Tensor Field Theory with local and nonlocal degrees of freedom: Phase Transition from the FRG Approach

Joseph Ben Geloun

LIPN, Univ. Sorbonne Paris Nord

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a joint work with
Andreas G A Pithis (Arnold Sommerfeld Center for TP, München)
and Johannes Thürigen (Mathematisches Institut der WW-Univ., Münster)

June 15, 2023
From perturbative to non-perturbative QFT
WWU Münster, Germany

Outline

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$$M_{ab} \rightsquigarrow \text{Random 2D geom/maps ;} \quad T_{abc} \rightsquigarrow \text{Random 3D geom.}$$

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• The Tensor Track for QG and random geometry ©Rivasseau.

Tensor fields and random discrete geometry

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You are never better served than by yourself !

Tensor fields and random discrete geometry

$\rightarrow$ Two hypothesis: Discreteness & Randomness

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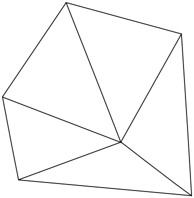

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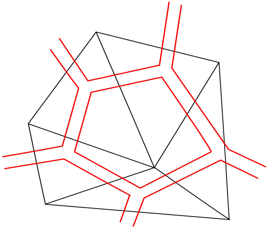

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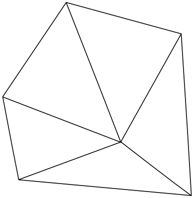

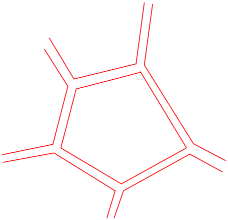

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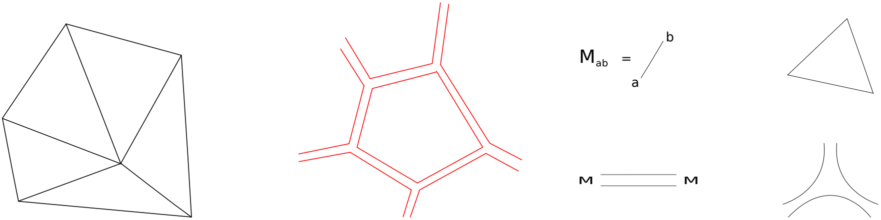

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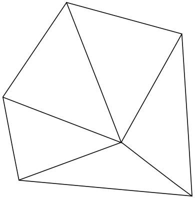

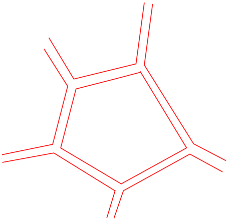

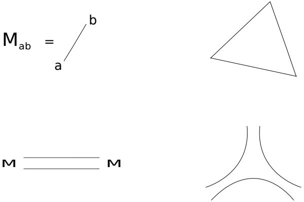

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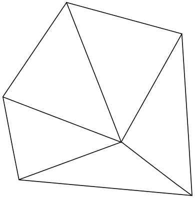

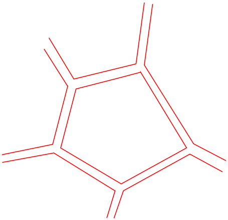

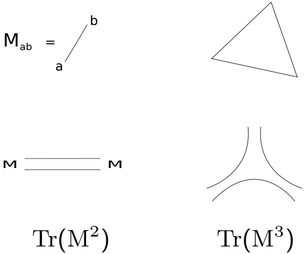

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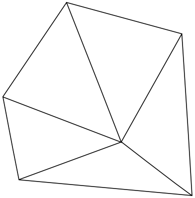

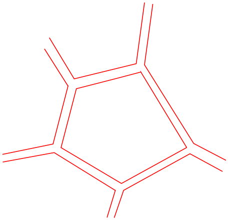

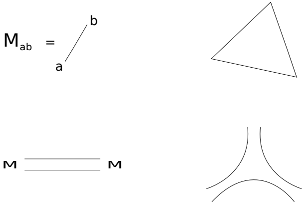

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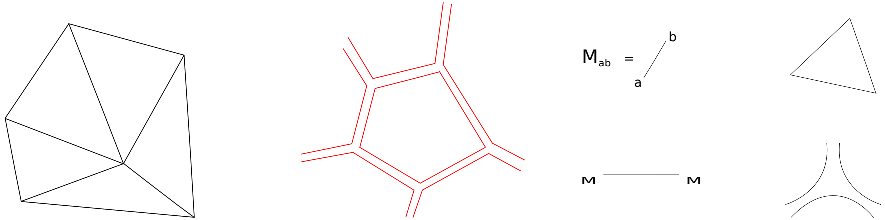

