U_c \in U(N)$ Haar distributed

- Very different results for these two zones
- Detects states satisfying:

$$\epsilon(\rho) > \beta(\rho)(D-1)$$

which are genuinely entangled!



Varying the rank of the observables

``…What about the choice of observables…
Are they all as good in the role of ``probe''? ''

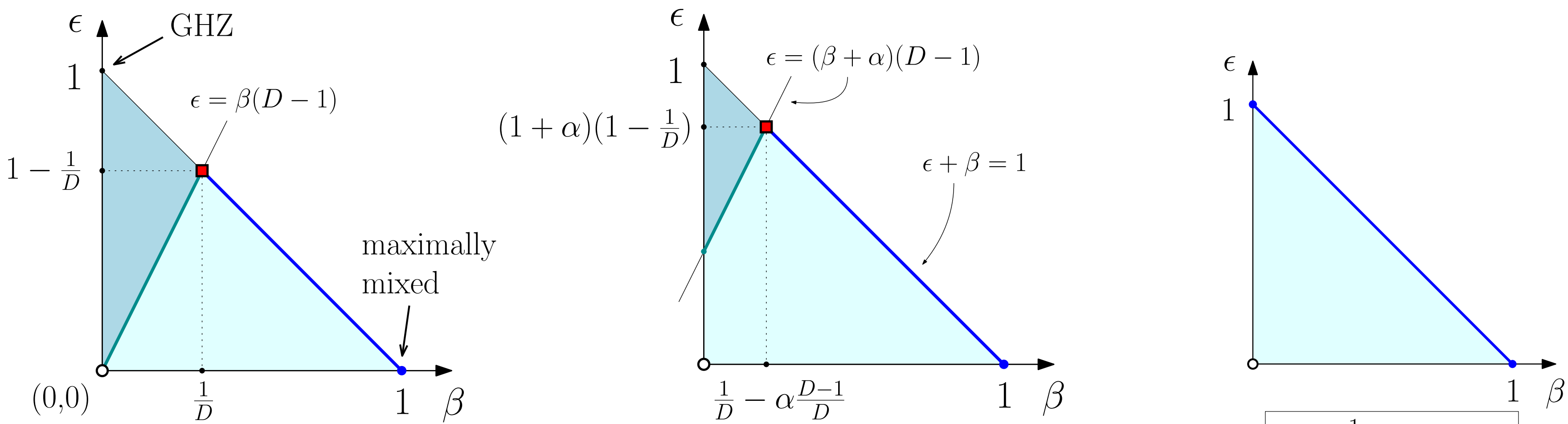
→ In our ensemble, let's try to see if the same result is obtained no matter the common rank of the observables.

Varying the ranks of the observables

$$\mathrm{Tr}(U^\dagger A U \rho) \qquad A = \bigotimes_{c=1}^{D} A_c, \quad \mathrm{Tr}(A_c^n) = N^\alpha$$

$$U = U_1 \otimes \cdots \otimes U_D$$

$U_c \in U(N)$ Haar distributed



a) $\alpha = 0$

b) $0 < \alpha < \frac{1}{D-1}$

d) $\frac{1}{D-1} < \alpha < 1$

→ The zone of detectability shrinks, and eventually disappears!

→ For the genuinely entangled states satisfying $\epsilon(\rho) > \beta(\rho)(D-1)$ we can extract both β(ρ) and ε(ρ) by performing two measurements with observables of different ranks: Get dominant contribution of Rényi entropies, mutual information, conditional entropies…

→ Other interesting results when observables of different ranks on different subsystems.

5 – Conclusions

Conclusions

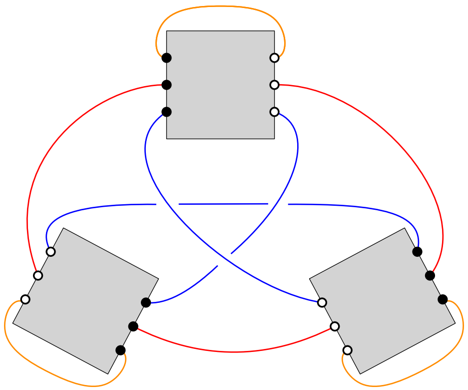
Ther tensor invariants that are the correlations and interactions of random tensor models appear very naturally in the study of entanglement (where they are called local unitary invariants), and in fact there is a growing literature on this, in the context of (black hole evaporation and) holography (including random tensor networks, see Sylvain's talk).
N.B: the topology/geometry of the dual triangulation seems to play a role there too.

