Old and new conformal field theories at large N

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June 15, 2023 - "From perturbative to non-perturbative QFT", Münster

Outline

Renormalization group and conformal field theories

- Renormalization group (RG) is crucial to modern understanding of QFT

  It singles out special theories: fixed points (scale invariant theories)

  However, mostly perturbative $\Rightarrow$ interacting FPs are hard to study

- Typically scale invariance is enhanced to conformal invariance

  $\Rightarrow$ Bootstrap strategy:

  Assume conformality and impose consistency relations to bound/isolate allowed CFT data
  $\Rightarrow$ many exact results in $d = 2$, and very precise numerical bounds in $d > 2$

  However, so far limited to unitary theories at zero temperature, and no direct link to path integral origin

Relating RG and CFT

$\Rightarrow$ It is useful to develop models and limits in which we can control both approaches

$\Downarrow$

Large-N methods

$\Rightarrow$ computable toy models (strict large-N)
and hopefully more (approach to finite N via series expansion in $1/N$)

They can allow us to show that (at least for some models):

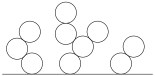
- RG admits interacting fixed point (FP)
- FP is a CFT
- Mix RG and CFT methods to understand properties of such a FP and flow to other FPs

Large-N limits in QFT

Two types of large-N limits have been extensively studied in the literature:

- Vector models: "$O(N)$ model" [Stanley (1968); Wilson (1972); . . .]

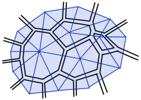
  $\Rightarrow$ Large-N limit: **cactus diagrams**

  - Solvable Schwinger-Dyson equation (mass gap equation) [Coleman, Jackiw, Politzer; Gross, Neveu (1974)]
  - Somewhat too simple (e.g. no wave function renormalization)

- Matrix models: e.g. adjoint rep. of $U(N)$ ['t Hooft (1974); ...]

  $\Rightarrow$ Large-N limit: **planar diagrams**

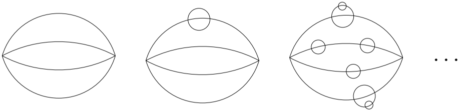
  - It plays an important role in AdS/CFT, integrability of $\mathcal{N} = 4$ SYM, Grosse-Wulkenhaar, etc ...
  - Many solvable models in zero dimension (2d QG), but increasingly hard in higher dimensions

In this talk:

Two ways to go between too simple and too difficult:

- Vector models at higher orders of the $1/N$ expansion
  [many people, but in particular: Vasil'ev, Pis'mak, Khonkonen (1981), η at $O(1/N^3)$]
  Work in progress with Maria Kallimani

- Large-N limit of **tensor models**: dominated by **melonic diagrams**



  Work in collaboration with Razvan Gurau, Sabine Harribey, Davide Lettera, Kenta Suzuki

CFT in a nutshell

- Impose invariance of n-point functions in $\mathbb{R}^d$ under $SO(d+1,1)$ (Euclidean) or $SO(d,2)$ (Lorentzian)

- Two-point functions of primary operators $\mathcal{O}_i(x)$ are completely determined by their scaling dimensions Δ_i (eigenvalue of dilations) and spin (rotations irrep.). E.g. for scalars:

$$\langle \mathcal{O}_i(x_i)\mathcal{O}_j(x_j)\rangle = \frac{\delta_{\Delta_i,\Delta_j}}{|x_{ij}|^{2\Delta_i}}\,, \qquad x_{ij} = x_i - x_j$$

  All local operators are either primaries or descendants (derivatives of primaries)

- Three-point functions of primary operators are fixed up to a constant. E.g. for scalars:

$$\langle \mathcal{O}_i(x_i)\mathcal{O}_j(x_j)\mathcal{O}_k(x_k)\rangle = \frac{c_{ijk}}{|x_{ij}|^{\Delta_i+\Delta_j-\Delta_k}|x_{ik}|^{\Delta_i+\Delta_k-\Delta_j}|x_{jk}|^{\Delta_j+\Delta_k-\Delta_i}}$$

- All higher n-point functions can in principle be reconstructed with these data, because with the operator product expansion (OPE), which depends only on them and it is convergent, we reduce to $(n-1)$-point functions:

$$\langle \mathcal{O}_i(x_i)\mathcal{O}_j(x_j)\ldots\rangle = \sum_k \frac{c_{ijk}}{|x_{ij}|^{\Delta_i+\Delta_j-\Delta_k}}\langle[\mathcal{O}_k(x_j) + \text{descendants}]\ldots\rangle$$

2PI formalism in a nutshell [Cornwall, Jackiw, Tomboulis - 1974]

Generalization of usual generating functionals to include bilocal composite operators.