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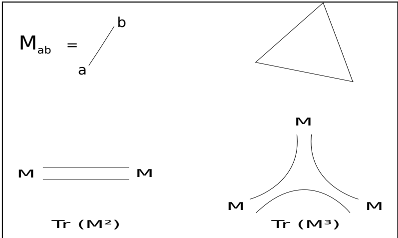

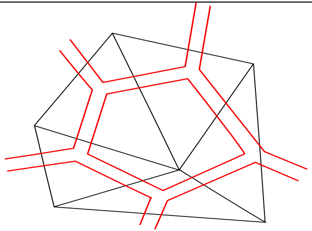

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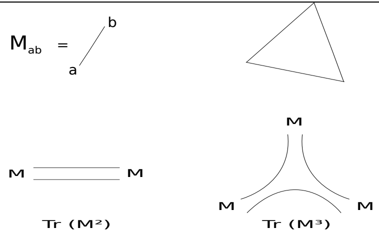	
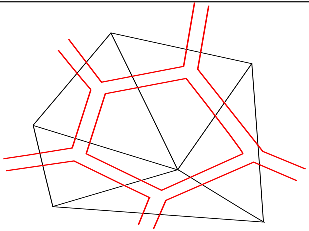	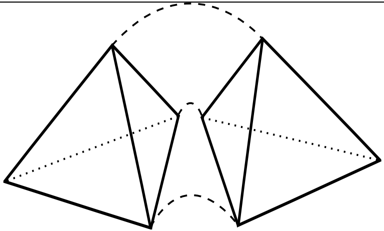
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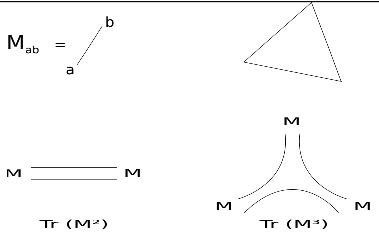
<tr><td>
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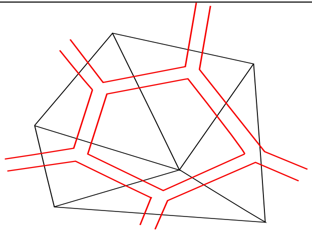
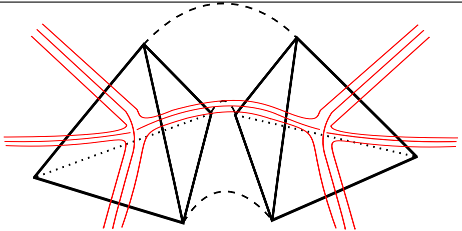
<tr><td></td><td></td></tr>
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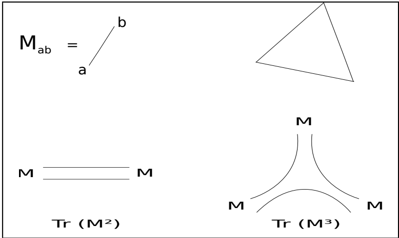	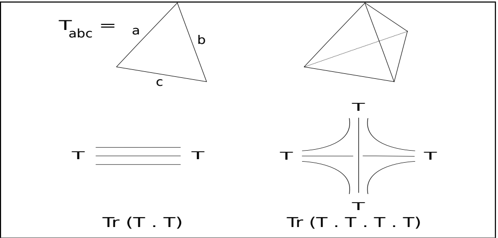
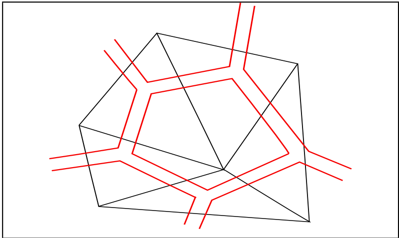	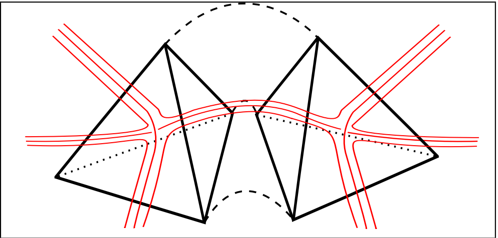
map ≡ triangulated surface	stranded graphs ≡ *d* simplicial complex

TFT/TGFT Renormalisation group (RG) analysis

• Nonlocal QFT with propagating tensor degrees of freedom: Tensor Field Theory

→ Renormalization perturbative have been worked out since 2011 [BG & Rivasseau 2011]

$T_{a_1 a_2 \ldots a_r}$ the indices are propagating themselves.

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Focus

• Nonperturbative study: FRG analysis was launched to understand the phase diagram of TFT [Benedetti, BG, Oriti, 2014].

