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Branching within Branching
—
A Stochastic Description of Host-Parasite Populations

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A Stochastic Description of Host-Parasite Populations

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Summary

In the present thesis, a theory of a discrete-time branching within branching process (BwBP) in a very general setting is developed. As a BwBP consists of two branching processes, one evolving in the individuals of the other, it describes host-parasite populations. More precisely, consider a cell population forming a Galton-Watson tree and proliferating parasites colonizing these cells. The two multiplication mechanisms of cells and parasites obey some dependence structure since cells and parasites influence each others reproduction in real biological settings.

We are interested in the long-time behavior of this process, particularly of the parasites and their distribution among the cells. The process $(\mathcal{Z}_n)_{n\geq 0}$, denoting the number of parasites per generation, satisfies an extinction-explosion principle. Almost sure extinction of parasites can be characterized in terms of the process of parasites evolving along a randomly picked cell line through the cell tree. This latter process and its different properties determine the behavior of the BwBP in the majority of the following results. If, on the one hand, parasites survive with positive probability, finer asymptotics for $(\mathcal{Z}_n)_{n\geq 0}$ and the process of contaminated cells $(\mathcal{T}_n^*)_{n\geq 0}$ are shown and their exponential rate of growth are identified. Furthermore, a Kesten-Stigum-type result is proved, giving us an equivalent condition for the normalized process of parasites to be uniformly integrable. In the case of a high parasite multiplication, we are able to construct an appropriate Heyde-Seneta norming for $(\mathcal{T}_n^*)_{n\geq 0}$. Additionally, when picking a contaminated cell in the far future, the distribution of the number of parasites in this cell is identified under different setups. If, on the other hand, parasites die out eventually, the decay rate of the survival probability is discussed, and under certain further assumptions, conditional limit theorems are proved. In particular, the law of the number of infected cells and the parasites they contain, conditioned upon survival of parasites up to the present time, converges to a quasi-stationary distribution. By letting parasites be still alive in the far future, we obtain a distributional convergence to a positive recurrent Markov chain.

One of the major tools used in the proofs of the mentioned results is the size-biased method. The constructed size-biased process has a connection to a branching process in random environment with immigration, whose few known theorems are extended in order to analyze the BwBP.

In the last part of this thesis, a bifurcating, two-type (A and B) cell division host-parasite model is studied in which cell type heredity is assumed to be unilateral, i.e. type B-cells cannot split into A-cells, whereas the converse is possible. This causes the before established theory to be applicable since the tree of A-cells and its parasites forms a BwBP. We study the proportion of contaminated A- and B-cells and present conditions under which the infected A-cells become negligible compared to all contaminated cells. Further limit theorems for the parasites and cells of the various types are shown, including asymptotics for the proportion of infected cells with a given number of parasites to all infected cells under various assumptions.

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Introduction

Branching models are prevalent for the stochastic description of population dynamics. During the last century, several different branching models have been established to analyze diverse population structures, but all these models are derived from or extensions of the classical *Galton-Watson process (GWP)*. This prototype branching model takes a genealogical perspective at a population with the inherent assumption that individuals reproduce independently of each other with the same offspring distribution. The GWP is well studied in numerous articles and the main results as well as further references are listed in the books of Asmussen and Hering [10], Athreya and Ney [14] and Jagers [46].

Via the parent-child relation of individuals, the GWP forms a random tree, the so-called *Galton-Watson tree (GWT)*. Suppose that the individuals of this GWT host smaller particles which multiply and share their offspring to the individual's children independently of each other. As they describe the evolution of small particles proliferating in the individuals of a population, that is for example host-parasite interactions over a period of time, processes of this kind are called *branching within branching processes*. Based on the mentioned biological context, from now on, we will refer to the individuals as *cells* and to the small particles as *parasites*. However, instead of parasites, one can also suppose the small particles to be some other biological or cell content, for example mitochondria.

In the host-parasite scenario, the cells are typically assumed to divide into two daughter cells at the end of their lifetime. Such bifurcating cell division processes have been studied, as one of the first, by Kimmel [50]. He modeled the situation with cells splitting after a randomly chosen continuous lifetime and a symmetric sharing of parasites into these two daughter cells. Bansaye [15] considered this model in discrete time and allowed asymmetric sharing of parasites. He extended his model in [16] by adding immigration of parasites and random environments, which means that parasites in a cell reproduce under the same but randomly chosen distribution. In [19] the authors considered a model in continuous time and parasites evolving according to a Feller diffusion. Moreover, the cell division rate depends on the quantity of parasites inside the cell and asymmetric sharing of parasites into the two daughter cells is assumed. Although asymmetric sharing of cell contents into the daughter cells seems to be a quite strange assumption at first glance, it is in fact a fundamental biological mechanism to generate cell diversity, see Jan and Jan [47] and Hawkins and Garriga [42]. The most convincing example in this context is the asymmetric division of a stem cell giving rise to a copy of itself and a second daughter cell which is coded to differentiate into cells with a particular functionality in the organism.

The above mentioned host-parasite models are restricted to a bifurcating cell division mechanism, determining the underlying GWT to be binary. In recent years, efforts were made to generalize the Galton-Watson cell tree to be non-deterministic. The greatest progress in this direction has been achieved by Delmas and Marsalle in [34] for a discrete-time model and in cooperation with Bansaye and Tran in [18] for a continuous-time model. Both articles consider a random splitting mechanism of cells and Markov chains operating on the resulting cell trees under ergodic hypotheses. Besides these articles the work of Guyon [41] is worth mentioning, who studied another discrete-time model with asymmetric sharing and ergodic suppositions. The states of the daughter cells, in our model the number of parasites in a cell, are described by the mentioned Markov chains and assumed to be picked asymmetrically in all of the three listed papers. However, the considered ergodicity excludes the possibility of parasite extinction, which is a fundamental property in our model. To the author's best knowledge, there is no fully elaborated theory considering a double structured branching process with a random cell tree and a parasite multiplication mechanism which allows extinction. The major part of this thesis is therefore devoted to the development of such a general theory in a discrete-time setting.

The extension of cell division into two daughter cells to a random splitting mechanism in a host-parasite situation is worth treating not only for mathematical reasons as the following discussion shows. Envision a cell biologist counting a cell population and checking their infection status in regular time periods. The population size at these points in time is not necessarily a power of two integer and might even be odd-numbered. This is the same situation when considering the model of Kimmel [50] only at discrete, periodic points in time. Hence, the GWP assumption of the underlying cell tree is justifiable. Besides cell diversity, another incentive for asymmetric sharing of parasites to the daughter cells arises from the appearance of the so-called cellular senescence, recently discovered even for several single-celled organisms (see [82]). Cellular senescence is the phenomenon that after cell division one of the two daughter cells can be recognized as the mother cell, for it accumulates age-related damage throughout its replication phases. It eventually loses the ability for cellular mitosis, the cell death occurs. This allows for another genealogical perspective by counting all cells spawned by a single cell during its lifetime and interpreting them as the succeeding generation. By proceeding with each of these new cells in the same manner, we get a Galton-Watson structure. As the infection level of the mother cell changes during its lifetime, this may result a different number of parasites in each daughter cell. Hence, the intended model with asymmetric sharing of parasites arises. Furthermore, a shorter lifetime of the mother cell implies a lower number of daughter cells as well as fewer parasite offspring, and thus it is also reasonable to link the number of daughter cells to the reproduction law of parasites.

In the following, we outline the organization and main results of this thesis. The first chapter is devoted to a rigorous definition of the *branching within branching process (BwBP)* studied in the present work. The underlying cell tree is assumed to be a GWT, and the number of a cell's daughters influences the offspring distribution of an accommodated parasite as well as the sharing of its progeny to those daughter cells. A short comparison with other branching models, appearing in special settings of the BwBP, is given. Three interesting processes emerge from the BwBP: the *associated branching process in random environment (ABPRE)* $(Z'_n)_{n\geq 0}$, describing the number of parasites in a randomly picked cell line through the cell tree, the *number of contaminated cells* $(\mathcal{T}^*_n)_{n\geq 0}$, counting the number of parasite infected cells, and the *process of parasites* $(\mathcal{Z}_n)_{n\geq 0}$, which describes the total number of parasites per generation. In the second part of this first chapter, these three processes are introduced and first results are proved. Due to the reproduction mechanism, the process of parasites does not follow a GWP structure. Still, it obeys an extinction-explosion principle, and one of the first main results is a complete characterization of almost certain extinction in terms of the ABPRE. Turning to the number of contaminated cells $(\mathcal{T}^*_n)_{n\geq 0}$, we obtain the almost sure convergence to infinity if the population of parasites explodes.

The proofs of most of the remaining results concerning the BwBP are based on the *size-biased method*, primarily used by Lyons et al. in their pioneering article [61]. In Chapter 2, we construct the *size-biased BwBP* by picking the *spine* along the parasites, and we show relations to the original BwBP. The cells containing the spinal parasites form a path through the cell tree, and the number of parasites along this cell line behaves like a *branching process in random environment with immigration*. Chapter 3 is devoted to the discussion of such processes in different regimes, and the rare known results from [16, 49, 72] are extended, especially in the supercritical case. These results will help us in the analysis of the BwBP.

In Chapter 4, we return to the study of the BwBP and focus on the case where parasites survive with positive probability. Normalizing $(\mathcal{Z}_n)_{n\geq 0}$ by its means leads to a non-negative martingale. We obtain an equivalent condition for the martingale limit to be positive on the set of parasite survival Surv by utilizing the size-biased method. This equivalent condition comprises the famous $(Z\log Z)$*-condition* and another one, which, roughly speaking, describes the partitioning of parasites over the cell tree. The problem of finding the proper normalization when the $(Z\log Z)$-condition fails is discussed thereafter. It is shown that such a norming sequence cannot differ much from the means. On Surv, we further determine the exponential factor in the rate of growth of $(\mathcal{T}^*_n)_{n\geq 0}$, which depends on the regimes of the ABPRE. In the case where the ABPRE survives with positive probability a suitable Heyde-Seneta norming is constructed. The last section of Chapter 4 is devoted to the proportion $F_n(k)$ of contaminated cells hosting k parasites to the total number of contaminated cells in generation n and its limit for $n\to\infty$. This limit highly depends on the behavior of the ABPRE. If the latter is supercritical, the number of parasites in a contaminated cell continuously rises. If, on the other hand, the ABPRE is strongly subcritical, we determine the limit of $(F_n(k))_{k\geq 1}$ as $n\to\infty$ to be a deterministic and quasi-stationary distribution derived from the ABPRE.

In Chapter 5, we analyze the BwBP in the case where parasites die out almost surely, and we identify decay rates of the survival probability. In particular, we give necessary and sufficient conditions for the survival probability to decrease with the same speed as the mean number of parasites. The final section of this chapter focuses on the case where the latter mentioned holds true. We show that, conditioned upon survival of parasites up to the present time, the distribution of the number of infected cells and the parasites they contain converges to a quasi-stationary distribution. This is an analog to the result of Yaglom for the ordinary GWP (see [14, Chapter I.8 and I.14]). Furthermore, given that parasites are still alive in the distant future, leads to a distributional convergence towards a positive recurrent Markov chain. The majority of the proofs will be carried out with the help of the size-biased process.

The last chapter deals with a host-parasite bifurcating cell division process with two cell types A and B. In this model, only unilateral cell type heredity is assumed. That is, daughter cells of a B-cell keep the type of their mother, whereas A-cells can split into cells of both types. Furthermore, parasites in cells having different cell types multiply with different reproduction laws. This forms a first basic model to study coevolutionary adaptations, here due to the presence of two different cell types. Host-parasite coevolution describes the reciprocal, adaptive genetic change of interacting species, which results from the selective pressure each antagonist can exert on the other one (see e.g. [57, 89]). The one-sided cell type heredity describes, inter alia, the situation where cells somehow may change, for example by irreversible mutation, and so develop some kind of immunity or resistance to the parasite infection. This influences the parasite reproduction and lowers their offspring rate. By the cell type heredity assumptions, the process of type-A cells together with its parasites forms a BwBP and the results of all previous chapters are applicable. Hence, we mainly focus on the B-cells and their parasites. Under the premise that infected A-cells survive with positive probability, asymptotic results for the proportion of contaminated B-cells to all contaminated cells are given as well as for the proportion of B-cells containing a fixed number of parasites to all infected cells of type B.

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Notation and the Ulam-Harris tree

Throughout this thesis, we denote by $\mathbb{N}$ the set of natural numbers $\{1, 2, 3, \dots\}$ and put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ as well as $\overline{\mathbb{N}}_0 := \mathbb{N}_0 \cup \{\infty\}$. In a classical manner, we will write $\mathcal{P}(\mathcal{X})$ for the power set and $\#\mathcal{X}$ for the cardinality of a non-empty set $\mathcal{X}$. For two real numbers $x, y \in \mathbb{R}$ we denote by δ_{xy} the ordinary Kronecker delta symbol, i.e. $\delta_{xy} = 1$ if $x = y$, and $= 0$ otherwise, and we write $x \wedge y$ for the minimum of these two numbers. Furthermore, we write $\mathcal{L}(X)$ for the law of a random variable X. As we will often deal with sequences of tuples for a denumerable index set I, we introduce the short notation $[x_i, y_i]_{i \in I}$ for the vector with the entries (x_i, y_i), $i \in I$.

Throughout this thesis,

$$\mathbb{V} := \bigcup_{n \in \mathbb{N}_0} \mathbb{N}^n$$

denotes the *infinite Ulam-Harris tree* with $\mathbb{N}^0 = \{\varnothing\}$ and root label $\varnothing$. To describe the lineage of vertices in $\mathbb{V}$ we use the usual Ulam-Harris labeling notation. A vertex $v = (v_1, ..., v_n) \in \mathbb{V}$ is understood to be the descendant v_n of the descendant v_{n-1} of $\dots$ of the descendant v_1 of the root $\varnothing$, and we will shortly write $v_1...v_n$. In other words, $v = v_1...v_n$ describes the unique path (or ancestral line)

$$\varnothing \to v_1 \to \cdots \to v_1...v_n$$

from the root $\varnothing$ to v. With $|v|$ we denominate the length of this path, i.e. $|v| = n$ for $v \in \mathbb{N}^n$, which means that v is in the n^{th} generation of the tree. For the set of vertices $\{v \in \mathbb{V} : |v| = n\}$ and $\{v \in \mathbb{V} : |v| \leq n\}$ in the n^{th} resp. in the first n generations, we will sometimes use the shorter notation $|v| = n$ resp. $|v| \leq n$. Furthermore, we write $v|k$ for the ancestor of $v = v_1...v_n$ in generation $k \leq n$ and $u < v$ if v is a descendant of the vertex u. Thus, $v|k = v_1...v_k$ and $v|k = u$ for some $k < n$ when $u < v$. Finally, the concatenation $uv = u_1...u_m v_1...v_n$ is identified to be the vertex $v = v_1...v_n$ in the tree rooted at $u = u_1...u_m$.

Chapter 1

The branching within branching model

In this first chapter, the branching within branching model is introduced. It is a special multi-type branching process with infinite many types and has connections to other branching models as explained in a later subsection. We close this chapter by introducing important processes arising from this model and proving first results.

1.1 The model

1.1.1 Description of the model

As mentioned in the Introduction, we develop in this thesis a general theory of discrete-time branching within branching processes which describe certain genealogical host-parasite coevolutions. To give an informal description of the *branching within branching process (BwBP)*, consider a cell population forming a standard *Galton-Watson tree (GWT)* $\mathbb{T}$ rooted in a single ancestor ($\varnothing$). Each of these cells contains proliferating parasites whose reproduction law is determined by the number of daughter cells spawning from their host cell. Given the daughter cells, the parasites multiply and share their offspring independently of each other to the cells in the next generation. More precisely, let $\varnothing$ contain a single parasite. First, the root cell divides into $T_\varnothing \in \mathbb{N}_0$ daughter cells, denoted by $1, \ldots, T_\varnothing$. Given $T_\varnothing = t_\varnothing$, the parasite in $\varnothing$ multiplies according to the law given by $(X^{(1,t_\varnothing)}, \ldots, X^{(t_\varnothing,t_\varnothing)})$, where $X^{(k,t_\varnothing)}$ describes the offspring number going in the k^{th} daughter cell. These new cells together with the parasites they contain then form the first generation of the BwBP. In the familiar Galton-Watson way, a cell v of this first generation splits into T_v daughter cells, and a parasite in v multiplies with the law of $(X^{(1,t_v)}, \ldots, X^{(t_v,t_v)})$ if $T_v = t_v$, independently of all other parasites and cells $u \neq v$, $|u| = 1$. All descendant cells and parasites of the first generation then form the second one which spawns the third generation in the same manner as just described and so on.

Host-parasite coevolution is a very complex procedure in which both participants, the cells and parasites, influence each other. Since we intend the cells to form a GWT, it is reasonable to consider the cell division before the reproduction of parasites. Potential applications and further motivations for the BwBP were already stated in the Introduction.

For a rigorous description of the branching within branching process we fix a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ assumed to be large enough to carry all random variables introduced hereafter. Let $\mathbb{V}$ be the infinite Ulam-Harris tree with root $\varnothing$ as introduced in the Introduction. Let further $(T_v)_{v\in\mathbb{V}}$ be independent and identically distributed (i.i.d.) copies of the $\mathbb{N}_0$-valued random variable T with distribution $(p_k)_{k\geq 0}$ and finite mean, viz. $\mathbb{P}(T = k) = p_k$ for all $k \in \mathbb{N}_0$ and $\mathbb{E}T < \infty$. This family of random variables describes a random subtree of $\mathbb{V}$ in a natural way. Put $\mathbb{T}_0 := \{\varnothing\}$ as the root and define for $n \in \mathbb{N}$ the n^{th} generation of this random tree recursively by

$$\mathbb{T}_n := \{v_1 \dots v_n \in \mathbb{V} \mid v_1 \dots v_{n-1} \in \mathbb{T}_{n-1} \text{ and } 1 \leq v_n \leq T_{v_1 \dots v_{n-1}}\}.$$

Hence, the random variable T_v for $v \in \mathbb{V}$ can be interpreted as the offspring number of cell v and due to the i.i.d. property of $(T_v)_{v\in\mathbb{V}}$, the union

$$\mathbb{T} := \bigcup_{n\in\mathbb{N}_0} \mathbb{T}_n \subseteq \mathbb{V}$$

forms a GWT with a single ancestor cell, reproduction law $(p_k)_{k\geq 0}$ and reproduction mean

$$\nu := \sum_{k\in\mathbb{N}} k p_k = \mathbb{E}T < \infty.$$

Moreover, let $(\mathsf{T}_v)_{v\in\mathbb{V}}$ be a family of random variables indicating which vertices of $\mathbb{V}$ belong to $\mathbb{T}$, i.e. for $n \in \mathbb{N}_0$ and $v \in \mathbb{V}$ with $|v| = n$

$$\mathsf{T}_v := \begin{cases} 1 & \text{if } v \in \mathbb{T}_n, \\ 0 & \text{if } v \notin \mathbb{T}_n. \end{cases} \tag{1.1}$$

In particular, $\mathsf{T}_\varnothing = 1$ almost surely (a.s.). If $\mathsf{T}_v = 1$, the cell $v \in \mathbb{V}$ is called *alive* and *dead* otherwise. For a cell $v = v_1...v_n \in \mathbb{V}$, we get $\{\mathsf{T}_v = 1\} = \{v \in \mathbb{T}_n\} = \{T_{v|n-1} \geq v_n,\ \mathsf{T}_{v|n-1} = 1\}$ a.s. and so

$$\mathsf{T}_v = \mathsf{T}_{v|n-1}\, \mathbb{1}_{\{T_{v|n-1} \geq v_n\}} = \mathsf{T}_\varnothing \prod_{i=0}^{n-1} \mathbb{1}_{\{T_{v|i} \geq v_{i+1}\}} \quad \text{a.s.} \tag{1.2}$$

Furthermore,

$$\begin{aligned} \mathbb{P}\left(\left(\sum_{u\geq 1} \mathsf{T}_{vu} \right)_{|v|=n} = (k_v)_{|v|=n} \;\middle|\; (\mathsf{T}_v)_{|v|=n} = (t_v)_{|v|=n} \right) &= \mathbb{P}\left((t_v T_v)_{|v|=n} = (k_v)_{|v|=n} \right) \\ &= \prod_{|v|=n, t_v=1} p_{k_v} \prod_{|v|=n, t_v=0} \delta_{0 k_v} \end{aligned}$$

for all $k_v \in \mathbb{N}_0$ and $t_v \in \{0, 1\}$ with $|v| = n$, where δ_{ij} denotes the ordinary Kronecker delta symbol.

We further put

$$\mathcal{T}_n := \#\, \mathbb{T}_n = \sum_{|v|=n} \mathsf{T}_v \tag{1.3}$$

for $n \in \mathbb{N}_0$ as the *number of (living) cells in the* n^{th} *generation.* It should be clear that $(\mathcal{T}_n)_{n\geq 0}$ is a standard *Galton-Watson process (GWP)* with reproduction law given by T and reproduction mean ν. For information on Galton-Watson processes, we refer to the books of Asmussen and Hering [10], Athreya and Ney [14] and Jagers [46].

Having defined the cell division process, we now focus on the parasites. Let us denote by Z_v the number of parasites in cell $v \in \mathbb{V}$, and we write $\mathbb{T}_n^*$ for the set of contaminated cells in generation $n \in \mathbb{N}_0$ and $\mathcal{T}_n^*$ for its cardinal number, i.e. for each $n \in \mathbb{N}_0$

$$\mathbb{T}_n^* := \{v \in \mathbb{T}_n : Z_v > 0\} \quad \text{and} \quad \mathcal{T}_n^* = \# \mathbb{T}_n^* . \tag{1.4}$$

As informally described at the beginning of this section, we postulate that parasites located in different cells multiply independently of each other, whereas parasites living in the same cell reproduce independently with the same law when the number of daughter cells is given. To model this situation, let for each $k \in \mathbb{N}$

$$\left(X_{i,v}^{(1,k)}, \ldots, X_{i,v}^{(k,k)}\right), \qquad i \in \mathbb{N}, v \in \mathbb{V},$$

be i.i.d. copies of the $\mathbb{N}_0^k$-valued random vector $X^{(\bullet,k)} := \left(X^{(1,k)}, \ldots, X^{(k,k)}\right)$ and we shortly write $X_{i,v}^{(\bullet,k)}$ instead of $\left(X_{i,v}^{(1,k)}, \ldots, X_{i,v}^{(k,k)}\right)$. Furthermore, the families $\left(X_{i,v}^{(\bullet,k)}\right)_{i\in\mathbb{N}, v\in\mathbb{V}}$, $k \in \mathbb{N}$, are assumed to be independent and independent of $(T_v)_{v\in\mathbb{V}}$. These random vectors indicate the reproduction and sharing of the various parasites living in the cell tree. In detail, let the cell $v \in \mathbb{V}$ have $k \in \mathbb{N}$ daughter cells. Then $X_{i,v}^{(u,k)}$, $1 \le u \le k$, describes the number of progeny from the i^{th} parasite in cell v which go in daughter cell u. In particular, the sum over all entries in $X_{i,v}^{(\bullet,k)}$ gives the total offspring number of this parasite. Since the families $(X_{i,v}^{(\bullet,k)})_{i\ge 1, v\in\mathbb{V}}$, $k \in \mathbb{N}$, are independent and each family consists of i.i.d. random variables, they fulfill all desired requirements for the multiplication behavior of the parasites. So the number of parasites in the cells can be defined recursively by putting $Z_\varnothing = 1$ (starting with a single parasite) and

$$Z_{vu} = \sum_{k\ge u} \mathbb{1}_{\{T_v = k\}} \sum_{i=1}^{Z_v} X_{i,v}^{(u,k)} = \sum_{i=1}^{Z_v} X_{i,v}^{(u,T_v)}, \qquad u \in \mathbb{N}, \tag{1.5}$$

where $X_{i,v}^{(u,t)} = 0$ a.s. if $u > t$, which will be a convention from now on. In particular, observe that by definition $\{\mathsf{T}_v = 0\} \subseteq \{Z_v = 0\}$ $\mathbb{P}$-a.s. and unless mentioned otherwise, *we assume the process starts with a unique cell containing a single parasite*, i.e.

$$\mathcal{T}_0 = 1 \quad \text{and} \quad Z_\varnothing = 1 \quad \text{a.s.}$$

Now, with keeping all the so far declared random variables in mind, the branching within branching process is defined as follows:

Definition 1.1. Given all the above defined random variables, we call the process $\boldsymbol{BP} = (\boldsymbol{BP}_n)_{n\ge 0}$ with $\boldsymbol{BP}_n = ((\mathsf{T}_v, Z_v))_{|v|=n} = [\mathsf{T}_v, Z_v]_{|v|=n}$ the *Branching within Branching process (BwBP)* and $\boldsymbol{BT} = (\boldsymbol{BT}_n)_{n\ge 0}$ with $\boldsymbol{BT}_n = [\mathsf{T}_v, Z_v]_{|v|\le n}$ the *Branching within Branching tree.*

Figure 1.1 shows a typical realization of the first three generations of a BwBP starting with one cell hosting one parasite. Only the living cells are displayed, i.e. the cells with $\mathsf{T}_v = 1$, and

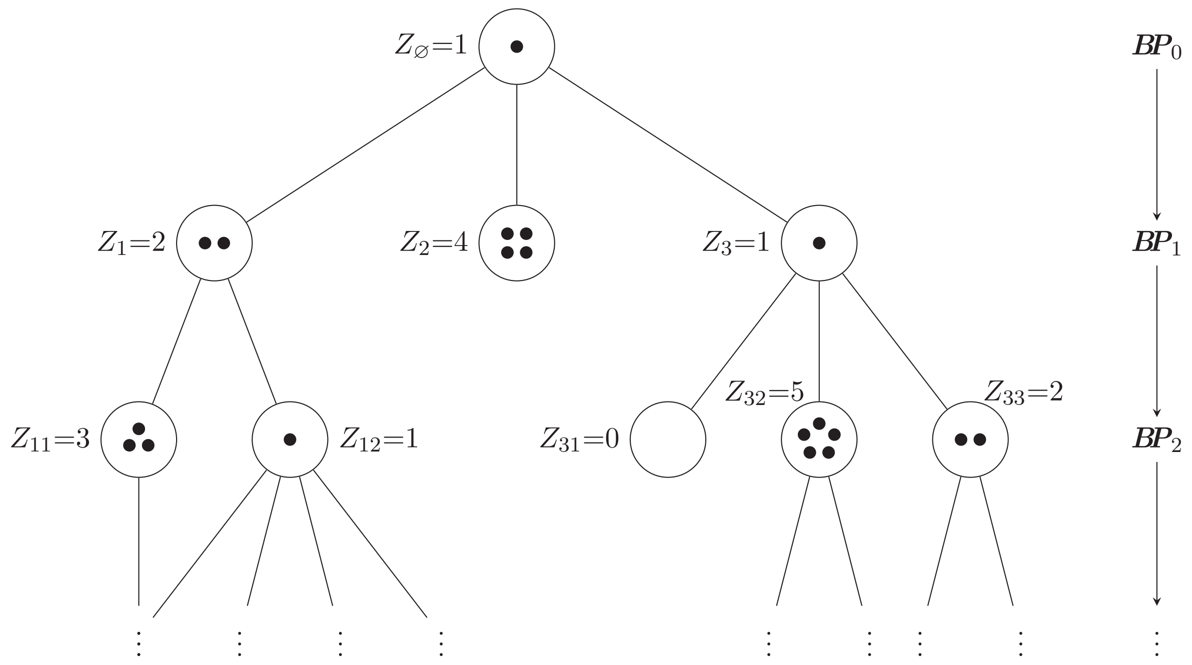

Figure 1.1: A typical realization of the first three generations of a BwBP.

in the shown realization, the first generation consists of three living cells ($\mathcal{T}_1 = 3$) hosting three, four and one parasite, respectively. The second cell reproduces no daughter cells, that is $T_2 = 0$, and so is a leave in the cell tree. Consequently, all four parasites living in this cell produce no offspring. The second generation then contains five cells ($\mathcal{T}_2 = 5$) of which four are contaminated and one is parasite free, hence $\mathcal{T}_2^* = 4$.

The definition of the BwBP model is kept as general as possible and therefore it comprises the following situation with multinomial repartition of parasites. Let every parasite in each generation multiply independently with the same distribution. After the parasite reproduction, the cell divides into a number of descendants with respect to $\mathcal{L}(T)$, and each of its containing parasites chooses independently the i^{th} daughter cell with probability $p_i(k) \in [0,1]$ when $T = k$. Thus,

$$\sum_{u=1}^{k} X^{(u,k)} \overset{d}{=} X^{(1,1)}$$

for all $k \in \mathbb{N}$, and given $\sum_{u=1}^{k} X^{(u,k)} = x$, the vector $(X^{(1,k)}, \ldots, X^{(k,k)})$ has a multinomial distribution with parameters x and $p_1(k), \ldots, p_k(k) \in [0,1]$.

Since we also intend the BwBP to start with several parasites in the root cell, we introduce for each $z \in \mathbb{N}_0$ a probability measure $\mathbb{P}_z$ on the measurable space $(\Omega, \mathfrak{F})$ such that (possibly after modifying the so far introduced random variables)

$$\mathbb{P}_z(\mathcal{T}_0 = 1, Z_\varnothing = z) = 1.$$

Furthermore, under $\mathbb{P}_z$ the $(T_v)_{v \in \mathbb{V}}$ are still i.i.d. random variables with distribution $(p_k)_{k \geq 0}$, and this family is independent of $(X_{i,v}^{(\bullet,k)})_{k \geq 1, i \geq 1, v \in \mathbb{V}}$. As before, all $X_{i,v}^{(\bullet,k)}$ for $i, k \in \mathbb{N}$, $v \in \mathbb{V}$, are independent of each other with

$$\mathbb{P}_z\left(X_{i,v}^{(\bullet,k)} \in \cdot\right) = \mathbb{P}\left(X^{(\bullet,k)} \in \cdot\right)$$

for each $i, k \in \mathbb{N}$ and $v \in \mathbb{V}$. Hence, under each $\mathbb{P}_z$ all parasites and cells have the same reproduction law, and the BwBP, as given in Definition 1.1, is a BwBP starting with a single cell hosting z parasites. Moreover, the processes $(\mathsf{T}_v)_{v\in\mathbb{V}}$ and $(\mathcal{T}_n)_{n\geq 0}$ keep their Markov chain resp. branching properties as their transition probability (1.2) resp. offspring distribution is independent from the parasite behavior.

We denote by $\mathbb{E}_z$, $z \in \mathbb{N}_0$, the corresponding expectation, and we omit the index in the case of the standard starting configuration, i.e $\mathbb{P} = \mathbb{P}_1$ and $\mathbb{E} = \mathbb{E}_1$, respectively. We further introduce a probability measure $\mathbb{P}_\dagger$ on underlying probability space under which the root cell, and thus every other cell, is dead, i.e. $\mathbb{P}_\dagger(\boldsymbol{BT} = [0,0]_{v\in\mathbb{V}}) = 1$ and $= 0$ otherwise. For later convenience, we will sometimes write $\mathbb{P}_{(1,z)}$ instead of $\mathbb{P}_z$ and $\mathbb{P}_{(0,z)}$ instead of $\mathbb{P}_\dagger$ for $z \in \mathbb{N}_0$. Of course, we will use the same corresponding notation for the expectation, viz. $\mathbb{E}_{(1,z)} = \mathbb{E}_z$ and $\mathbb{E}_\dagger = \mathbb{E}_{(0,z)}$.

We further introduce the canonical filtration $(\mathcal{F}_n)_{n\geq 0}$, that is $\mathcal{F}_0 := \sigma(\mathsf{T}_\varnothing, Z_\varnothing)$ and for $n \geq 1$

$$\mathcal{F}_n := \sigma\left(\mathsf{T}_v, Z_v, T_v, X_{i,v}^{(\bullet,k)} \;:\; |v| \leq n-1, k \geq 1, i \geq 1\right),$$

and let $\mathcal{F} = \sigma\left(\bigcup_{n\geq 0} \mathcal{F}_n\right)$. It is obvious by definition that $\boldsymbol{BP}$ and $\boldsymbol{BT}$ are $(\mathcal{F}_n)_{n\geq 0}$ adapted and $\mathcal{F}$-measurable and that $\mathcal{F}_n$ and $X_{i,v}^{(\bullet,T_v)}$ are independent for all $n \geq 0$, $|v| \geq n$ and $i \geq 1$.