Introduce a bilocal source in path integral:

$$\mathbf{W}[\mathcal{J}] = \ln Z[\mathcal{J}] = \ln \int d\mu_C[\phi] \exp\left\{-S_{\text{int}}[\phi] + \frac{1}{2}\int_{x,y} \phi(x)\mathcal{J}(x,y)\phi(y)\right\}$$

The 2PI effective action is defined by the Legendre transform:

$$\mathbf{\Gamma}[G] = \left(-\mathbf{W}[\mathcal{J}] + \frac{1}{2}\,\text{Tr}[\mathcal{J}G]\right)\Big|_{\frac{\delta \mathbf{W}}{\delta \mathcal{J}} = \frac{1}{2}G} = \underbrace{\frac{1}{2}\,\text{Tr}[C^{-1}G] + \frac{1}{2}\,\text{Tr}[\ln G^{-1}]}_{\text{free theory part}} + \underbrace{\mathbf{\Gamma}_2[G]}_{\text{interactions}}$$

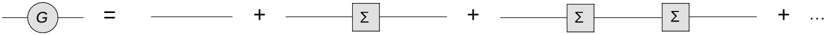
$\mathbf{\Gamma}_2[G]$: $-$ sum of two-particle irreducible (2PI) diagrams constructed from the vertices of $S[\phi]$, but with G as propagator (instead of free propagator C)

Two key geometric series

- The field equations of $\mathbf{\Gamma}[G]$ are the Schwinger-Dyson equations for 2-point function:

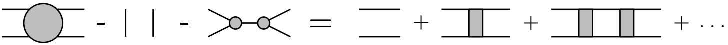
$$\frac{\delta \mathbf{\Gamma}}{\delta G(x,y)} = 0 \qquad \Leftrightarrow \qquad G^{-1}(x,y) = C^{-1}(x,y) - \Sigma(x,y)$$

with the self energy given by $\Sigma[G] = -2\,\delta\mathbf{\Gamma}_2/\delta G$



- Bethe-Salpeter kernel is defined by

$$\left(\frac{\delta^2\Gamma}{\delta G \delta G}\right)^{-1} = (\mathbb{1} - K)^{-1}\, GG$$



$$K(x_1,x_2,x_3,x_4) = -2\int_{y_1,y_2} G(x_1,y_1)G(x_2,y_2)\,\frac{\delta^2\mathbf{\Gamma}_2[G]}{\delta G(y_1,y_2)\delta G(x_3,x_4)}$$

Combining CFT and 2PI formalism

- Suppose that the Schwinger-Dyson equation has a conformal solution

$$G(x,y) \sim \frac{1}{|x-y|^{2\Delta}}$$

and higher n-point functions of the fundamental field are conformal as well

- $\Rightarrow$ Bethe-Salpeter kernel is diagonalized by conformal 3-point functions

$$\int_{x_3,x_4} K(x_1,x_2,x_3,x_4)\langle\phi(x_3)\phi(x_4)\mathcal{O}_h^{\mu_1\cdots\mu_J}(z)\rangle_{\text{c.s.}} = k(h,J)\langle\phi(x_1)\phi(x_2)\mathcal{O}_h^{\mu_1\cdots\mu_J}(z)\rangle_{\text{c.s.}}$$

(e.g. for $J=0$: $\langle\phi(x_1)\phi(x_2)\mathcal{O}_h(x_0)\rangle_{\text{c.s.}} = \frac{1}{|x_{34}|^{2\Delta-h}|x_{30}|^h|x_{40}|^h}$)

$h = \frac{d}{2} + \mathrm{i}\alpha$, with $\alpha \in \mathbb{R}^+ \Rightarrow$ complete basis [Dobrev et al. 1976]

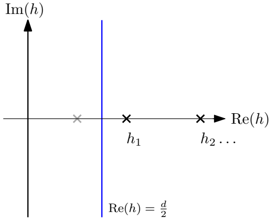
OPE spectrum

Conformal partial wave expansion:

$$\left(\frac{\delta^2\Gamma}{\delta G\delta G}\right)^{-1}(x_1,x_2,x_3,x_4) = \sum_{J\geq 0}\int_{d/2}^{d/2+\mathrm{i}\infty}\frac{\mathrm{d}h}{2\pi\mathrm{i}}\,\frac{2\,\rho(h,J)}{1-k(h,J)}$$

$$\times\int \mathrm{d}^d z\,\langle\phi(x_1)\phi(x_2)\mathcal{O}_h^{\mu_1\cdots\mu_J}(z)\rangle\langle\mathcal{O}_{\tilde{h}}^{\mu_1\cdots\mu_J}(z)\phi(x_3)\phi(x_4)\rangle$$