Functional Renormalisation Group analysis of TFT/TGFT

• Consider G a compact group and $T : G^r \to \mathbb{K}$

• No possible phase transition as long as G is compact [Benedetti 2014]; (in the limit of infinite radius, yes).

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• $T : U(1)^3 \to \mathbb{R}$
$\to$ The system of β-functions was non-autonomous: explicit k in the eq.
$\to$ due to an external scale: the radius of the compact manifold

• Making the system autonomous and finding good notion of scaling dimension of coupling constants
$\to$ large N mode limit (UV) (decompactify the space);
$\to$ small mode limit (IR)
$\rightsquigarrow$ Phase diagram: strong evidence of fixed points

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• Computation at an intermediate/interpolated regime.

TFT/TGFT: FRG analysis

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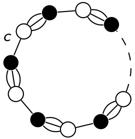

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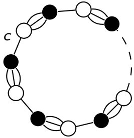

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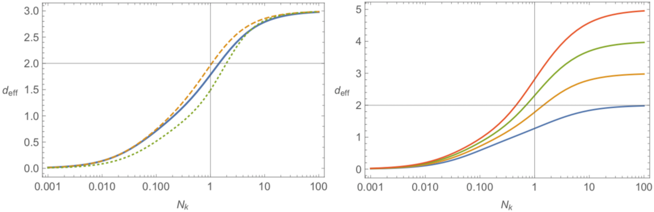

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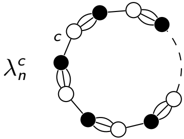

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- The fields: G a Lie group

$$\Phi : \mathbb{R}^d \times G^r \to \mathbb{K} = \mathbb{C}, \mathbb{R} \qquad (1)$$

$$(\boldsymbol{x}, \boldsymbol{g}) \mapsto \Phi(\boldsymbol{x}, \boldsymbol{g}) \qquad (2)$$

- G is chosen compact $\rightarrow$ Peter-Weyl transform of the field

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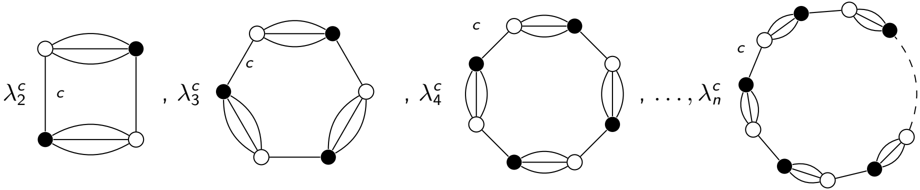

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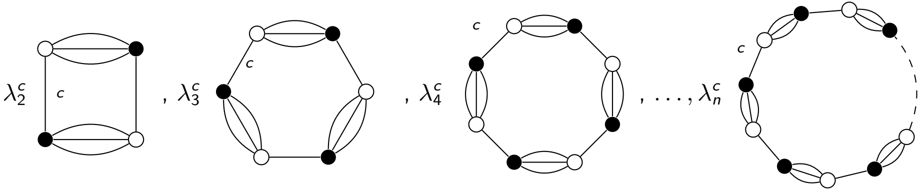

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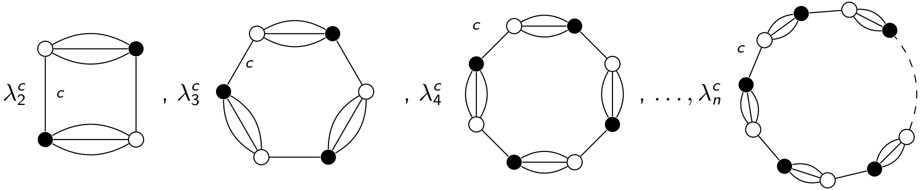

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- Nonlocal and a tractable combinatorics: computable at arbitrary valence.
- $S_{int}(\phi, \bar{\phi}) = \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{x} \left[\sum_{n=2}^{n_{\max}} \sum_{c=1}^{r} \lambda_n^c \, \mathrm{Tr}_{n;c}(\phi, \bar{\phi})(\boldsymbol{x}) \right]$

TFT model: action

- The action

$$S(\phi, \bar{\phi}) = S_{kin}(\phi, \bar{\phi}) + S_{int}(\phi, \bar{\phi})$$

$$S_{kin}(\phi, \bar{\phi}) = (\bar{\phi}, K\phi) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{x}' \int_{G^r \times G^r} \mathrm{d}\boldsymbol{g}\mathrm{d}\boldsymbol{g}' \;\; \bar{\phi}(\boldsymbol{x}, \boldsymbol{g}) K(\boldsymbol{x}, \boldsymbol{g}; \boldsymbol{x}', \boldsymbol{g}'))\phi(\boldsymbol{x}', \boldsymbol{g}')$$

$$K(\boldsymbol{x}, \boldsymbol{g}; \boldsymbol{x}', \boldsymbol{g}') = \delta(\boldsymbol{x} - \boldsymbol{x}')\delta(\boldsymbol{g}\boldsymbol{g}'^{-1})\Big[\Big(- \Delta_x - \kappa^2 \sum_{c=1}^{r} (\Delta_g^{(c)})^{\zeta}\Big) + \mu_k\Big] \quad (5)$$

where

Δ_x is the Laplacian on $\mathbb{R}^d$,
$\Delta_g^{(c)}$ the (colored) Laplacian on G,
$\zeta \in]0, 1]$
κ restores the dimension balance.