We define the *process of parasites* by

$$\mathcal{Z}_n := \sum_{v\in\mathbb{T}_n} Z_v, \quad n \in \mathbb{N}_0,$$

which will be one of the main investigated processes in this thesis, see Subsection 1.2.4. For each $1 \leq l \leq k$, we further set

$$\mu_{l,k} := \mathbb{E}X^{(l,k)}$$

and put

$$\gamma := \mathbb{E}\mathcal{Z}_1 = \sum_{k=0}^{\infty} \mathbb{P}(T = k) \sum_{l=1}^{k} \mu_{l,k}$$

as the mean number of offspring parasites, which is assumed to be positive and finite, i.e.

$$0 < \gamma < \infty. \tag{A1}$$

In particular, this implies the existence of all $\mu_{l,k}$, $l \leq k$, and $\mathbb{P}(T = 0) < 1$. To avoid trivial cases, we assume that

$$\mathbb{P}(T = 1) < 1 \quad \text{and} \quad \mathbb{P}(\mathcal{Z}_1 = 1) < 1, \tag{A2}$$

for otherwise, if the first assumption fails, the cell tree would just be a cell line and $(\mathcal{Z}_n)_{n\geq 0}$ a standard GWP with reproduction law $\mathcal{L}(X^{(1,1)})$. If, on the other hand, the second assumption is violated, the number of parasites in each generation is the same and thus $\mathcal{T}_n^* = \mathcal{T}_0^*$ a.s. for all

$n \in \mathbb{N}_0$ or $\mathcal{T}_n^* = \mathcal{Z}_0$ eventually. To rule out the simple case where every daughter cell contains the same number of parasites as the root cell, we further assume that

$$p_t \mathbb{P}(X^{(u,t)} \neq 1) > 0 \quad \text{for at least one } 1 \leq u \leq t < \infty. \tag{A3}$$

We shortly mentioned at the beginning of this chapter that the BwBP can be interpreted as a multi-type branching process (MTBP) having countably many types. In a MTBP each individual (here cell) is marked with a type (here number of parasites) from a set of types $\mathcal{X}$ (here $\mathcal{X} = \mathbb{N}_0$). Multiplying independently, each individual produces offspring of various types determined by a reproduction law depending on their own type. The case of a finite type-space, i.e. $\#\mathcal{X} < \infty$, is well studied and results are transfered from the classical theory of GWPes (see e.g. [14, Chapter V] or [46, Chapter 4]). If, on the other hand, the state space is infinite (countable or uncountable) a variety of behaviors can be expected based on the reproduction mechanism of individuals. For example, letting the type-space transition have the form of a random walk, leads to the famous *branching random walk* (see Subsection 1.1.3). Other MTBPes are studied in the articles [11, 38, 48, 64, 65], just to mention a few, and we refer to Kimmel and Axelrod [51, Chapter 7] for a series of examples of MTBPes with applications in biology. We further mention [27] in which a MTBP in a very general setting is studied and conditions for martingale mean convergence are derived. This model comprises the BwBP, but the conditions given in the article are much weaker than those presented in Chapter 4 for our model. For more articles dealing with models related to the BwBP, we refer to the references listed in the Introduction.

1.1.2 The space of host-parasite trees

In this short subsection, we formally introduce the set of host-parasite trees and construct a suitable σ-algebra such that $\boldsymbol{BT}$ is measurable. We thereby follow the approaches in [29, 55, 66].

Put $\mathsf{S} := \{0,1\} \times \mathbb{N}_0$ and denote the *set of host-parasite trees* by

$$\mathbb{S} := \mathsf{S}^{\mathbb{V}} = \{\{0,1\} \times \mathbb{N}_0\}^{\mathbb{V}},$$

consisting of elements $[s_v, x_v]_{v \in \mathbb{V}}$. Each of these elements represents a host-parasite cell tree, which can also be identified by a mapping tr from $\mathbb{V}$ to S with $tr(v) = (s_v, x_v)$ for $v \in \mathbb{V}$. Let t_v and z_v be the projection on the first resp. second component of vertex $v \in \mathbb{V}$, viz.

$$\mathsf{t}_v : \mathbb{S} \to \{0,1\},\ [s_v, x_v]_{v \in \mathbb{V}} \mapsto s_v \quad \text{and} \quad \mathsf{z}_v : \mathbb{S} \to \mathbb{N}_0,\ [s_v, x_v]_{v \in \mathbb{V}} \mapsto x_v.$$

We further define a filtration $(\mathcal{S}_n)_{n \geq 0}$ generated by the projections t_v and z_v

$$\mathcal{S}_n := \sigma\left(\mathsf{t}_v, \mathsf{z}_v : |v| \leq n\right),$$

and let $\mathcal{S} = \sigma(\bigcup_{n \geq 0} \mathcal{S}_n)$. Obviously, the random host-parasite tree $\boldsymbol{BP} = \boldsymbol{BT} = [\mathsf{T}_v, Z_v]_{v \in \mathbb{V}}$ is $\mathbb{S}$-valued and $\mathcal{S}$-measurable by definition, for each (T_v, Z_v) is a random vector with values in S. Furthermore, observe that $(\mathbb{S}, \mathcal{S})$ is polish as a denumerable product of discrete spaces (see [28, Chapter IX §6]), and its open sets form a generator of the σ-algebra $\mathcal{S}$.

Let tr_n and $tr_{|n}$ for $n \in \mathbb{N}_0$ denote the restriction of a host-parasite tree to the n^{th} resp. the first n generations. Formally speaking, for $\mathbb{S}_n := \mathsf{S}^{|v| \leq n}$, $n \in \mathbb{N}_0$, endowed with the canonical σ-algebra $\mathcal{S}_{|n}$,

$$tr_n : \mathbb{S} \to \mathbb{S}^{|v|=n}, \ \ [s_v, x_v]_{v \in \mathbb{V}} \mapsto [s_v, x_v]_{|v|=n} \quad \text{and} \quad tr_{|n} : \mathbb{S} \to \mathbb{S}_n, \ \ [s_v, x_v]_{v \in \mathbb{V}} \mapsto [s_v, x_v]_{|v| \leq n},$$

which are of course surjective mappings and $tr_{|n}$ is $\mathcal{S}$-$\mathcal{S}_{|n}$-measurable. Then we can describe the n^{th} resp. first n generations of the BwBP as follows:

$$\boldsymbol{BP}_n = tr_n(\boldsymbol{BT}) \qquad \text{and} \qquad \boldsymbol{BT}_n = tr_{|n}(\boldsymbol{BT}).$$

Evidently, $\boldsymbol{BT}_n$ is $(\mathbb{S}_n, \mathcal{S}_{|n})$-measurable and for each $A \in \mathcal{S}_n$ there exists a set $B \in \mathcal{S}_{|n}$ such that $tr_{|n}(A) = B$ and

$$\mathbb{P}_z(\boldsymbol{BT} \in A) = \mathbb{P}_z(\boldsymbol{BT}_n \in B) \quad \text{for all } z \in \mathbb{N}_0\,. \tag{1.6}$$

Let t_n, t_n^* and z_n for $n \in \mathbb{N}_0$ be the measurable functions counting the number of living resp. contaminated cells and alive parasites in the n^{th} generation. More precisely,

$$t_n : (\mathbb{S}, \mathcal{S}) \to (\overline{\mathbb{N}}_0, \mathcal{P}(\overline{\mathbb{N}}_0)), \quad [s_v, x_v]_{v \in \mathbb{V}} \mapsto \sum_{|v|=n} \mathsf{t}_v([s_v, x_v]_{v \in \mathbb{V}}) = \sum_{|v|=n} s_v,$$

and

$$t_n^* : (\mathbb{S}, \mathcal{S}) \to (\overline{\mathbb{N}}_0, \mathcal{P}(\overline{\mathbb{N}}_0)), \quad [s_v, x_v]_{v \in \mathbb{V}} \mapsto \sum_{|v|=n} s_v(1 - \delta_{0 x_v}),$$

as well as

$$z_n : (\mathbb{S}, \mathcal{S}) \to (\overline{\mathbb{N}}_0, \mathcal{P}(\overline{\mathbb{N}}_0)), \quad [s_v, x_v]_{v \in \mathbb{V}} \mapsto \sum_{|v|=n} \mathsf{z}_v([s_v, x_v]_{v \in \mathbb{V}})\, \mathsf{t}_v([s_v, x_v]_{v \in \mathbb{V}}) = \sum_{|v|=n} z_v s_v.$$

Hence,

$$\mathcal{T}_n = t_n(\boldsymbol{BT}), \quad \mathcal{T}_n^* = t_n^*(\boldsymbol{BT}) \quad \text{and} \quad \mathcal{Z}_n = \sum_{|v|=n} Z_v\, \mathsf{T}_v = z_n(\boldsymbol{BT}) \quad \mathbb{P}_{(t,z)}\text{-a.s.}$$

for all $(t, z) \in \mathsf{S}$.

1.1.3 Comparison to other branching models

The process of parasites generally disobeys known branching structures. However, in some setups, it forms a standard GWP or other famous branching processes.

Galton-Watson branching process

$(\mathcal{T}_n)_{n \geq 0}$ is a standard GWP with reproduction law $(p_k)_{k \geq 0}$ by definition. If for all $k \geq 1$

$$X^{(1,k)} = \cdots = X^{(k,k)} = 1 \quad \text{a.s.}$$

and $\mathcal{Z}_\varnothing = \mathcal{T}_\varnothing = 1$, then $\mathcal{Z}_n = \mathcal{T}_n$ $\mathbb{P}$-a.s. for all $n \geq 0$, and therefore $(\mathcal{Z}_n)_{n \geq 0}$ is a standard GWP starting with a single individual, reproduction law $(p_k)_{k \geq 0}$ and reproduction mean ν.

There is another situation in which the process of parasites forms a GWP, namely when $\mathbb{T}$ is t-adic for a $t \in \mathbb{N}$. This means that $T = t$ a.s., and thus every cell in each generation divides into t daughter cells. So each parasite in the BwBP produces offspring according to the distribution of $X^{(\bullet,t)}$ and the cell tree structure is irrelevant for parasite multiplication. This is exactly the situation in the model studied by Bansaye in [15] for $t = 2$. More precisely, we get

$$\mathcal{Z}_{n+1} = \sum_{v \in \mathbb{T}_{n+1}} Z_v = \sum_{v \in \mathbb{T}_n} \sum_{i=1}^{Z_v} \sum_{u=1}^{t} X_{i,v}^{(u,t)} \quad \text{a.s.}$$

for all $n \geq 0$. Since the $\sum_{u=1}^{t} X_{i,v}^{(u,t)}$, $i \geq 1$, $v \in \mathbb{V}$, are i.i.d., the offspring of the parasites is chosen independently and with the same distribution and thus

$$\mathbb{E}s^{\mathcal{Z}_{n+1}} = \mathbb{E}\left(\prod_{v \in \mathbb{T}_n} \prod_{i=1}^{Z_v} \mathbb{E}\left(s^{\sum_{u=1}^{t} X_{i,v}^{(u,t)}} \middle| \mathcal{F}_n \right) \right) = \mathbb{E}\left(\mathbb{E}\left(s^{\sum_{u=1}^{t} X^{(u,t)}} \right)^{\mathcal{Z}_n} \right) = \mathbb{E}(\varphi(s)^{\mathcal{Z}_n}),$$

where $\varphi(s) := \mathbb{E}s^{\mathcal{Z}_1}$, $s \in [0,1]$, is the generating function of $\mathcal{Z}_1$. This shows that $(\mathcal{Z}_n)_{n \geq 0}$ has a Galton-Watson branching process structure with reproduction law $\mathcal{L}(\sum_{u=1}^{t} X^{(u,t)})$. By the classical theory of GWPes, it follows that $(\mathcal{Z}_n)_{n \geq 0}$ dies out almost surely if and only if $\varphi'(1) = \gamma \leq 1$ (recall that $\mathbb{P}(\mathcal{Z}_1 = 1) < 1$ by (A2)). As it turns out, the condition $\gamma \leq 1$ is still sufficient but not necessary for the process of parasites to die out almost surely in a general BwBP setting, see Theorem 1.10. For more background information on GWPes, we refer once again to the books [10, 14, 46].

Branching process in random environment

If all parasites are in the same cell in each generation, the process of parasites forms another well-known branching process, the *branching process in random environment (BPRE)*. This follows from the property that the number of daughter cells determine the parasite offspring distribution.

Consider the BwBP in which at most one daughter cell has positive probability for being contaminated. So, let $1 \leq l_k \leq k$ for $k \geq 1$ be the index such that $X^{(l,k)} = 0$ a.s. for all $l \neq l_k$. Without loss of generality (w.l.o.g.) we can assume that $l_k = 1$ for all $k \geq 1$. That is,

$$X^{(2,k)} = \cdots = X^{(k,k)} = 0 \quad \text{a.s.}$$

for all $k \geq 2$, which means that only $X^{(1,k)}$, $k \geq 1$, contributes to the total number of parasites in the next generation. Hence, when starting with one contaminated ancestor cell, the number of contaminated cells in each generation is at most 1, viz.

$$\mathbb{P}(\mathcal{T}_n^* \leq 1) = 1 \quad \text{for all } n \geq 0.$$

Furthermore, $\mathcal{Z}_n = Z_{1^{*n}}$ a.s. for each $n \geq 0$, where $1^{*n} = 1 \ldots 1$ (n-times) is the left most cell in the n^{th} generation in $\mathbb{V}$, and thus

$$\mathcal{Z}_{n+1} = \sum_{v \in \mathbb{T}_n^*} \sum_{i=1}^{Z_v} \sum_{u \geq 1} X_{i,v}^{(u,T_v)} = \sum_{i=1}^{\mathcal{Z}_n} X_{i,1^{*n}}^{(1,T_{1^{*n}})} \quad \text{for } n \geq 0.$$

Observe that the offspring distribution of parasites in the n^{th} generation depends on T_{1*n}, but given the value of T_{1*n}, the parasites multiply independently with the same distribution. So, $(\mathcal{Z}_n)_{n\geq 0}$ forms a branching process in random environment with environmental sequence $(T_{1*n})_{n\geq 0}$, which consists of i.i.d. random variables giving the offspring distribution for each generation.

It is further remarked that in this situation the set of possible reproduction laws is countable and the environmental sequence consists of i.i.d. random variables, which is an essential restriction in the setting of branching processes in random environment. In many works concerning branching processes in random environment, the environmental sequence is assumed to be stationary and ergodic taking values in the set of all probability measures on $\mathbb{N}_0$. See e.g. [1–3,12,13,17,31,40,81,83–85] for a detailed description of the BPRE and its basic and more advanced properties.

Weighted branching process and branching random walk

Consider a standard GWT in which each edge carries a random weight, and each individual in the population is assigned the product of all weights along his unique path to the root. Such a process is called a *weighted branching process (WBP)*, firstly introduced by Rösler [73] and treated in various articles afterwards, see for example [8, 9, 54, 55, 63, 74, 75] and the references given there. The multiplicative structure appears in the BwBP in the degenerated case where parasites in a cell beget the same number of descendants. For all $1 \leq u \leq t < \infty$ let $a_{u,t} \in \mathbb{N}_0$ and further

$$X^{(u,t)} = a_{u,t} \quad \mathbb{P}\text{-a.s.}$$

as well as $X^{(u,t)} = 0$ a.s. if $u > t$. So given the number of daughter cells, every parasite in the mother cell reproduces via a Dirac-measure. This implies

$$Z_v = \sum_{i=1}^{Z_{v|n-1}} X_{i,v|n-1}^{(v_n,T_{v|n-1})} = Z_{v|n-1} \cdot a_{v_n,T_{v|n-1}} = \cdots = \prod_{i=1}^{n} a_{v_i,T_{v|i-1}} \tag{1.7}$$

for $v \in \mathbb{V}$ with $v = v_1 \dots v_n$ and $Z_\varnothing = 1$, and thus

$$\mathcal{Z}_{n+1} = \sum_{v\in\mathbb{T}_n} \sum_{i=1}^{Z_v} \sum_{u\geq 1} X_{i,v}^{(u,T_v)} = \sum_{v\in\mathbb{T}_n} Z_v \sum_{u\geq 1} a_{u,T_v}$$

for $n \geq 0$. Since $\sum_{u\geq 1} a_{u,T_v}$, $v \in \mathbb{V}$, are i.i.d., $(\mathcal{Z}_n)_{n\geq 0}$ forms a WBP in the exceptional case where $\mathbb{N}_0$-valued weights are considered.

Taking the logarithm in (1.7) provides an additive structure along a cell line, i.e. for $|v| = n$

$$\log Z_v = \sum_{i=1}^{n} \log a_{v_i,T_{v|i-1}},$$

and the family of point processes $(\mathcal{N}_n)_{n\in\mathbb{N}_0}$ with $\mathcal{N}_n = \sum_{|v|=n} \delta_{\log Z_v}(\cdot \cap \mathbb{R})$ forms a *branching random walk (BRW)*, where the $\log Z_v$ $(> -\infty)$, $v \in \mathbb{V}$, give the position of an individual on the real line. Roughly speaking, a BRW is a GWP in which individuals are residing on $\mathbb{R}$, multiply in an i.i.d. manner and their children are moved on the real line relative to their mother according to a point process (here $\mathcal{N}_1$). Via the just described logarithmic relation, a WBP can be uniquely associated with a BRW and vice versa. See [7, 21–24, 26, 52, 54] for properties of the BRW and its relation to the WBP.

1.1.4 The branching property and the model with multiple root cells

Recall that $\mathbb{T}$ forms a GWT and that the number of parasites in each cell depends only on the number of parasites in the mother cell (given the number of daughter cells). So, the distribution of the daughter cells and the parasites they contain of a cell $v \in \mathbb{V}$ with $|v| = n$ given the past $\boldsymbol{BT}_n$ depends only on (T_v, Z_v), i.e.

$$\begin{aligned}\mathbb{P}\left([\mathsf{T}_{vu}, Z_{vu}]_{u\geq 1} \in A \mid \boldsymbol{BT}_n = [s_w, x_w]_{|w|\leq n}\right) &= \mathbb{P}\left([\mathsf{T}_{vu}, Z_{vu}]_{u\geq 1} \in A \mid (\mathsf{T}_v, Z_v) = (s_v, x_v)\right) \\ = \mathbb{P}_{(s_v,x_v)}(\boldsymbol{BP}_1 \in A) &= \begin{cases}\mathbb{P}_{x_v}\left(\boldsymbol{BP}_1 \in A\right) & \text{if } s_v = 1, \\ \mathbb{P}_{\dagger}(\boldsymbol{BP}_1 \in A) & \text{if } s_v = 0,\end{cases}\end{aligned}$$

for all $[s_w, x_w]_{|w|\leq n} \in \mathbb{S}_n$ and $A \in \bigotimes_{u\in\mathbb{N}} \mathcal{P}(\mathsf{S})$, where the second equality holds true since the reproduction law of cells and parasites is independent of v by definition. Applying this Markov property successively yields the evident result that a BwBP on the subtree rooted in a cell $v \in \mathbb{V}$ with $\mathsf{T}_v = 1$ behaves as the original BwBP with Z_v ancestor parasites. Additionally, the i.i.d. property of the families $(X_{i,v}^{(\bullet,T_v)})_{i\geq 1}$, $v \in \mathbb{V}$, provides that subtrees having different ancestor cells in the same generation are independent. This forms some kind of branching property for the BwBP, which is summarized in detail in the next proposition. To formally state this observation, let us denote by

$$\boldsymbol{BT}^{(v)} := [\mathsf{T}_{vu}, Z_{vu}]_{u\in\mathbb{V}} \tag{1.8}$$

the BwBP on the subtree rooted in cell $v \in \mathbb{V}$.

Proposition 1.2 (Branching property)**.** *For every $n \in \mathbb{N}_0$, given $\boldsymbol{BT}_n$ the host-parasite processes $(\boldsymbol{BT}^{(v)})_{|v|=n}$ on the subtrees rooted in the cells of the n^{th} generation are independent and each $\boldsymbol{BT}^{(v)}$ is distributed as $\boldsymbol{BT}$ under $\mathbb{P}_{(\mathsf{T}_v,Z_v)}$. More precisely,*

$$\mathbb{P}_{(t,z)}\left(\left(\boldsymbol{BT}^{(v)}\right)_{|v|=n} \in \cdot \mid \boldsymbol{BT}_n = [s_w, x_w]_{|w|\leq n}\right) = \left(\bigotimes_{|v|=n} Q_{(s_v,x_v)}\right)(\cdot)$$

for every $n \in \mathbb{N}_0$, $(t, z) \in \mathsf{S}$ and $[s_w, x_w]_{|w|\leq n} \in \mathbb{S}_n$, with $Q_{(s_v,x_v)}$ denoting the measure of $\boldsymbol{BT}$ under $\mathbb{P}_{(s_v,x_v)}$, i.e. $Q_{(s_v,x_v)}(\cdot) = \mathbb{P}_{(s_v,x_v)}(\boldsymbol{BT} \in \cdot)$.

The branching property particularly says that, conditioned under $\boldsymbol{BT}_n$, the process evolving from the cells of the n^{th} generation onwards can be interpreted as a BwBP with multiple ancestor cells, in which every root cell starts a BwBP independent of the other ones. As a matter of course, a BwBP starting with a dead cell, i.e. $\mathsf{T}_{\varnothing}^{(i)} = 0$, does not contribute to the number of living cells and parasites in the succeeding generations, and thus these BwBPes can be ignored. But since in each generation of a BwBP only a finite number of cells are alive, the process rooted in the cells of generation n behaves as a BwBP starting with a finite, but multiple, number of cells. So it is reasonable to allow the BwBP to start with several ancestor cells. Let

$$S := \{(0,0)\} \cup \bigcup_{n\in\mathbb{N}} \{\{n\} \times \mathbb{N}_0^n\} \tag{1.9}$$

be the *set of all possible root configurations.* We write $\mathbb{P}_{(t,z)}$ with $(t,z) \in S$ and $z = (z_1, \dots, z_t)$ for the probability measure under which the BwBP starts with t cells having $z_1, \dots, z_t$ parasites, i.e.

$$\mathbb{P}_{(t,z)} \left(\mathcal{T}_0 = t, Z_\varnothing^{(1)} = z_1, \dots, Z_\varnothing^{(t)} = z_t \right) = 1,$$

with $Z_\varnothing^{(1)}, \dots, Z_\varnothing^{(t)}$ denoting the number of parasites in the root cells. Let $\boldsymbol{BT}^{(1)}, \dots, \boldsymbol{BT}^{(t)}$ be the $t \in \mathbb{N}$ independent BwBPes starting from these root cells, and let $\mathbb{T}^{(i)}$, $\mathbb{T}_n^{(i)}$, $\mathcal{T}_n^{(i)}$, $\mathcal{Z}_n^{(i)}$, $\mathsf{T}_v^{(i)}$ and $Z_v^{(i)}$ be the random variables describing the obvious. In particular,

$$\mathcal{T}_n^{(i)} = \sum_{|v|=n} t_n(\boldsymbol{BT}^{(i)}) \quad \text{and} \quad \mathcal{Z}_n^{(i)} = \sum_{|v|=n} z_n(\boldsymbol{BT}^{(i)}).$$

Then the number of living resp. contaminated cells in the n^{th} generation of the BwBP is

$$\mathcal{T}_n := \sum_{i=1}^t \mathcal{T}_n^{(i)} \qquad \text{and} \qquad \mathcal{T}_n^* := \sum_{i=1}^t t_n^*(\boldsymbol{BT}^{(i)}) \quad \mathbb{P}_{(t,z)}\text{-a.s.},$$

and the process of parasites is the sum of all parasites in the corresponding generation, viz.

$$\mathcal{Z}_n := \sum_{i=1}^t \mathcal{Z}_n^{(i)} \quad \mathbb{P}_{(t,z)}\text{-a.s.}$$

As before, we write $\mathbb{P}_\dagger$ for $\mathbb{P}_{(0,z)}$, $z \in \mathbb{N}_0$, and note that the new defined probability measures are consistent with the notation of the measures $\mathbb{P}_{(t,z)}$ with $(t,z) \in \mathsf{S}$, viz. in both cases $\mathbb{P}_{(1,z)}$ denotes that we start with a living single cell hosting z parasites. We further use $\mathbb{E}_{(t,z)}$, $(t,z) \in S$, for the expectation under $\mathbb{P}_{(t,z)}$ and set $\mathbb{P}_z = \mathbb{P}_{(1,z)}$ as well as $\mathbb{E}_z = \mathbb{E}_{(1,z)}$. Needless to say, we omit the index, i.e. $\mathbb{P} = \mathbb{P}_1$ and $\mathbb{E} = \mathbb{E}_1$, if we start with one alive cell and one parasite, which describes the standard configuration.

1.2 Important processes and first results

In this section, we introduce some important processes arising from the BwBP, namely the *associated branching process in random environment, the process of contaminated cells* and *the process of parasites*, which were curtly touched in the Introduction. Furthermore, we introduce a Markov chain representing the set of contaminated cells and the number of parasites they contain in each generation. A large part of the presented results in this section has been published in [6] in the special case, where a cell has at most two daughter cells.

1.2.1 The associated branching process in random environment

One of the first steps when dealing with the BwBP is to reveal properties of an *infinite random cell line* through the cell tree $\mathbb{T}$. This approach was first used by Bansaye in [15]. In his article, a random cell line was obtained by simply picking a random path in the infinite binary Ulam-Harris tree representing the cell population. Since in our case the cell tree is of a general Galton-Watson structure and therefore random, we must proceed in a different manner. Here we will not pick a path in $\mathbb{V}$ uniformly but according to a size-biased distribution. The resulting path can be seen as a so-called *spine* in a size-biased tree. The spine cell at generation n then gives us a "typical" cell in the ordinary cell tree in generation n. The parasites along the thus obtained spine form a branching process in an i.i.d. random environment and its behavior is highly related to the one of $(\mathcal{T}_n^*)_{n\geq 0}$.

Spinal trees or *size-biased trees* have turned out to be of great use to prove convergence results of various branching processes. This is not different in our setup, see Chapter 2, 4 and 5. The concept of size-biasing goes back to Lyons et al. in [61], who used it to show classical limit theorems for the GWP. We refer to this article for a detailed construction of a spinal GWT and to Chapter 2 for further references.

Shortly speaking, the spine in a GWT is constructed successively by picking in each generation the next vertex in the spine uniformly from the offspring of a size-biased reproducing individual. For the formal definition, let $\{(\breve{T}_n, C_n) : n \in \mathbb{N}_0\}$ be a family of i.i.d. random vectors independent of $(T_v)_{v\in\mathbb{V}}$ and $(X_{i,v}^{(\bullet,k)})_{k\geq 1, i\geq 1, v\in\mathbb{V}}$. Thereby, each $\breve{T}_n$ has a size-biased distribution of T, i.e. for each $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$

$$\mathbb{P}(\breve{T}_n = k) = \frac{kp_k}{\nu},$$

and for $1 \leq l \leq k$

$$\mathbb{P}(C_n = l \mid \breve{T}_n = k) = \frac{1}{k},$$

which means that C_n is uniformly distributed on $\{1, \dots, k\}$ given $\breve{T}_n = k$. The *spine* $(V_n)_{n\geq 0}$ is then recursively defined by $V_0 = \varnothing$ and for $n \geq 1$ by

$$V_n := V_{n-1}C_{n-1}.$$

Then

$$\varnothing =: V_0 \to V_1 \to V_2 \to \cdots \to V_n \to \dots$$

provides us with a random cell line (not picked uniformly) in $\mathbb{V}$.

Figure 1.2 illustrates a typical realization of a random path through a cell tree. Only living cells are shown, and the cells in the spine are indicated by the symbol □ and all other cells by ◯. In that particular realization, we have $V_0 = \varnothing$, $V_1 = 1$, $V_2 = 12$, $V_3 = 122$ and looking at the number of parasites along the spine $Z_{V_0} = 1$, $Z_{V_1} = 2$, $Z_{V_2} = 1$ and $Z_{V_3} = 3$.

Concentrating now on the number of parasites along $(V_n)_{n\geq 0}$, we get $Z_{V_0} = Z_\varnothing$ and for $n \geq 0$ the recursive formula

$$Z_{V_{n+1}} = \sum_{t=1}^{\infty}\sum_{u=1}^{t} \mathbb{1}_{\{\breve{T}_n = t, C_n = u\}} \sum_{i=1}^{Z_{V_n}} X_{i,V_n}^{(u,t)} = \sum_{i=1}^{Z_{V_n}} X_{i,V_n}^{(C_n,\breve{T}_n)}.$$

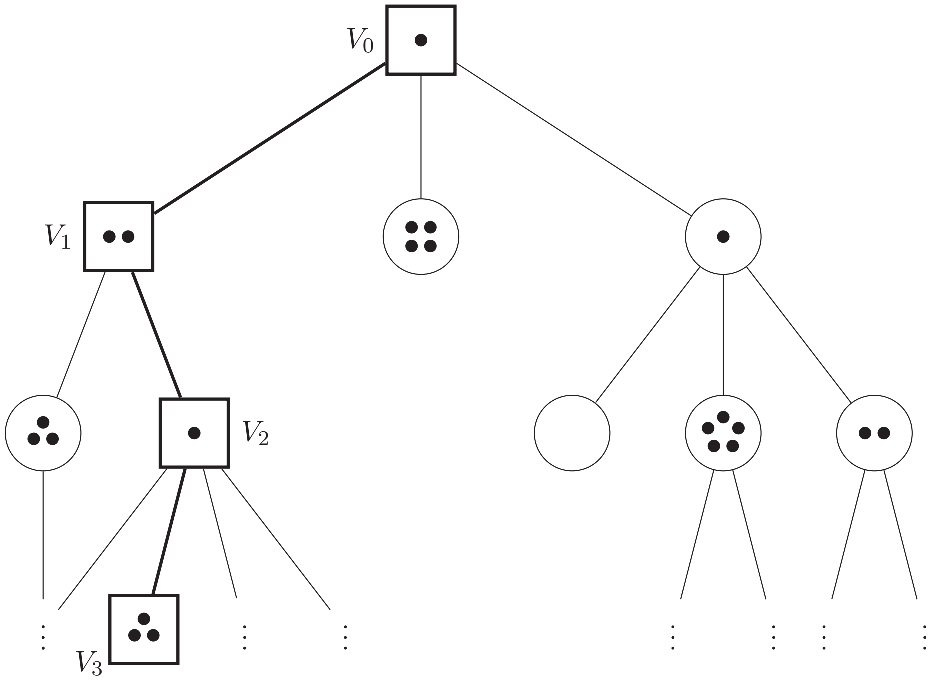

Figure 1.2: Typical realization of a spine in the size-biased cell tree.

Thus, given $(\breve{T}_n, C_n)$ all parasites in generation n multiply independently with the same distribution $\mathcal{L}(X^{(C_n,\breve{T}_n)})$. Since $(\breve{T}_n, C_n)_{n\geq 0}$ are i.i.d. and independent of $(X_{i,v}^{(\bullet,k)})_{k\geq 1, i\geq 1, v\in\mathbb{V}}$, we infer that the process of parasites along the spine forms a branching process in random environment. More precisely, for $n \in \mathbb{N}_0$ we calculate

$$\begin{aligned}\mathbb{E}\left(s^{Z_{V_{n+1}}} \mid [\breve{T}_k, C_k]_{k\geq 0}, (Z_{V_k})_{k\leq n}\right) &= \prod_{i=1}^{Z_{V_n}} \mathbb{E}\left(s^{X_{i,V_n}^{(C_n,\breve{T}_n)}} \mid [\breve{T}_k, C_k]_{k\leq n}, (Z_{V_k})_{k\leq n}\right) \\ &= \prod_{i=1}^{Z_{V_n}} \mathbb{E}\left(s^{X^{(C_n,\breve{T}_n)}} \mid [\breve{T}_k, C_k]_{k\leq n}, (Z_{V_k})_{k\leq n}\right) \\ &= \mathbb{E}\left(s^{X^{(C_n,\hat{T}_n)}} \mid (\breve{T}_n, C_n)\right)^{Z_{V_n}} \quad \text{a.s.,}\end{aligned}$$

where in the last equation the independence of $(\breve{T}_n, C_n)$ and $\sigma([\breve{T}_k, C_k]_{k<n}, (Z_{V_k})_{k\leq n})$ was used. Thus, the process of parasites along the spine behaves like a branching process with an i.i.d. environmental sequence $(\breve{T}_k, C_k)_{n\in\mathbb{N}_0}$ determining the reproduction laws (see [13, 81] for the definition of a BRPE). We summarize this observation in the following theorem.