$$= \sum_{J\geq 0}\int_{d/2-\mathrm{i}\infty}^{d/2+\mathrm{i}\infty}\frac{\mathrm{d}h}{2\pi\mathrm{i}}\,\frac{2\,\hat{\rho}_\Delta(h,J)}{1-k(h,J)}\,\mathcal{G}^\Delta_{h,J}(x_1\ldots x_4)$$



poles at solutions of $k(h,J)=1$,
or at poles of $\hat{\rho}_\Delta(h,J)$ in free theory ($k=0$)

$$\Rightarrow \quad = \sum_J\sum_n \underbrace{c^2_{h_n(J),J}}_{\text{OPE coeff.}}\,\underbrace{\mathcal{G}^\Delta_{h_n(J),J}(x_1\ldots x_4)}_{\text{Conformal blocks}}$$

$O(N)$ model

Generating functional:

$$\mathcal{Z}[J] = \int d\mu_C[\phi] e^{-S_{\text{int}}[\phi] + J\cdot\phi}$$

where

$$C(x,y) = \frac{c(\Delta)}{|x-y|^{2\Delta}}$$

$$S_{\text{int}}[\phi] = \int \mathrm{d}^d x \left(\frac{\lambda_2}{2}\phi^2 + \frac{\lambda}{4N}(\phi^2)^2 \right)$$

- $\Delta = 2/d - \zeta$, with either $\zeta = 1$ (short-range) or $d/4 < \zeta < 1$ (long-range)
- J-dependence useful for generating n-point functions of ϕ, and for studying spontaneous symmetry breaking of the $O(N)$ symmetry (i.e. $\lim_{J\to 0^+} \langle \phi_a \rangle_J \neq 0$)

Introduce intermediate (Hubbard-Stratonovich) field:

$$e^{-S_{\text{int}}[\phi]} = \int d\mu_{2\lambda \mathbb{1}}[\sigma]\; e^{\frac{\mathrm{i}}{2\sqrt{N}}\,\sigma\cdot\left(\phi^2 + \frac{\lambda_2 N}{\lambda}\right)}$$

Integrating out ϕ we arrive at formulation in which large-N limit is saddle-point approximation for σ

2PI effective action in mixed ϕ-σ representation

Introduce only sources for $O(N)$-invariant (composite, bilocal) operators:

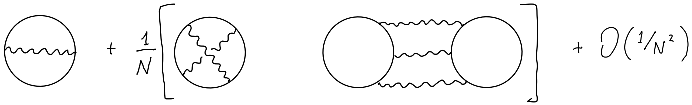
$$e^{W[\mathcal{J},\mathcal{K},\mathcal{L}]} = \int d\mu_C[\phi] d\mu_{2\lambda\mathbb{1}}[\sigma]\; e^{\frac{\mathrm{i}}{2\sqrt{N}}\,\sigma\cdot\left(\phi^2+\frac{\lambda_2 N}{\lambda}\right)+\mathcal{J}\cdot\sigma+\frac{1}{2}\sigma\cdot\mathcal{K}\cdot\sigma+\frac{1}{2}\phi_a\cdot\mathcal{L}\cdot\phi_a}$$

After Legendre transform, we find

$$\Gamma[\mathbf{s},\mathbf{D},\mathbf{G}] = \int \mathrm{d}^d x\,\left(\frac{1}{4\lambda}\mathbf{s}^2 - \mathrm{i}\frac{\lambda_2\sqrt{N}}{2\lambda}\mathbf{s}\right) + \frac{N}{2}\,\mathrm{Tr}[(C^{-1} - \frac{\mathrm{i}}{\sqrt{N}}\mathbf{s}\mathbb{1})\cdot\mathbf{G}] + \frac{N}{2}\,\mathrm{Tr}[\ln(\mathbf{G}^{-1})] + \frac{1}{4\lambda}\,\mathrm{Tr}[\mathbb{1}\cdot\mathbf{D}] + \frac{1}{2}\,\mathrm{Tr}[\ln(\mathbf{D}^{-1})] + \Gamma_2[\mathbf{G},\mathbf{D}]\,,$$

where

$$\Gamma_2[\mathbf{s},\mathbf{D},\mathbf{G}] \equiv -\ln\int_{2\mathrm{PI}} d\mu_{\mathbf{G}}[\phi] d\mu_{\mathbf{D}}[\sigma]\; e^{\frac{\mathrm{i}}{2\sqrt{N}}\,\sigma\cdot\phi^2}\,,$$