Outline

FRG formalism for TFT: Wetterich-Morris equation

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• Introduce a scale k and an IR (cut-off) regulator $\mathcal{R}_k$ that projects only on field modes relevant to that scale

$$Z_k[J,\bar{J}] = \mathrm{e}^{W_k[J,\bar{J}]} = \int \mathcal{D}\varphi \mathcal{D}\bar{\varphi} \mathrm{e}^{-S[\varphi,\bar{\varphi}]-(\varphi,\mathcal{R}_k\varphi)+(J,\varphi)+(\varphi,J)}. \tag{6}$$

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$\mathcal{R}_k$ should satisfy specific conditions;

• Scale dependent effective action

$$\Gamma_k[\varphi,\bar{\varphi}] = \sup_{J,\bar{J}} \big[(\varphi,J)+(J,\varphi)-W_k[J,\bar{J}]\big] - (\varphi,\mathcal{R}_k\varphi). \qquad (7)$$

• Expansion for TFT:

$$\Gamma_k[\varphi,\bar{\varphi}] = (\varphi,\mathcal{K}_k\varphi) + \sum_\gamma \lambda_{\gamma;k}\mathrm{Tr}_\gamma[\varphi,\bar{\varphi}],$$

$$\mathcal{K}_k = Z_k\Big(-\Delta_x - \kappa^2\sum_{c=1}^{r}(\Delta_g^{(c)})^\zeta\Big) + \mu_k \qquad (8)$$

FRG formalism for TFT: Wetterich-Morris equation

• Flow equation for the effective average action: The Wetterich-Morris equation

$$(k\partial_k)\,\Gamma_k[\varphi,\bar\varphi] = \frac{1}{2}\mathrm{STr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\mathbb{I}_2\right)^{-1}(k\partial_k)\,\mathcal{R}_k\right], \tag{9}$$

where STr is a supertrace (all configuration space variables integrated), $\Gamma_k^{(2)}$ is the Hessian matrix of Γ_k

$$\begin{aligned}
\Gamma_k^{(2)}[\varphi,\bar\varphi](\boldsymbol{x},\boldsymbol{g};\boldsymbol{y},\boldsymbol{h}) &:= \frac{\delta^2\Gamma_k[\varphi,\bar\varphi]}{\delta\varphi(\boldsymbol{x},\boldsymbol{g})\delta\bar\varphi(\boldsymbol{y},\boldsymbol{h})}\\
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\Gamma_k^{(2)}[\bar\varphi,\bar\varphi](\boldsymbol{x},\boldsymbol{g};\boldsymbol{y},\boldsymbol{h}) &:= \dots
\end{aligned} \tag{10}$$

• Results are dependent on $\mathcal{R}_k$ and the ansatz for Γ_k;
$\Rightarrow$ Prove that the results holds for classes of regulators and an enlarged truncation helps in gaining confidence in the results.

Outline

The cyclic melonic potential approximation

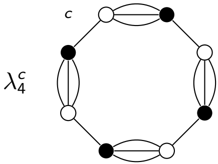

Figure: Rank $d = 4$ cyclic-melonic interaction with valence $2n = 8$.

- Second field derivative of the interacting part:

$$F_2[\varphi, \bar{\varphi}](\boldsymbol{x}, \boldsymbol{g}; \boldsymbol{y}, \boldsymbol{h}) = \sum_{c=1}^{r} \sum_{n=2}^{n_{\max}} \frac{n}{n!} \lambda^c_{n,k} \Big[$$

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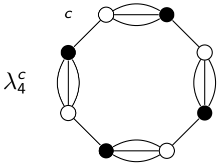

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$$\Big[\prod_{b \neq c} \delta(g_b, h_b) \Big] \left(\bar{\varphi} \cdot_{\hat{c}} \varphi \right)^{n-1} (g_c, h_c)$$

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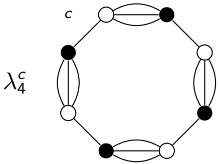

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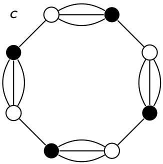