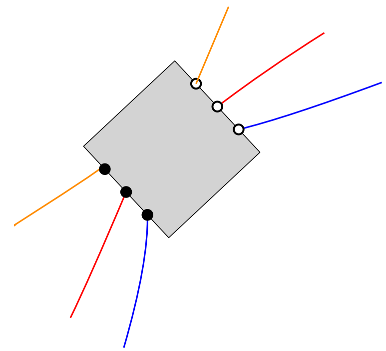
In the limit where N is very large, the dominant exponent of N of the tensor invariants already contain some information on entanglement (because it is the leading term of the associated ``entropies'').
I've illustrated this for the Rényi entropies (a.k.a. cyclic melonic and necklaces) using an ``ensemble'' of density matrices with a dominant exponent that depends on two parameters.

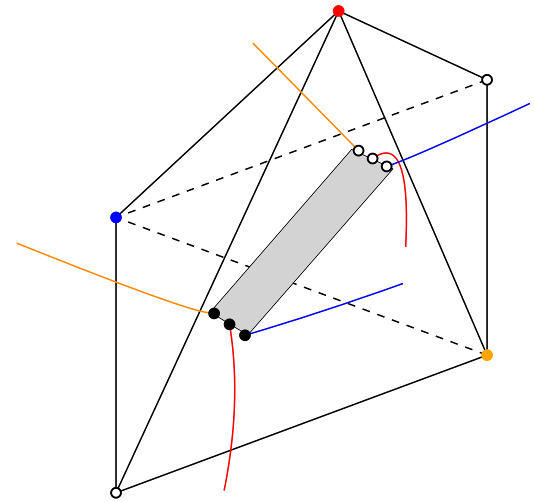
The quantities that appear from the correlations of randomized measurements (tensor HCIZ integral) for density matrices in this ensembles are sums over SYK-melonic graphs.
This leads to the Renyi mutual information for two tensors but not for more.

>>> What is the information contained in these quantities? More generally, do melonic graphs contain more information than Renyi entropies?

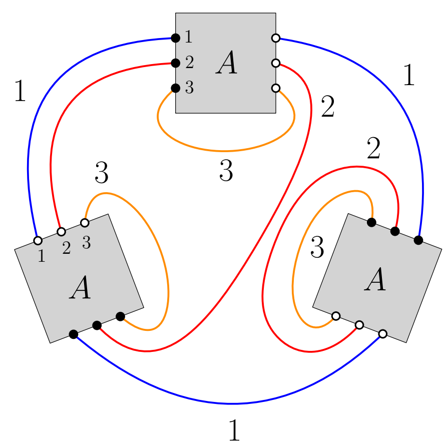
Many things computed in random tensor models have an interpretation in this context…
Including anlyticity properties of generating functions / partition functions etc.







Thank you!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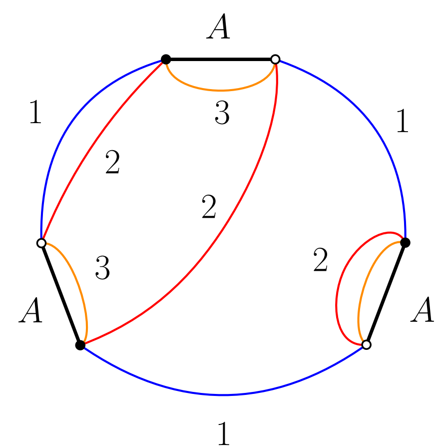
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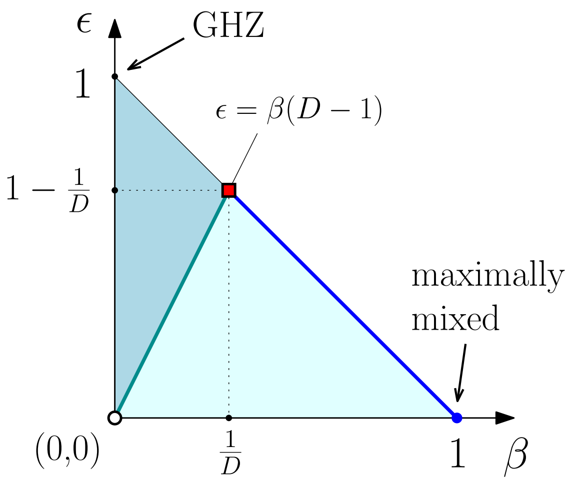
ϵ
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$\epsilon = \beta(D-1)$
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maximally mixed
(0,0)
$\frac{1}{D}$
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β