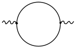
The field equations

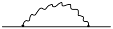
$$0 = \frac{\delta\Gamma}{\delta\mathbf{s}} = \frac{\delta\Gamma}{\delta\mathbf{D}} = \frac{\delta\Gamma}{\delta\mathbf{G}}$$

$\Rightarrow$ self-consistent Schwinger-Dyson equations (SDE):

$$\begin{aligned} s(x) &= \mathrm{i}\sqrt{N}\big(\lambda_2 + \lambda\, G(x,x)\big)\,, \qquad (\lambda_2 = -\lambda\, G(x,x) \quad \Rightarrow \quad s(x) = 0) \\ D^{-1}(x,y) &= \frac{1}{2\lambda}\delta(x-y) + 2\frac{\delta\Gamma_2}{\delta\mathbf{D}}[\mathbf{s}, G, D]\,, \\ G^{-1}(x,y) &= C^{-1}(x,y) - \frac{\mathrm{i}}{\sqrt{N}} s(x)\delta(x-y) + \frac{2}{N}\frac{\delta\Gamma_2}{\delta\mathbf{G}}[\mathbf{s}, G, D]\,. \end{aligned}$$

Notice:

$$\frac{\delta\Gamma_2}{\delta\mathbf{D}} =$$  $$+ O(1/N)$$

$$\frac{\delta\Gamma_2}{\delta\mathbf{G}} =$$  $$+ O(1/N)$$

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$$D^{-1}(x,y) = \frac{1}{2\lambda}\delta(x-y) + 2\frac{\delta\Gamma_2}{\delta\mathbf{D}}[\mathbf{s}, G, D]\,,$$

$$G^{-1}(x,y) = C^{-1}(x,y) - \frac{\mathrm{i}}{\sqrt{N}} s(x)\delta(x-y) + \frac{2}{N}\frac{\delta\Gamma_2}{\delta\mathbf{G}}[\mathbf{s}, G, D]\,.$$

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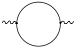
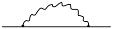
$$\frac{\delta\Gamma_2}{\delta\mathbf{D}} =$$  $$+ O(1/N)$$

$$\frac{\delta\Gamma_2}{\delta\mathbf{G}} =$$  $$+ O(1/N)$$

Leading order

In the strict $N \to \infty$ limit we have:

- Consistent solution of SDE in the conformal limit $\lambda \to +\infty$ (or $p \to 0$)

$$G(x,y) = C(x,y) \equiv \frac{c(d/2-\zeta)}{|x-y|^{d-2\zeta}}\,, \qquad D(x,y) = \frac{b(d/2-\zeta)}{|x-y|^{4\zeta}}$$

$\Rightarrow \quad \Delta = d/2 - \zeta$ and $\Delta_\sigma = 2\zeta$

- Bethe-Salpeter kernel has vanishing eigenvalues at leading order

$\Rightarrow$ OPE spectrum of free theory

$$h_{[\phi_a \partial_{\mu_1} \cdots \partial_{\mu_J} \phi_a]} = 2\Delta + J$$

except for $J = 0$, as $2\Delta = d - 2\zeta < d/2 \quad \Rightarrow \quad \phi^2$ replaced by σ

Next-to-leading order: self-consistent SDE

- Keep order $1/N$ term in SDE of G
- Plug in conformal ansatz for G and D with $\Delta = d/2 - \zeta + \eta/2$ and $\Delta_\sigma = 2\zeta - \eta$:

$$G(x,y) = \frac{A}{|x-y|^{d-2\zeta+\eta}}\,, \qquad D(x,y) = \frac{B}{|x-y|^{4\zeta-2\eta}}$$

- Find η by solving SDE self-consistently

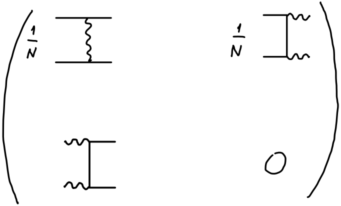
Two cases:

- $\zeta = 1$ (short-range model):
  $\eta_1 > 0$ requires discarding C^{-1} term in SDE, which is justified in IR limit ($p^2 \ll p^{2-\eta}$)

  $G^{-1}(x) = \frac{p(\Delta)}{A|x|^{2(d-\Delta)}} \quad \Rightarrow \quad$ SDE: $\quad p(\Delta) + A^2B/N = 0,\ 2p(\Delta_\sigma) + A^2B = 0$
  $\Rightarrow$ the two equations fix η_1 (from $\eta = \sum_{q>0} \eta_q/N^q$) and A^2B