Theorem 1.3. *Let $(Z'_n)_{n\geq 0}$ be a BPRE with $Z_\varnothing$ ancestors and i.i.d. environmental sequence $\Lambda := (\Lambda_n)_{n\geq 0}$ taking values in $\{\mathcal{L}(X^{(u,t)}) \mid 1 \leq u \leq t < \infty\}$ with*

$$\mathbb{P}\left(\Lambda_0 = \mathcal{L}(X^{(u,t)})\right) = \frac{p_t}{\nu}$$

for all $1 \leq u \leq t < \infty$. Then $(Z_{V_n})_{n\geq 0}$ and $(Z'_n)_{n\geq 0}$ equal in law.

Proof. Obviously $Z'_0 = Z_\varnothing$ a.s. by definition. Furthermore,

$$\mathbb{P}\left(\Lambda_0 = \mathcal{L}(X^{(u,t)})\right) = \frac{p_t}{\nu} = \mathbb{P}\left((\breve{T}_0, C_0) = (t,u)\right)$$

for all $1 \le u \le t < \infty$, and thus, the reproduction laws in each generation are chosen according to the same distribution. Hence, $(Z_{V_n})_{n\ge 0}$ and $(Z'_n)_{n\ge 0}$ equal in law. □

We call the BPRE $(Z'_n)_{n\ge 0}$ from the above theorem with environmental sequence Λ the *associated branching process in random environment* and we will refer to it with *ABPRE*. For $n \in \mathbb{N}$ and $s \in [0,1]$, let

$$f_n(s|\Lambda) := \mathbb{E}(s^{Z'_n}|\Lambda) \quad \text{and} \quad f_n(s) := \mathbb{E}s^{Z'_n} = \mathbb{E}f_n(s|\Lambda)$$

denote the quenched and annealed generating function of Z'_n, respectively. Then the theory of branching processes in random environment (see Subsection 1.1.3 for references) provides us with the following facts: For each $n \in \mathbb{N}$,

$$f_n(s|\Lambda) = g_{\Lambda_0} \circ ... \circ g_{\Lambda_{n-1}}(s), \quad g_\lambda(s) := \mathbb{E}(s^{Z'_1}|\Lambda_0 = \lambda) = \sum_{n\ge 0} \lambda_n s^n$$

for any distribution $\lambda = (\lambda_n)_{n\ge 0}$ on $\mathbb{N}_0$. Moreover, the g_{Λ_n} are i.i.d. with

$$\mathbb{E}g'_{\Lambda_0}(1) = \mathbb{E}Z'_1 = \sum_{1\le u\le t} \frac{p_t}{\nu}\, \mathbb{E}X^{(u,t)} = \frac{\mathbb{E}\mathcal{Z}_1}{\nu} = \frac{\gamma}{\nu} \quad (< \infty), \tag{1.10}$$

where we recall that $\gamma = \mathbb{E}\mathcal{Z}_1$. As a consequence,

$$\mathbb{E}Z'_n = f'_n(1) = \prod_{k=0}^{n-1} \mathbb{E}g'_{\Lambda_k}(1) = \left(\frac{\gamma}{\nu}\right)^n$$

for each $n \in \mathbb{N}$. If the process starts with $k \ge 1$ parasites in a single cell, i.e. $Z_\varnothing = k$ $\mathbb{P}_k$-a.s., then

$$\mathbb{E}_k(s^{Z'_n}|\Lambda) = (f_n(s|\Lambda))^k \quad \mathbb{P}_k\text{-a.s.} \tag{1.11}$$

It is also well-known that $(Z'_n)_{n\ge 0}$ survives with positive probability (w.p.p.) if and only if

$$\mathbb{E}\log g'_{\Lambda_0}(1) > 0 \quad \text{and} \quad \mathbb{E}\log^-(1 - g_{\Lambda_0}(0)) < \infty, \tag{1.12}$$

see e.g. [13, 81] and recall that $\gamma < \infty$ is assumed by (A1). Furthermore, by (A3), there exists $1 \le u \le t < \infty$ such that $p_t > 0$ and $\mathbb{P}(X^{(u,t)} \ne 1) > 0$, which ensures that $\Lambda_0 \ne \delta_1$ w.p.p. As usual, we call the ABPRE *supercritical, critical* or *subcritical* if $\mathbb{E}\log g'_{\Lambda_0}(1) > 0, = 0$ or < 0, respectively. In the subcritical case there exist three sub-regimes, $\mathbb{E}g'_{\Lambda_0}(1)\log g'_{\Lambda_0}(1) < 0, = 0, >$ 0, in which the process behaves differently. They are called *strongly, intermediate* and *weakly subcritical.* See [40] for detailed limiting results in the three cases.

The connection between the distribution of Z'_n and the expected number of cells in generation n with a fixed number of parasites is stated in the next result.

Proposition 1.4. *For all* $n,k,z \in \mathbb{N}_0$,

$$\mathbb{P}_z\left(Z'_n = k\right) = \nu^{-n}\, \mathbb{E}_z\left(\#\{v \in \mathbb{T}_n : Z_v = k\}\right), \tag{1.13}$$

in particular

$$\mathbb{P}_z\left(Z'_n > 0\right) = \nu^{-n}\, \mathbb{E}_z \mathcal{T}^*_n. \tag{1.14}$$

Proof. For all $n,k \in \mathbb{N}$, vertices $v = v_1 \dots v_n$ and $t_0 \geq v_1, \dots, t_{n-1} \geq v_n$, we find that

$$\begin{aligned}
\mathbb{E}\left(s^{Z_v} \mid T_\varnothing = t_0, \dots, T_{v|n-1} = t_{n-1}\right) &= \mathbb{E}\left(\prod_{i=1}^{Z_{v|n-1}} \mathbb{E}\left(s^{X^{(v_n,t_{n-1})}_{i,v|n-1}}\right) \mid T_\varnothing = t_0, \dots, T_{v|n-2} = t_{n-2}\right) \\
&= \mathbb{E}\left(g_{\mathcal{L}\left(X^{(v_n,t_{n-1})}\right)}(s)^{Z_{v|n-1}} \mid T_\varnothing = t_0, \dots, T_{v|n-2} = t_{n-2}\right) \\
&= \dots = g_{\mathcal{L}\left(X^{(v_1,t_0)}\right)} \circ g_{\mathcal{L}\left(X^{(v_2,t_1)}\right)} \circ \dots \circ g_{\mathcal{L}\left(X^{(v_n,t_{n-1})}\right)}(s) \\
&= \mathbb{E}\left(s^{Z'_n} \mid \Lambda_0 = \mathcal{L}\big(X^{(v_1,t_0)}\big), \dots, \Lambda_{n-1} = \mathcal{L}\big(X^{(v_n,t_{n-1})}\big)\right),
\end{aligned}$$

and thus by (1.11)

$$\begin{aligned}
&\mathbb{P}_z(Z_v = k, \mathsf{T}_v = 1) = \sum_{t_0 \geq v_1} \cdots \sum_{t_{n-1} \geq v_n} \mathbb{P}_z(Z_v = k, T_{v|0} = t_0, \dots, T_{v|n-1} = t_{n-1}) \\
&= \sum_{t_0 \geq v_1} \cdots \sum_{t_{n-1} \geq v_n} \mathbb{P}_z(T_{v|0} = t_0, \dots, T_{v|n-1} = t_{n-1}) \mathbb{P}_z(Z_v = k \mid T_{v|0} = t_0, \dots, T_{v|n-1} = t_{n-1}) \\
&= \sum_{t_0 \geq v_1} \cdots \sum_{t_{n-1} \geq v_n} \left(\prod_{i=0}^{n-1} p_{t_i}\right) \mathbb{P}_z\left(Z'_n = k \mid \Lambda_0 = \mathcal{L}\big(X^{(v_1,t_0)}\big), \dots, \Lambda_{n-1} = \mathcal{L}\big(X^{(v_n,t_{n-1})}\big)\right) \\
&= \nu^n \sum_{t_0 \geq v_1} \cdots \sum_{t_{n-1} \geq v_n} \mathbb{P}_z\left(Z'_n = k, \Lambda_0 = \mathcal{L}\big(X^{(v_1,t_0)}\big), \dots, \Lambda_{n-1} = \mathcal{L}\big(X^{(v_n,t_{n-1})}\big)\right)
\end{aligned}$$

for all $z \in \mathbb{N}_0$. Finally, we get for each $z \in \mathbb{N}_0$

$$\begin{aligned}
&\mathbb{E}_z\left(\#\{v \in \mathbb{T}_n : Z_v = k\}\right) = \sum_{|v|=n} \mathbb{P}_z(Z_v = k, \mathsf{T}_v = 1) \\
&= \nu^n \sum_{|v|=n} \sum_{t_0 \geq v_1} \cdots \sum_{t_{n-1} \geq v_n} \mathbb{P}_z\left(Z'_n = k, \Lambda_0 = \mathcal{L}\big(X^{(v_1,t_0)}\big), \dots, \Lambda_{n-1} = \mathcal{L}\big(X^{(v_n,t_{n-1})}\big)\right) \\
&= \nu^n \sum_{t_0, \dots, t_{n-1} = 1}^{\infty} \sum_{\substack{v_i \leq t_{i-1}, \\ i = 1, \dots, n}} \mathbb{P}_z\left(Z'_n = k, \Lambda_0 = \mathcal{L}\big(X^{(v_1,t_0)}\big), \dots, \Lambda_{n-1} = \mathcal{L}\big(X^{(v_n,t_{n-1})}\big)\right) \\
&= \nu^n \mathbb{P}_z\left(Z'_n = k\right).
\end{aligned}$$

Summation over all $k \in \mathbb{N}$ then gives (1.14). □

1.2.2 A Markov chain arising from the tree of infected cells

To know exactly which cells are alive ($\mathsf{T}_v = 1$) in $\mathbb{V}$ is unnecessary for certain analysis, for example when dealing only with the number of cells $\mathcal{T}_n$ resp. contaminated cells $\mathcal{T}^*_n$. Since $\mathbb{T}$ is a

GWT and non-infected cells influence neither the future behavior of parasite multiplication nor the partition onto the daughter cells, the behavior of the BwBP depends only on the number of contaminated cells and the parasite number in each of them. Roughly speaking, we look at ***BP*** generation-wise, erase the cell tree structure and ignore all "healthy" cells. In this subsection, we introduce a process ***BPG*** that meets the afore described heuristic.

For a formal definition of this process, we first denote by

$$S^* := \{(s,(z_1,\ldots,z_s)) \in S \mid 1 \le z_1 \le z_2 \le \cdots \le z_s\} \tag{1.15}$$

the *set of configurations of contaminated cells in a generation* and put $S_0^* := \{(0,0)\} \cup S^*$. For each $n \in \mathbb{N}_0$ denote by χ_n the measurable mapping which maps a vector of host-parasite trees $(\tau^{(1)},\ldots,\tau^{(k)})$, $k \in \mathbb{N}$, to a vector providing the total number of contaminated cells over all trees in the n^{th} generation and a vector having non-decreasing entries giving the number of parasites in them. That is, with $t(k) := \sum_{i=1}^{k} t_n^*(\tau^{(i)})$ for a vector $(\tau^{(1)},\ldots,\tau^{(k)})$,

$$\chi_n : \left(\bigcup_{k\ge1}\mathbb{S}^k, \sigma\Big(\bigcup_{k\ge1}\mathcal{S}^k\Big)\right) \to (S_0^*, \mathcal{P}(S_0^*)), \quad (\tau^{(i)})_{1\le i\le k} \mapsto \begin{cases} (t,z) & \text{if } t := t(k) > 0,\\ (0,0) & \text{if } t = 0, \end{cases} \tag{1.16}$$

where $z = (z_1,\ldots,z_t)$ is the t-dimensional vector of increasing entires $z_j = \mathsf{z}_{v_j}(\tau^{(i_j)})$, $1 \le j \le t$, for distinct tuples $(i_1,v_1),\ldots,(i_t,v_t) \in \{1,\ldots,k\} \times \{|v| = n\}$, denoting the number of parasites in the alive cells over all trees in generation n, i.e. $\mathsf{t}_{v_j}(\tau^{(i_j)})\,\mathsf{z}_{v_j}(\tau^{(i_j)}) > 0$ for each $1 \le j \le t$ and $z_1 \le z_2 \le \cdots \le z_t$. In particular, t gives the number of contaminated cells in the n^{th} generation. We define the process $\boldsymbol{BPG} = (\boldsymbol{BPG}_n)_{n\ge0}$ generation-wise by

$$\boldsymbol{BPG}_n := \chi_n(\boldsymbol{BT}), \quad n \in \mathbb{N}_0 .$$

So $\boldsymbol{BPG}_n = (s,(z_1,\ldots,z_s))$ means that the n^{th} generation of ***BT*** has s infected cells containing $z_1,\ldots,z_s$ parasites.

As each cell and its parasites multiply independently of all other cells and their parasites in the same generation, the exact positions of the infected cells in a generation are unimportant for the number of contaminated cells in the next generation and the number of parasites they contain. So, for each $n \in \mathbb{N}_0$ and $[s_v^{(i)}, x_v^{(i)}]_{|v|=n,i\in\mathbb{N}} \in \chi^{-1}((s,x))$, $(s,x) \in S_0^*$, we obtain

$$\mathbb{P}\left(\boldsymbol{BPG}_{n+1} \in \cdot \mid \boldsymbol{BP}_n = [s_v^{(i)}, x_v^{(i)}]_{|v|=n,i\in\mathbb{N}}\right) = \mathbb{P}_{(s,x)}\left(\boldsymbol{BPG}_1 \in \cdot\right)$$

by utilizing the branching property (see Proposition 1.2). Consequently, ***BPG*** is a Markov chain with state space S_0^* and transition probabilities

$$p((s,x),(t,z)) := \mathbb{P}_{(s,x)}(\boldsymbol{BPG}_1 = (t,z)) = \mathbb{P}_{(s,x)}\left(\boldsymbol{BP}_1 \in \chi_1^{-1}((t,z))\right) \tag{1.17}$$

for $(s,x),(t,z) \in S_0^*$. We note this in the following proposition.

Proposition 1.5. *The process* ***BPG*** *is a homogeneous Markov chain with state space* S_0^* *and transition probabilities defined by* (1.17). *Moreover, all states in* S^* *are transient.*

Proof. We have already seen in the discussion above the proposition that $\boldsymbol{BPG}$ is a homogeneous Markov chain with state space S_0^* and transition probabilities given by (1.17). So, it is left to prove that all states in S^* are transient. First, we point out that for each $z \in \mathbb{N}_0$

$$\{\mathcal{Z}_1 = 0\} \;=\; \{T_\varnothing = 0\} \cup \bigcup_{t\in\mathbb{N}} \left\{ T_\varnothing = t, \sum_{i=1}^{z}\sum_{u=1}^{t} X_{i,\varnothing}^{(u,t)} = 0 \right\} \qquad \mathbb{P}_z\text{-a.s.}$$

So if $\mathbb{P}(\mathcal{Z}_1 = 0) > 0$, then $\mathbb{P}(T = 0) > 0$ or there exists a $t \geq 1$ such that $\mathbb{P}(T = t)\mathbb{P}(\sum_{u=1}^{t} X^{(u,t)} = 0) > 0$, hence,

$$\mathbb{P}_z(\mathcal{Z}_1 = 0) \geq \begin{cases} \mathbb{P}(T = 0) & \text{if } \mathbb{P}(T = 0) > 0, \\ \sum_{t=1}^{\infty} \mathbb{P}(T = t)\mathbb{P}\left(\sum_{u=1}^{t} X^{(u,t)} = 0\right)^z & \text{if } \mathbb{P}(T = 0) = 0, \end{cases} \Bigg\} > 0$$

for all $z \in \mathbb{N}_0$. Using the branching property, this implies for each $(s,x) \in S^*$ with $x = (x_1, \ldots, x_s)$

$$\mathbb{P}_{(s,x)}(\boldsymbol{BPG}_n \neq (s,x) \text{ for all } n \geq 1) \geq \begin{cases} \mathbb{P}_{(s,x)}(\mathcal{Z}_1 = 0) & \text{if } \mathbb{P}(\mathcal{Z}_1 = 0) > 0, \\ 1 - \mathbb{P}_{(s,x)}(\mathcal{Z}_1 = \sum_{i=1}^{s} x_i) & \text{if } \mathbb{P}(\mathcal{Z}_1 = 0) = 0, \end{cases}$$

$$\geq \begin{cases} \prod_{i=1}^{s} \mathbb{P}_{x_i}(\mathcal{Z}_1 = 0) & \text{if } \mathbb{P}(\mathcal{Z}_1 = 0) > 0, \\ 1 - \prod_{i=1}^{s} \mathbb{P}_{x_i}(\mathcal{Z}_1 = x_i) & \text{if } \mathbb{P}(\mathcal{Z}_1 = 0) = 0, \end{cases} \Bigg\} > 0,$$

where $\mathbb{P}_z(\mathcal{Z}_1 = z) < 1$ for all $z \geq 1$ by (A2). Thus, $(s,x) \in S^*$ is a transient state. $\square$

As an immediate consequence of the above proposition, we deduce the *extinction-explosion principle* saying that the population of parasites either dies out or tends to infinity.

Corollary 1.6 (Extinction-explosion principle). *The parasite population of a BwBP either extincts or explodes, i.e. for all $(t,z) \in S$*

$$\mathbb{P}_{(t,z)}(\mathcal{Z}_n \to 0) + \mathbb{P}_{(t,z)}(\mathcal{Z}_n \to \infty) = 1.$$

Proof. Since non-infected root cells have no effect on parasite survival, we can assume without the loss of generality that $(t,z) \in S^*$. But the transience of all states in S^* for the process $\boldsymbol{BPG}$ implies

$$\lim_{n\to\infty} \mathbb{P}_{(t,z)}(1 \leq \mathcal{Z}_n \leq K) \;\leq\; \lim_{n\to\infty} \sum_{s=1}^{K} \sum_{x\in\{1,..,K\}^s} \mathbb{P}_{(t,z)}(\boldsymbol{BPG}_n = (s,x)) \;=\; 0$$

for all $K \in \mathbb{N}$. $\square$

We denote by

$$\mathsf{Ext} := \{\mathcal{Z}_n \to 0\} \quad \text{and} \quad \mathsf{Surv} := \mathsf{Ext}^c = \{\mathcal{Z}_n \to \infty\}$$

the *set of extinction* and *survival of parasites*, respectively. Furthermore, we put for $(t,z) \in S$

$$\mathbb{P}^*_{(t,z)} := \mathbb{P}_{(t,z)}(\cdot|\,\mathsf{Surv}) \quad \text{and} \quad \mathbb{E}^*_{(t,z)} := \mathbb{E}_{(t,z)}(\cdot|\,\mathsf{Surv}),$$

and in consistency of our notation, we write $\mathbb{P}_z^*$ and $\mathbb{E}_z^*$, $z \in \mathbb{N}$, for the corresponding probability measures $\mathbb{P}_z$ and expectations $\mathbb{E}_z$ conditioned under Surv. To round up these definitions, we put $\mathbb{P}^* := \mathbb{P}_1^*$ and $\mathbb{E}^* := \mathbb{E}_1^*$.

As a final thought in this subsection, we find for all $(t, z) \in S^*$ by utilizing the branching property

$$\mathbb{P}_{(t,z)}(\mathcal{Z}_n = 0) = \prod_{i=1}^{t} \mathbb{P}_{z_i}(\mathcal{Z}_n = 0),$$

and thus $\mathbb{P}_{(t,z)}(\mathsf{Ext}) = 1$ if and only if $\mathbb{P}_{z_i}(\mathsf{Ext}) = 1$ for all $1 \leq i \leq t$. Now, let the BwBP start with a unique cell and $z \in \mathbb{N}$ parasites, and let $\mathcal{Z}_{n,i}$ denote the descendants of parasite i in the n^{th} generation. Since $\{(X_{i,v}^{(\bullet,k)})_{k\geq 1} : i \geq 1, v \in \mathbb{V}\}$ is an i.i.d. family and the cell tree is independent of the parasites, $\mathcal{Z}_{n,i}$, $1 \leq i \leq t$, are i.i.d. given the cell tree $\mathbb{T}$. That is, for a tree $\tau \subseteq \mathbb{V}$ with one root cell and $A = \times_{i=1}^{z} A_i \subseteq \mathbb{N}_0^z$

$$\mathbb{P}_z(\mathcal{Z}_n \in A |\, \mathbb{T} = \tau) = \prod_{i=1}^{z} \mathbb{P}(\mathcal{Z}_{n,i} \in A_i |\, \mathbb{T} = \tau),$$

and thus for all $z \in \mathbb{N}$

$$\mathbb{P}_z(\mathsf{Ext}) = 1 \quad \text{iff} \quad \mathbb{P}(\mathsf{Ext}) = 1. \tag{1.18}$$

1.2.3 The process of contaminated cells

We proceed to the statement of results for the *process of contaminated cells* $(\mathcal{T}_n^*)_{n\geq 0}$ and its asymptotic behavior. Since the extinction-explosion principle holds for the process of parasites $(\mathcal{Z}_n)_{n\geq 0}$ (see Corollary 1.6), a natural question arising is the following: In the case of non-extinction of the parasite population, are these parasites concentrated in only a finite number of cells or do they spread over the whole cell tree. In other words, does $\mathcal{T}_n^*$ tend to infinity for $n \to \infty$ if $\mathcal{Z}_n$ does? This would lead to an extinction-explosion principle for the process of contaminated cells, i.e. for $(t, z) \in S$

$$\mathbb{P}_{(t,z)}(\mathcal{T}_n^* \to 0) + \mathbb{P}_{(t,z)}(\mathcal{T}_n^* \to \infty) = 1.$$

It turns out that this is in fact true besides some degenerated cases. Due to the branching property (Proposition 1.2), it is enough to consider a single root cell. Hence, we just prove the above relation under the measures $\mathbb{P}_z$, $z \in \mathbb{N}$.

Theorem 1.7. *Let* $\mathbb{P}(\mathsf{Surv}) > 0$ *and* $z \in \mathbb{N}$.

(a) If $\mathbb{P}_2(\mathcal{T}_1^* \geq 2) > 0$, *then* $\mathbb{P}_z(\mathcal{T}_n^* \to \infty \mid \mathsf{Surv}) = 1$.

(b) If $\mathbb{P}_2(\mathcal{T}_1^* \geq 2) = 0$, *then* $\mathbb{P}_z(\mathcal{T}_n^* = 1 \;\forall\; n \geq 0 \mid \mathsf{Surv}) = 1$.

Proof. Let $z \in \mathbb{N}$. We first prove the easier case (b) and note that

$$\mathbb{P}_2(\mathcal{T}_1^* \geq 2) \;\geq\; p_t \mathbb{P}_2(X_{1,\varnothing}^{(u,t)} > 0, X_{2,\varnothing}^{(v,t)} > 0) \;=\; p_t \mathbb{P}(X^{(u,t)} > 0)\mathbb{P}(X^{(v,t)} > 0)$$

for all $t \geq 1$ and $1 \leq u < v \leq t$. Thus, for all $t \geq 1$ with $p_t > 0$ there exists at most one $1 \leq u \leq t$ such that $\mathbb{P}(X^{(u,t)} > 0) > 0$. Consequently, $\mathbb{P}_z(\mathcal{T}_n^* \leq 1 \ \forall\ n \geq 0) = 1$. But since $\mathsf{Surv} = \{\mathcal{T}_n^* \geq 1 \ \forall\ n \geq 0\}$ $\mathbb{P}_z$-a.s., (b) is proved.

The proof of (a) is a bit more complicated and uses the Markov chain $\boldsymbol{BPG}$ introduced in Subsection 1.2.2 to show that $(\mathcal{T}_n^*)_{n\geq 0}$ visits each $t \geq 1$ only finitely often. If this holds true, we can conclude that for all $t \geq 1$

$$\mathbb{P}_z(1 \leq \mathcal{T}_n^* \leq t \text{ infinitely often}) = 0$$

and thus the extinction-explosion principle for $(\mathcal{T}_n^*)_{n\geq 0}$. But since $\mathsf{Ext} = \{\mathcal{T}_n^* \to 0\}$ $\mathbb{P}_z$-a.s., (a) follows.

So after these preliminaries, it is left to prove that $\mathcal{T}_n^* = t$ for at most finitely many $n \in \mathbb{N}$, for each $t \geq 1$. To verify this, we define

$$A_t := \{(t,(z_1,\ldots,z_t)) \in S^* \mid z_t \geq 2\} \subseteq \mathbb{N}^t$$

for $t \geq 1$ and note that for $n \geq 0$

$$\{\mathcal{T}_n^* = t\} = \{\boldsymbol{BPG}_n \in A_t\} \cup \{\boldsymbol{BPG}_n = (t, \underbrace{(1,\ldots,1)}_{t\text{-times}})\} \quad \mathbb{P}_z\text{-a.s.}$$

Since $(t,(1,\ldots,1)) \in S^*$ is transient by Proposition 1.5, we get

$$\mathbb{P}_z(\mathcal{T}_n^* = t \text{ infinitely often}) \ = \ \mathbb{P}_z(\boldsymbol{BPG}_n \in A_t \text{ infinitely often}),$$

and it remains to prove that the Markov chain $\boldsymbol{BPG}$ visits the set A_t only finitely often with probability 1. For $(t,x) \in A_t$ with $x = (x_1,\ldots,x_t)$, we get by using the branching property

$$\begin{aligned}\mathbb{P}_{(t,x)}\left(\boldsymbol{BPG}_n \notin A_t \text{ for all } n \geq 1\right) &\geq \mathbb{P}_{(t,x)}\left(\mathcal{T}_n^* > t \text{ for all } n \geq 1\right)\\ &\geq \mathbb{P}_{x_t}\left(\mathcal{T}_1^* \geq 2\right)\mathbb{P}(\mathsf{Surv})^2 \prod_{i=1}^{t-1} \mathbb{P}_{x_i}\left(\mathcal{T}_1^* \geq 1\right)\mathbb{P}(\mathsf{Surv})\\ &\geq \mathbb{P}_2\left(\mathcal{T}_1^* \geq 2\right)\mathbb{P}\left(\mathcal{T}_1^* \geq 1\right)^{t-1}\mathbb{P}(\mathsf{Surv})^{t+1} > 0\end{aligned}$$

due to our assumptions in (a). It is remarked that the established lower bound does not depend on the special choice of (t,x) anymore. Let $\tau_0 = 0$ and for $n \geq 0$

$$\tau_{n+1} := \inf\{k > \tau_n \mid \boldsymbol{BPG}_k \in A_t\}$$

be the successive entry times of $\boldsymbol{BPG}$ into the set A_t. Then the inequality just achieved above and the strong Markov property of $\boldsymbol{BPG}$ imply the existence of a constant $c < 1$ such that for all $(t,x) \in A_t$ and $n \geq 0$

$$\mathbb{P}_z\left(\tau_{n+1} - \tau_n < \infty \middle| \boldsymbol{BPG}_{\tau_n} = (t,x), \tau_n < \infty\right) \ = \ \mathbb{P}_{(t,x)}\left(\tau_1 < \infty\right) \ \leq \ c \ < \ 1.$$

Using this inequality and iteration, we conclude for $n \geq 1$

$$\mathbb{P}_z(\tau_n < \infty) \ = \ \sum_{(t,x)\in A_t} \mathbb{P}_z(\boldsymbol{BPG}_{\tau_{n-1}} = (t,x), \tau_n - \tau_{n-1} < \infty, \tau_{n-1} < \infty)$$

$$
\begin{aligned}
&= \sum_{(t,x)\in A_t} \mathbb{P}_z(\tau_n-\tau_{n-1}<\infty \,|\, \boldsymbol{BPG}_{\tau_{n-1}}=(t,x),\tau_{n-1}<\infty)\mathbb{P}_z(\boldsymbol{BPG}_{\tau_{n-1}}=(t,x),\tau_{n-1}<\infty)\\
&\le c\,\mathbb{P}_z(\tau_{n-1}<\infty)\\
&\le c^{n-1}\mathbb{P}_z(\tau_1<\infty)\\
&\le c^{n-1}
\end{aligned}
$$

and finally

$$
\begin{aligned}
\mathbb{P}_z\left(\boldsymbol{BPG}_n\in A_t \text{ infinitely often}\right) &= \mathbb{P}_z\left(\tau_n<\infty \text{ for all } n\ge 1\right) = \mathbb{P}_z\left(\bigcap_{n\ge 1}\{\tau_n<\infty\}\right)\\
&= \lim_{n\to\infty}\mathbb{P}_z(\tau_n<\infty) \le \lim_{n\to\infty} c^{n-1} = 0.
\end{aligned}
$$

□

The next result provides us with a geometric rate at which the number of contaminated cells tends to infinity.

Theorem 1.8. *$(\nu^{-n}\mathcal{T}_n^*)_{n\ge 0}$ is a non-negative supermartingale with respect to $(\mathcal{F}_n)_{n\ge 0}$ under each $\mathbb{P}_z$, $z\in\mathbb{N}_0$, and therefore almost surely convergent to an integrable random variable L as $n\to\infty$. Furthermore, for $z\in\mathbb{N}$,*

(a) $L=0$ $\mathbb{P}_z$-a.s. if and only if one of the following conditions hold true:

  - *(i) $\nu\le 1$*
  - *(ii) $\mathbb{E}T\log T=\infty$*
  - *(iii) $\mathbb{E}\log g'_{\Lambda_0}(1)\le 0$ or $\mathbb{E}\log^-(1-g_{\Lambda_0}(0))=\infty$.*

  In particular, $\mathbb{P}_z(L=0)=1$ for $z\in\mathbb{N}$ if and only if $\mathbb{P}(L=0)=1$.

(b) $\mathbb{P}_z(L=0)<1$ implies $\{L=0\}=\mathsf{Ext}$ $\mathbb{P}_z$-a.s.

Proof. Let $z\in\mathbb{N}$. That $(\nu^{-n}\mathcal{T}_n^*)_{n\ge 0}$ forms a supermartingale follows by an easy calculation. For $n\ge 0$, we get

$$
\begin{aligned}
\mathbb{E}_z\left(\mathcal{T}_{n+1}^* \mid \mathcal{F}_n\right) &= \sum_{v\in\mathbb{T}_n^*}\mathbb{E}_z\left(\sum_{u=1}^{T_v}\mathbb{1}_{\{Z_{vu}>0\}} \mid \mathcal{F}_n\right)\\
&\le \sum_{v\in\mathbb{T}_n^*}\mathbb{E}_z\left(T_v \mid \mathcal{F}_n\right)\\
&= \sum_{v\in\mathbb{T}_n^*}\mathbb{E}\left(T_v\right)\\
&= \nu\mathcal{T}_n^* \quad \mathbb{P}_z\text{-a.s.},
\end{aligned}
$$

where in the penultimate equation we used the independence of T_v and $(\mathcal{F}_k)_{k\le n}$ for every $v\in\mathbb{T}_n$. This confirms the supermartingale property as well as the integrability. The adaptivity is obvious.

Thus $(\nu^{-n}\mathcal{T}_n^*)_{n\geq 0}$ forms a non-negative supermartingale and therefore almost sure convergence to an integrable random variable L is ensured.

If $\nu > 1$ and $\mathbb{E}T\log T < \infty$, the supermartingale $(\nu^{-n}\mathcal{T}_n^*)_{n\geq 0}$ is even uniformly integrable. This follows, since the obvious majorant $(\nu^{-n}\mathcal{T}_n)_{n\geq 0}$ is a normalized supercritical GWP having the $(Z\log Z)$-condition of the Kesten-Stigum theorem fulfilled (see [14, Section I.10]). Consequently,

$$\mathbb{E}_z L = \lim_{n\to\infty} \mathbb{E}_z\left(\frac{\mathcal{T}_n^*}{\nu^n}\right) = \lim_{n\to\infty} \mathbb{P}_z(Z_n' > 0), \tag{1.19}$$

where the last equality follows from (1.14) in Proposition 1.4. Now (1.12) implies that in this case $L = 0$ $\mathbb{P}_z$-a.s. if and only if condition (iii) holds true. If, on the other hand, $\mathbb{E}T\log T = \infty$, Fatou's lemma and the theorem of Kesten and Stigum imply

$$\mathbb{E}_z L \leq \liminf_{n\to\infty} \mathbb{E}_z\left(\frac{\mathcal{T}_n^*}{\nu^n}\right) \leq \liminf_{n\to\infty} \mathbb{E}\left(\frac{\mathcal{T}_n}{\nu^n}\right) = 0,$$

and if $\nu \leq 1$, then $\mathcal{T}_n^* \leq \mathcal{T}_n = 0$ eventually. In both cases we obtain $L = 0$ $\mathbb{P}_z$-a.s. Since the conditions $(i)-(iii)$ do not depend on $z \in \mathbb{N}$, it follows $\mathbb{P}_z(L = 0) = 1$ if and only if $\mathbb{P}(L = 0) = 1$, which completes the proof of (a).