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The cyclic melonic potential approximation: Projection on local fields

- $G = U(1)$
- Projection on local fields after derivation: $\varphi(\boldsymbol{x}, \boldsymbol{g}) = \chi$ and $\rho = a_G \chi^2$

$$
\begin{aligned}
& F_2[\bar{\chi}, \chi](\boldsymbol{x}, \boldsymbol{g}; \boldsymbol{y}, \boldsymbol{h}) \\
& = a_{\mathbb{R}}^d a_{\mathrm{G}}^{-r} \sum_{c=1}^{r} \left[\left(a_G \prod_{b \neq c} \delta(g_b, h_b) + a_G \delta(g_c, h_c) - 1 \right) V_k^{c\,\prime}(\rho) + \rho V''(\rho) \right] \\
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- Regulator in momentum space

$$
\mathcal{R}_k(\boldsymbol{p}, \boldsymbol{j}) = Z_k \left(k^2 - p^2 - \kappa^2 \frac{j^{2\zeta}}{a_{\mathrm{G}}^{2\zeta}} \right) \theta \left(k^2 - p^2 - \kappa^2 \frac{j^{2\zeta}}{a_{\mathrm{G}}^{2\zeta}} \right) \tag{13}
$$

where $j^{2\zeta} = \sum_c j_c^2$ spectrum of the Laplacian on $U(1)^r$.

The cyclic melonic potential approximation: isotropic sector

- We consider the isotropic sector: $\lambda^{c}_{n,k} = \lambda_{n,k}/r\ ,\ \forall c = 1,\ldots,r.$
- Scale $t = \log k$ then $\partial_t = k\partial_k$

$$U_k(\rho) = \mu_k\rho + \sum_{n=2}^{\infty} \frac{1}{n!}\lambda_{n,k}\rho^n \tag{14}$$

- The FRG equation becomes:

$$\frac{\partial_t U_k(\rho)}{k^2 Z_k} = \frac{F^{(0)}(k)}{k^2 Z_k + U_k'(\rho) + 2\rho\, U_k''(\rho)} + \frac{F^{(0)}(k) + 2r\ F^{(1)}(k)}{k^2 Z_k + U_k'(\rho)} + 2\sum_{s=2}^{r}\binom{r}{s}\frac{F^{(s)}(k)}{k^2 Z_k + \mu_k + \frac{r-s}{r} V_k'(\rho)} \tag{15}$$

Beta-functions

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$\rightarrow$ How do you deal with a generic inverse potentials (and their derivatives) with arbitrary valence ? Ans: expansion in Bell-polynomials (that I cannot discuss !)

$$\frac{1}{f(\rho)} = \frac{1}{f(0)} + \sum_{n=1}^{\infty} \frac{\rho^n}{n!} \sum_{l=1}^{n} (-1)^l \frac{l!}{f(0)^{l+1}} B_{n,l}\left((f'(0), f''(0), ..., f^{(n-l+1)}(0) \right) ,$$

which is given in terms of partial (exponential) Bell polynomials

$$B_{n,l}(x_1, x_2, \dots, x_{n-l+1}) = \sum_{\substack{\sigma \vdash n \\ |\sigma| = l}} \binom{n}{s_1, \dots, s_n} \prod_{j=1}^{n-l+1} \left(\frac{x_j}{j!}\right)^{s_j} .$$

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$\rightarrow$ How do you deal with the spectral sums on subvolumes of $\mathbb{R}^d \times \mathbb{Z}^r$? Ans: Approximation ...

Threshold spectral sums in rank $s \leq r$

- The master: $\eta_k = -\partial_t \log Z_k$

$$F^{(s)}(k) = \left(1 - \frac{\eta_k}{2}\right) I_1^{(d,s)} + \frac{\eta_k}{2k^2} \left(I_{p^2}^{(d,s)}(k) + \bar{\kappa} I_{j^{2\zeta}}^{(d,s)}(k)\right) \qquad (17)$$

where the threshold functions are defined by, for all $f : \mathbb{R}^d \times \mathbb{Z}^s \to \mathbb{R}$

$$I_f^{(d,s)}(k) = \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{p} \sum_{\boldsymbol{j} \in (\mathbb{Z} \setminus \{0\})^s} \theta\left(k^2 - p^2 - \bar{\kappa} j^{2\zeta}\right) f(\boldsymbol{p}, \boldsymbol{j}), \qquad (18)$$

for all $s > 0$, and $I_f^{(d,0)}(k) = 0$.