- $d/4 < \zeta < 1$ (long-range model):
  $\eta = 0 \;\Rightarrow\;$ keep $C^{-1} \;\Rightarrow\;$ only need to compute constant

Next-to-leading order: Bethe-Salpeter diagonalization

$K =$ 

Eigenfunctions: linear combinations of $\langle \phi\phi\mathcal{O}_{h,J} \rangle$ and $\langle \sigma\sigma\mathcal{O}_{h,J} \rangle$
Eigenvalues: ratios of gamma functions with poles at

$$h_{[\phi_a \partial_{\mu_1} \cdots \partial_{\mu_J} (\partial^2)^n \phi_a]} = 2\Delta + J + 2n$$
$$h_{[\sigma \partial_{\mu_1} \cdots \partial_{\mu_J} (\partial^2)^n \sigma]} = 2\Delta_\sigma + J + 2n$$

Next-to-leading order: Bethe-Salpeter diagonalization

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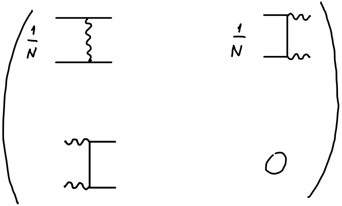


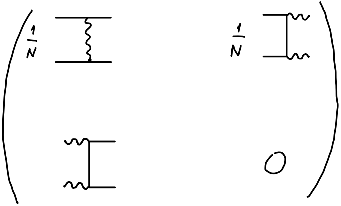
Eigenfunctions: linear combinations of $\langle \phi\phi\mathcal{O}_{h,J}\rangle$ and $\langle \sigma\sigma\mathcal{O}_{h,J}\rangle$
Eigenvalues: ratios of gamma functions with poles at
Solutions of $k(h,J) = 1$:

$$h_{[\phi_a \partial_{\mu_1}\cdots\partial_{\mu_J}(\partial^2)^n \phi_a]} = 2\Delta + J + 2n + \frac{1}{N}\gamma_1^{[\phi\phi]_{[n,J]}}$$

$$h_{[\sigma \partial_{\mu_1}\cdots\partial_{\mu_J}(\partial^2)^n \sigma]} = 2\Delta_\sigma + J + 2n + \frac{1}{N}\gamma_1^{[\sigma\sigma]_{[n,J]}}$$

Anomalous dimensions γ_1 again given by ratios of gamma functions
(previously known for $[\phi\phi]_{[n,J]}$ at $\zeta = 1$, the rest is new)

Next-to-leading order: Bethe-Salpeter diagonalization

$K =$ 

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$$h_{[\phi_a \partial_{\mu_1}\cdots\partial_{\mu_J}(\partial^2)^n \phi_a]} = 2\Delta + J + 2n + \tfrac{1}{N}\gamma_1^{[\phi\phi]_{[n,J]}}$$

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Anomalous dimensions γ_1 again given by ratios of gamma functions
(previously known for $[\phi\phi]_{[n,J]}$ at $\zeta = 1$, the rest is new)

NNLO: old partial results for $\zeta = 1$, future work for $\zeta < 1$ and other operators

Tensor models, basics

Characteristics of tensor models:

- Fields (e.g. Lorentz scalars) that transform as tensors of rank ≥ 3 under a global symmetry group, e.g. in the tri-fundamental representation of $O(N)^3$:

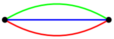
$$\phi_{abc} \to \sum_{a',b',c'}^{1\dots N} R^{(1)}_{aa'} R^{(2)}_{bb'} R^{(3)}_{cc'} \phi_{a'b'c'} \ , \quad R^{(i)} \in O(N)$$

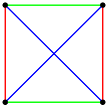
- An invariant action, containing at least one term having no larger (continuous) symmetry than the one above:

$\phi_{abc}\phi_{abc}$ has in fact $O(N^3)$ invariance

$\phi_{abc}\phi_{ab'c'}\phi_{a'bc'}\phi_{a'b'c}$ is only invariant under $O(N)^3$

It is useful to represent invariants as colored graphs. E.g. for the two above we have:



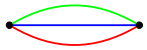


A case study: the quartic $O(N)^3$ model

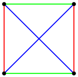
[$d = 0$: Carrozza, Tanasa (2015); $d = 1$: Klebanov, Tarnopolsky (2016);
$d > 1$: Giombi, Klebanov, Tarnopolsky (2017, short-range), DB, Gurau, Harribey (2019, long-range)]

- Interacting part of the action (or full action if $d = 0$)
  $\Leftrightarrow$ all $O(N)^3$ invariants with up to 4 fields:

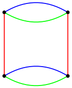
$$S_{\text{int}}[\phi] = \int_x \Bigg(\frac{1}{2} m^{2\zeta}$$



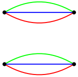
$$+ \frac{\lambda_t}{4N^{3/2}}$$



$$+ \frac{\lambda_p}{12N^2} \sum_{\text{col.perm.}}$$



$$+ \frac{\lambda_d}{4N^3}$$



$$\Bigg)$$

  $t =$ tetrahedron; $p =$ pillow; $d =$ double-trace.