(b) First, we note the obvious property that for each $x, y \in \mathbb{N}_0$ with $x \leq y$ and $k \in \mathbb{N}_0$

$$\mathbb{P}_x(\mathcal{T}_n^* \geq k) \;\leq\; P_y(\mathcal{T}_n^* \geq k) \quad \text{for each } n \in \mathbb{N}_0\,. \tag{1.20}$$

Defining $\tau_n = \inf\{m \in \mathbb{N} : \mathcal{T}_m^* \geq n\}$, we find that for $z \in \mathbb{N}$ and all $n \in \mathbb{N}$

$$\begin{aligned}
\mathbb{P}_z(L = 0) \;&\leq\; \mathbb{P}_z(L = 0|\tau_n < \infty) + \mathbb{P}_z(\tau_n = \infty)\\
&=\; \mathbb{P}_z\left(\frac{1}{\nu^{\tau_n}}\sum_{v\in\mathbb{T}_{\tau_n}^*} \nu^{-(m-\tau_n)} t_m^*(\boldsymbol{BT}^{(v)}) \xrightarrow{m\to\infty} 0 \;\middle|\; \tau_n < \infty\right) + \mathbb{P}_z(\tau_n = \infty)\\
&\leq\; \mathbb{P}_z\left(\bigcap_{k=1}^{\mathcal{T}_{\tau_n}^*}\{\mathcal{T}_{m,k}^*/\nu^m \to 0\}\middle|\tau_n < \infty\right) + \mathbb{P}_z(\tau_n = \infty)\\
&\leq\; \mathbb{P}\left(\bigcap_{k=1}^{n}\{\mathcal{T}_{m,k}^*/\nu^m \to 0\}\right) + \mathbb{P}_z(\tau_n = \infty)\\
&\leq\; \mathbb{P}(L = 0)^n + \mathbb{P}_z(\tau_n = \infty),
\end{aligned}$$

where $t_m^*(\boldsymbol{BT}^{(v)}) = \sum_{|u|=n} \mathsf{T}_{vu}\,\mathbb{1}_{\{Z_{vu}>0\}}$ is the number of contaminated cells in the m^{th} generation of the subtree rooted in cell $v \in \mathbb{T}_n^*$, and the $\mathcal{T}_{m,k}^*$, $k \geq 1$, are independent copies of $\mathcal{T}_m^*$ starting with one ancestor cell hosting one parasite. Since $\mathbb{P}(L = 0) < 1$, Theorem 1.7 implies

$$\mathbb{P}_z(L = 0) \leq \lim_{n\to\infty} \mathbb{P}_z(\tau_n = \infty) = \mathbb{P}_z\left(\sup_{n\geq 1} \mathcal{T}_n^* < \infty\right) = \mathbb{P}_z(\mathsf{Ext}),$$

which in combination with $\mathsf{Ext} \subseteq \{L = 0\}$ $\mathbb{P}_z$-a.s. proves the assertion. □

At last, we look at the situation of almost sure extinction of parasites when starting with multiple ancestor cells. We show the auxiliary result that in this case only one subtree of an ancestor cell contributes to the total number of contaminated cells in a generation.

Lemma 1.9. *Let* $\mathbb{P}(\mathsf{Ext}) = 1$. *Then for all* $(s,z) \in S^*$

$$\lim_{n\to\infty} \mathbb{P}_{(s,z)}\left(\exists\, 1 \le i < j \le s \text{ s.t. } \min\{t_n^*(\boldsymbol{BT}^{(i)}), t_n^*(\boldsymbol{BT}^{(j)})\} > 0 \;\middle|\; \sum_{k=1}^{s} t_n^*(\boldsymbol{BT}^{(k)}) > 0\right) = 0,$$

where $\boldsymbol{BT}^{(1)}, \ldots, \boldsymbol{BT}^{(s)}$ *denote the* s *independent BwBP starting from the root cells.*

Proof. Let $(s,z) \in S^*$ with $z = (z_1, \ldots, z_s)$. First, we note that for all $n \ge 0$

$$\mathbb{P}_{(s,z)}\left(\sum_{k=1}^{s} t_n^*(\boldsymbol{BT}^{(k)}) > 0\right) \;\ge\; \mathbb{P}_{z_s}\left(\mathcal{T}_n^* > 0\right).$$

Thanks to the branching property of $\boldsymbol{BT}$ and (1.20), we conclude for $n \ge 0$

$$\begin{aligned}
&\mathbb{P}_{(s,z)}\left(\exists\, 1 \le i < j \le s \text{ s.t. } \min\{t_n^*(\boldsymbol{BT}^{(i)}), t_n^*(\boldsymbol{BT}^{(j)})\} > 0\right)\\
&\le \sum_{1\le i<j\le s} \mathbb{P}_{(s,z)}\left(t_n^*(\boldsymbol{BT}^{(i)}) > 0, t_n^*(\boldsymbol{BT}^{(j)}) > 0\right)\\
&= \sum_{1\le i<j\le s} \mathbb{P}_{z_i}\left(\mathcal{T}_n^* > 0\right)\mathbb{P}_{z_j}\left(\mathcal{T}_n^* > 0\right)\\
&\le s^2 \mathbb{P}_{z_s}\left(\mathcal{T}_n^* > 0\right)^2,
\end{aligned}$$

where it is recalled that the elements in z are increasing. This now yields

$$\begin{aligned}
&\mathbb{P}_{(s,z)}\left(\exists\, 1 \le i < j \le s \text{ s.t. } \min\{t_n^*(\boldsymbol{BT}^{(i)}), t_n^*(\boldsymbol{BT}^{(j)})\} > 0 \;\middle|\; \sum_{k=1}^{s} t_n^*(\boldsymbol{BT}^{(k)}) > 0\right)\\
&= \frac{\mathbb{P}_{(s,z)}\left(\exists\, 1 \le i < j \le s \text{ s.t. } \min\{t_n^*(\boldsymbol{BT}^{(i)}), t_n^*(\boldsymbol{BT}^{(j)})\} > 0\right)}{\mathbb{P}_{(s,z)}\left(\sum_{k=1}^{s} t_n^*(\boldsymbol{BT}^{(k)}) > 0\right)}\\
&\le\; s^2\, \mathbb{P}_{z_s}\left(\mathcal{T}_n^* > 0\right)\\
&\to\; 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

□

1.2.4 The process of parasites

We continue by turning to the *process of parasites* $(\mathcal{Z}_n)_{n\ge0}$, and this subsection is devoted to the derivation of first results. We will begin proving a full characterization of almost sure extinction of parasites, which is the main result in this subsection. Thereafter we concentrate on the normalized process $(\gamma^{-n}\mathcal{Z}_n)_{n\ge0}$, which forms a non-negative martingale, and give conditions for L^2-boundedness. Finer results, including an equivalent condition for the mean convergence, are later presented in Chapter 4. Referring to the branching property it is enough to consider the process starting with one root cell, i.e. under the measures $(\mathbb{P}_z)_{z\geq 0}$.

Recalling that $(\mathcal{Z}_n)_{n\geq 0}$ satisfies the extinction-explosion principle (see Corollary 1.6), a natural problem in hand is under which conditions almost sure extinction occurs. Compared to the standard GWP, which dies out almost surely if and only if the reproduction mean is less or equal to 1 (see [14, Section I.5]), $(\mathcal{Z}_n)_{n\geq 0}$ has a slightly different behavior. Although $\mathbb{E}\mathcal{Z}_n = \gamma^n$ is the n^{th} power of the reproduction mean (see Proposition 1.12), as in the standard Galton-Watson case, we find that $\gamma > 1$ is not sufficient for parasites to survive. To see that, consider the case where sharing of parasites into daughter cells is highly unequal. In this situation, there are only a few cells containing most of the parasites. If one of this "highly" infected cells dies, a large number of parasites disappears and so the probability of extinction is high in every generation. This becomes especially evident in the setting when only one daughter cell can be contaminated, i.e. $\mathbb{P}_2(\mathcal{T}_1^* \geq 2) = 0$. Hence, a condition warranting some kind of equal sharing must be in force too for parasites to survive w.p.p.

The next theorem gives us a full characterization of almost sure extinction. By (1.18) it is enough to concentrate on the process starting with one root cell hosting one parasite.

Theorem 1.10.

(a) If $\mathbb{P}_2(\mathcal{T}_1^* \geq 2) = 0$, *then* $\mathbb{P}(\mathsf{Ext}) = 1$ *if and only if*

$$\mathbb{E}\left(\log \mathbb{E}\left(\mathcal{Z}_1 | T_0\right)\right) \leq 0 \quad \textit{or} \quad \mathbb{E}\log^- \mathbb{P}\left(\mathcal{Z}_1 > 0 \mid T_0\right) = \infty.$$

(b) If $\mathbb{P}_2(\mathcal{T}_1^* \geq 2) > 0$, *then the following statements are equivalent:*

- *(i)* $\mathbb{P}(\mathsf{Ext}) = 1$
- *(ii)* $\mathbb{E}\mathcal{T}_n^* \leq 1$ *for all* $n \in \mathbb{N}_0$
- *(iii)* $\sup_{n\in\mathbb{N}_0} \mathbb{E}\mathcal{T}_n^* < \infty$
- *(iv)* $\nu \leq 1$, *or*

$$\nu > 1, \quad \mathbb{E}\log g'_{\Lambda_0}(1) < 0 \quad \textit{and} \quad \inf_{0\leq\theta\leq 1} \mathbb{E}g'_{\Lambda_0}(1)^\theta \leq \frac{1}{\nu}.$$

Remark 1.11. Let us point out the following useful facts about subcritical branching processes in random environment, that is $\mathbb{E}\log g'_{\Lambda_0}(1) < 0$, before proceeding to the proof of the theorem. We take the subcriticality in this remark as granted. We first note that, if $\mathbb{E}g'_{\Lambda_0}(1)\log g'_{\Lambda_0}(1) \leq 0$, the convexity of $\theta \mapsto \mathbb{E}g'_{\Lambda_0}(1)^\theta$ implies that

$$\mathbb{E}g'_{\Lambda_0}(1) = \inf_{0\leq\theta\leq 1} \mathbb{E}g'_{\Lambda_0}(1)^\theta.$$

Under the assumptions

$$\mathbb{P}(Z'_1 \leq C) = 1 \quad \text{and} \quad \mathbb{P}\left(0 < g'_{\Lambda_0}(1) < \varepsilon\right) = 0 \tag{AsGe}$$

for suitable constants $C > 0$ and $\varepsilon > 0$, Geiger et al. [40, Theorems 1.1–1.3] showed that

$$\mathbb{P}(Z'_n > 0) \simeq cn^{-\kappa} \left(\inf_{0\leq\theta\leq 1} \mathbb{E}g'_{\Lambda_0}(1)^\theta\right)^n \quad \text{as } n \to \infty \tag{1.21}$$

for some $c \in (0,\infty)$, where

$$\kappa = \begin{cases} 0 & \text{if } \mathbb{E}g'_{\Lambda_0}(1)\log g'_{\Lambda_0}(1) < 0 \qquad \text{(strongly subcritical case)},\\ 1/2 & \text{if } \mathbb{E}g'_{\Lambda_0}(1)\log g'_{\Lambda_0}(1) = 0 \qquad \text{(intermediately subcritical case)},\\ 3/2 & \text{if } \mathbb{E}g'_{\Lambda_0}(1)\log g'_{\Lambda_0}(1) > 0 \qquad \text{(weakly subcritical case)}. \end{cases}$$

The condition (AsGe) can be strongly weakened for the asymptotic relation of the survival probability in (1.21) to hold true, see for example [40,86]. But (AsGe) is enough for our purposes.

A combination of (1.14) and (1.21) provides us with the asymptotic relation

$$\mathbb{E}\mathcal{T}_n^* \simeq cn^{-\kappa}\nu^n \left(\inf_{0\le\theta\le1} \mathbb{E}g'_{\Lambda_0}(1)^\theta\right)^n \quad \text{as } n\to\infty, \tag{1.22}$$

in particular (with (AsGe) still in force)

$$\inf_{0\le\theta\le1} \mathbb{E}g'_{\Lambda_0}(1)^\theta \le \frac{1}{\nu} \quad \text{if} \quad \sup_{n\ge1}\mathbb{E}\mathcal{T}_n^* < \infty. \tag{1.23}$$

Proof. (*a*) If $\mathbb{P}_2(\mathcal{T}_n^* \ge 2) = 0$, we have seen in the proof of Theorem 1.7 that for all $t \ge 1$ with $p_t > 0$ there exists at most one $1 \le u \le t$ such that $\mathbb{P}(X^{(u,t)} > 0) > 0$. This implies that $(\mathcal{Z}_n)_{n\ge0}$ is a branching process in an i.i.d. random environment (see Subsection 1.1.3) and thus dies out almost surely if and only if $\mathbb{E}(\log\mathbb{E}(\mathcal{Z}_1|T_0)) \le 0$ or $\mathbb{E}\log^-\mathbb{P}(\mathcal{Z}_1 > 0 \mid T_0) = \infty$ (see e.g. [81]).

Let us now focus on (*b*) and assume $\mathbb{P}_2(\mathcal{T}_1^* \ge 2) > 0$.

"(*i*) ⇒ (*ii*)" (by contraposition) Fix $m \in \mathbb{N}$ such that $\mathbb{E}(\mathcal{T}_m^*) > 1$ and consider a supercritical GWP $(S_n)_{n\ge0}$ with $S_0 = 1$ and offspring distribution

$$\mathbb{P}(S_1 = k) = \mathbb{P}(\mathcal{T}_m^* = k), \quad k \in \mathbb{N}_0.$$

Obviously,

$$\mathbb{P}(S_n > k) \le \mathbb{P}(\mathcal{T}_{nm}^* > k)$$

for all $k, n \in \mathbb{N}_0$, hence

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{T}_{nm}^* > 0) \ge \lim_{n\to\infty} \mathbb{P}(S_n > 0) > 0,$$

i.e. parasites survive w.p.p.

"(*ii*) ⇒ (*iii*)" Here is nothing to prove.

"(*iii*) ⇒ (*i*)" Recall that $\liminf_{n\to\infty}\mathcal{T}_n^* = \infty$ a.s. on Surv by Theorem 1.7. Thus, if $\sup_{n\ge0}\mathbb{E}\mathcal{T}_n^* < \infty$, Fatou's lemma implies

$$\infty > \liminf_{n\to\infty}\mathbb{E}\mathcal{T}_n^* \ge \mathbb{E}\left(\liminf_{n\to\infty}\mathcal{T}_n^*\right),$$

giving $\mathbb{P}(\mathsf{Surv}) = 0$.

"(*iv*) ⇒ (*i*), (*ii*)" If $\nu \le 1$, then $\mathbb{E}\mathcal{T}_n^* \le \mathbb{E}\mathcal{T}_n = \nu^n \le 1$ for all $n \in \mathbb{N}$. So let us consider the situation when

$$\nu > 1, \quad \mathbb{E}\log g'_{\Lambda_0}(1) < 0 \quad \text{and} \quad \inf_{0\le\theta\le1}\mathbb{E}g'_{\Lambda_0}(1)^\theta \le \frac{1}{\nu}$$

is valid. By (1.14),

$$\mathbb{E}\mathcal{T}_n^* = \nu^n \mathbb{P}(Z_n' > 0)$$

for all $n \in \mathbb{N}$. We have to distinguish three cases:

Case A. $\mathbb{E}g_{\Lambda_0}'(1)\log g_{\Lambda_0}'(1) \le 0$. By what has been pointed out in the above remark, we then infer

$$\frac{\gamma}{\nu} = \mathbb{E}g_{\Lambda_0}'(1) = \inf_{0\le\theta\le 1} \mathbb{E}g_{\Lambda_0}'(1)^\theta \le \frac{1}{\nu}$$

and thus $\gamma \le 1$, which in turn entails

$$\mathbb{E}\mathcal{T}_n^* \le \mathbb{E}\mathcal{Z}_n = \gamma^n \le 1$$

for all $n \in \mathbb{N}$ as required.

Case B. $\mathbb{E}g_{\Lambda_1}'(1)\log g_{\Lambda_1}'(1) > 0$ and (AsGe). Then, by (1.21),

$$\mathbb{P}(Z_n' > 0) \;\simeq\; cn^{-3/2}\left(\inf_{0\le\theta\le 1} \mathbb{E}g_{\Lambda_1}'(1)^\theta\right)^n \quad \text{as } n \to \infty$$

holds true for a suitable constant $c \in (0,\infty)$ and therefore, using Fatou's lemma,

$$0 = \lim_{n\to\infty} \nu^n \mathbb{P}(Z_n' > 0) = \liminf_{n\to\infty} \mathbb{E}\mathcal{T}_n^* \ge \mathbb{E}\left(\liminf_{n\to\infty} \mathcal{T}_n^*\right).$$

Consequently, $\mathbb{P}(\mathsf{Surv}) = 0$ since $\inf_{n\ge 0}\mathcal{T}_n^* \ge 1$ a.s. on Surv.

Case C. $\mathbb{E}g_{\Lambda_1}'(1)\log g_{\Lambda_1}'(1) > 0$. Using contraposition, suppose that $\sup_{n\in\mathbb{N}} \mathbb{E}\mathcal{T}_n^* > 1$. Fix any vector $\alpha = (\alpha^{(u,t)})_{1\le u\le t<\infty}$ of distributions on $\mathbb{N}_0$ satisfying

$$\alpha_x^{(u,t)} \;\le\; \mathbb{P}\left(X^{(u,t)} = x\right) \quad \text{for } x \ge 1$$

and u, t as stated, hence

$$\alpha_0^{(u,t)} \;\ge\; \mathbb{P}\left(X^{(u,t)} = 0\right) \quad \text{and} \quad \sum_{x\ge n} \alpha_x^{(u,t)} \;\le\; \mathbb{P}\left(X^{(u,t)} \ge n\right)$$

for each $n \ge 0$. Possibly after enlarging the underlying probability space, we can then construct a BwBP $\boldsymbol{BP}(\alpha) = [\mathsf{T}_v, Z_{\alpha,v}]_{v\in\mathbb{V}}$ coupled with and of the same kind as $\boldsymbol{BP}$ such that

$$\mathbb{P}\left(X_{\alpha,i,v}^{(u,t)} = x\right) \;=\; \alpha_x^{(u,t)} \quad \text{and} \quad X_{\alpha,i,v}^{(u,t)} \;\le\; X_{i,v}^{(u,t)} \quad \text{a.s.}$$

for each $1 \le u \le t < \infty$, $v \in \mathbb{V}$, $i \ge 1$ and $x \ge 1$. Then $Z_{\alpha,v} \le Z_v$ a.s. for all $v \in \mathbb{V}$ and since the choice of α has no affect on the cell splitting process, we have $\nu_\alpha = \nu > 1$ and thus for $\theta \in [0,1]$

$$\begin{aligned}\mathbb{E}g_{\alpha,\Lambda_0}'(1)^\theta \;&=\; \mathbb{E}\left(\mathbb{E}(Z_{\alpha,1}'|\Lambda_0)^\theta\right) \;=\; \sum_{1\le u\le t<\infty} \frac{p_t}{\nu}\left(\mathbb{E}X_\alpha^{(u,t)}\right)^\theta \\ &\le\; \sum_{1\le u\le t<\infty} \frac{p_t}{\nu}\mu_{u,t}^\theta \;\le\; \mathbb{E}g_{\Lambda_0}'(1)^\theta \qquad (1.24)\end{aligned}$$

where ν_α, $Z'_{\alpha,n}$, $X_\alpha^{(u,t)}$ and g_{α,Λ_0} have the obvious meaning. It is recalled that $\mu_{u,t} = \mathbb{E}X^{(u,t)}$. With a similar calculation as in (1.24), we establish

$$\mathbb{E}\log g'_{\alpha,\Lambda_0}(1) \leq \mathbb{E}\log g'_{\Lambda_0}(1) < 0. \tag{1.25}$$

For $N \in \mathbb{N}$ let $\alpha(N) = (\alpha^{(u,t)}(N))_{1\leq u\leq t<\infty}$ be the vector specified by

$$\alpha_x^{(u)}(N) := \begin{cases} \mathbb{P}\left(X^{(u,t)} = x\right) & \text{if } 1 \leq x \leq N, \\ 0 & \text{if } x > N, \end{cases}$$

if $\mu_{u,t} \geq 1/N$, and $\alpha_0^{(u,t)} = 1$ if $\mu_{u,t} < 1/N$. Then $\boldsymbol{BP}(\alpha(N))$ satisfies the condition (AsGe) and we can fix $N \in \mathbb{N}$ such that $\sup_{n\in\mathbb{N}} \mathbb{E}\mathcal{T}^*_{\alpha(N),n} > 1$, because $\mathcal{T}^*_{\alpha(N),n} \uparrow \mathcal{T}^*_n$ as $N \to \infty$. Then, by what has already been proved under Case B in combination with (1.24), (1.25) and $\nu_{\alpha(N)} > 1$, we infer

$$\inf_{0\leq\theta\leq 1} \mathbb{E}g'_{\Lambda_0}(1)^\theta \geq \inf_{0\leq\theta\leq 1} \mathbb{E}g'_{\alpha(N),\Lambda_0}(1)^\theta > \frac{1}{\nu}. \tag{1.26}$$

and thus violation of (iv).

"$(ii) \Rightarrow (iv)$" Suppose that $\mathbb{E}\mathcal{T}^*_n \leq 1$ for all $n \in \mathbb{N}_0$ and further $\nu > 1$ which, by (1.14), entails $\lim_{n\to\infty} \mathbb{P}(Z'_n > 0) = 0$ and thus $\mathbb{E}\log g'_{\Lambda_0}(1) \leq 0$ or $\mathbb{E}\log^-(1 - g_{\Lambda_0}(0)) = \infty$. We must show that $\mathbb{E}\log g'_{\Lambda_0}(1) < 0$ and $\inf_{0\leq\theta\leq 1} \mathbb{E}g'_{\Lambda_0}(1)^\theta \leq \nu^{-1}$. But given $\mathbb{E}\log g'_{\Lambda_0}(1) < 0$, the second condition follows from (1.23) if (AsGe) is valid. With $\mathbb{E}\log g'_{\Lambda_0}(1) < 0$ still be in force, suppose the contrary, i.e. $\inf_{0\leq\theta\leq 1} \mathbb{E}g'_{\Lambda_0}(1)^\theta > \nu^{-1}$. Then we can find a $N \geq 1$ and construct a suitable "$\alpha(N)$-coupling" as described under Case C above, such that

$$\inf_{0\leq\theta\leq 1} \mathbb{E}g'_{\Lambda_0}(1)^\theta \geq \inf_{0\leq\theta\leq 1} \mathbb{E}g'_{\alpha(N),\Lambda_0}(1)^\theta > \frac{1}{\nu}.$$

Since (AsGe) is fulfilled by $\boldsymbol{BP}(\alpha(N))$, we get

$$\sup_{n\in\mathbb{N}} \mathbb{E}\mathcal{T}^*_n \geq \sup_{n\in\mathbb{N}} \mathbb{E}\mathcal{T}^*_{\alpha(N),n} > 1$$

by referring to (1.25) and by what has already be established for a BwBP with a subcritical ABPRE, i.e. $\mathbb{E}\log g'_{\Lambda_0}(1) < 0$. Thus we get the statement under the circumstance that the ABPRE is subcritical and Assumption (AsGe) is invalid.

Hence, it remains to rule out that $\mathbb{E}\log g'_{\Lambda_0}(1) \geq 0$. Assuming the latter, we find with the help of Jensen's inequality that

$$\inf_{0\leq\theta\leq 1} \log \mathbb{E}g'_{\Lambda_0}(1)^\theta \geq \inf_{0\leq\theta\leq 1} \theta\, \mathbb{E}\log g'_{\Lambda_0}(1) \geq 0$$

or, equivalently,

$$\inf_{0\leq\theta\leq 1} \mathbb{E}g'_{\Lambda_0}(1)^\theta \geq 1 > \frac{1}{\nu}$$

(which implies $\inf_{0\leq\theta\leq 1} \mathbb{E}g'_{\Lambda_0}(1)^\theta = 1$). Use once more a suitable "α-coupling" (α is not necessary to be of the $\alpha(N)$ structure) and fix α in such a way that

$$1 = \inf_{0\leq\theta\leq 1} \mathbb{E}g'_{\Lambda_0}(1)^\theta > \inf_{0\leq\theta\leq 1} \mathbb{E}g'_{\alpha,\Lambda_0}(1)^\theta > \frac{1}{\nu},$$

which implies subcriticality of the ABPRE $(Z'_{\alpha,n})_{n\geq 0}$ by taking the logarithm and using Jensen's inequality. As above, we thus arrive at the contradiction

$$\sup_{n\in\mathbb{N}} \mathbb{E}\mathcal{T}_n^* \geq \sup_{n\in\mathbb{N}} \mathbb{E}\mathcal{T}_{\alpha,n}^* > 1$$

by using the already established results for a BwBP with a subcritical ABPRE. This completes the proof of (b). □

After having proved characteristics for almost certain extinction, we turn to the mean normalized process $(\gamma^{-n}\mathcal{Z}_n)_{n\geq 0}$. As mentioned at the beginning of this subsection, this process is a martingale. We will state this in the next proposition.

Proposition 1.12. *The process $W_n := \gamma^{-n}\mathcal{Z}_n$, $n \geq 0$, is a non-negative martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$ under each $\mathbb{P}_z$, $z \in \mathbb{N}_0$, and therefore converges almost surely to an integrable random variable W. In particular $\mathbb{E}_z\mathcal{Z}_n = z\gamma^n$ for all $n \geq 0$.*

Proof. The adaptivity is clear by the definition of $(\mathcal{F}_n)_{n\geq 0}$. For $n, z \in \mathbb{N}_0$ we calculate

$$\begin{aligned}\mathbb{E}_z\left(\mathcal{Z}_{n+1} \mid \mathcal{F}_n\right) &= \sum_{v\in\mathbb{T}_n}\sum_{i=1}^{Z_v}\mathbb{E}_z\left(\sum_{u=1}^{T_v} X_{i,v}^{(u,T_v)} \mid \mathcal{F}_n\right) = \sum_{v\in\mathbb{T}_n}\sum_{i=1}^{Z_v}\mathbb{E}_z\left(\sum_{u=1}^{T_v} X_{i,v}^{(u,T_v)}\right)\\ &= \sum_{v\in\mathbb{T}_n}\sum_{i=1}^{Z_v}\mathbb{E}\left(\mathcal{Z}_1\right) = \gamma\mathcal{Z}_n \qquad \mathbb{P}_z\text{-a.s.,}\end{aligned}$$

where in the second equation the independence of $\sum_{u=1}^{T_v} X_{i,v}^{(u,T_v)}$ and $(\mathcal{F}_k)_{k\leq n}$ and in the third equation the fact that $\sum_{u=1}^{T_v} X_{i,v}^{(u,T_v)}$ is independent of z and distributed as $\mathcal{Z}_1$ (under $\mathbb{P}$) for all $v \in \mathbb{V}$ was used. This states the martingale property as well as the integrability since $\mathbb{E}_z\mathcal{Z}_n = z\gamma^n$ by iteration. The convergence then follows from the martingale convergence theorem. □

After having confirmed the martingale property of $(W_n)_{n\geq 0}$, we wonder under which conditions the almost sure limit W of this martingale is positive w.p.p. and when $(W_n)_{n\geq 0}$ is uniformly integrable. Since $(W_n)_{n\geq 0}$ is a martingale by Proposition 1.12 this is the case, when the process is L^2-bounded, i.e. $\sup_{n\geq 0}\mathbb{E}W_n^2 < \infty$, by standard martingale theory. In this context it is worthwhile to calculate the variance of $\mathcal{Z}_n$, which may be done in a straightforward but tedious computation. We therefore just state the variance in the next lemma and give the proof and the exact constants in the variance formula in Appendix A.

Lemma 1.13. *Let $\sigma^2 := \mathbb{V}ar\mathcal{Z}_1 < \infty$. Then for $n \geq 1$*

$$\mathbb{V}ar\mathcal{Z}_n = \sigma^2\gamma^{n-1}\sum_{k=0}^{n-1}\gamma^k + c\frac{\gamma^{2(n-1)}}{\tilde{\gamma}}\sum_{k=0}^{n-1}\left(\tilde{\gamma}\gamma^{-2}\right)^k\sum_{j=0}^{k-1}\left(\frac{\gamma}{\tilde{\gamma}}\right)^j$$

for a constant $0 \leq c < \infty$ and $\tilde{\gamma} := \nu\,\mathbb{E}g'_{\Lambda_0}(1)^2 = \sum_{t=1}^{\infty} p_t \sum_{u=1}^{t} \mu_{u,t}^2$.

It is immediately deduced from the lemma that the martingale $(W_n)_{n\geq 0}$ is L^2-bounded if and only if

$$\sigma^2 < \infty,\ \gamma > 1 \quad \text{and} \quad \tilde{\gamma} \leq \gamma^2. \tag{1.27}$$

Hence, parasites survive w.p.p. and $\mathbb{E}W = 1$ by standard martingale theory if these three conditions in (1.27) are fulfilled. This particularly implies $\mathbb{P}(W > 0) > 0$ which is equivalent to $\mathbb{P}_z(W > 0) > 0$ for each $z \in \mathbb{N}$, i.e. for $z \in \mathbb{N}$

$$\mathbb{P}_z(W = 0) = 1 \quad \text{iff} \quad \mathbb{P}(W = 0) = 1, \tag{1.28}$$

by a similar argumentation as the one given to prove (1.18).

Theorem 1.14. *Assuming $\mathbb{P}(\mathsf{Surv}) > 0$ and thus particularly $\gamma > 1$, the following assertions hold true:*

(a) $(W_n)_{n\geq 0}$ is L^2-bounded if and only if $\sigma^2 < \infty$ and $\tilde{\gamma} \leq \gamma^2$. In this case $(W_n)_{n\geq 0}$ is uniformly integrable and $\mathbb{P}(W > 0) > 0$ as well as $\mathbb{E}W = 1$.

(b) If $\mathbb{P}_z(W = 0) < 1$ for $z \in \mathbb{N}$, then $\mathsf{Ext} = \{W = 0\}$ $\mathbb{P}_z$-a.s.

Proof. (a) As pointed out above this theorem, $(W_n)_{n\geq 0}$ is a L^2-bounded martingale if and only if $\sigma^2 < \infty$ and $\tilde{\gamma} \leq \gamma^2$ and thus uniformly integrable. It therefore converges in L^1 to its limit W satisfying $\mathbb{E}W = 1$ as well as $\mathbb{P}(W > 0) > 0$.

(b) follows in the same manner as Theorem 1.8(b). Let $\tau_n = \inf\{m \in \mathbb{N} \mid \mathcal{T}_m^* \geq n\}$. Then we get the following inequality

$$\begin{aligned}
\mathbb{P}_z(W = 0) &\leq \mathbb{P}_z(W = 0 \mid \tau_n < \infty) + \mathbb{P}_z(\tau_n = \infty) \\
&\leq \mathbb{P}_z\left(\bigcup_{k=1}^{\mathcal{T}_{\tau_n}^*} \{\mathcal{Z}_{m,k}/\gamma^m \to 0\} \mid \tau_n < \infty\right) + \mathbb{P}_z(\tau_n = \infty) \\
&\leq \prod_{k=1}^{n} \mathbb{P}(\mathcal{Z}_{m,k}/\gamma^m \to 0) + \mathbb{P}_z(\tau_n = \infty) \\
&= \mathbb{P}(W = 0)^n + \mathbb{P}_z(\tau_n = \infty),
\end{aligned}$$

where $\mathcal{Z}_{m,k}$ are i.i.d. and distributed as $\mathcal{Z}_m$ when starting with one parasite in a single cell. Because of $\mathbb{P}(W = 0) < 1$ (by the considerations above the theorem) and Theorem 1.7 it follows

$$\mathbb{P}_z(W = 0) \leq \lim_{n\to\infty} \mathbb{P}_z(\tau_n = \infty) = \mathbb{P}_z(\mathcal{T}_n^* \text{ is bounded}) = \mathbb{P}_z(\mathsf{Ext}).$$

Since $\mathsf{Ext} \subseteq \{W = 0\}$ $\mathbb{P}_z$-a.s. the theorem is proved. □

It is remarked that no further conditions despite survival of parasites w.p.p. and $\mathbb{P}(W = 0) < 1$ are assumed in part (b) of the above theorem. In particular, $(W_n)_{n\geq 0}$ does not need to be L^2-bounded or uniformly integrable for now. However, we will see later in Section 4.1 that $\mathbb{P}(W = 0) < 1$ can only be true if and only if $(W_n)_{n\geq 0}$ is uniformly integrable. Furthermore, as in many other branching processes, L^2-boundedness is a too strict condition for uniform integrability. In the standard Galton-Watson case this leads to the famous theorem from Kesten and Stigum [14, Chapter I.10, Theorem 1] and in Theorem 4.6 we will state a similar result for the BwBP. But before proving the mentioned theorem we need to introduce a different kind of size-biased process, which is done in the next chapter, and some asymptotic properties of branching processes in random environment with immigration, which will be introduced in Chapter 3.