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→ Difficult to handle in full generality.
→ Hopefully: no need of an explicit expression, but just their asymptotic behavior !
→ Approximation at large k: Lejeune-Dirichlet sums (1839's paper)

$$I_1^{(d,s)} \sim k^{d+s/\zeta}$$

$$I_{p^2}^{(d,s)}(k) \sim I_{j^{2\zeta}}^{(d,s)}(k) \sim k^{2+d+s/\zeta} \tag{19}$$

The full β-functions

- Look like this

$$\beta_{n,k}(\mu,\lambda_i) = \mathit{Coeff}(\mu,\lambda_i)F^{(0)}(k) + \sum_{l=1}^{n} \mathit{Coeff}_{n,l}(\mu,\lambda_i)F_l(k) \tag{20}$$

$$\mathit{Coeff}(\mu,\lambda_i) = \frac{(-1)^l l!}{(Z_k k^2 + \mu_k)^{l+1}} B_{n,l}\left(3\lambda_2, 5\lambda_3, \dots, (2n-2l+3)\lambda_{n-l+2}\right)$$

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- Example: the flow equation at the first three orders ($n = 1, 2, 3$) are

$$\frac{\partial_t \mu_k}{Z_k k^2} = \frac{-\lambda_2}{(Z_k k^2+\mu_k)^2}\left(3F^{(0)}+F_1\right)(k), \tag{22}$$

$$\frac{\partial_t \lambda_2}{Z_k k^2} = \frac{-\lambda_3}{(Z_k k^2+\mu_k)^2}\left(5F^{(0)}+F_1\right)(k) + \frac{2\lambda_2^2}{(Z_k k^2+\mu_k)^3}\left(9F^{(0)}+F_2\right)(k), \tag{23}$$

$$\frac{\partial_t \lambda_3}{Z_k k^2} = \frac{-\lambda_4}{(Z_k k^2+\mu_k)^2}\left(7F^{(0)}+F_1\right)(k) + \frac{6\lambda_2\lambda_3}{(Z_k k^2+\mu_k)^3}\left(15F^{(0)}+F_2\right)(k)$$
$$+ \frac{-6\lambda_2^3}{(Z_k k^2+\mu_k)^4}\left(27F^{(0)}+F_3\right)(k). \tag{24}$$

Outline

$O(N)^r$-invariant TFT

→ No dynamics on the j's: $\kappa = 0$ (same types of models Benedetti, Gurau, Harribey...)
→ Spectral sums, $|j_c| < N_c$

$$F^{(s)}(k) = v_d Z_k k^d \left(1 - \frac{\eta_k}{d+2}\right)(N-1)^s \qquad N = 2N_c + 1 \tag{25}$$

$v_d =$ volume of the d-dimensional unit ball
→ Dimensionless couplings (ordinary for local field theory)

$$\mu_k = Z_k k^2 \tilde{\mu}_k \quad , \quad \lambda_{n;k} = Z_k^n k^{2n} (v_d\, k^d)^{1-n} \tilde{\lambda}_{n;k} \quad \text{for } n \geq 2\,. \tag{26}$$

→ FRG equation for the potential at the large N limit

$$\partial_t u_k(\tilde{\rho}) + d u_k(\tilde{\rho}) - (d - 2 + \eta_k)\tilde{\rho}\, u_k'(\tilde{\rho}) = \frac{1 - \frac{\eta_k}{d+2}}{1 + \frac{r-1}{r}\tilde{\mu}_k + u_k'(\tilde{\rho})}\,. \tag{27}$$

→ $r = 1$, $O(N)$-vector model: $(\eta_k = 0\ (LPA), \tilde{\mu}_* < 0) \Rightarrow$ Wilson-Fisher fixed point for $2 < d < 4$ (a single relevant direction);
→ $r > 1$, $\eta_k = 0$, $\tilde{\mu}_* < 0$: minor modifications by r factors.

n	$10\tilde{\mu}$	$10^2\tilde{\lambda}_2$	$10^3\tilde{\lambda}_3$	$10^4\tilde{\lambda}_4$	$10^5\tilde{\lambda}_5$	$10^6\tilde{\lambda}_6$	$10^7\tilde{\lambda}_7$	$10^8\tilde{\lambda}_8$	$10^9\tilde{\lambda}_9$	$10^{10}\tilde{\lambda}_{10}$
6	-6.5649	5.1643	9.4342	15.067	7.9684	-54.935				
7	-6.5541	5.1883	9.4629	14.916	6.0346	-73.574	-229.55			
8	-6.5563	5.1834	9.4570	14.947	6.4366	-69.694	-181.66	797.55		
9	-6.5576	5.1806	9.4538	14.964	6.6554	-67.584	-155.63	1230.5	8760.4	
10	-6.5575	5.1808	9.4540	14.963	6.6390	-67.743	-157.59	1198.0	8102.3	-15350.
11	-6.5573	5.1811	9.4544	14.961	6.6164	-67.961	-160.28	1153.3	7198.1	-36441.
12	-6.5573	5.1811	9.4544	14.961	6.6157	-67.967	-160.35	1152.0	7172.4	-37040.