- Large-N expansion governed by a non-negative half-integer, the degree ω:
  [Carrozza, Tanasa (2015)]

$$\ln Z = \sum_{\omega \in \mathbb{N}/2} N^{3-\omega} F_\omega$$

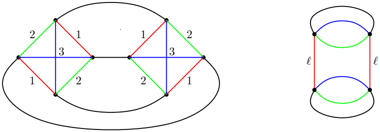
  Note: ω is not a topological invariant

- Complete classification of diagrams up to $\omega = 3/2$ [Bonzom, Nador, Tanasa (2019)]

Edge-colored graphs and Feynman diagrams

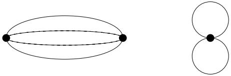
Perturbative expansion:

- Represent Wick contraction of two tensors by black line, obtaining 4-colored graphs , e.g.:



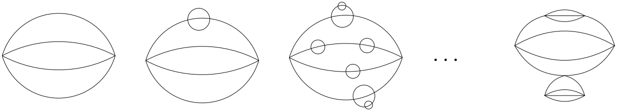
$\Rightarrow$ useful for keeping track of powers of N $\Rightarrow$ determine dominant graphs at large-N

- Ordinary Feynman diagrams are obtained by shrinking interaction bubbles to a point:



$\Rightarrow$ useful for Feynman integrals $\Rightarrow$ crucial in $d > 0$ QFT

- This model has **melon-tadpole Feynman diagrams** at leading order ($\omega = 0$):



Benefits of melonic dominance in QFT

Some important features provided by the melonic limit:

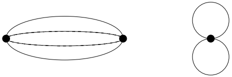
- Closed Schwinger-Dyson equation for two-point function
- Bethe-Salpeter kernel obtained from few (2PI) diagrams
- No radiative corrections for tetrahedron(-like) coupling

Best seen through lens of 2PI formalism

2PI formalism and melonic limit [DB, Gurau - 2018]

- 2PI formalism is well suited for melonic limit, because melons are built via 2-point insertions
- It effectively replaces the bilocal formalism used in $O(N)$ model and SYK model

Leading order in $1/N$ of quartic $O(N)^3$ model:
there are only two melon-tadpole diagrams that are also 2PI



$\Rightarrow$ At leading order in $1/N$ we can write its full 2PI effective action in closed form:

$$\Gamma[G] = N^3 \left(\frac{1}{2} \int_{x,y} C^{-1}(x,y) G(y,x) + \frac{1}{2} \int_{x,y} \ln(G^{-1})(x,y) + \frac{\lambda_p + \lambda_d}{4} \int_x G(x,x)^2 - \frac{\lambda_t^2}{8} \int_{x,y} G(x,y)^4 \right)$$

Our case study (again): long-range $O(N)^3$ model

[DB, Gurau, Harribey, Suzuki, Lettera (2019 - 2021)]

- Free part of the action (short-range: $\zeta = 1$; long-range: $0 < \zeta < 1$):

$$S_{\text{free}}[\phi] = \int_x \phi_{abc}(x)(-\partial^2)^\zeta \phi_{abc}(x)$$

Fractional Laplacian defined trivially in Fourier space ($p^{2\zeta}$).

In x-space it is an integral operator: $S_{\text{free}}[\phi] = \int_{x,y} \phi_{abc}(x) C^{-1}(x,y) \phi_{abc}(y)$ with

$$C^{-1}(x,y) \propto 1/|x-y|^{d+2\zeta}$$

- Long-range model: $0 < \zeta < 1 \Rightarrow$ well-defined and reflection positive

- canonical dimension $\Delta_\phi = \frac{d-2\zeta}{2}$ $\quad C(x,y) \propto 1/|x-y|^{d-2\zeta}$

$\Rightarrow \boxed{\zeta = d/4 \ \Rightarrow \ \Delta_\phi = d/4} \Rightarrow$ the quartic interactions are canonically marginal