Chapter 2

The size-biased process

As mentioned at the beginning of Subsection 1.2.1, the *method of size-biasing* is a very powerful tool to prove limit results of various branching processes. This technique uses a change of measure on the set of (marked) trees to construct *size-biased trees*, in which a randomly picked line of descendants occurs, the *spine*. Transferring properties of the original to the size-biased process then makes it possible to show results with purely probabilistic arguments. The concept goes back to Lyons et al. [61], who used it to prove classical limit theorems for the simple GWP. It is worth mentioning that Waymire and Williams [87] developed a similar construction in a different scenery simultaneously with, and independently of Lyons et al.

The publication of this method led to a couple of articles, see e.g. [11, 27, 55, 56, 60, 67, 68], in which Kesten-Stigum-type theorems for branching processes in various settings were deduced by using adjusted versions of the new size-biasing technique. Furthermore, subcritical branching processes were studied in [2, 58] with the help of this method. Let us further mention that a slightly different construction was used by Geiger [39] to prove classical limit results for critical and subcritical GWPes.

Unlike the size-biased construction used in Subsection 1.2.1 to derive the ABPRE, which was just concentrated on the cell process, we here introduce a size-biased process $\widehat{\boldsymbol{BP}}$ of the whole BwBP by picking the spine along the parasites and not the cells. This approach gives us a connection between $\widehat{\boldsymbol{BP}}$ and a *branching process in random environment with immigration*, and with the help of these two processes, we will analyze the BwBP in the later Chapters 4 and 5. For similar size-biased constructions see [27, 54, 55].

2.1 Construction of the size-biased process

We assume throughout this chapter that the BwBP starts with at least one parasite. Let us first consider the case when starting with a single ancestor cell. For a formal description of the size-biased process $\widehat{\boldsymbol{BP}}$, let us consider the random vector $\left(\hat{X}^{(\bullet,\hat{T})}, \hat{T}, \hat{C}\right)$, where $\hat{T}$ and $\hat{C}$ are $\mathbb{N}_0$-valued and $\hat{X}^{(\bullet,\hat{T})} := (X^{(1,\hat{T})}, \ldots, X^{(\hat{T},\hat{T})})$ is a vector of the random length $\hat{T}$. The distribution of these random variables is the same under each $\mathbb{P}_z$, $z \in \mathbb{N}$, and supposed to fulfill the following:

For all $t \geq 1$

$$\mathbb{P}\left(\hat{T} = t\right) \;=\; \mathbb{P}\left(T = t\right) \frac{\sum_{u=1}^{t} \mu_{u,t}}{\gamma} \tag{2.1}$$

and for $x = (x_1, \ldots, x_t) \in \mathbb{N}_0^t$

$$\mathbb{P}\left(\hat{X}^{(\bullet,\hat{T})} = x \mid \hat{T} = t\right) \;=\; \frac{\sum_{u=1}^{t} x_u}{\sum_{u=1}^{t} \mu_{u,t}} \, \mathbb{P}\left(X^{(\bullet,t)} = x\right). \tag{2.2}$$

Furthermore, for all $t \in \mathbb{N}$, $x = (x_1, \ldots, x_t) \in \mathbb{N}_0^t$ and $1 \leq k \leq \sum_{u=1}^{t} x^{(u,t)}$

$$\mathbb{P}\left(\hat{C} = k \mid \hat{X}^{(\bullet,\hat{T})} = x, \hat{T} = t\right) \;=\; \frac{1}{\sum_{u=1}^{t} x^{(u,t)}}, \tag{2.3}$$

hence, $\hat{C}$ is uniformly distributed on $\{1, \ldots, \sum_{u=1}^{t} x_u\}$ given $\hat{X}^{(\bullet,\hat{T})} = x$ and $\hat{T} = t$. In particular,

$$\mathbb{P}\left(\hat{X}^{(\bullet,\hat{T})} = x, \hat{T} = t, \hat{C} = k\right) \;=\; \frac{p_t}{\gamma} \, \mathbb{P}\left(X^{(\bullet,t)} = x\right).$$

The random vector $\hat{X}^{(\bullet,\hat{T})}$ can be seen as a partition of the line of natural numbers into several finite random sections

$$\left\{1, \ldots, \hat{X}^{(1,\hat{T})}\right\}, \; \left\{1 + \hat{X}^{(1,\hat{T})}, \ldots, \sum_{u=1}^{2} \hat{X}^{(u,\hat{T})}\right\}, \; \ldots \;, \; \left\{1 + \sum_{u=1}^{\hat{T}-1} \hat{X}^{(u,\hat{T})}, \sum_{u=1}^{\hat{T}} \hat{X}^{(u,\hat{T})}\right\},$$

and we therefore put

$$\hat{U} := \sum_{u=1}^{\hat{T}} \mathbb{1}_{\left\{\hat{C} \leq \sum_{i=1}^{u} X^{(i,\hat{T})}\right\}}$$

as the random variable indicating in which of these random sections $\hat{C}$ is located, i.e. $\hat{U} = u$ given that $\hat{T} = t, \hat{X}^{(\bullet,t)} = (x_1, \ldots, x_t)$ and $\sum_{i=1}^{u-1} x_i < \hat{C} \leq \sum_{i=1}^{u} x_i$.

Let

$$\left(\hat{X}_n^{(\bullet,\hat{T}_n)}, \hat{T}_n, \hat{C}_n\right), \quad n \in \mathbb{N}_0,$$

be i.i.d. copies of $\left(\hat{X}^{(\bullet,\hat{T})}, \hat{T}, \hat{C}\right)$ independent of $(X_{i,v}^{(\bullet,k)})_{k\geq 1, i\geq 1, v\in\mathbb{V}}$ and $(T_v)_{v\in\mathbb{V}}$. Let further $\hat{C}_{-1}$ be independent of all other random variables and uniformly distributed on the number of ancestor parasites, i.e. for $z \in \mathbb{N}$

$$\mathbb{P}_z(\hat{C}_{-1} = y) \;=\; \frac{1}{z} \qquad \text{for each } y \in \{1, \ldots, z\}.$$

These random variables give us a random path through the parasites in the following way: As described above $\hat{C}_{-1}$ picks a parasite uniformly from all ancestor parasites which we call the *spinal root parasite*. The root cell then multiplies according to $\hat{T}_0$ with distribution (2.1). Given $\hat{T}_0 = t$, the *spine parasite* produces an offspring given by $\hat{X}_0^{(\bullet,t)}$ from which a parasite is chosen uniformly by $\hat{C}_0$ to be the spine parasite in the first generation. All other parasites in the root cell, however, reproduce according to the ordinary law $\mathcal{L}(X^{(\bullet,t)})$. Now, the cell containing the marked parasite chosen by $\hat{C}_0$, multiplies according to $\hat{T}_1$, and the spinal parasite produces an

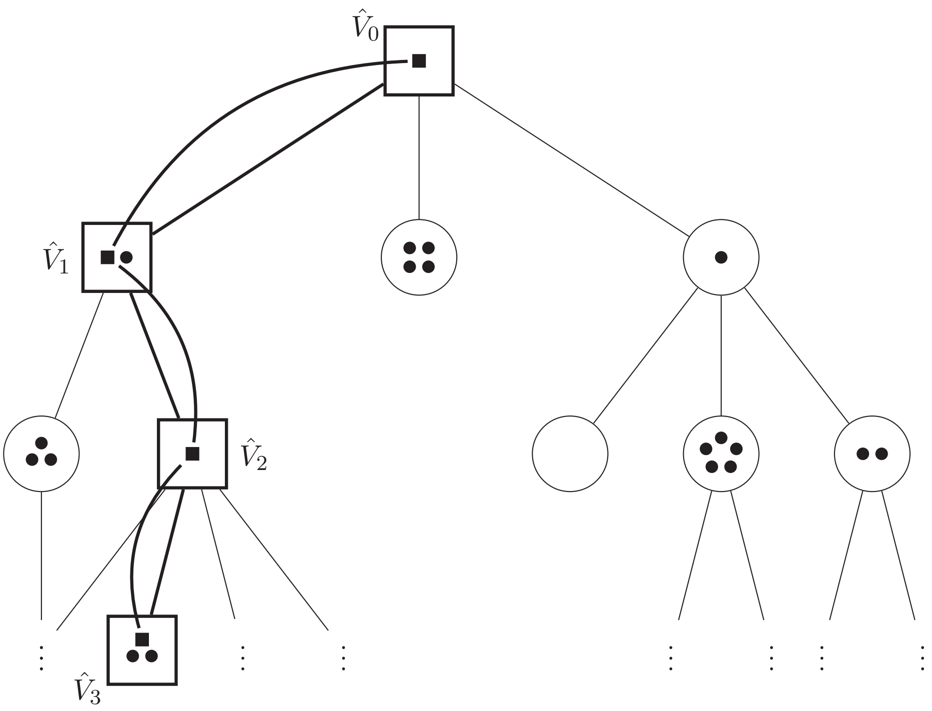

Figure 2.1: A typical realization of a size-biased BwBP

offspring given by $X_1^{(\bullet,\hat{T}_1)}$ from which the next spinal parasite is picked by $\hat{C}_1$, and all other parasites in this cell multiply with distribution $\mathcal{L}(X^{(\bullet,\hat{T}_1)})$, given $\hat{T}_1$. This routine is repeated in the next generation and so on. This procedure gives us a random cell line $(\hat{V}_n)_{n\geq 0}$ through the cell tree indicating the cells which host the spine parasite. That is, $\hat{V}_0 := \varnothing$ and

$$\hat{V}_{n+1} \ := \ \hat{V}_n \hat{U}_n$$

for $n \geq 0$, where $\hat{U}_n$ denotes the *daughter cell containing the next generation spinal parasite.* Since the $\left(\hat{X}_n^{(\bullet,\hat{T}_n)}, \hat{T}_n, \hat{C}_n\right)$, $n \in \mathbb{N}_0$, are i.i.d., so are the $\hat{U}_n$, $n \in \mathbb{N}_0$, and distributed as $\hat{U}$.

All parasites and cells not in the spine keep their known behavior. Thus, following the definitions in (1.1) and (1.5), we set $\hat{\mathsf{T}}_\varnothing = 1$ as well as $\hat{Z}_\varnothing = Z_\varnothing$. Furthermore, we put for $v \in \mathbb{V}$ with $|v| = n$ and $u \in \mathbb{N}$

$$\hat{\mathsf{T}}_{vu} \ := \ \begin{cases} \mathbb{1}_{\{u \leq \hat{T}_n\}} & \text{if } v = \hat{V}_n, \\ \mathbb{1}_{\{u \leq T_v\}} & \text{if } v \neq \hat{V}_n, \end{cases}$$

and

$$\hat{Z}_{vu} \ := \ \begin{cases} \sum_{i=1}^{\hat{Z}_v - 1} X_{i,v}^{(u,\hat{T}_n)} \ + \ \hat{X}_n^{(u,\hat{T}_n)} & \text{if } v = \hat{V}_n, \\ \sum_{i=1}^{\hat{Z}_v} X_{i,v}^{(u,T_v)} & \text{if } v \neq \hat{V}_n. \end{cases}$$

Then the *size-biased branching within branching process* $\widehat{\boldsymbol{BP}} := (\widehat{\boldsymbol{BP}}_n)_{n\geq 0}$ is given by $\widehat{\boldsymbol{BP}}_n := [\hat{\mathsf{T}}_v, \hat{Z}_v]_{|v|=n}$ for $n \in \mathbb{N}_0$. Let further $\widehat{\boldsymbol{BT}} := (\widehat{\boldsymbol{BT}}_n)_{n\geq 0}$ with $\widehat{\boldsymbol{BT}}_n = [\hat{\mathsf{T}}_v, \hat{Z}_v]_{|v|\leq n}$ denote the *size-biased branching within branching tree* and $\hat{\mathbb{T}}$, $\hat{\mathbb{T}}_n$, $\hat{T}_v$ as well as $\hat{\mathcal{Z}}_n$ and $\hat{W}_n$ the obvious, that is the analogous variables in $\widehat{\boldsymbol{BT}}$ to the ones in $\boldsymbol{BT}$.

Figure 2.1 above illustrates a realization of a size-biased branching within branching process. Spinal parasites are symboled by ■ and cells hosting these parasites by □. As in all figures, "non-spinal parasites" and usual cells are denoted by • and ○, respectively. The spine of the parasites is indicated by the bended edges. So in this particular realization we have $\hat{V}_0 = \varnothing$, $\hat{V}_1 = 1$, $\hat{V}_2 = 12$, $\hat{V}_3 = 122$ and $\hat{Z}_{\hat{V}_0} = 1$, $\hat{Z}_{\hat{V}_1} = 2$, $\hat{Z}_{\hat{V}_2} = 1$, $\hat{Z}_{\hat{V}_3} = 3$.

Taking a close look, the definition of the size-biased BwBP yields for a vertex $v \in \mathbb{V}$ of the n^{th} generation

$$\hat{T}_v = \begin{cases} \hat{T}_n & \text{if } v = \hat{V}_n, \\ T_v & \text{if } v \neq \hat{V}_n, \end{cases}$$

where $\hat{T}_v$ denotes the number of daughter cells of cell v. Moreover, we obtain that by the definition of $\widehat{\boldsymbol{BP}}$ and particularly the i.i.d. properties of the spinal random vectors $(\hat{X}_n^{(\bullet,\hat{T}_n)}, \hat{T}_n, \hat{C}_n)$, $n \in \mathbb{N}_0$, the size-biased BwBP also features a branching property. Furthermore, since "non-spinal cells" multiply as in the original BwBP, subtrees spawning from such cells behave like $\boldsymbol{BT}$. Recapitulating, let $\widehat{\boldsymbol{BT}}^{(v)} = [\hat{\mathsf{T}}_{vu}, \hat{Z}_{vu}]_{u\in\mathbb{V}}$ be the subtree of cell $v \in \mathbb{V}$, then for all $n \in \mathbb{N}_0$, $[s_w, x_w]_{|w|\leq n} \in \mathbb{S}_n$ and $\hat{V}_n = \hat{v}$ we get

$$\mathbb{P}_z\left(\left(\widehat{\boldsymbol{BT}}^{(v)}\right)_{|v|=n} \in \cdot \;\middle|\; \widehat{\boldsymbol{BT}}_n = [s_w, x_w]_{|w|\leq n}, \hat{V} = \hat{v}\right) \;=\; \left(\bigotimes_{|v|=n} \hat{Q}_{(s_v,x_v)}\right)(\cdot)$$

with $\hat{Q}_{(s_v,x_v)} = Q_{(s_v,x_v)}$ denoting the measure of $\boldsymbol{BT}$ under $\mathbb{P}_{(s_v,x_v)}$ if $v \neq \hat{v}$, and $\hat{Q}_{(1,x_{\hat{v}})}$ denoting the measure of $\widehat{\boldsymbol{BT}}$ under $\mathbb{P}_{x_{\hat{v}}}$, i.e. $Q_{(s_v,x_v)}(\cdot) = \mathbb{P}_{(s_v,x_v)}(\boldsymbol{BT} \in \cdot)$ and $\hat{Q}_{(1,x_{\hat{v}})}(\cdot) = \mathbb{P}_{x_{\hat{v}}}(\widehat{\boldsymbol{BT}} \in \cdot)$ (see Proposition 1.2 for comparison).

This size-biased BwBP $\widehat{\boldsymbol{BT}}$ can be easily extended to a size-biased process starting with multiple root cells. For that purpose, recall that S defined in (1.9) gives the configurations of the ancestor generation and $\mathbb{P}_{(t,z)}$, $(t,z) \in S$, the measures under which $\boldsymbol{BT}$ starts with configuration (t,z). The spine through $\boldsymbol{BT}$ is then constructed by picking a root parasite at random, which particularly gives the ancestor spine cell, and the process starting from this cell behaves like $\widehat{\boldsymbol{BT}}$ independent of processes starting from the other cells. However, the other BwBPes evolve in the same manner as $\boldsymbol{BT}$. More precisely, let $\hat{C}_{-1}$ be a random variable choosing a parasite uniformly from the ancestor parasites, i.e.

$$\mathbb{P}_{(t,z)}(\hat{C}_{-1} = y) \;=\; \frac{1}{\sum_{u=1}^{t} z_u}$$

for $(t,z) = (t,(z_1,\ldots,z_t)) \in S$ with $z_t > 0$ and $y \in \{1,\ldots,\sum_{u=1}^{t} z_u\}$. Then for $(t,z) = (t,(z_1,\ldots,z_t)) \in S$ the size-biased BwBP $\widehat{\boldsymbol{BT}}$ given $\hat{C}_{-1} = y$ with $\sum_{i=1}^{u-1} z_i < y \leq \sum_{i=1}^{u} z_i$ has the distribution

$$\mathbb{P}_{(t,z)}\left(\widehat{\boldsymbol{BT}} \in \cdot \mid \hat{C}_{-1} = y\right) \;=\; \left(\bigotimes_{i=1}^{t} \hat{Q}_{(1,z_i)}\right)(\cdot),$$

where $\hat{Q}_{1,z_u}(\cdot) = \mathbb{P}_{z_u}(\widehat{\boldsymbol{BT}} \in \cdot)$ and $\hat{Q}_{(1,z_v)}(\cdot) = Q_{(1,z_v)}(\cdot) = \mathbb{P}_{z_v}(\boldsymbol{BT} \in \cdot)$ for $v \neq u$.

2.2 Auxiliary results

In this section, we give helpful results for the size-biased BwBP and present the relation between $\widehat{\boldsymbol{BT}}$ and the original process $\boldsymbol{BT}$. Our investigations will end by a dichotomy between the martingale limit W and its analog $\widehat{W}$ in $\widehat{\boldsymbol{BP}}$.

The offspring distribution of parasites in a spinal cell $\hat{V}_n$, $n \geq 0$, is quite different from those in regular cells, for in this case not only the spinal parasite reproduces progeny with a different reproduction law, but also all other parasites in that cell. This is due to the fact that daughter cells are spawned according to the law given by $\hat{T}$, which in particular fulfills $\hat{T} \geq 1$ a.s., although $\mathbb{P}(T = 0) > 0$ might be possible. The next lemma provides us with some useful probabilities and especially states the reproduction distribution of a spinal cell and the parasites it contains.

Lemma 2.1. *Let* $n \in \mathbb{N}_0$.

(a) For all $v \in \mathbb{V}$ *such that* $|v| = n$ *and* $u \in \mathbb{N}$

$$\mathbb{P}\left(\hat{V}_{n+1} = vu \mid \hat{V}_n = v\right) = \mathbb{P}\left(\hat{V}_1 = u\right) = \mathbb{P}\left(\hat{U} = u\right) = \frac{\sum_{t\geq 1} p_t \mu_{u,t}}{\gamma}.$$

(b) For all $t \in \mathbb{N}$, $u \leq t$ *and* $x = (x_1, \ldots, x_t) \in \mathbb{N}_0^t$

$$\mathbb{P}\left(\hat{T}_n = t, \hat{U}_n = u, \hat{X}_n^{(\bullet,t)} = x\right) = \frac{p_t x_u}{\gamma} \mathbb{P}\left(X^{(\bullet,t)} = x\right).$$

In particular

$$\mathbb{P}\left(\hat{T}_n = t, \hat{U}_n = u\right) = \frac{p_t \mu_{u,t}}{\gamma}$$

as well as

$$\mathbb{P}\left(\hat{X}_n^{(\bullet,t)} = x \mid \hat{T}_n = t, \hat{U}_n = u\right) = \frac{x_u}{\mu_{u,t}} \mathbb{P}(X^{(\bullet,t)} = x).$$

(c) For all $t \in \mathbb{N}$, $v \leq t$, $(z_u)_{1\leq u\leq t} \in \mathbb{N}_0^t$ *and* $z \in \mathbb{N}$

$$\mathbb{P}_z\left((\hat{Z}_u)_{1\leq u\leq t} = (z_u)_{1\leq u\leq t}, \hat{T}_0 = t, \hat{U}_0 = v\right) = \frac{p_t z_v}{z\gamma} \mathbb{P}_z\left((Z_u)_{1\leq u\leq t} = (z_u)_{1\leq u\leq t}\right).$$

In particular,

$$\mathbb{P}_z\left((\hat{Z}_u)_{1\leq u\leq t} = (z_u)_{1\leq u\leq t}, \hat{T}_0 = t\right) = \frac{p_t \sum_{u=1}^t z_u}{z\gamma} \mathbb{P}_z\left((Z_u)_{1\leq u\leq t} = (z_u)_{1\leq u\leq t}\right).$$

Proof. (*a*) Since the $\hat{U}_n$, $n \geq 0$, are i.i.d. and distributed as $\hat{V}_1$, we get from the branching property

$$\mathbb{P}\left(\hat{V}_{n+1} = vu \mid \hat{V}_n = v\right) = \mathbb{P}\left(\hat{U}_n = u\right) = \mathbb{P}\left(\hat{V}_1 = u\right)$$

and further

$$\mathbb{P}\left(\hat{V}_1 = u\right) = \sum_{s\geq u} \sum_{x_1,\ldots,x_s=1}^{\infty} \sum_{k=\sum_{i=1}^{u-1} x_i+1}^{\sum_{i=1}^{u} x_i} \mathbb{P}\left(\hat{X}_0^{(1,s)} = x_1, \ldots, \hat{X}_0^{(s,s)} = x_s, \hat{T}_0 = u, \hat{C}_0 = k\right)$$

$$\begin{aligned}
&= \sum_{s\geq u}\sum_{x_1,\dots,x_s=1}^{\infty}\sum_{k=\sum_{i=1}^{u-1}x_i+1}^{\sum_{i=1}^{u}x_i}\frac{p_s}{\gamma}\mathbb{P}\left(X^{(1,s)}=x_1,\dots,X^{(s,s)}=x_s\right)\\
&= \sum_{s\geq u}\frac{p_s}{\gamma}\sum_{x_1,\dots,x_s=1}^{\infty}x_u\mathbb{P}\left(X^{(1,s)}=x_1,\dots,X^{(s,s)}=x_s\right)\\
&= \frac{1}{\gamma}\sum_{s\geq u}p_s\mathbb{E}X^{(u,s)}.
\end{aligned}$$

(b) follows with a similar calculation as in (a). For $t\in\mathbb{N}$, $u\leq t$ and $x=(x_1,\dots,x_t)\in\mathbb{N}_0^t$, obtain that

$$\begin{aligned}
\mathbb{P}\left(\hat{X}_n^{(\bullet,t)}=x,\hat{T}_n=t,\hat{U}_n=u\right) &= \sum_{k=\sum_{i=1}^{u-1}x_i+1}^{\sum_{i=1}^{u}x_i}\mathbb{P}\left(\hat{X}_n^{(\bullet,t)}=x,\hat{T}_n=t,\hat{C}_n=k\right)\\
&= \frac{p_tx_u}{\gamma}\mathbb{P}\left(X^{(\bullet,t)}=x\right).
\end{aligned}$$

The second equation then follows by summation over all x and the third by dividing the two just discovered relations.

(c) Let $t\in\mathbb{N}$, $v\leq t$, $(z_u)_{1\leq u\leq t}\in\mathbb{N}_0^t$ and $z\in\mathbb{N}$. Then due to the independence of $\hat{X}^{(\bullet,\hat{T}_0)}$ and $(X_{i,v}^{(\bullet,k)})_{k\geq1,i\geq1,v\in\mathbb{V}}$

$$\begin{aligned}
&\mathbb{P}_z\left((\hat{Z}_u)_{1\leq u\leq t}=(z_u)_{1\leq u\leq t},\hat{T}_0=t,\hat{U}_0=v\right)\\
&= \mathbb{P}_z\left(\left(\sum_{i=1}^{z-1}X_{i,\varnothing}^{(u,t)}\ +\ \hat{X}_0^{(u,t)}\right)_{1\leq u\leq t}=(z_u)_{1\leq u\leq t},\hat{T}_0=t,\hat{U}_0=v\right)\\
&= \sum_{x_u\leq z_u}\mathbb{P}\left(\hat{X}_0^{(\bullet,t)}=(x_1,\dots,x_t),\hat{T}_0=t,\hat{U}_0=v\right)\mathbb{P}\left(\left(\sum_{i=1}^{z-1}X_{i,\varnothing}^{(u,t)}\right)_{1\leq u\leq t}=(z_u-x_u)_{1\leq u\leq t}\right)\\
&= \frac{p_t}{\gamma}\sum_{x_u\leq z_u}x_v\mathbb{P}\left(X^{(\bullet,t)}=(x_1,\dots,x_t)\right)\mathbb{P}\left(\left(\sum_{i=1}^{z-1}X_{i,\varnothing}^{(u,t)}\right)_{1\leq u\leq t}=(z_u-x_u)_{1\leq u\leq t}\right)\\
&= \frac{p_t}{\gamma}\sum_{x_u\leq z_u}x_v\mathbb{P}\left(\left(\sum_{i=1}^{z}X_{i,\varnothing}^{(u,t)}\right)_{1\leq u\leq t}=(z_u)_{1\leq u\leq t},\left(X_{z,\varnothing}^{(u,t)}\right)_{1\leq u\leq t}=(x_u)_{1\leq u\leq t}\right)\\
&= \frac{p_t}{\gamma}\mathbb{E}\left(X_{z,\varnothing}^{(v,t)}\ \middle|\ \left(\sum_{i=1}^{z}X_{i,\varnothing}^{(u,t)}\right)_{1\leq u\leq t}=(z_u)_{1\leq u\leq t}\right)\mathbb{P}_z\left((Z_u)_{1\leq u\leq t}=(z_u)_{1\leq u\leq t}\right).
\end{aligned}$$

Because a random walk $(S_n)_{n\geq0}$ with $S_0=0$ and i.i.d. increments $(X_n)_{n\geq1}$ satisfies $\mathbb{E}(X_1|S_n)=S_n/n$ a.s., we conclude for our above equation

$$\mathbb{P}_z\left((\hat{Z}_u)_{1\leq u\leq t}=(z_u)_{1\leq u\leq t},\hat{T}_0=t,\hat{U}_0=v\right)\ =\ \frac{p_tz_v}{\gamma z}\mathbb{P}_z\left((Z_u)_{1\leq u\leq t}=(z_u)_{1\leq u\leq t}\right).$$

Summation over $|v|=1$ completes the proof of (c). □

Before proceeding to the lemma comparing the distributions of the size-biased and original BwBP, recall from Subsection 1.1.2 that $(\mathbb{S},\mathcal{S})$ denotes the space of host parasite trees and $\mathcal{S}_n \subseteq \mathcal{S}$ is the σ-algebra generated by the projections on the first n levels of the trees. Furthermore, $(\mathbb{S}_n,\mathcal{S}_{|n})$ for $n \in \mathbb{N}_0$ denotes the space of the finite host-parasite trees up to generation n. In particular, $\widehat{\boldsymbol{BT}}$ and $\widehat{\boldsymbol{BT}}_n$ are $\mathcal{S}$ and $\mathcal{S}_{|n}$-measurable, respectively. Furthermore, the mapping $z_n : \mathbb{S} \to \mathbb{N}_0$ describes the number of parasites in the n^{th} generation of a host-parasite tree, thence $\hat{\mathcal{Z}}_n = z_n(\widehat{\boldsymbol{BT}})$ for each $n \in \mathbb{N}_0$.

For each $x \in \mathbb{N}$, we define the measures

$$\hat{Q}_x(\cdot) := \mathbb{P}_x\left(\widehat{\boldsymbol{BT}} \in \cdot\right) \quad \text{and} \quad Q_x(\cdot) := \mathbb{P}_x\left(\boldsymbol{BT} \in \cdot\right) \tag{2.4}$$

on $(\mathbb{S},\mathcal{S})$, and the subsequent lemma provides us with a relation between $\hat{Q}_x$ and Q_x. As a matter of fact, size biasing has the effect that for each $n \in \mathbb{N}_0$ the restricted probability measure $\hat{Q}_{x|\mathcal{S}_n}$ is dominated by $Q_{x|\mathcal{S}_n}$. The corresponding Radon-Nikodym densities can be determined as the mappings

$$w_n : \mathbb{S} \to [0,\infty), \quad w_n(\tau) = \frac{1}{\gamma^n} z_n(\tau) \tag{2.5}$$

for $n \in \mathbb{N}_0$. We further put $w := \limsup_{n\to\infty} w_n$. Thus w_n is $\mathcal{S}_n$-measurable by definition and we have the representations

$$W_n = w_n \circ \boldsymbol{BT}_n \quad \text{and} \quad \hat{W}_n = w_n \circ \widehat{\boldsymbol{BT}}_n.$$

As a consequence of the following lemma, the question of uniform integrability of $(W_n)_{n\geq 0}$ is transfered into the one of almost sure finiteness of $\hat{W}$. This forms the final part of the lemma and utilizes a measure theoretic result due to Durrett [36].

Lemma 2.2.

(a) For all $n \in \mathbb{N}_0$, $[t_v, z_v]_{|v|\leq n} \in \mathbb{S}_n$, $u \in \mathbb{V}$ with $|u| = n$ and $x \in \mathbb{N}$

$$\mathbb{P}_x\left(\widehat{\boldsymbol{BT}}_n = [t_v, z_v]_{|v|\leq n}, \hat{V}_n = u\right) = \frac{z_u}{\gamma^n x}\,\mathbb{P}_x\left(\boldsymbol{BT}_n = [t_v, z_v]_{|v|\leq n}\right).$$

(b) For all $x \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $A \in \mathcal{S}_n$

$$\hat{Q}_x(A) = \frac{1}{x}\mathbb{E}_x\left(W_n \mathbb{1}_{\{\boldsymbol{BT}\in A\}}\right) = \int_A \frac{w_n(\tau)}{x} Q_x(d\tau).$$

In particular, $\hat{Q}_{x|\mathcal{S}_n} \ll Q_{x|\mathcal{S}_n}$ for all $n \in \mathbb{N}_0$.

(c) We have the dichotomy

(i) $\mathbb{P}_x(\hat{W} < \infty) = 1 \quad \Leftrightarrow \quad \mathbb{E}_x W = x$,

(ii) $\mathbb{P}_x(\hat{W} = \infty) = 1 \quad \Leftrightarrow \quad \mathbb{P}_x(W = 0) = 1$.