n	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	θ_9	θ_{10}
6	0.50915	-1.7691	-5.5429	-9.9919	-16.288	-28.526				
7	0.51807	-1.7196	-4.4455	-8.5409	-12.944	-21.296	-34.652			
8	0.51817	-1.7601	-3.9621	-7.3798	-11.061	-17.086	-26.710	-41.022		
9	0.51716	-1.7723	-3.8661	-6.5101	-9.8464	-14.329	-21.803	-32.301	-47.464	
10	0.51704	-1.7673	-3.9116	-6.0278	-8.9458	-12.485	-18.399	-26.781	-38.014	-53.954
11	0.51714	-1.7650	-3.9374	-5.9025	-8.2795	-11.246	-15.945	-22.858	-31.940	-43.840
12	0.51716	-1.7654	-3.9317	-5.9493	-7.8900	-10.401	-14.165	-19.931	-27.550	-37.247

Table: Values of the coupling constants and scaling exponents (eigenvalues of the stability matrix) at the Wilson-Fisher type fixed point for the $d = 3$ dimensional $O(N)^{r=3}$-invariant local field theory in $(\bar{\varphi}\varphi)^n$ truncation. Convergence with higher orders n justifies to draw conclusions from results at finite n.

The large k and autonomous limit

- Case $\kappa > 0$ (presence of j^{ζ}): Non autonomous system difficult to handle.

- Large momentum makes autonomous the system

$$\tilde{k} = a_G \left(\frac{k}{\sqrt{\bar{\kappa}}} \right)^{\frac{1}{\zeta}} \tag{28}$$

- We consider the large $\tilde{k}$-limit and its interpretations:

$\to$ large momentum limit: UV

$\to$ large volume a_G limit (kind of thermodynamic limit)

- Spectral sum approximation

$$F_k^{(s)} \sim_{\tilde{k}\to\infty} \frac{1}{2} v_{d,r,\zeta} k^d \tilde{k}^s \left(2 - \eta_k \left(1 - \frac{d + \frac{s}{\zeta}}{d + \frac{s}{\zeta} + 2} \right) \right) \tag{29}$$

The matter of dimension and (re-)scaling

- Dimensionless couplings

$$\mu_k = Z_k k^2 \tilde{\mu}_k \qquad \lambda_{n;k} = r Z_k^n k^{2n} \left(V_{d,r,\zeta} k^{d+\frac{r-1}{\zeta}} \right)^{1-n} \tilde{\lambda}_{n;k} \quad \text{for } n \geq 2 \tag{30}$$

- Effective dimension

$$d_{\mathrm{eff}} := d + \frac{r-1}{\zeta} , \quad r > 1$$
$$d_{\mathrm{eff}} := d + \frac{1}{\zeta} , \quad r = 1 \tag{31}$$

- Flow equation $n \geq 2, r > 0$,

$$\partial_t u_k(\tilde{\rho}) + d_{\mathrm{eff}} u_k(\tilde{\rho}) - (d_{\mathrm{eff}} - 2 + \eta_k)\tilde{\rho}\, u_k'(\tilde{\rho}) = \frac{1 - \frac{\eta_k}{d_{\mathrm{eff}}+2}}{1 + \frac{r-1}{r}\tilde{\mu}_k + u_k'(\tilde{\rho})} \tag{32}$$

→ Same as for the $O(N)^r$ model but exchange $d \leftrightarrow d_{\mathrm{eff}}$.
→ Noticed in [Marchetti et al, 2021] in the Gaussian approx.

- The analysis is similar: solutions are linked, critical dimensions shifted around: $d_{\mathrm{eff}} = d + \frac{r-1}{\zeta} < d_{crit} = 4$ and valid only for restricted couples (d, r).
- Existence of WF-fixed points with minor quantitative modifications.