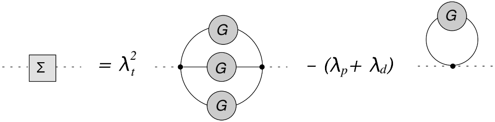
Melonic Schwinger-Dyson equation

The standard Schwinger-Dyson equation (SDE) for the 2-point function,

$$\boxed{G^{-1} = C^{-1} - \Sigma}$$

is obtained from $\frac{\delta \mathbf{\Gamma}}{\delta G(x,y)} = 0$

$\Rightarrow$ it simplifies in large-N limit, as $\mathbf{\Gamma}$ has only two diagrams $\Rightarrow$ the self-energy becomes:



i.e. $\Sigma(x,y) = \lambda_t^2\, G(x,y)^3 - (\lambda_p + \lambda_d)\, G(x,x)\delta(x-y)$

$\Rightarrow$ SDE $=$ closed equation for $G(x,y)$

Solution of the Schwinger-Dyson equation

$\zeta = d/4$:

Long range kinetic term $\Rightarrow$ **no wave function renormalization** (locality of counterterms)

Setting renormalized mass to zero, melonic SDE is solved exactly by

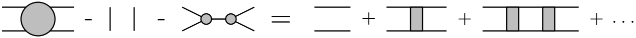
$$\boxed{G(p) = \frac{\mathcal{Z}}{p^{d/2}} = \mathcal{Z} C(p)} , \qquad 1 = \mathcal{Z} + \mathcal{Z}^4 \lambda_t^2 \frac{1}{(4\pi)^d} \frac{\Gamma\left(1 - \frac{d}{4}\right)}{\frac{d}{4}\Gamma\left(3\frac{d}{4}\right)}$$

$\mathcal{Z}$ resums all the melonic insertions in the propagator
(it is the generating function of 4-Catalan numbers)

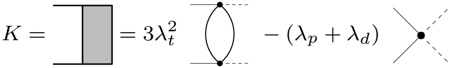
$\Rightarrow$ convergent series, up to a critical value of λ_t

Four-point function and Bethe-Salpeter kernel

- General result (from 2PI formalism):
  4-point function connected in the s-channel $(12 \to 34)$ is obtained as geometric series in Bethe-Salpeter kernel K:



- For $O(N)^3$ model at large-N:



$\Rightarrow$ chain/ladder decorated by melons (not drawn, they are resummed in propagator)

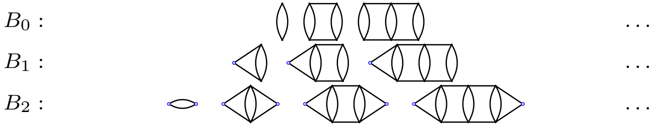
Four-point function and beta functions [DB, Gurau, Harribey (2019)]

Introduce renormalized couplings g_d, g_p, and g_t (finite renormalization), and define $g_1 = g_p/3$, $g_2 = g_p + g_d$

- No vertex correction to the tetrahedron $\Rightarrow$ $\boxed{\beta_t = 0}$

  $\Rightarrow$ g_t is an exactly marginal coupling, it can be used as **small parameter**

- Other beta functions: $\boxed{\beta_1 = B_0(-g_t^2) - 2B_1(-g_t^2)g_1 + B_2(-g_t^2)g_1^2}$, and similar for g_2.



$\Rightarrow$ Other couplings have a g_t-dependent fixed point, which is real and with real critical exponents for **imaginary tetrahedron coupling**: $g_t^2 < 0$:

$$g_{1\pm} = \frac{B_1 \pm \sqrt{(B_1)^2 - B_0 B_2}}{B_2} = \pm\sqrt{-g_t^2} + \mathcal{O}(g_t^2)$$

$$\beta_1'(g_{p\pm}) = \pm 2\sqrt{(B_1)^2 - B_0 B_2} = \pm\sqrt{-g_t^2}\left(4\frac{\Gamma(\frac{d}{4})^2}{\Gamma(\frac{d}{2})}\right)\left(1 + \mathcal{O}(g_t^2)\right)$$

Imaginary tetrahedron coupling

- The $O(N)^3$ model's action is unbounded from below, due to the tetrahedron interaction
  $\Rightarrow$ trouble for Euclidean path integral (at nonperturbative level)

- In principle it still makes sense to study the model in the large-N limit,
  but in fact complex operator dimensions are found for $\zeta = 1$ in $d = 4 - \epsilon$
  [Klebanov et al. (2017-2019)] or for $\zeta < 1$ if $\lambda_t \in \mathbb{R}$ [DB, Gurau, Harribey (2019)]

  $\Rightarrow$ thermodynamic instability of the conformal solution [DB (2021)]
  (dual of Breitenlohner-Freedman instability in AdS)

$$\Downarrow$$

- We choose a purely imaginary tetrahedron coupling: $\boxed{\lambda_t = \mathrm{i}|\lambda_t|}$

  (similarly to Lee-Yang model with $\mathrm{i}\lambda\phi^3$ interaction)

  Note: in long-range model we can choose it at will, because it is a marginal coupling

$$\Downarrow$$

FPs with real critical exponents

Melonic CFTs

We expect the scaling symmetry of the fixed points to be enhanced to conformal symmetry, at least for local unitary theories

- Not obvious for a long-range model with imaginary coupling!