Proof. Fix $x \in \mathbb{N}$. (a) In the case $n = 0$ the statement is obviously true by the definitions of the random variables. Let $n \in \mathbb{N}$ and $u = u'u_n$ for a vertex u' with $|u'| = n - 1$ and $u_n \in \mathbb{N}$. Then by induction, the branching property and Lemma 2.1 we get for each $[t_v, z_v]_{|v|\le n} \in \mathbb{S}_n$

$$\begin{aligned}
&\mathbb{P}_x\left(\widehat{\boldsymbol{BT}}_n = [t_v, z_v]_{|v|\le n}, \hat{V}_n = u\right)\\
&= \mathbb{P}_x\left(\widehat{\boldsymbol{BT}}_{n-1} = [t_v, z_v]_{|v|\le n-1}, \hat{V}_{n-1} = u'\right)\\
&\qquad \cdot \mathbb{P}_x\left([\hat{\mathsf{T}}_v, \hat{Z}_v]_{|v|=n} = [t_v, z_v]_{|v|=n}, \hat{V}_n = u \;\middle|\; \widehat{\boldsymbol{BT}}_{n-1} = [t_v, z_v]_{|v|\le n-1}, \hat{V}_{n-1} = u'\right)\\
&= \frac{z_{u'}}{\gamma^{n-1}x}\mathbb{P}_x\left(\boldsymbol{BT}_{n-1} = [t_v, z_v]_{|v|\le n-1}\right) \prod_{|v|=n-1, v\ne u'} \mathbb{P}_{(t_v,z_v)}\left([\mathsf{T}_{v'}, Z_{v'}]_{v'\in\mathbb{N}} = [t_{vv'}, z_{vv'}]_{v'\in\mathbb{N}}\right)\\
&\qquad \cdot \mathbb{P}_{z_{u'}}\left([\hat{\mathsf{T}}_{v'}, \hat{Z}_{v'}]_{v'\in\mathbb{N}} = [t_{u'v'}, z_{u'v'}]_{v'\in\mathbb{N}}, \hat{V}_1 = u_n\right)\\
&= \frac{z_{u'}}{\gamma^{n-1}x}\mathbb{P}_x\left(\boldsymbol{BT}_{n-1} = [t_v, z_v]_{|v|\le n-1}\right) \prod_{|v|=n-1, v\ne u'} \mathbb{P}_{(t_v,z_v)}\left(\boldsymbol{BP}_1 = [t_{vv'}, z_{vv'}]_{v'\in\mathbb{N}}\right)\\
&\qquad \cdot \frac{z_u}{z_{u'}\gamma}\mathbb{P}_{z_{u'}}\left(\boldsymbol{BP}_1 = [t_{u'v'}, z_{u'v'}]_{v'\in\mathbb{N}}\right)\\
&= \frac{z_u}{\gamma^n x}\mathbb{P}_x\left(\boldsymbol{BT}_n = [t_v, z_v]_{|v|\le n}\right)
\end{aligned}$$

(b) Summation over all u with $|u| = n$ in (a) yields for all $[t_v, z_v]_{|v|\le n} \in \mathbb{S}_n$

$$\mathbb{P}_x\left(\widehat{\boldsymbol{BT}}_n = [t_v, z_v]_{|v|\le n}\right) = \frac{\sum_{|u|=n} z_u}{\gamma^n x}\mathbb{P}_x\left(\boldsymbol{BT}_n = [t_v, z_v]_{|v|\le n}\right).$$

Thus, by an appeal to (1.6) we infer for all $A \in \mathcal{S}_n$

$$\begin{aligned}
\hat{Q}_x(A) &= \mathbb{P}_x(\widehat{\boldsymbol{BT}} \in A) = \mathbb{P}_x(\widehat{\boldsymbol{BT}}_n \in tr_{|n}(A))\\
&= \int_{tr_{|n}(A)} \mathbb{P}_x\left(\widehat{\boldsymbol{BT}}_n \in d[t_v, z_v]_{|v|\le n}\right)\\
&= \int_{tr_{|n}(A)} \frac{\sum_{|u|=n} z_u}{\gamma^n x}\mathbb{P}_x\left(\boldsymbol{BT}_n \in d[t_v, z_v]_{|v|\le n}\right)\\
&= \int_A \frac{\sum_{|u|=n} z_u}{\gamma^n x}\mathbb{P}_x\left(\boldsymbol{BT} \in d[t_v, z_v]_{v\in\mathbb{V}}\right)\\
&= \int_A \frac{w_n}{x} dQ_x = \frac{1}{x}\mathbb{E}_x\left(W_n \mathbb{1}_{\{\boldsymbol{BT}\in A\}}\right).
\end{aligned}$$

(c) Part (b) and [36, Theorem 5.3.3] imply for all $A \in \mathcal{S}$

$$\hat{Q}_x(A) = \int_A \frac{w}{x}\, dQ_x + \hat{Q}_x\left(A \cap \{w = \infty\}\right),$$

which leads to

$$\frac{1}{x}\mathbb{E}_x W = \int_{\mathbb{S}} \frac{w}{x}\, dQ_x = 1 - \hat{Q}_x\left(w = \infty\right).$$

This ensures the dichotomy stated in (c). □

Remark 2.3. *(a)* Lemma 2.2(*b*) can be easily extended by applying measurable functions on $\widehat{\boldsymbol{BT}}$ and $\boldsymbol{BT}$. So let $h : (\mathbb{S}, \mathcal{S}) \to (\mathbb{X}, \mathcal{X})$ be a measurable function, $B \in \mathcal{X}$ such that $h^{-1}(B) \in \mathcal{S}_n$ for a $n \in \mathbb{N}_0$. Then

$$\begin{aligned} \mathbb{P}\left(h(\widehat{\boldsymbol{BT}}) \in B\right) &= \mathbb{P}\left(\widehat{\boldsymbol{BT}} \in h^{-1}(B)\right) \\ &= \mathbb{E}\left(W_n \mathbb{1}_{\{\boldsymbol{BT} \in h^{-1}(B)\}}\right) = \mathbb{E}\left(W_n \mathbb{1}_{\{h(\boldsymbol{BT}) \in B\}}\right) \end{aligned}$$

for a $\mathcal{S}_n$-$\mathcal{X}$-measurable function h.

(b) The change of measure relation in Lemma 2.2(*b*) is not restricted to $\mathbb{P}_x$, $x \in \mathbb{N}$, and can be formulated in the same manner and proved for a BwBP with multiple ancestor cells and parasites. That is, for $n \in \mathbb{N}_0$ and $(t, z) \in S$ with $z_t > 0$

$$\mathbb{P}_{(t,z)}\left(\widehat{\boldsymbol{BT}} \in A\right) = \frac{1}{\sum_{i=1}^{z} z_i} \mathbb{E}_{(t,z)}\left(W_n \mathbb{1}_{\{\boldsymbol{BT} \in A\}}\right) \quad \text{for } A \in \mathcal{S}_n^t.$$

This follows directly from the branching property and Lemma 2.2. Let $(t, z) \in S$ with $z = (z_1, \ldots, z_t)$ and $z_t > 0$. Then for all $A^{(i)} \in \mathcal{S}$, $1 \leq i \leq t$ we get

$$\begin{aligned} \mathbb{P}_{(t,z)}\left(\widehat{\boldsymbol{BT}} \in \times_{i=1}^{t} A^{(i)}\right) &= \frac{1}{\sum_{i=1}^{z} z_i} \sum_{j=1}^{t} z_j \mathbb{P}_{z_j}\left(\widehat{\boldsymbol{BT}} \in A^{(j)}\right) \prod_{j \neq k \leq t} \mathbb{P}_{z_k}\left(\boldsymbol{BT} \in A^{(k)}\right) \\ = \frac{1}{\sum_{i=1}^{z} z_i} \sum_{j=1}^{t} \mathbb{E}_{z_j}\left(W_n \mathbb{1}_{\{\boldsymbol{BT} \in A^{(j)}\}}\right) &\prod_{j \neq k \leq t} \mathbb{P}_{z_k}\left(\boldsymbol{BT} \in A^{(k)}\right) \\ = \frac{1}{\sum_{i=1}^{z} z_i} \sum_{j=1}^{t} \mathbb{E}_{(t,z)}\left(W_n^{(j)} \mathbb{1}_{\left\{\boldsymbol{BT} \in \times_{i=1}^{t} A^{(j)}\right\}}\right) &= \frac{1}{\sum_{i=1}^{z} z_i} \mathbb{E}_{(t,z)}\left(W_n \mathbb{1}_{\left\{\boldsymbol{BT} \in \times_{i=1}^{t} A^{(j)}\right\}}\right). \end{aligned}$$

Of course, the remarks made in (*a*) also apply for the multiple rooted BwBP.

(c) Is h real valued and additionally non-negative or bounded, it can be integrated with respect to $\hat{Q}$ and the resulting integral is

$$\mathbb{E}\left(h(\widehat{\boldsymbol{BT}})\right) = \mathbb{E}\left(W_n h(\boldsymbol{BT})\right).$$

Putting $h(\cdot) = \log(w_1(\cdot))$, then h is $\mathcal{S}_1$-$\mathbb{B}_{[0,\infty)}$-measurable and we deduce

$$\mathbb{E}\left(h(\widehat{\boldsymbol{BT}})\right) = \mathbb{E} \log \hat{W}_1 = \mathbb{E}\left(W_1 \log W_1\right).$$

Hence, the $(Z \log Z)$-condition, that is $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 < \infty$, is transfered to an integrability condition in the size-biased process, namely $\mathbb{E} \log \hat{\mathcal{Z}}_1 < \infty$.

2.3 Connection to a branching process in random environment with immigration

The size-biased process has a connection to a branching process with an immigrational component. More precisely, the parasite process along the spinal cells behaves like a *branching* *process in random environment with immigration (BPREI)*. Because of this connection, we will use the next chapter to specify such a branching process and prove limiting results. Shortly speaking, the individuals of a BPREI behave as the ones in a BPRE, but in each generation new individuals immigrate into the population. The randomly picked immigration component and offspring distribution of each generation are dependent and are given by the environmental sequence. However, at this point we would like to skip a detailed description of a BPREI and refer to Chapter 3 for a precise definition and bibliographical references.

To verify the just mentioned behavior of the *process of parasites along the spinal cells* $(\hat{Z}_{\hat{V}_n})_{n\geq 0}$, we take a look at its recursive formula. For $n \geq 0$ we find that

$$\hat{Z}_{\hat{V}_{n+1}} = \sum_{i=1}^{\hat{Z}_{\hat{V}_{n-1}}-1} X_{i,\hat{V}_n}^{(\hat{U}_n,\hat{T}_n)} + \hat{X}_n^{(\hat{U}_n,\hat{T}_n)}. \tag{2.6}$$

Thus, all but one parasite in a spinal cell multiply with the same distribution while the last one, the *spinal parasite*, produces offspring according to a size-biased law. Imagine the spinal parasite to be outside the cell. Then all remaining $\hat{Z}_{\hat{V}_n} - 1$ parasites in the cell reproduce with the same offspring distribution and the progeny from the spinal parasite immigrates into cell $\hat{V}_{n+1}$ of the next generation. All offspring combined forms the new parasite population hosted in the spinal cell. By repeating the same procedure just described for all generations, namely assuming the spinal parasite to proliferate outside the cell and its children to immigrate into the next generation cell, it is justifiable to claim a branching behavior in random environment with immigration of $(\hat{Z}_{\hat{V}_n})_{n\geq 0}$.

Theorem 2.4. *Let $(\hat{Z}'_n)_{n\geq 0}$ be a BPREI with an i.i.d. environmental sequence $\Delta = (\Delta_n)_{n\geq 0}$ taking values in $\{\mathcal{L}((X^{(u,t)}, \hat{X}^{(u,t)} - 1)|(\hat{U}, \hat{T}) = (u,t)) : 1 \leq u \leq t < \infty\}$ such that*

$$\mathbb{P}\left(\Delta_0 = \mathcal{L}\big((X^{(u,t)}, \hat{X}^{(u,t)} - 1)|(\hat{U}, \hat{T}) = (u,t)\big)\right) = \frac{p_t \mu_{u,t}}{\gamma}$$

for all $1 \leq u \leq t < \infty$. Then the distribution of the process $(\hat{Z}_{\hat{V}_n} - 1)_{n\geq 0}$ starting with $z \in \mathbb{N}$ parasites equals the law of the BPREI $(\hat{Z}'_n)_{n\geq 0}$ starting with the same number of individuals.

Proof. Let $z \in \mathbb{N}$. Obviously, the assertion holds for $n = 0$ since both processes start with the same number of ancestors. By the definition of $(\hat{Z}'_n)_{n\geq 0}$, Equation (3.2) in the next chapter yields

$$\begin{aligned} \mathbb{E}_z\left(s^{\hat{Z}'_{n+1}} \mid \hat{Z}'_n = y, \Delta_n = \mathcal{L}\big((X^{(u,t)}, \hat{X}^{(u,t)} - 1)|(\hat{U}, \hat{T}) = (u,t)\big)\right) \\ = \mathbb{E}\left(s^{X^{(u,t)}}\right)^y \mathbb{E}\left(s^{\hat{X}^{(u,t)}-1} \mid (\hat{U}, \hat{T}) = (u,t)\right) \end{aligned}$$

for all $n, y \in \mathbb{N}_0$ and $1 \leq u \leq t < \infty$. But (2.6) and the i.i.d. assumption of the engaged random variables imply

$$\mathbb{E}_z\left(s^{\hat{Z}_{\hat{V}_{n+1}}-1} \mid \hat{Z}_{\hat{V}_n} - 1 = y, (\hat{U}_n, \hat{T}_n) = (u,t)\right) = \mathbb{E}_z\left(s^{\sum_{i=1}^{y} X_{i,\varnothing}^{(u,t)} + \hat{X}_n^{(u,t)} - 1} \mid (\hat{U}_n, \hat{T}_n) = (u,t)\right)$$

$$= \mathbb{E}\left(s^{X^{(u,t)}}\right)^y \mathbb{E}\left(s^{\hat{X}^{(u,t)}-1} \mid (\hat{U},\hat{T}) = (u,t)\right)$$

for all $1 \leq u \leq t < \infty$, and thus

$$\mathbb{E}_z\left(s^{\hat{Z}_{\hat{V}_{n+1}}-1} \mid \hat{Z}_{\hat{V}_n} - 1 = y\right) = \mathbb{E}_z\left(s^{\hat{Z}'_{n+1}} \mid \hat{Z}'_n = y\right)$$

for all $n \in \mathbb{N}_0$ and $y \in \mathbb{N}_0$. Since the process of parasites along a cell line is Markovian, we infer by induction

$$\begin{aligned}
&\mathbb{P}_z\left(\hat{Z}_{\hat{V}_1} - 1 = y_1, \ldots, \hat{Z}_{\hat{V}_n} - 1 = y_n\right) \\
&= \mathbb{P}_z\left(\hat{Z}_{\hat{V}_1} - 1 = y_1, \ldots, \hat{Z}_{\hat{V}_{n-1}} - 1 = y_{n-1}\right) \mathbb{P}\left(\hat{Z}_{\hat{V}_n} - 1 = y_n \mid \hat{Z}_{\hat{V}_{n-1}} - 1 = y_{n-1}\right) \\
&= \mathbb{P}_z\left(\hat{Z}'_1 = y_1, \ldots, \hat{Z}'_{n-1} = y_{n-1}\right) \mathbb{P}\left(\hat{Z}'_n = y_n \mid \hat{Z}'_{n-1} = y_{n-1}\right) \\
&= \mathbb{P}_z\left(\hat{Z}'_1 = y_1, \ldots, \hat{Z}'_n = y_n\right)
\end{aligned}$$

for all $n \in \mathbb{N}$ and $y_1, \ldots, y_n \in \mathbb{N}_0$, which finishes the proof. □

We call a BPREI $(\hat{Z}'_n)_{n\geq 0}$ with an i.i.d. environmental sequence Δ as introduced in the previous theorem an *associated branching process in random environment with immigration (ABPREI)* and denote by

$$\hat{g}_{\Delta_n}(s) = \sum_{u \leq t} \mathbb{E}\left(s^{X^{(u,t)}}\right) \mathbb{1}_{\left\{\Delta_n = \mathcal{L}((X^{(u,t)}, \hat{X}^{(u,t)}-1) | (\hat{U},\hat{T})=(u,t))\right\}}$$

the generating function of the first marginal distribution given by Δ_n. In the context of a branching process in random environment without immigration, we categorize the behavior of the ABPREI and thus of $(\hat{Z}_{\hat{V}_n} - 1)_{n\geq 0}$ in three different cases depending on the reproduction law of the parasites in the spinal cells. We call this process *supercritical, critical* or *subcritical* if $\mathbb{E} \log \hat{g}'_{\Delta_0}(1) > 0, = 0$ or < 0, respectively.

Remark 2.5. There is a highly connection in the behavior between the ABPRE and the ABPREI. Namely, if $\mu_{u,t} \neq 1$ for at least one $1 \leq u \leq t < \infty$ satisfying $p_t > 0$, then

$$\text{ABPREI} \begin{cases} \textit{subcritical}, \\ \textit{critical}, \\ \textit{supercritical}, \end{cases} \quad \text{iff} \quad \text{ABPRE} \begin{cases} \textit{strongly subcritical}, \\ \textit{intermediate subcritical}, \\ \textit{weakly subcritical or non-subcritical}. \end{cases}$$

This can be easily seen by exploiting the equation

$$\mathbb{E} \log \hat{g}'_{\Delta_0}(1) = \sum_{1 \leq u \leq t < \infty} \frac{p_t \mu_{u,t}}{\gamma} \log \mu_{u,t} = \frac{\nu}{\gamma} \mathbb{E} g'_{\Lambda_0}(1) \log g'_{\Lambda_0}(1), \tag{2.7}$$

which can be derived from Subsection 1.2.1 and the definition of the generating function $g_{\Lambda_0}(s)$. Since the function $x \mapsto x \log x$ is strictly convex and $g'_{\Lambda_0}(1) \neq 1$ w.p.p., Jensen's inequality provides

$$\mathbb{E} g'_{\Lambda_0}(1) \log g'_{\Lambda_0}(1) > \mathbb{E} g'_{\Lambda_0}(1) \log \mathbb{E} g'_{\Lambda_0}(1) > \mathbb{E} g'_{\Lambda_0}(1) \mathbb{E} \log g'_{\Lambda_0}(1),$$

and in combination with (2.7) the assertion follows.

Chapter 3

The branching process in random environment with immigration

At the end of the previous chapter, we have seen in Theorem 2.4 that the process of parasites along the spinal cell line $(\hat{Z}_{\hat{V}_n})_{n\geq 0}$ can be interpreted as a branching process in a random environment with immigration. Since the behavior of $(\hat{Z}_{\hat{V}_n})_{n\geq 0}$ will play a crucial role in our further analysis, we dedicate this chapter to the formal introduction and study of the *branching process in random environment with immigration (BPREI)*. In contrast to the BPRE (see Subsection 1.1.3) an immigration component is added in the BPREI model. This means that in each generation new individuals may immigrate into the population according to a law which depends on the environmental sequence.

Galton-Watson processes with immigration but without random environment have been already studied in various articles. See [43, 44, 69, 70, 78–80] for the most important results, and we refer to the books [10, 62] for probabilistic proofs. Key [49] and Roitershtein [72] then added random environments and considered a multi-type setting, and they proved limiting results in the subcritical case. Recently, Bansaye in [16] showed theorems for the single-type process in random environment with immigration for all three (supercritical, critical and subcritical) regimes. However, the results stated in the mentioned articles are not sufficient for our later analysis, and thus we need to formulate finer results resp. present some new findings in the different regimes, especially in the supercritical case.

3.1 The model

The environmental sequence $\mathcal{U} = (\mathcal{U}_n)_{n\geq 0}$ consists of i.i.d. random variables taking values in the set

$$\mathbb{M} = \left\{ (p_{ij})_{i,j\geq 0} \in [0,1]^{\mathbb{N}_0^2} \;\middle|\; \sum_{i,j\geq 0} p_{ij} = 1,\ \sum_{i\geq 0} i \sum_{j\geq 0} p_{ij} < \infty \right\}$$

of probability measures on $\mathbb{N}_0^2$ such that the first marginal has finite mean. $\mathbb{M}$ is endowed with the σ-algebra $\mathcal{M}$ generated by the usual topology induced by the metric of the total variation distance. We denote by $p_{ij}(\mathcal{U}_n)$ the random probability of $(i,j) \in \mathbb{N}_0^2$ of the probability measure given by $\mathcal{U}_n$, $n \in \mathbb{N}_0$. Furthermore, we write

$$p_{i\star}(\mathcal{U}_n) := \sum_{j=1}^{\infty} p_{ij}(\mathcal{U}_n) \quad \text{and} \quad p_{\star j}(\mathcal{U}_n) := \sum_{i=1}^{\infty} p_{ij}(\mathcal{U}_n)$$

for the marginal probabilities of the elementary events $i, j \in \mathbb{N}_0$.

Let $(X_{i,n})_{i\geq 1, n\geq 0}$ and $(\xi_n)_{n\geq 0}$ be $\mathbb{N}_0$-valued random variables all independent conditioned under $\mathcal{U}$ with the following distributions: For all $i \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $x, z \in \mathbb{N}_0$

$$\mathbb{P}\left(X_{i,n} = x \mid \mathcal{U}\right) \;=\; p_{x\star}(\mathcal{U}_n) \quad \text{and} \quad \mathbb{P}\left(\xi_n = z \mid \mathcal{U}\right) \;=\; p_{\star z}(\mathcal{U}_n) \quad \text{a.s.}$$

The *branching process in random environment with immigration (BPREI)* $(Z_n)_{n\geq 0}$ with environmental sequence $\mathcal{U}$ is then defined as $Z_0 = 0$ a.s. and for $n \in \mathbb{N}_0$ recursively

$$Z_{n+1} \;=\; \sum_{i=1}^{Z_n} X_{i,n} \;+\; \xi_n. \tag{3.1}$$

Here, the $(X_{i,n})_{i\geq 1, n\geq 0}$ describe the offspring of the individuals at generation n and the sequence $(\xi_n)_{n\geq 0}$ gives the number of individuals entering the population in the several generations. Both, the reproduction law of the population and the one of the immigrational component, are picked at random and may therefore differ in each generation. Since the $(X_{i,n})_{i\geq 1, n\geq 0}$ and $(\xi_n)_{n\geq 0}$ are independent conditioned under the i.i.d. sequence $\mathcal{U}$, we infer the independence of Z_n and $(X_{i,m}, \xi_m)_{m\geq n}$ for all $n \geq 0$. This in turn ensures the *Markov property* for $(Z_n)_{n\geq 0}$.

Let $g_{1,\mathcal{U}_n}(s)$ and $g_{2,\mathcal{U}_n}(s)$ for $s \in [0,1]$ and $n \in \mathbb{N}_0$ be the generating functions of the marginal distributions of $\mathcal{U}_n$, i.e. for a distribution $\bar{p} = (p_{ij})_{i,j\geq 0} \in \mathbb{M}$

$$g_{1,\bar{p}}(s) = \sum_{i=0}^{\infty} s^i \sum_{j=0}^{\infty} p_{ij} = \mathbb{E}\left(s^{X_{1,n}} | \mathcal{U}_n = \bar{p}\right) \quad \text{and} \quad g_{2,\bar{p}}(s) = \sum_{j=0}^{\infty} s^j \sum_{i=0}^{\infty} p_{ij} = \mathbb{E}\left(s^{\xi_n} | \mathcal{U}_n = \bar{p}\right).$$

With this notation, the generating function of Z_{n+1}, $n \in \mathbb{N}_0$, can be represented by

$$\mathbb{E}\left(s^{Z_{n+1}} \;\middle|\; Z_0, \ldots, Z_n, \mathcal{U}\right) \;=\; \mathbb{E}\left(s^{Z_{n+1}} \;\middle|\; Z_n, \mathcal{U}_n\right) \;=\; g_{2,\mathcal{U}_n}(s) g_{1,\mathcal{U}_n}(s)^{Z_n} \quad \text{a.s.} \tag{3.2}$$

under usage of the recursive formula (3.1) and the independence assumptions made in this model. We further denote by

$$\mu_{\mathcal{U}_n} \;:=\; g'_{1,\mathcal{U}_n}(1) = \mathbb{E}\left(X_{1,n} \mid \mathcal{U}_n\right)$$

the mean of the first marginal distribution of $\mathcal{U}_n$. As in the branching in random environment without immigration setting, we consider three different cases, namely the *supercritical case* ($\mathbb{E} \log \mu_{\mathcal{U}_0} > 0$), the *critical* case ($\mathbb{E} \log \mu_{\mathcal{U}_0} = 0$) and the *subcritical* case ($\mathbb{E} \log \mu_{\mathcal{U}_0} < 0$). Before stating asymptotic results in each of the three regimes, we explore $(Z_n)_{n\geq 0}$ as a Markov chain.

Throughout this chapter, we assume that immigration is actual possible, i.e.

$$\mathbb{P}\left(\xi_0 > 0\right) > 0,$$

as otherwise we are in the well-known BPRE case without immigration.

Since we intend to start the process with multiple ancestors, we denote by $\mathbb{P}_x$, $x \in \mathbb{N}_0$, the measure under which the BPREI has x initial individuals, i.e. $\mathbb{P}_x(Z_0 = x) = 1$. If we start with 0 ancestors, which is our usual setting, we write $\mathbb{P}$ instead of $\mathbb{P}_0$. Of course, the behavior of all random variables introduced in this section remains the same under each of these probability measures.

3.2 The BPREI as a Markov chain

As already pointed out in Section 3.1, $(Z_n)_{n\geq 0}$ forms a Markov chain with state space $\mathbb{N}_0$ under each $\mathbb{P}_x$,$x \in \mathbb{N}_0$, since the reproduction laws and independence assumptions persist. Since the environmental sequence is assumed to be i.i.d., the BPREI is further homogeneous with transition probabilities

$$p(x,z) \;:=\; \mathbb{P}(Z_1 = z \mid Z_0 = x) \;=\; \mathbb{P}_x(Z_1 = z), \quad x, z \in \mathbb{N}_0\,.$$

It is well-known that the branching process with immigration is a Markov chain and that there exists an irreducible, aperiodic subset of the state space, which is hit by the process with probability 1 (see for example [71,88]). We show that the same holds true when adding a random environment. For this purpose, we introduce some notation. Put

$$\kappa := \inf\{z \in \mathbb{N}_0 \mid \mathbb{P}(X_{1,0} = 0, \xi_0 = z) > 0\}$$

with $\kappa := \infty$ if the set is empty. For $i \in \mathbb{N}_0$, let

$$C_i := \{j \in \mathbb{N}_0 \mid \text{ex. } n \in \mathbb{N}_0 \text{ such that } \mathbb{P}_i(Z_n = j) > 0\}$$

be the set of states that can be reached from i.

Lemma 3.1. *Let $\mathbb{P}(\xi_0 = 0) < 1$ and $\mathbb{P}(X_{1,0} = 0) > 0$. Then $\kappa \in C_i$ for all $i \in \mathbb{N}_0$ and C_κ is irreducible, aperiodic and hit by Z_n eventually, i.e. $\mathbb{P}_x(Z_n \notin C_\kappa \ \forall\, n \geq 1) = 0$ for all $x \in \mathbb{N}_0$.*

Proof. Let $i \in \mathbb{N}_0$. Because of $\mathbb{P}(X_{1,0} = 0) > 0$, we find that $\mathbb{P}(X_{1,0} = 0, \xi_0 = \kappa) > 0$ for a $\kappa < \infty$ and, recalling the conditional independence of $X_{i,0}$, $i \in \mathbb{N}$, and ξ_0, we establish

$$0 \;<\; \mathbb{P}(X_{1,0} = 0, \xi_0 = \kappa) \;=\; \mathbb{E}\left(\mathbb{P}(X_{1,0} = 0, \xi_0 = \kappa | \mathcal{U})\right) \;=\; \mathbb{E}\left(p_{0\star}(\mathcal{U}_0) p_{\star\kappa}(\mathcal{U}_0)\right).$$

Hence, $p_{0\star}(\mathcal{U}_0)p_{\star\kappa}(\mathcal{U}_0) > 0$ as well as $p_{0\star}(\mathcal{U}_0) > 0$ and $p_{\star\kappa}(\mathcal{U}_0) > 0$ w.p.p.

For each $i \in \mathbb{N}_0$ we get from the conditional independence and the above observations

$$\begin{aligned}\mathbb{P}_i(Z_1 = \kappa) \;&=\; \mathbb{P}\left(\sum_{j=1}^{i} X_{j,0} + \xi_0 = \kappa\right) \;\geq\; \mathbb{P}\left(\sum_{j=1}^{i} X_{j,0} = 0, \xi_0 = \kappa\right) \\ &=\; \mathbb{E}\left(\mathbb{P}\left(\sum_{j=1}^{i} X_{j,0} = 0, \xi_0 = \kappa \,\middle|\, \mathcal{U}\right)\right) \;=\; \mathbb{E}\left(p_{\star\kappa}(\mathcal{U}_0) p_{0\star}(\mathcal{U}_0)^i\right) \;>\; 0 \qquad (3.3)\end{aligned}$$

and thus $\kappa \in C_i$. If $i \in C_\kappa$, by definition $\mathbb{P}_\kappa(Z_n = i) > 0$ for a $n \in \mathbb{N}$ and due to the above calculation we get $\mathbb{P}_i(Z_1 = \kappa) > 0$, implying irreducibility of C_κ. Setting $i = \kappa$ in the above inequality yields the aperiodicity of κ and thus of C_κ.

So it is left to prove that the BPREI hits C_κ with probability 1. For that purpose, we first note that $C_\kappa = C_{\kappa'}$ for each $\kappa' \in \mathbb{N}_0$ with $\mathbb{P}(p_{0\kappa'}(\mathcal{U}_0) > 0) > 0$. This can be seen by an analogous calculation as in (3.3):

$$\mathbb{P}_\kappa(Z_1 = \kappa') \;\geq\; \mathbb{P}\left(\sum_{j=1}^{\kappa} X_{j,0} = 0, \xi_0 = \kappa'\right) \;=\; \mathbb{E}\left(p_{\star\kappa'}(\mathcal{U}_0) p_{0\star}(\mathcal{U}_0)^\kappa\right) \;\geq\; \mathbb{E}\left(p_{0\kappa'}(\mathcal{U}_0)^{\kappa+1}\right) \;>\; 0.$$

Assume now that there exists a constant $c > 0$ such that

$$\inf_{x \in \mathbb{N}_0} \mathbb{P}_x(Z_1 \in C_\kappa) \geq c > 0. \tag{3.4}$$

Then the Markov property and iteration yield for all $x \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$

$$\begin{aligned}
\mathbb{P}_x(Z_n \notin C_\kappa) &= \mathbb{P}_x(Z_1 \notin C_\kappa, \ldots, Z_n \notin C_\kappa) \\
&= \sum_{z_1, \ldots, z_{n-1} \notin C_\kappa} \mathbb{P}_x(Z_1 = z_1, \ldots, Z_{n-1} = z_{n-1}, Z_n \notin C_\kappa) \\
&= \sum_{z_1, \ldots, z_{n-1} \notin C_\kappa} \mathbb{P}_x(Z_1 = z_1, \ldots, Z_{n-1} = z_{n-1}) \mathbb{P}_{z_{n-1}}(Z_1 \notin C_\kappa) \\
&\leq (1-c)\mathbb{P}_x(Z_1 \notin C_\kappa, \ldots, Z_{n-1} \notin C_\kappa) \;\leq\; \ldots \;\leq\; (1-c)^n.
\end{aligned}$$

Now, the Borel-Cantelli lemma provides $\mathbb{P}_x(Z_n \notin C_\kappa \text{ infinitely often}) = 0$ and the statement is proved.

Thus, it is left to verify (3.4) for a suitable constant $c > 0$. If $\mathbb{P}(p_{0\star}(\mathcal{U}_0) = 1) > 0$, then for all $x \in \mathbb{N}_0$

$$\mathbb{P}_x(Z_1 \in C_\kappa) \;\geq\; \mathbb{P}\left(\sum_{i=1}^{x} X_{i,0} = 0, \xi_0 \in C_\kappa\right) \;\geq\; \mathbb{P}\left(p_{0\star}(\mathcal{U}_0) = 1\right) \;>\; 0,$$

where we recall that $C_\kappa = C_{\kappa'}$ if $\mathbb{P}(p_{0\kappa'}(\mathcal{U}_0) > 0) > 0$. If, on the other hand, $\mathbb{P}(p_{0\star}(\mathcal{U}_0) = 1) = 0$, and thus particularly $p_{00}(\mathcal{U}_0) < 1$ a.s., we find that

$$\mathbb{P}(p_{0\star}(\mathcal{U}_0) p_{\star\kappa'}(\mathcal{U}_0) p_{k\star}(\mathcal{U}_0) > 0) > 0$$

for suitable $\kappa' \geq 1$ and $k \geq 1$. Since $C_\kappa = C_{\kappa'}$, we can assume w.l.o.g. that $\kappa' = \kappa$. Following the transformations in the Inequality (3.3), we establish for each $z \in \mathbb{N}_0$

$$\begin{aligned}
\mathbb{P}_{zk+\kappa}(Z_1 = (z+1)k + \kappa) &\geq \mathbb{E}\left(\mathbb{P}\left(X_{1,0} = \ldots = X_{z+1,0} = k, \sum_{j=z+2}^{zk+\kappa} X_{j,0} = 0, \xi_0 = \kappa \;\middle|\; \mathcal{U}\right)\right) \\
&\geq \mathbb{E}\left(p_{k\star}(\mathcal{U}_0)^{z+1} p_{0\star}(\mathcal{U}_0)^{z(k-1)+\kappa-1} p_{\star\kappa}(\mathcal{U}_0)\right) \;>\; 0.
\end{aligned}$$

This particularly implies $\kappa + k\,\mathbb{N}_0 \subseteq C_\kappa$, since $(z+1)k+\kappa$ can be reached from the state $zk+\kappa$ for all $z \in \mathbb{N}_0$. Fix a $x \in \mathbb{N}_0$. If $x \le \kappa$, then by (3.3)

$$\mathbb{P}_x(Z_1 \in C_\kappa) \;\ge\; \mathbb{P}_x(Z_1 = \kappa) \;\ge\; \mathbb{E}\left(p_{\star\kappa}(\mathcal{U}_0)p_{0\star}(\mathcal{U}_0)^x\right) \;\ge\; \mathbb{E}\left(p_{\star\kappa}(\mathcal{U}_0)p_{0\star}(\mathcal{U}_0)^\kappa\right) \;>\; 0.$$

If otherwise $x > \kappa$, there exists a $z \in \mathbb{N}_0$ such that $zk+\kappa \le x < (z+1)k+\kappa$. But this implies

$$\begin{aligned}\mathbb{P}_x(Z_1 \in C_\kappa) \;&\ge\; \mathbb{P}\left(\sum_{i=1}^{\kappa+zk} X_{i,0} + \xi_0 \in C_\kappa,\ \sum_{i=\kappa+zk+1}^{x} X_{i,0} = 0\right) \;=\; \mathbb{P}\left(\sum_{i=\kappa+zk+1}^{x} X_{i,0} = 0\right)\\ &\ge\; \mathbb{P}\left(\sum_{i=1}^{k} X_{i,0} = 0\right) \;=\; \mathbb{E}(p_{0\star}(\mathcal{U}_0)^k) \;>\; 0,\end{aligned}$$

where in the first equation the fact that $\kappa + zk \in C_\kappa$ and the irreducibility of C_κ was used. Thus (3.4) holds true in all cases and the lemma is proved. □

By classical Markov theory and the above lemma, we obtain conditions for convergence to a stationary distribution (see for example [5, Chapter 2 and Theorem 2.33]).