Non autonomous limit: Explicit k integration

• Even more complicated: v_G kept finite not possible to obtain a dimensionless flow equation using only natural coupling rescaling;

• Use $F_1(k)$ to define the scaling of the couplings

$$\mu_k = Z_k k^2 \tilde{\mu}_k \qquad \lambda_{n;k} = Z_k^n k^{2n} \left(F_1(k)\right)^{1-n} \tilde{\lambda}_{n;k} \quad \text{for } n \geq 2 \tag{33}$$

• The effective dimension is then defined

$$d_{\text{eff}}(k) := k\partial_k \log F_1(k)\,.$$

• Flow equation

$$\partial_t \tilde{\lambda}_{n;k} + d_{\text{eff}}(k)\tilde{\lambda}_{n;k} - n(d_{\text{eff}}(k) - 2 + \eta_k)\tilde{\lambda}_{n;k} =$$

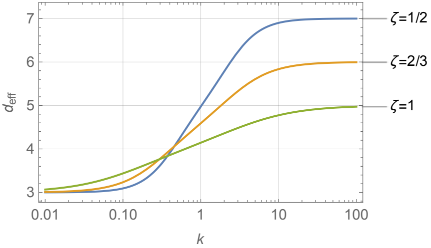
$$\frac{F^{(0)}}{F_1}(k)\beta^{\text{V1}}_{n;k}(\tilde{\mu}_k, \tilde{\lambda}_{i;k}) + \sum_{l=1}^{n} \frac{F_l}{F_1}(k)\beta^{\text{V2}}_{n,l;k}(\tilde{\mu}_k, \tilde{\lambda}_{i;k}) \tag{34}$$

Flow of dimension

- Limits

$$d_{\text{eff}}(k \gg 1) = d + \frac{r-1}{\zeta} \qquad d_{\text{eff}}(k \ll 1) = d \tag{35}$$

- At finite k: $F_1^{(d,r)}(k)$ is a polynomial in k;



Left: Comparing the flow of effective dimension for different values of ζ in the case $d = r = 3$ (with $\bar{\kappa} = 1, \eta_k = 0$).

Fixed points, phase transition and symmetry broken

- Fixed points: hints that we recover the structure of fixed of a ϕ^4 in the IR;
- Numerics: symmetry may be restored in the IR, for a choice of $\mu_k < 0$

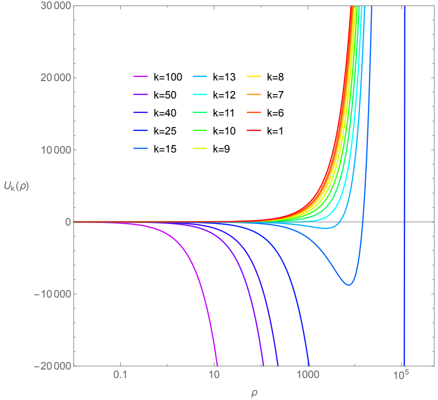


Figure: Symmetry restoration in the IR for $d = r = 3$ for φ^6-model.

Fixed points, phase transition and symmetry broken

• Numerics: we see symmetry is still broken in the IR (thus phase transition): for another choice $\mu_k < 0$ (15% off the previous choice)

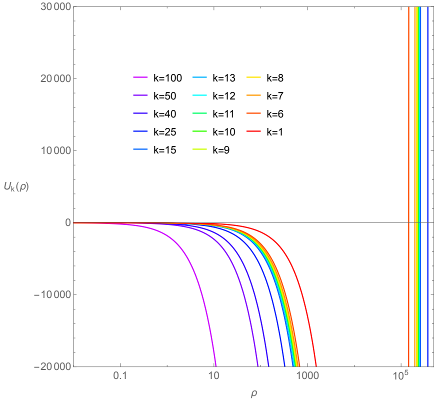


Figure: Symmetry remains broken in the IR for $d = r = 3$ for φ^6-model.

Outline

Conclusion

- $TFT(x)$ with local dimension $x \in \mathbb{R}^d$ and nonlocal dimensions $g \in G^r$,

$\rightarrow$ in the cyclic melonic approx and LPA: strong phase transition

$\rightarrow$ allows to identify a flow of an effective dimension;

- Effective dimension $d_{\rm eff}(k)$ flows from $d - (r-1)/\zeta \rightarrow d$

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$\rightarrow$ Dramatic approximation: LPA making $\eta_k = 0$
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$$\mathcal{R}_k(\boldsymbol{p},\boldsymbol{j}) = Z_k^1 \left(k^2 - p^2 - \left(\kappa_k^2 = \frac{Z_k^2}{Z_k^1} \right) \frac{j^{2\zeta}}{a_{\rm G}^{2\zeta}} \right) \theta \left(k^2 - p^2 - \kappa_k^2 \frac{j^{2\zeta}}{a_{\rm G}^{2\zeta}} \right) \tag{36}$$

$\rightarrow$ Alternative regulator: Buccio and Percacci '22 [arXiv:2207.10596[hep-th]]
$Z_1 \left(k^2 - p^2\right) \theta \left(k^2 - p^2\right) + Z_2 \left(k^{2\zeta} - j^{2\zeta}\right) \theta \left(k^{2\zeta} - j^{2\zeta}\right)$
$\rightarrow$ Talk of Robero: fields with scaling dimension interpolating between 0 to 1.

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Thank you !