  Nevertheless, formal proof of conformal invariance of n-point functions is obtained by embedding in higher dimensions such that theory becomes local [DB, Gurau, Suzuki (2020)] (as in long-range Ising model [Paulos, Rychkov, van Rees, Zan (2015)])

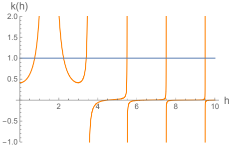
- We thus have an interacting CFT in $d > 2$ constructed and controlled thanks to the melonic limit: a **melonic CFT**

  $\Rightarrow$ we can obtain CFT data and more by combining melonic limit and conformal methods

OPE spectrum, results

For the long-range model we find: $k(h,J) = \frac{3g^2}{(4\pi)^d} \frac{\Gamma(-\frac{d}{4}+\frac{h+J}{2})\Gamma(\frac{d}{4}-\frac{h-J}{2})}{\Gamma(\frac{3d}{4}-\frac{h-J}{2})\Gamma(\frac{d}{4}+\frac{h+J}{2})}$

and solutions of $\boxed{k(h,J)=1}$ can be found analytically at small g, or numerically at finite coupling:



We find

$$\boxed{h_{n,J} = d/2 + J + 2n + z_{n,J}(-g_t^2)} \quad n \in \mathbb{N}_0$$

(conformal dimension of $O_{h,J} \sim \phi_{abc}\partial_{\mu_1}\dots\partial_{\mu_J}(\partial^2)^n\phi_{abc}$)

$$z_{0,0} \sim \sqrt{-g_t^2}\,(1+\mathcal{O}(g_t^2)) \,, \quad z_{n,J\neq 0,0} \sim g_t^2 + \mathcal{O}(g_t^4)$$

real for $-g_c^2(d) < g_t^2 < 0$, consistent with unitarity bounds, and with real OPE coefficients $C_{\phi\phi\,\mathcal{O}_{n,J}}$ [DB, Gurau, Harribey, Suzuki (2019)]

⇒ With imaginary coupling, the large-N model appears to be a unitarity CFT

Recap and further results

- A real and unitary CFT is found at large N in the long-range $O(N)^3$ model with imaginary tetrahedron coupling [DB, Gurau, Harribey, Suzuki (2019-2020)]

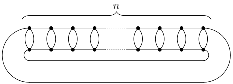
- Similar results found for long-range $U(N)^3$ model with (marginal) sextic interaction, but with real exactly-marginal coupling [DB, Delporte, Harribey, Sinha (2019)]

- And similarly for Amit-Roginsky model (multi-scalar vector model with cubic interaction mediated by Wigner $3jm$ symbol) [DB, Delporte (2020)]

- Non-unitarity is suppressed in $1/N$, e.g. conformal dimensions acquire imaginary part at subleading orders [DB, Gurau, Harribey (2020); Harribey (2021)]

  Alternative: sextic ("prismatic") model [Giombi et al. (2018)],
  or supersymmetric models [Popov (2019); Lettera, Vichi (2020)]

- Conformal partial wave basis very useful to study CFT data, but also free energy on the sphere (application: F-theorem) [DB, Gurau, Harribey, Lettera (2021)]
  $\Leftrightarrow$ resum infinite series of vacuum diagrams!



Conclusions and outlook

- The large-N limit is a powerful method for studying interacting CFTs (fixed points of RG) and combine conformal and RG methods
- The large-N limit of the $O(N)$ model is very simple, but a richer structure arises at next orders in the $1/N$ expansion
- The melonic large-N limit has a rich structure already at leading order
- For the scalar $O(N)^3$ long-range model (our case study) we find interacting IR fixed points in $d < 4$, for which CFT data and sphere free energy can be computed from first principles (no bootstrap). Similar results are found in other models with melonic limits.

More work to be done, especially for $1/N$ corrections, both for vector and tensor models, e.g.:

- Higher orders in $1/N$ for $O(N)$ model: can it become numerically competitive with respect to other methods? (by Padé-Borel summation)
- Unitarity at higher orders of $1/N$ expansion in prismatic or supersymmetric tensor models?