Corollary 3.2. *Let* $\mathbb{P}(\xi_0 > 0) > 0$, $\mathbb{P}(X_{1,0} = 0) > 0$ *and set* $\tau := \inf\{n \in \mathbb{N} \mid Z_n = \kappa\}$. *Then for* $n \to \infty$*:*

(a) If $\mathbb{P}_\kappa(\tau < \infty) < 1$, *then* $Z_n \to \infty$ $\mathbb{P}_x$*-a.s. for all* $x \in \mathbb{N}_0$

(b) If $\mathbb{E}_\kappa \tau = \infty$, *then* $Z_n \xrightarrow{\mathbb{P}_x} \infty$ *for all* $x \in \mathbb{N}_0$.

(c) If $\mathbb{E}_\kappa \tau < \infty$, *then* $Z_n \xrightarrow{d} Z_\infty$ *for a finite random variable* Z_∞ *independent of the number of ancestors.*

Let us now analyze the three above mentioned cases (supercritical, critical, subcritical) separately and specify asymptotic behavior.

3.3 The supercritical regime

In the supercritical regime, that is when $\mathbb{E}\log\mu_{\mathcal{U}_0} > 0$, the multiplication of individuals is high. This causes the convergence to infinity of $(Z_n)_{n\ge0}$ and thus transience or null-recurrence of the set C_κ.

Theorem 3.3. *Let* $\mathbb{E}\log\mu_{\mathcal{U}_0} > 0$ *and* $\mathbb{E}\log^-(1-g_{1,\mathcal{U}_0}(0)) < \infty$. *Then* $Z_n \to \infty$ $\mathbb{P}_x$*-a.s. for all* $x \in \mathbb{N}_0$ *as* $n \to \infty$.

Proof. It is enough to prove almost sure convergence for $x = 0$. A coupling argument then provides the assertion for all $x \in \mathbb{N}_0$. Let $(\bar{Z}_n)_{n\ge0}$ be the BPRE starting with a single ancestor, environmental sequence $\mathcal{U}$ and no immigration. Since $\mathbb{E}\log\mu_{\mathcal{U}_0} > 0$ and $\mathbb{E}\log^-(1-g_{1,\mathcal{U}_0}(0)) < \infty$, this process is supercritical and consequently $\mathbb{P}_1(\bar{Z}_n \to \infty) > 0$ (see [81]). We define the following stopping times

$$\sigma_0 := \inf\{n \ge 0 \mid \xi_n > 0\}$$

and for $i \geq 0$

$$\tau_i := \inf\left\{n > \sigma_i \mid \bar{Z}_{n-\sigma_i}(i) = 0\right\} \quad \text{and} \quad \sigma_{i+1} := \inf\left\{n > \tau_i \mid \xi_n > 0\right\},$$

where $(\bar{Z}_{n-\sigma_i}(i))_{n\geq\sigma_i}$ denotes the progeny process of the first immigrant of generation σ_i. Defining

$$Y_n := \sum_{i=0}^{\infty} \bar{Z}_{n-\sigma_i}(i)\, \mathbb{1}_{\{\sigma_i \leq n < \tau_i\}} \tag{3.5}$$

for $n \geq 0$, it follows immediately that $Y_n \leq Z_n$ $\mathbb{P}$-a.s. for all $n \in \mathbb{N}_0$. By the Markov property for each i the process $(\bar{Z}_{n-\sigma_i}(i))_{n\geq\sigma_i}$ behaves like $(\bar{Z}_n)_{n\geq 0}$ and thus survives w.p.p. Furthermore, the Markov property provides that the increments $\sigma_{i+1} + \tau_{i+1} - (\sigma_i + \tau_i)$ are i.i.d. and since the offspring of a parasite survives w.p.p. the Borel-Cantelli lemma gives

$$\mathbb{P}\left(\sigma_i + \tau_i < \infty \text{ infinitely often}\right) = 0.$$

But this implies $\mathbb{P}(Y_n \to \infty) = 1$ and thus the same holds true for Z_n. □

We can find a more precise asymptotic behavior of the BPREI in the supercritical regime which depends on the immigrational component. If the number of immigrants is small, that is $\mathbb{E}\log^+\xi_0 < \infty$, the normalized BPREI converges almost surely to a finite random variable, whereas no proper geometric normalization can be found, if the immigration rate is high, i.e. $\mathbb{E}\log^+\xi_0 = \infty$. These are analogs to the results found by Seneta in [79, 80] for the process without random environment. To prove the mentioned assertions, we follow the proof given by Asmussen an Hering in [10] in the case without random environment. For this purpose, we need the following lemma about non-negative i.i.d. sequences of random variables.

Lemma 3.4 (Lemma 1.1 in [61]). *Let $(X_n)_{n\geq 0}$ be a sequence of i.i.d. and non-negative random variables. Then almost surely*

$$\limsup_{n\to\infty} \frac{X_n}{n} = \begin{cases} 0 & \text{if } \mathbb{E}X_0 < \infty, \\ \infty & \text{if } \mathbb{E}X_0 = \infty. \end{cases}$$

Proof. The assertion follows with an easy Borel-Cantelli argument. □

Theorem 3.5. *Let $\mathbb{E}\log\mu_{\mathcal{U}_0} > 0$ (and $\mu_{\mathcal{U}_0} < \infty$ a.s.).*

(a) If $\mathbb{E}\log^+\xi_0 < \infty$, then for every $x \in \mathbb{N}_0$ there exists a finite random variable Z_∞ such that

$$\lim_{n\to\infty} \frac{Z_n}{\prod_{i=0}^{n-1}\mu_{\mathcal{U}_i}} = Z_\infty \qquad \mathbb{P}_x\text{-a.s.}$$

Furthermore,

$$\mathbb{P}_x(Z_\infty > 0) = 1 \quad \Leftrightarrow \quad \mathbb{P}_x(Z_\infty > 0) > 0 \quad \Leftrightarrow \quad \mathbb{E}((X_{1,0}\log^+ X_{1,0})/\mu_{\mathcal{U}_0}) < \infty.$$

(b) If $\mathbb{E}\log^+\xi_0 = \infty$, then $\limsup_{n\to\infty} c^{-n} Z_n = \infty$ $\mathbb{P}_x$-a.s. for every $x \in \mathbb{N}_0$ and $c \in (0, \infty)$.

Proof. (a) Let $x \in \mathbb{N}_0$ and define for $n \in \mathbb{N}_0$ the filtration

$$\mathcal{F}_n := \sigma(Z_0, Z_1, \ldots, Z_n, (\xi_k)_{k\geq 0}, \mathcal{U}),$$

which in particular means that $\mathcal{F}_0 = \sigma((\xi_k)_{k\geq 0}, \mathcal{U})$. We show that $((\prod_{i=0}^{n-1} \mu_{\mathcal{U}_i})^{-1} Z_n)_{n\geq 0}$ is a L^1-bounded submartingale when conditioned under $\mathcal{F}_0$. Then the martingale convergence theorem provides the convergence assertion. First, note that this process is $(\mathcal{F}_n)_{n\geq 0}$-adapted. Furthermore, we establish for $n \in \mathbb{N}_0$

$$\mathbb{E}_x(Z_{n+1}|\mathcal{F}_n) \;=\; \sum_{i=1}^{Z_n} \mathbb{E}(X_{i,n}|\mathcal{F}_n) + \xi_n \;\geq\; \sum_{i=1}^{Z_n} \mathbb{E}(X_{i,n}|\mathcal{U}) \;=\; Z_n \mu_{\mathcal{U}_n} \qquad \mathbb{P}_x\text{-a.s.}$$

and

$$\begin{aligned}
\mathbb{E}_x(Z_{n+1}|\mathcal{F}_0) &= \mathbb{E}_x\left(\mathbb{E}(Z_{n+1}|\mathcal{F}_n)|\mathcal{F}_0\right) \\
&= \mathbb{E}_x\left(\sum_{i=1}^{Z_n} \mathbb{E}(X_{i,n}|\mathcal{F}_n) + \xi_n \middle| \mathcal{F}_0\right) \\
&= \mathbb{E}_x(Z_n|\mathcal{F}_0)\mu_{\mathcal{U}_n} + \xi_n \qquad \mathbb{P}_x\text{-a.s.}
\end{aligned}$$

Hence by iteration, we get

$$\begin{aligned}
\mathbb{E}_x\left(\frac{Z_{n+1}}{\prod_{i=0}^{n} \mu_{\mathcal{U}_i}} \middle| \mathcal{F}_0\right) &= \mathbb{E}_x\left(\frac{Z_n}{\prod_{i=0}^{n-1} \mu_{\mathcal{U}_i}} \middle| \mathcal{F}_0\right) + \frac{\xi_n}{\prod_{i=0}^{n} \mu_{\mathcal{U}_i}} \;=\; x + \sum_{k=0}^{n} \frac{\xi_k}{\prod_{i=0}^{k} \mu_{\mathcal{U}_i}} \qquad (3.6) \\
&\leq x + \sum_{k=0}^{\infty} \frac{\xi_k}{\prod_{i=0}^{k} \mu_{\mathcal{U}_i}} \\
&\leq x + \sum_{k=0}^{\infty} \exp\left(\log^+ \xi_k - \sum_{i=0}^{k} \log \mu_{\mathcal{U}_i}\right) \\
&= x + \sum_{k=0}^{\infty} \exp\left(\frac{1}{k+1}\log^+ \xi_k - \frac{1}{k+1}\sum_{i=0}^{k} \log \mu_{\mathcal{U}_i}\right)^{k+1} \qquad \mathbb{P}_x\text{-a.s.} \qquad (3.7)
\end{aligned}$$

for each $n \in \mathbb{N}_0$. Since $(\xi_n)_{n\geq 0}$ and $(\mu_{\mathcal{U}_n})_{n\geq 0}$ are i.i.d. families and $\mathbb{E}\log^+ \xi_0 < \infty$, Lemma 3.4 and the strong law of large numbers yield

$$\limsup_{k\to\infty} \left(\frac{1}{k+1}\log^+ \xi_k - \frac{1}{k+1}\sum_{i=0}^{k} \log \mu_{\mathcal{U}_i}\right) \;=\; -\mathbb{E}\log \mu_{\mathcal{U}_0} < 0 \quad \mathbb{P}_x\text{-a.s.}$$

and thus almost sure finiteness of the sum in (3.7). Therefore, $((\prod_{i=0}^{n-1} \mu_{\mathcal{U}_i})^{-1} Z_n)_{n\geq 0}$ is a submartingale and L^1-bounded conditioned under $\mathcal{F}_0$, and the convergence statement in (a) follows.

So it remains to prove the characterization of non-degeneracy of Z_∞ and note that it is enough to consider the case when $x = 0$. As in the proof of Theorem 3.3, we denote by $(\bar{Z}_n)_{n\geq 0}$ the BPRE starting with a single ancestor, environmental sequence $\mathcal{U}$ and no immigration. Now [85, Theorem 2] yields that $(\bar{Z}_n/\mathbb{E}\bar{Z}_n)_{n\geq 0}$ conditioned under $\mathcal{U}$ converges to a limit $\bar{W}$ $\mathbb{P}_1$-a.s.

for $n \to \infty$, which is non-degenerated, i.e. $q(\mathcal{U}) := \mathbb{P}_1(\bar{W} = 0|\mathcal{U}) < 1$ a.s., if and only if $\mathbb{E}((X_{1,0} \log^+ X_{1,0})/\mu_{\mathcal{U}_0}) < \infty$. So, we have to verify the implications

$$\mathbb{P}(Z_\infty > 0) > 0 \quad \Rightarrow \quad \mathbb{P}(q(\mathcal{U}) < 1) > 0 \quad \Rightarrow \quad \mathbb{P}(Z_\infty > 0) = 1. \tag{3.8}$$

We show the first implication by contraposition and assume that $q(\mathcal{U}) = 1$ a.s. Note that

$$Z_n \overset{d}{=} \sum_{k=0}^{n} \sum_{i=1}^{\xi_k} Z_k(i), \tag{3.9}$$

where the random variables $(Z_k(i))_{i,k\in\mathbb{N}}$ are independent conditioned under $\mathcal{U}$, and $Z_k(i)$ has the generating function $g_{\mathcal{U}_{k-1}} \circ \cdots \circ g_{\mathcal{U}_0}$ for $k \in \mathbb{N}$ and id if $k = 0$ (see [49]). In particular, $Z_k(i)$ is distributed as $\bar{Z}_k$ due to the i.i.d. environmental sequence and thus

$$\frac{Z_{n-k}(i)}{\prod_{j=0}^{n-1} \mu_{\mathcal{U}_j}} \;=\; \frac{1}{\prod_{j=n-k}^{n-1} \mu_{\mathcal{U}_j}} \frac{Z_{n-k}(i)}{\prod_{j=0}^{n-k-1} \mu_{\mathcal{U}_j}} \;\to\; 0 \quad \text{a.s.}$$

as $n \to \infty$ for each $i \in \mathbb{N}$ and all $k \in \mathbb{N}$, by recalling that $q(\mathcal{U}) = 1$ a.s. Furthermore, obtain that $Z_k(i)$ is independent of the sequence $(\mathcal{U}_{k+1}, \mathcal{U}_{k+2}, \dots)$ for each k by definition. From this, (3.9) and the already in this proof deduced convergence results, we then infer

$$\frac{Z_n}{\prod_{j=0}^{n-1} \mu_{\mathcal{U}_j}} \overset{d}{=} \sum_{k=n-K+1}^{n} \sum_{i=1}^{\xi_k} \frac{Z_k(i)}{\prod_{j=0}^{n-1} \mu_{\mathcal{U}_j}} + \frac{1}{\prod_{j=n-K}^{n-1} \mu_{\mathcal{U}_j}} \sum_{k=0}^{n-K} \sum_{i=1}^{\xi_k} \frac{Z_k(i)}{\prod_{j=0}^{n-K-1} \mu_{\mathcal{U}_j}} \overset{d}{\to} \frac{Z_\infty}{\prod_{j=1}^{K} \mu_{\mathcal{U}'_j}}$$

for each $K \in \mathbb{N}$, where $(\mu_{\mathcal{U}'_1}, \dots, \mu_{\mathcal{U}'_K})$ is a copy of $(\mu_{\mathcal{U}_0}, \dots, \mu_{\mathcal{U}_{K-1}})$ and independent of Z_∞. Hence,

$$Z_\infty \overset{d}{=} \frac{Z_\infty}{\prod_{j=1}^{K} \mu_{\mathcal{U}'_j}}$$

for all $K \in \mathbb{N}$, and since $Z_\infty < \infty$ and $\prod_{j=1}^{K} \mu_{\mathcal{U}'_j} = \exp(\sum_{j=1}^{K} \log \mu_{\mathcal{U}'_j}) \to \infty$ a.s. as $K \to \infty$ by the law of large numbers, this yields $Z_\infty = 0$ a.s.

For the second implication in (3.8) let now be $\mathbb{P}(q(\mathcal{U}) < 1) > 0$. This in particular implies $\bar{Z}_n \to \infty$ w.p.p. Following the proof of Theorem 3.3, this deduces $Z_n \to \infty$ $\mathbb{P}$-a.s. Fix $\varepsilon > 0$ and choose $\eta > 0$ such that

$$\mathbb{P}_1(q(\mathcal{U}) < 1 - \eta) \geq 1 - \varepsilon.$$

For each $k \in \mathbb{N}$, we find that

$$\begin{aligned}
\mathbb{P}(Z_\infty = 0 \mid Z_k, \mathcal{U}) &= \mathbb{P}\left(\lim_{n\to\infty} \frac{Z_n}{\prod_{i=0}^{n-1} \mu_{\mathcal{U}_i}} = 0 \,\middle|\, Z_k, \mathcal{U} \right) \\
&\leq \mathbb{P}\left(\lim_{n\to\infty} \sum_{j=1}^{Z_k} \frac{\bar{Z}_n(j)}{\prod_{i=k}^{n-1} \mu_{\mathcal{U}_i}} = 0 \,\middle|\, Z_k, \mathcal{U} \right) \\
&= \mathbb{P}_1\left(W = 0 \mid (\mathcal{U}_i)_{i\geq k}\right)^{Z_k} \quad \text{a.s.}
\end{aligned}$$

where $\bar{Z}_n(j)$ describes the offspring in generation n stemming from the j^{th} individual in generation k and thus behaves like the BPRE $\bar{Z}_n$. Since the population of the BPREI explodes almost surely and $\mathcal{U}$ consists of i.i.d. random variables, we finally conclude

$$\mathbb{P}(Z_\infty = 0) \;\le\; \mathbb{E}\left(q((\mathcal{U}_i)_{i\ge k})^{Z_k}\right) \;\le\; \mathbb{E}\left((1-\eta)^{Z_k}\right) + \varepsilon \;\to\; \varepsilon \quad \text{as } k \to \infty.$$

Now, (a) is proved because $\varepsilon > 0$ is arbitrary.

(b) Let $c > 0$. By (3.1) it follows that $Z_{n+1} \ge \xi_n$ $\mathbb{P}$-a.s. for all $n \ge 0$. Since $\mathbb{E}\log^+ \xi_0 = \infty$, Lemma 3.4 implies

$$\limsup_{n\to\infty} \frac{Z_n}{c^n} \;\ge\; \limsup_{n\to\infty} \frac{\xi_n}{c^n} \;=\; \limsup_{n\to\infty} \left(\frac{1}{c}\exp\left(\frac{\log \xi_n}{n}\right)\right)^n \;=\; \infty \quad \mathbb{P}\text{-a.s.}$$

By a coupling argument, the assertion follows for all $x \in \mathbb{N}_0$. □

Remark 3.6. The supercriticality or almost sure finiteness of $\mu_{\mathcal{U}_0}$ was not needed in the proof of (b) in the above theorem. Thus, this statement holds true in the critical and subcritical case too, as long as $\mathbb{E}\log^+ \xi_0 = \infty$.

In the situation of a classical GWP with immigration (and no random environment) the limit Z_∞ is non-degenerated if and only if $\mathbb{E}Z \log Z < \infty$ is valid. This was proved by Seneta in [79] and follows also directly from the above theorem. In a subsequent article [80], Seneta showed the existence of a norming sequence for the GWP with immigration to converge to a non-degenerated limit if $\mathbb{E}Z\log Z = \infty$. The next theorem states that a suitable norming sequence for the BPREI cannot differ much from the mean normalization and thus the population explodes at an exponential rate on the set of survival.

Theorem 3.7. *Let* $\mathbb{E}\log \mu_{\mathcal{U}_0} > 0$, $\mathbb{E}\log^+ \xi_0 < \infty$ *and* $\mathbb{E}\log^-(1-g_{1,\mathcal{U}_0}(0)) < \infty$. *Then* $\frac{1}{n}\log Z_n \to \mathbb{E}\log \mu_{\mathcal{U}_0}$ $\mathbb{P}_x$*-a.s. for all* $x \in \mathbb{N}_0$ *as* $n \to \infty$.

Proof. Let $x \in \mathbb{N}_0$. By our assumptions and Theorem 3.5 $((\prod_{i=0}^{n-1} \mu_{\mathcal{U}_i})^{-1} Z_n)_{n\ge 0}$ converges $\mathbb{P}_x$-a.s. to a finite random variable Z_∞. Hence, we find that

$$\limsup_{n\to\infty} \frac{1}{n}\log Z_n \;\le\; \limsup_{n\to\infty} \frac{1}{n}\log\left(\frac{Z_n}{\prod_{i=0}^{n-1}\mu_{\mathcal{U}_i}}\right) + \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}\log \mu_{\mathcal{U}_i} \;\le\; \mathbb{E}\log \mu_{\mathcal{U}_0} \quad \mathbb{P}_x\text{-a.s.}$$

by the law of large numbers. By [83, Theorem 5.5], we get for a BPRE (without immigration) $(\bar{Z}_n)_{n\ge 0}$ with a single ancestor and environmental sequence $\mathcal{U}$

$$\lim_{n\to\infty} \frac{1}{n}\log \bar{Z}_n \;=\; \mathbb{E}\log \mu_{\mathcal{U}_0} \quad \mathbb{P}_1\text{-a.s.}$$

on the event $\{\bar{Z}_n \to \infty\}$, and thus we get for the process $(Y_n)_{n\ge 0}$ defined in (3.5)

$$\lim_{n\to\infty} \frac{1}{n}\log Y_n \;=\; \mathbb{E}\log \mu_{\mathcal{U}_0} \quad \mathbb{P}\text{-a.s.},$$

since the progeny of an immigrant survives eventually. Thanks to $Y_n \le Z_n$ $\mathbb{P}$-a.s. for each n

$$\liminf_{n\to\infty} \frac{1}{n}\log Z_n \;\ge\; \liminf_{n\to\infty} \frac{1}{n}\log Y_n \;=\; \mathbb{E}\log \mu_{\mathcal{U}_0} \quad \mathbb{P}\text{-a.s.}$$

and thus for all $\mathbb{P}_x$, $x \in \mathbb{N}_0$. This finishes the proof of the theorem. □

3.4 The critical regime

Compared to the supercritical case, the population size Z_n in the critical regime tends to infinity in probability under some integrability assumptions.

Proposition 3.8. *Let* $\mathbb{E}\log\mu_{\mathcal{U}_0} = 0$*,*

$$0 < \mathbb{E}(\log(g'_{1,\mathcal{U}_0}(1))^2) < \infty \quad and \quad \mathbb{E}\left((1+\log g'_{1,\mathcal{U}_0}(1))\frac{g''_{1,\mathcal{U}_0}(1)}{g'_{1,\mathcal{U}_0}(1)}\right) < \infty.$$

Then $Z_n \xrightarrow{\mathbb{P}_x} \infty$ *for every* $x \in \mathbb{N}_0$ *as* $n \to \infty$*.*

Proof. We reproduce the proof of Bansaye in [16]. Since $\mathbb{E}\log\mu_{\mathcal{U}_0} = 0$, it follows that $\mathbb{P}(X_{1,0} = 0) > 0$. By Corollary 3.2 it is enough to show that $\mathbb{E}_\kappa\tau = \infty$. Thus, we consider starting with κ ancestors. It can be easily seen by definition that κ is the minimal element in C_κ. Thus w.o.l.g. we can assume that $\kappa = 0$, for otherwise we look at the process $(Z_n - \kappa)_{n\geq 0}$, and that τ describes the first hitting time of the BPREI to 0. Let $(\bar{Z}_n)_{n\geq 0}$ be the BPRE with reproduction law given by $X_{1,0}$ and no immigration. Due to our assumptions and [53], there exists a positive constant $c > 0$ such that for every $n \geq 0$

$$\mathbb{P}_1(\bar{Z}_n > 0) \;\geq\; \frac{c}{\sqrt{n}}.$$

Since Z_n is stochastically larger than $\bar{Z}_n$ for every $n \geq 0$ we obtain

$$\mathbb{P}_1(\tau > n) \;\geq\; \mathbb{P}_1(\bar{Z}_n > 0) \;\geq\; \frac{c}{\sqrt{n}}$$

which ensures

$$\mathbb{E}_1\tau = \infty.$$

Since the BPREI starting from 1 individual is stochastically smaller than starting from $k \geq 1$ ancestors, we get by using the Markov property

$$\mathbb{E}_0\tau \;=\; 1 + \sum_{z=1}^{\infty}\mathbb{P}_0(Z_1 = z)\mathbb{E}_z\tau \;\geq\; 1 + \mathbb{P}_0(Z_1 > 0)\mathbb{E}_1\tau \;=\; \infty$$

and thus the assertion. □

The above theorem particularly states that, under the given assumptions, the BPREI is not positive recurrent. However, there are cases under which the critical BPREI is positive recurrent and thus converges in distribution to a finite random variable. See [77] for an example of a positive recurrent branching process with immigration in a constant environment.

As in the situation without random environment (see [69, 78]), the population of a critical BPREI grows slower than every geometric rate if the immigration rate is low, that is $\mathbb{E}\log^+\xi_0 < \infty$. This is stated in the next theorem.

Theorem 3.9. *Let* $\mathbb{E}\log\mu_{\mathcal{U}_0} = 0$ *and* $\mathbb{E}\log^+\xi_0 < \infty$*. Then* $c^{-n}Z_n \to 0$ *in probability under each* $\mathbb{P}_x$*,* $x \in \mathbb{N}_0$*, and* $c \in (1,\infty)$*.*

Proof. Let $c > 1$ and put $\mathcal{F}_0 := \sigma\left((\xi_n)_{n\geq 0}, \mathcal{U}\right)$. By (3.6), we get

$$\mathbb{E}_x\left(Z_{n+1} \mid \mathcal{F}_0\right) \;=\; x + \sum_{k=0}^{n} \xi_k \prod_{i=k+1}^{n} \mu_{\mathcal{U}_i} \quad \mathbb{P}_x\text{-a.s.}$$

for $n \in \mathbb{N}_0$ and $x \in \mathbb{N}_0$. Since x/c^n vanishes for $n \to \infty$, we can assume from now on that $x = 0$. Fix $\varepsilon \in (0, c-1)$ and put $\tilde{\mu}_{\mathcal{U}_i} := \mu_{\mathcal{U}_i}(1+\varepsilon)$. Then

$$0 < \mathbb{E}\log\tilde{\mu}_{\mathcal{U}_0} = \log(1+\varepsilon) < \log c \tag{3.10}$$

and thus

$$\begin{aligned}\frac{1}{c^{n+1}}\mathbb{E}\left(Z_{n+1} \mid \mathcal{F}_0\right) \;&\leq\; \frac{1}{c^{n+1}}\sum_{k=0}^{n} \xi_k \prod_{i=k+1}^{n} \tilde{\mu}_{\mathcal{U}_i} \\ &=\; \left(\prod_{k=0}^{n} \frac{\tilde{\mu}_{\mathcal{U}_k}}{c}\right)\sum_{k=0}^{n} \frac{\xi_k}{\prod_{i=0}^{k}\tilde{\mu}_{\mathcal{U}_i}} \;\leq\; \left(\prod_{k=0}^{n} \frac{\tilde{\mu}_{\mathcal{U}_k}}{c}\right)\sum_{k=0}^{\infty} \frac{\xi_k}{\prod_{i=0}^{k}\tilde{\mu}_{\mathcal{U}_i}} \quad \text{a.s.}\end{aligned}$$

Since $\mathbb{E}\log\mu_{\tilde{\mu}_{\mathcal{U}_0}} > 0$, the means $\tilde{\mu}_{\mathcal{U}_k}$, $k \in \mathbb{N}_0$, can be interpreted as given by a supercritical BPREI with immigration sequence $(\xi_n)_{n\geq 0}$. In the proof of Theorem 3.5, we have already seen that the sum on the right side is almost surely finite. Because of (3.10) and the i.i.d. property of $\tilde{\mu}_{\mathcal{U}_n}$, $n \in \mathbb{N}_0$, we further get

$$\limsup_{n\to\infty} \prod_{k=0}^{n} \frac{\tilde{\mu}_{\mathcal{U}_k}}{c} \;=\; \limsup_{n\to\infty} \exp\left(\sum_{k=0}^{n} \log\frac{\tilde{\mu}_{\mathcal{U}_k}}{c}\right) \;=\; 0$$

by the law of large numbers and hence

$$\limsup_{n\to\infty} \frac{1}{c^{n+1}}\mathbb{E}\left(Z_{n+1} \mid \mathcal{F}_0\right) = 0 \quad \text{a.s.}$$

But this implies for each $\eta > 0$

$$\limsup_{n\to\infty} \mathbb{P}\left(c^{-n}Z_n > \eta | \mathcal{F}_0\right) \;\leq\; \limsup_{n\to\infty} \frac{1}{\eta c^n}\mathbb{E}\left(Z_n \mid \mathcal{F}_0\right) \;=\; 0 \quad \text{a.s.}$$

and thus dominated convergence ensures

$$\limsup_{n\to\infty} \mathbb{P}\left(c^{-n}Z_n > \eta\right) \;=\; \limsup_{n\to\infty} \mathbb{E}\left(\mathbb{P}\left(c^{-n}Z_n > \eta | \mathcal{F}_0\right)\right) \;=\; 0,$$

which finishes the proof. □

Remark 3.10. *(a)* By an easy coupling argument or Theorem 3.11 below, the above theorem also holds true for the subcritical BPREI.

(b) As a consequence of the proof, we conclude that

$$\limsup_{n\to\infty} \frac{1}{c^n}\mathbb{E}_x\left(Z_{n+1} \mid \mathcal{F}_0\right) = 0 \qquad \mathbb{P}_x\text{-a.s.}$$

for each $x \in \mathbb{N}_0$ and $c > 1$ such that $\mathbb{E}\log\mu_{\mathcal{U}_0} < \log c$, where $\mathcal{F}_0 := \sigma((\xi_n)_{n\geq 0}, \mathcal{U})$ as defined in the proof above. This holds true in all three regimes.

3.5 The subcritical regime

Suppose that $\mathbb{E}\log\mu_{\mathcal{U}_0} < 0$. In contrast to both previous cases, $(Z_n)_{n\geq 0}$ only converges to infinity (in probability) if the immigration rate is high, that is when $\mathbb{E}\log^+\xi_0 = \infty$. If $\mathbb{E}\log^+\xi_0 < \infty$, then the population of the subcritical BPREI stabilizes, meaning that $(Z_n)_{n\geq 0}$ converges in distribution to a finite random variable. These two results have been proved in [16] and [49] under quite similar assumptions, which is why we omit a detailed proof here and refer to these articles for precise arguments.

Theorem 3.11. *Let* $\mathbb{E}\log\mu_{\mathcal{U}_0} < 0$.

(a) If $\mathbb{E}\log^+\xi_0 < \infty$*, then there exists a finite random variable* Z_∞ *such that* Z_n *converges in distribution to* Z_∞ *as* $n\to\infty$ *for every ancestor number* $x\in\mathbb{N}_0$.

(b) If $\mathbb{E}\log^+\xi_0 = \infty$ *and* $\mathbb{E}\log^-(1-g_{1,\mathcal{U}_0}(0)) < \infty$*, then* $Z_n\to\infty$ $\mathbb{P}_x$*-stochastically for each* $x\in\mathbb{N}_0$ *as* $n\to\infty$.

Proof. Since $\mathbb{E}\log\mu_{\mathcal{U}_0} < 0$ is assumed, the descendants of every individual die out eventually as they form a subcritical BPRE (see [81]). Hence, the number of ancestors has no influence on the limiting distribution. So it is enough to show the results under $\mathbb{P}$. Now, (3.9) provides

$$Z_n \xrightarrow{d} \sum_{k=0}^{\infty}\sum_{i=1}^{\xi_k} Z_k(i) =: Z_\infty \quad \text{for } n\to\infty,$$

where conditioned under $\mathcal{U}$ the random variables $(Z_k(i))_{i,k\in\mathbb{N}}$ are independent and $Z_k(i)$ has generating function $g_{\mathcal{U}_{k-1}}\circ\cdots\circ g_{\mathcal{U}_0}$. Let $\mathcal{F}_0 = \sigma((\xi_k)_{k\geq 0},\mathcal{U})$.

(a) In [49, Theorem 3.3] it is shown that $\mathbb{E}(Z_\infty|\mathcal{F}_0) < \infty$ a.s. if $\mathbb{E}\log^+\xi_0 < \infty$, and (a) follows.

(b) Let $\mathbb{E}\log^+\xi_0 = \infty$. We infer from the Borel-Cantelli lemma

$$\mathbb{P}(Z_\infty = \infty|\mathcal{F}_0) = 1 \quad \text{a.s.} \qquad \text{iff} \qquad \sum_{k=0}^{\infty}\xi_k\mathbb{P}(Z_k(1)>0|\mathcal{U}) = \infty \quad \text{a.s.}$$

The convexity of the generating functions, the i.i.d. property of $\mathcal{U}$ and the law of large numbers with the assumption $\mathbb{E}\log^-(1-g_{\mathcal{U}_i}(0)) < \infty$ ensure

$$\begin{aligned}\mathbb{P}(Z_k(1)>0|\mathcal{U}) &= 1-g_{\mathcal{U}_{k-1}}\circ\cdots\circ g_{\mathcal{U}_0}(0) \;\geq\; \prod_{i=0}^{k-1}(1-g_{\mathcal{U}_i}(0))\\ &= \exp\left(\sum_{i=0}^{k-1}\log(1-g_{\mathcal{U}_i}(0))\right) \;\geq\; \exp(k\alpha) \quad \text{a.s.}\end{aligned}$$

for all $k\in\mathbb{N}_0$ and a constant $\alpha\in(-\infty,0)$. Hence,

$$\sum_{k=0}^{\infty}\xi_k\mathbb{P}(Z_k(1)>0|\mathcal{U}) \;\geq\; \sum_{k=0}^{\infty}\xi_k\exp(k\alpha) \quad \text{a.s.,}$$

and the second sum is almost surely infinite due to Lemma 3.4. This completes the proof. □

Remark 3.12. If $\mathbb{E}\log^+\xi_0 = \infty$ and $\mathbb{E}\log^-(1-g_{1,\mathcal{U}_0}(0)) < \infty$, then Theorem 3.11$(b)$ and a coupling argument gives $Z_n \xrightarrow{\mathbb{P}} \infty$ in the supercritical and critical case.

Chapter 4

Limit theorems for the BwBP in the case $\mathbb{P}(\mathsf{Surv}) > 0$

In the present chapter, we return back to the BwBP and analyze it under the premise of non-certain extinction, i.e. $\mathbb{P}(\mathsf{Surv}) > 0$, *which is assumed from now on for the entire chapter.* We deduce equivalent conditions for the martingale $(W_n)_{n\geq 0}$ to be uniformly integrable. Afterwards, we focus on the question of finding an appropriate Heyde-Seneta norming for $(\mathcal{Z}_n)_{n\geq 0}$ and $(\mathcal{T}_n^*)_{n\geq 0}$. In particular, we identify the geometric rate at which these processes grow. The last part of this chapter is devoted to the partition of parasites to the cells. Thereby, diverse cases have to be considered, for this behavior depends on the regimes of the ABPRE.

4.1 Conditions for the number of parasites to grow like its means: A Kesten - Stigum theorem

By Proposition 1.12, $(W_n)_{n\geq 0}$ forms a non-negative martingale and therefore converges almost surely to a finite random variable W. In this section, we give equivalent conditions for this convergence to hold in mean too. We have already seen in Theorem 1.14 that $\mathbb{E}W = 1$ under certain second moment assumptions. However, there are weaker conditions, namely

$$\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 < \infty \quad \text{and} \quad \mathbb{E}\left(\frac{g'_{\Lambda_0}(1)}{\gamma} \log \frac{g'_{\Lambda_0}(1)}{\gamma}\right) < 0, \tag{4.1}$$

under which uniform integrability still holds true; in fact, these conditions are equivalent to $\mathbb{E}W = 1$, besides in some trivial cases, as will be shown in Theorem 4.6. (4.1) comprises the well-known $(Z \log Z)$-condition, i.e. $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 < \infty$, which is equivalent to uniform integrability in the classical Galton-Watson setting (see [14, Theorem 10.1 in Chapter I]), and a second condition saying that the parasites are more or less uniformly spread over all cells and not concentrated in a few cell lines. This second condition is similar to the one obtained for the weighted branching model (see [55]). As in nearly all probabilistic proofs of Kesten-Stigum-type theorems, we use the size-biased method and the dichotomy stated in Lemma 2.2 to verify the mentioned results.

For analog proceedings, we refer to [27, 55] and recall that our conditions are similar to, but stronger than the ones given in [27] (see the end of Subsection 1.1.1).

Before proving the afore mentioned results, we show that $\mathbb{P}(W > 0) > 0$ and uniform integrability are two sides of the same medal, meaning that these two properties are equivalent. Note that by the considerations in Subsection 1.2.4 and (1.28), it is enough to consider the BwBP when starting with one cell containing a single parasite.

Theorem 4.1. *Let $\mathbb{P}(\mathsf{Ext}) < 1$. The following statements are equivalent:*

- *(i)* $\mathbb{P}(W > 0) > 0$
- *(ii)* $\mathbb{E}W = 1$
- *(iii)* $(W_n)_{n\geq 0}$ *is uniformly integrable*
- *(iv)* $\mathbb{E}\left(\sup_{n\geq 0} W_n\right) < \infty$.

Proof. The implications "*(iv)* ⇒ *(iii)* ⇒ *(ii)* ⇒ *(i)*" follow directly from standard martingale theory. So it is left to deduce that $\mathbb{P}(W > 0) > 0$ implies $\mathbb{E}\left(\sup_{n\geq 0} W_n\right) < \infty$. Besides slight modifications, we follow the argumentations in the proofs of [23, Lemma 2] and [10, Lemma 2.6 in Chapter II], and we estimate the tail probabilities of $\sup_{n\geq 0} W_n$.

Let $\mathbb{P}(W > 0) > 0$. Assuming the existence of constants $\delta^* > 0$ and $B > 0$ such that

$$\mathbb{P}\left(W > \delta^* t\right) \ \geq \ B\mathbb{P}\left(\sup_{n\geq 0} W_n > t\right) \tag{4.2}$$

for all $t \in [1, \infty)$, we can conclude

$$\begin{aligned}
\mathbb{E}\left(\sup_{n\geq 0} W_n\right) \ &= \ \int_0^\infty \mathbb{P}\left(\sup_{n\geq 0} W_n > t\right) dt \\
&\leq \ 1 + \int_1^\infty \mathbb{P}\left(\sup_{n\geq 0} W_n > t\right) dt \\
&\leq \ 1 + \frac{1}{B}\int_1^\infty \mathbb{P}\left(W > \delta^* t\right) dt \\
&\leq \ 1 + \frac{\mathbb{E}W}{\delta^* B} \ < \ \infty.
\end{aligned}$$

This proves the implication "*(i)* ⇒ *(iv)*". Thus, it is left to verify (4.2).

Proof of (4.2): Clearly, $\mathbb{P}(W > 0) > 0$ implies $\mathbb{E}W > 0$. By the monotone convergence theorem, we find for each $a \in (0, \mathbb{E}W)$ a constant $K \geq a$ such that $\mathbb{E}(W \wedge K) \geq a$. Fix $t \in [1, \infty)$ and define for $n \in \mathbb{N}_0$

$$E_n := \left\{ W_n > t, \ \sup_{0\leq k<n} W_k \leq t \right\}.$$

Then for all $\delta > 0$, we get

$$\mathbb{P}(W > \delta t) \ \geq \ \mathbb{P}\left(W > \delta t, \ \sup_{n\geq 0} W_n > t\right) \ = \ \sum_{n\in\mathbb{N}_0} \mathbb{P}(W > \delta t | E_n)\mathbb{P}(E_n). \tag{4.3}$$

For $v \in \mathbb{V}$ and $n \in \mathbb{N}_0$ let $\mathcal{Z}_n^{(v)}$ denote the number of parasites in the n^{th} generation of the subtree rooted in cell v, which contains Z_v parasites. Since $(W_n)_{n\geq 0}$ is a martingale under each $\mathbb{P}_z$, $z \in \mathbb{N}$, by Proposition 1.12, we obtain the almost sure convergence of $\gamma^{-n}\mathcal{Z}_n^{(v)}$ conditioned under Z_v and denote its limit by $W^{(v)}$. Then for all $n \in \mathbb{N}_0$, we get the representation

$$W \;=\; \frac{1}{\gamma^n}\lim_{k\to\infty}\sum_{v\in\mathbb{T}_n^*}\frac{\mathcal{Z}_k^{(v)}}{\gamma^k} \;=\; \frac{1}{\gamma^n}\sum_{v\in\mathbb{T}_n^*} W^{(v)} \quad \text{a.s.}$$

and consequently

$$\begin{aligned}
\mathbb{P}(W > \delta t | E_n) &= \mathbb{P}\left(\frac{1}{\gamma^n}\sum_{v\in\mathbb{T}_n^*} W^{(v)} > \delta t \;\middle|\; E_n\right) \\
&= \mathbb{P}\left(\frac{1}{\gamma^n W_n}\sum_{v\in\mathbb{T}_n^*} W^{(v)} > \frac{\delta t}{W_n} \;\middle|\; E_n\right) \\
&\geq \mathbb{P}\left(\frac{1}{\mathcal{Z}_n}\sum_{v\in\mathbb{T}_n^*} W^{(v)} > \delta \;\middle|\; E_n\right) \\
&\geq \mathbb{P}\left(\frac{1}{\mathcal{Z}_n}\sum_{v\in\mathbb{T}_n^*} (W^{(v)} \wedge Z_v K) > \delta \;\middle|\; E_n\right) \\
&= \mathbb{P}(E_n)^{-1}\int_{E_n}\mathbb{P}\left(\frac{1}{\mathcal{Z}_n}\sum_{v\in\mathbb{T}_n^*} (W^{(v)} \wedge Z_v K) > \delta \;\middle|\; \mathcal{F}_n\right) d\mathbb{P}. \qquad (4.4)
\end{aligned}$$

For $Z_\varnothing = z \in \mathbb{N}_0$ let $\mathcal{Z}_{k,j}$ denote the number of parasites in generation $k \in \mathbb{N}_0$ stemming from the ancestor parasite $j \in \{1, \ldots, z\}$. If for all $1 \leq j \leq z$ the offspring number in generation k is at most $\gamma^k K$, then the sum over all offspring parasites is at most $z\gamma^k K$, i.e.

$$\sum_{j=1}^{z}\left(\mathcal{Z}_{k,j} \wedge \gamma^k K\right) \;\leq\; \left(\sum_{j=1}^{z}\mathcal{Z}_{k,j}\right) \wedge z\gamma^k K \quad \mathbb{P}_z\text{-a.s.}$$

This consideration implies

$$\begin{aligned}
\mathbb{E}_z(W \wedge zK) &= \mathbb{E}_z\left(\lim_{k\to\infty}\left(\frac{1}{\gamma^k}\sum_{j=1}^{z}\mathcal{Z}_{k,j}\right) \wedge zK\right) \\
&\geq \mathbb{E}_z\left(\sum_{j=1}^{z}\left(\lim_{k\to\infty}\frac{1}{\gamma^k}\mathcal{Z}_{k,j} \wedge K\right)\right) \;=\; z\mathbb{E}(W \wedge K) \;\geq\; za
\end{aligned}$$

for all $z \in \mathbb{N}_0$. From this, we infer

$$\mathbb{E}\left(\frac{1}{\mathcal{Z}_n}\sum_{v\in\mathbb{T}_n^*}\left(W^{(v)} \wedge Z_v K\right) \;\middle|\; \mathcal{F}_n\right) \;=\; \frac{1}{\mathcal{Z}_n}\sum_{v\in\mathbb{T}_n^*}\sum_{z=1}^{\infty}\mathbb{E}_z\left(W \wedge zK\right)\mathbb{1}_{\{Z_v=z\}}$$

$$\geq \frac{1}{\mathcal{Z}_n}\sum_{v\in\mathbb{T}_n^*} Z_v a \;=\; a \quad \text{a.s.}$$

for all $n \in \mathbb{N}_0$. Let us put $W_n(K) := \frac{1}{\mathcal{Z}_n}\sum_{v\in\mathbb{T}_n^*}(W(v)\wedge Z_v K)$ for $n \in \mathbb{N}_0$ and note that $W_n(K) \leq K$ a.s. for all $n \in \mathbb{N}_0$. Keeping this in mind, we find that for all $\delta \in (0,a)$

$$\begin{aligned} a \;&\leq\; \mathbb{E}\left(W_n(K) \mid \mathcal{F}_n\right) \;=\; \int_0^K \mathbb{P}\left(W_n(K) > x \mid \mathcal{F}_n\right) dx \\ &\leq\; \delta + \int_\delta^K \mathbb{P}\left(W_n(K) > x \mid \mathcal{F}_n\right) dx \;\leq\; \delta + (K-\delta)\mathbb{P}\left(W_n(K) > \delta \mid \mathcal{F}_n\right) \end{aligned}$$

and thus

$$\mathbb{P}\left(W_n(K) > \delta \mid \mathcal{F}_n\right) \;\geq\; \frac{a-\delta}{K-\delta} \qquad \text{a.s.}$$

Plugging this inequality into (4.4) for $\delta^* := a/2$ and $B := a/(2K-a)$ yields

$$\mathbb{P}(W > \delta^* t | E_n) \;\geq\; B$$

for all $n \in \mathbb{N}_0$ and $t \in [1,\infty)$. By using (4.3), this finally implies

$$\mathbb{P}(W > \delta^* t) \;\geq\; \sum_{n\geq 0}\mathbb{P}(W > \delta^* t|E_n)\mathbb{P}(E_n) \;\geq\; B\sum_{n\geq 0}\mathbb{P}(E_n) \;=\; B\mathbb{P}\left(\sup_{n\geq 0} W_n > t\right)$$

for all $t \in [1,\infty)$, and thus (4.2). □

Next, we verify that the conditions (4.1) imply uniform integrability of $(W_n)_{n\geq 0}$. To shorten the arising formulas, we extend the notation of the means in a natural way and put

$$\mu_{u,M} := \mathbb{E}\left(X^{(u,M)}|M\right) \quad \text{and} \quad \mu_{N,M} := \mathbb{E}\left(X^{(N,M)}|N,M\right) \tag{4.5}$$

for all $u \in \mathbb{N}$ and $\mathbb{N}_0$-valued random variables N, M.

Theorem 4.2. *If* $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 < \infty$ *and* $\mathbb{E}\left(\frac{g'_{\Lambda_0}(1)}{\gamma}\log\frac{g'_{\Lambda_0}(1)}{\gamma}\right) < 0$, *then* $\mathbb{E}W = 1$.

Proof. To prove the stated result, we use the size-biased tree introduced in Chapter 2 and show that $\hat{W} := \limsup_{n\to\infty}\hat{W}_n$ is almost surely finite. Then $\mathbb{E}W = 1$ follows by the dichotomy in Lemma 2.2(*c*).

Recalling the notation of the size-biased process, we have the recursive representation

$$\hat{\mathcal{Z}}_{n+1} \;=\; \sum_{v\in\hat{\mathbb{T}}_n}\sum_{u=1}^{\infty}\hat{Z}_{vu} \;=\; \sum_{u=1}^{\hat{T}_n}\hat{Z}_{\hat{V}_n u} + \sum_{v\in\hat{\mathbb{T}}_n\setminus\{\hat{V}_n\}}\sum_{u=1}^{T_v}\sum_{i=1}^{\hat{Z}_v} X_{i,v}^{(u,T_v)}, \qquad n \in \mathbb{N}_0\,.$$

Let us further define the σ-algebra

$$\mathcal{G} := \sigma\left((\hat{T}_n)_{n\geq 0}, (\hat{X}_n^{(\bullet,t)})_{n\geq 0, t\geq 1}, (\hat{V}_n)_{n\geq 0}\right). \tag{4.6}$$

Then we get from the above recursive formula for each $n \in \mathbb{N}_0$

$$\mathbb{E}\left(\hat{\mathcal{Z}}_{n+1}\,\middle|\,\mathcal{G}\right) \;=\; \mathbb{E}\left(\sum_{u=1}^{\hat{T}_n}\hat{Z}_{\hat{V}_n u}\,\middle|\,\mathcal{G}\right) \;+\; \mathbb{E}\left(\sum_{v\in\hat{\mathbb{T}}_n\setminus\{\hat{V}_n\}}\sum_{u=1}^{T_v}\sum_{i=1}^{\hat{Z}_v} X_{i,v}^{(u,T_v)}\,\middle|\,\mathcal{G}\right)$$

$$
\begin{aligned}
&= \mathbb{E}\left(\sum_{u=1}^{\hat{T}_n} \hat{Z}_{\hat{V}_n u} \;\middle|\; \mathcal{G}\right) + \mathbb{E}\left(\sum_{v \in \hat{\mathbb{T}}_n \setminus \{\hat{V}_n\}} \sum_{i=1}^{\hat{Z}_v} \underbrace{\mathbb{E}\left(\sum_{u=1}^{T_v} X_{i,v}^{(u,T_v)}\right)}_{=\gamma} \;\middle|\; \mathcal{G}\right) \\
&\leq \mathbb{E}\left(\sum_{u=1}^{\hat{T}_n} \hat{Z}_{\hat{V}_n u} \;\middle|\; \mathcal{G}\right) + \gamma \mathbb{E}\left(\hat{\mathcal{Z}}_n \mid \mathcal{G}\right) \\
&\leq \ldots \leq \sum_{k=0}^{n} \gamma^{n-k} \mathbb{E}\left(\sum_{u=1}^{\hat{T}_k} \hat{Z}_{\hat{V}_k u} \;\middle|\; \mathcal{G}\right) \quad \text{a.s.}
\end{aligned}
$$

Using the definition of the size-biased variables and the fact that for fixed $1 \leq u \leq t < \infty$ the random variables $(X_{i,v}^{(u,t)})_{i \in \mathbb{N}, v \in \mathbb{V}}$ are i.i.d., we further obtain

$$
\begin{aligned}
\mathbb{E}\left(\hat{\mathcal{Z}}_{n+1} \;\middle|\; \mathcal{G}\right) &\leq \sum_{k=0}^{n} \gamma^{n-k} \sum_{u=1}^{\hat{T}_k} \left(\hat{X}_k^{(u,\hat{T}_k)} + \mathbb{E}\left(\sum_{i=1}^{\hat{Z}_{\hat{V}_k}-1} X_{i,\hat{V}_k}^{(u,\hat{T}_k)} \;\middle|\; \mathcal{G}\right)\right) \\
&= \sum_{k=0}^{n} \gamma^{n-k} \sum_{u=1}^{\hat{T}_k} \left(\hat{X}_k^{(u,\hat{T}_k)} + \mathbb{E}(\hat{Z}_{\hat{V}_k} - 1 | \mathcal{G}) \underbrace{\mathbb{E}\left(X^{(u,\hat{T}_k)} \mid \hat{T}_k\right)}_{=\mu_{u,\hat{T}_k}}\right) \quad \text{a.s.} \qquad (4.7)
\end{aligned}
$$

Thus, letting n tend to infinity on the right hand side, leads to

$$
\mathbb{E}\left(\hat{W}_{n+1} \;\middle|\; \mathcal{G}\right) \leq \underbrace{\sum_{k=0}^{\infty} \frac{1}{\gamma^k} \sum_{u=1}^{\hat{T}_k} \hat{X}_k^{(u,\hat{T}_k)}}_{(*)} + \underbrace{\sum_{k=0}^{\infty} \frac{1}{\gamma^k} \mathbb{E}(\hat{Z}_{\hat{V}_k} - 1 | \mathcal{G}) \sum_{u=1}^{\hat{T}_k} \mu_{u,\hat{T}_k}}_{(**)} \qquad (4.8)
$$

a.s. for all $n \in \mathbb{N}_0$. Recall that $\gamma > 1$ by Theorem 1.10 and $\mathbb{P}(\mathsf{Ext}) < 1$. Next, we show that both sums $(*)$ and $(**)$ are almost surely finite.

Finiteness of $(*)$: By definition, the family $(\sum_{u=1}^{\hat{T}_k} \hat{X}_k^{(u,\hat{T}_k)})_{k \geq 0}$ consists of i.i.d. random variable distributed as $\hat{\mathcal{Z}}_1$. As pointed out in Remark 2.3, $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 < \infty$ is equivalent to $\mathbb{E} \log \hat{\mathcal{Z}}_1 < \infty$, and thus Lemma 3.4 implies

$$
\lim_{k \to \infty} \frac{1}{k} \log \left(\sum_{u=1}^{\hat{T}_k} \hat{X}_k^{(u,\hat{T}_k)}\right) = 0 \quad \text{a.s.} \qquad (4.9)
$$

This guarantees for almost every $\omega \in \Omega$ the existence of a $k_\omega \in \mathbb{N}$ such that

$$
\sum_{u=1}^{\hat{T}_k(\omega)} \hat{X}_k^{(u,\hat{T}_k(\omega))}(\omega) \leq \left(\frac{\gamma}{2}\right)^k
$$

for all $k \geq k_\omega$. Hence, $(*) < \infty$ a.s.

Finiteness of $(**)$: First, recall that by Theorem 2.4 $(\hat{Z}_{\hat{V}_n} - 1)_{n \geq 0}$ is a BPREI with i.i.d. environmental sequence $[\hat{U}_n, \hat{T}_n]_{n \geq 0}$ and immigration sequence $(\hat{X}_n^{(\hat{U}_n,\hat{T}_n)} - 1)_{n \geq 0}$. Consequently,

$\mu_{\hat{U}_i,\hat{T}_i}$, $i \in \mathbb{N}_0$, is the (random) reproduction mean of parasites in cell $\hat{V}_i$, and thus of the first marginal distribution of individuals in the i^{th} generation of the ABPREI (see Subsection 2.3). As previously pointed out, $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 < \infty$ implies $\mathbb{E} \log \hat{\mathcal{Z}}_1 < \infty$, and thus the immigration components satisfy

$$\mathbb{E} \log^+ \left(\hat{X}_0^{(\hat{U}_0,\hat{T}_0)} - 1 \right) \ \le \ \mathbb{E} \log \hat{\mathcal{Z}}_1 \ < \ \infty.$$

Using the assumptions in the theorem and $\mathbb{E}g'_{\Lambda_0}(1) = \gamma/\nu$ (see (1.10)), we get

$$\mathbb{E} \left(g'_{\Lambda_0}(1) \log \frac{g'_{\Lambda_0}(1)}{\gamma} \right) \ = \ \mathbb{E}g'_{\Lambda_0}(1) \log g'_{\Lambda_0}(1) - \frac{\gamma}{\nu} \log \gamma \ < 0$$

and by an appeal to (2.7)

$$\mathbb{E} \log \mu_{\hat{U}_0,\hat{T}_0} \ = \ \frac{\nu}{\gamma} \mathbb{E}g'_{\Lambda_0}(1) \log g'_{\Lambda_0}(1) \ < \ \log \gamma. \tag{4.10}$$

Thus, we find a constant $c \in (1, \gamma)$ such that $\mathbb{E} \log \mu_{\hat{U}_0,\hat{T}_0} < \log c$, and by Remark 3.10($b$), we get

$$\hat{Z}_\infty := \sup_{n \in \mathbb{N}_0} \frac{1}{c^n} \mathbb{E} \left(\hat{Z}_{\hat{V}_k} - 1 | \mathcal{G} \right) \ < \ \infty \quad \text{a.s.}$$

This consideration now leads to a new upper bound for $(**)$, namely

$$(**) \ \le \ \hat{Z}_\infty \sum_{k=0}^{\infty} \left(\frac{c}{\gamma} \right)^k \sum_{u=1}^{\hat{T}_k} \mu_{u,\hat{T}_k} \ \le \ \hat{Z}_\infty \sum_{k=0}^{\infty} \exp \left(\log \frac{c}{\gamma} + \frac{1}{k} \log^+ \left(\sum_{u=1}^{\hat{T}_k} \mu_{u,\hat{T}_k} \right) \right)^k \tag{4.11}$$

a.s. Using Jensen's inequality and (2.1), we estimate

$$\begin{aligned}
\mathbb{E} \log^+ \left(\sum_{u=1}^{\hat{T}_0} \mu_{u,\hat{T}_0} \right) \ &= \ \sum_{t \ge 1} \mathbb{P}(\hat{T}_0 = t) \log^+ \mathbb{E} \left(\sum_{u=1}^{t} X^{(u,t)} \right) \\
&= \ \sum_{t \ge 1} \frac{p_t}{\gamma} \mathbb{E} \left(\sum_{u=1}^{t} X^{(u,t)} \right) \log^+ \mathbb{E} \left(\sum_{u=1}^{t} X^{(u,t)} \right) \\
&\le \ \frac{1}{\gamma} \sum_{t \ge 1} p_t \mathbb{E} \left(\sum_{u=1}^{t} X^{(u,t)} \log^+ \sum_{u=1}^{t} X^{(u,t)} \right) \\
&= \ \frac{1}{\gamma} \mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 \ < \ \infty,
\end{aligned}$$

and since the $\sum_{u=1}^{\hat{T}_k} \mu_{u,\hat{T}_k}$, $k \ge 0$, are i.i.d., Lemma 3.4 yields

$$\limsup_{n \to \infty} \frac{1}{k} \log^+ \left(\sum_{u=1}^{\hat{T}_0} \mu_{u,\hat{T}_0} \right) \ = \ 0 \quad \text{a.s.} \tag{4.12}$$

Hence, $(**) < \infty$ a.s. by (4.11) and the just discovered asymptotic behavior.

Having verified that the sums $(*)$ and $(**)$ are almost surely finite, inequality (4.8) gives

$$\sup_{n \in \mathbb{N}_0} \mathbb{E} \left(\hat{W}_n \mid \mathcal{G} \right) \ < \ \infty \quad \text{a.s.}$$

and Fatou's lemma ensures almost sure finiteness of $\liminf_{n\to\infty} \hat{W}_n$, i.e.

$$\mathbb{P}(\liminf_{n\to\infty} \hat{W}_n < \infty) \;=\; \hat{Q}(\liminf_{n\to\infty} w_n < \infty) \;=\; 1,$$

where $\hat{Q} = \mathbb{P}(\widehat{\boldsymbol{BT}} \in \cdot)$ and $\hat{W}_n = w_n \circ \hat{\boldsymbol{BT}}$ (see (2.4) and (2.5)) should be recalled. It is left to prove that $(w_n)_{n\geq 0}$ converges $\hat{Q}$-a.s., since then $\hat{W} = \liminf_{n\to\infty} \hat{W}_n$ and $\hat{W}$ is almost surely finite, which completes the proof of the theorem.

We show that $(1/w_n)_{n\geq 0}$ is a $\hat{Q}$-supermartingale with respect to the filtration $(\mathcal{S}_n)_{n\geq 0}$ as defined in Subsection 1.1.2. The adaptivity is clear by definition. For each $n \in \mathbb{N}_0$, note that

$$\hat{Q}(w_n = 0) \;=\; \int_{\{w_n=0\}} w_n dQ \;=\; 0$$

by Lemma 2.2(b). For a probability measure $\mathcal{Q}$ let $\mathbb{E}_{\mathcal{Q}}$ denote the expectation with respect to $\mathcal{Q}$. Then for each $A \in \mathcal{S}_n \subseteq \mathcal{S}_{n+1}$, we establish by using Lemma 2.2 and Remark 2.3

$$\begin{aligned}
\int_A \mathbb{E}_{\hat{Q}}\left(\frac{1}{w_{n+1}}|\mathcal{F}_n\right) d\hat{Q} \;&=\; \int_A \frac{1}{w_{n+1}} d\hat{Q} \;=\; \mathbb{E}_{\hat{Q}}\left(\frac{1}{w_{n+1}}\mathbb{1}_{\{A\cap\{w_{n+1}>0\}\}}\right)\\
&=\; \mathbb{E}_Q\left(\frac{1}{w_{n+1}} w_{n+1}\,\mathbb{1}_{\{A\cap\{w_{n+1}>0\}\}}\right)\\
&=\; Q\left(A\cap\{w_{n+1}>0\}\right)\\
&\leq\; Q\left(A\cap\{w_n>0\}\right)\\
&=\; \int_A \frac{1}{w_n} d\hat{Q},
\end{aligned}$$

where the last equality results by following the before made transformations backwards. Hence, for each $n \in \mathbb{N}_0$

$$\mathbb{E}_{\hat{Q}}\left(\frac{1}{w_{n+1}}|\mathcal{F}_n\right) \;\leq\; \frac{1}{w_n} \quad \hat{Q}\text{-a.s.}$$

and the supermartingale property as well as the integrability are confirmed. The martingale convergence theorem ensures almost sure convergence of $(1/w_n)_{n\geq 0}$ under $\hat{Q}$ as $n \to \infty$, and thus almost sure convergence for $(w_n)_{n\geq 0}$. This completes the proof of the theorem. □

Given $\mathbb{P}(g'_{\Lambda_0}(1) \in \{\gamma, 0\}) < 1$, the conditions of the above theorem are not only sufficient but also necessary for uniform integrability of $(W_n)_{n\geq 0}$. This is stated in the next theorem.

Theorem 4.3. *Let* $\mathbb{P}(g'_{\Lambda_0}(1) \in \{\gamma, 0\}) < 1$. *If* $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 = \infty$ *or* $\mathbb{E}\left(\frac{g'_{\Lambda_0}(1)}{\gamma}\log\frac{g'_{\Lambda_0}(1)}{\gamma}\right) \geq 0$, *then* $W = 0$ *a.s.*

Proof. We use again the size-biased tree introduced in Chapter 2 and show that $\mathbb{P}(\hat{W} = \infty) = 1$ for $\hat{W} := \limsup_{n\to\infty} \hat{W}_n$. Then Lemma 2.2($c$) provides $\mathbb{P}(W = 0) = 1$.

First, note that

$$\hat{W}_n \;=\; \frac{1}{\gamma^n}\sum_{v\in\hat{\mathbb{T}}_n} \hat{Z}_v \;\geq\; \frac{1}{\gamma^n}\sum_{u=1}^{\hat{T}_{n-1}} \hat{Z}_{\hat{V}_{n-1}u} \;\geq\; \frac{1}{\gamma^n}\sum_{u=1}^{\hat{T}_{n-1}} \hat{X}_{n-1}^{(u,\hat{T}_{n-1})} \quad \text{a.s.}$$

for $n \in \mathbb{N}$. Since $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 = \infty$ gives $\mathbb{E}\log \hat{\mathcal{Z}}_1 = \infty$ by Remark 2.3 and the random sums $\sum_{u=1}^{\hat{T}_{n-1}} \hat{X}_{n-1}^{(u,\hat{T}_{n-1})}$, $n \in \mathbb{N}$, are independent and identically distributed as $\hat{\mathcal{Z}}_1$, we infer

$$\begin{aligned}\limsup_{n\to\infty} \hat{W}_n &\geq \limsup_{n\to\infty} \frac{1}{\gamma^n} \sum_{u=1}^{\hat{T}_{n-1}} \hat{X}_{n-1}^{(u,\hat{T}_{n-1})} \\ &= \limsup_{n\to\infty} \exp\left(\frac{1}{n}\log \sum_{u=1}^{\hat{T}_{n-1}} \hat{X}_{n-1}^{(u,\hat{T}_{n-1})} - \log\gamma\right)^n = \infty \quad \text{a.s.}\end{aligned}$$

by using Lemma 3.4. Hence, the assertion is proved in the case when $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 = \infty$.

Let now be $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 < \infty$. Once again, by the definition of $\hat{W}_n$, we get

$$\hat{W}_n \;=\; \frac{1}{\gamma^n}\sum_{v\in\hat{\mathbb{T}}_n} \hat{Z}_v \;\geq\; \frac{1}{\gamma^n}\hat{Z}_{\hat{V}_n} \;\geq\; \frac{1}{\gamma^n}(\hat{Z}_{\hat{V}_n} - 1) \quad \text{a.s.} \tag{4.13}$$

for $n \in \mathbb{N}_0$. As stated in the part FINITENESS OF $(**)$ in the proof of the previous theorem, $(\hat{Z}_{\hat{V}_n} - 1)_{n\geq 0}$ forms a BPREI with i.i.d. environmental sequence $[\hat{U}_n, \hat{T}_n]_{n\geq 0}$ and immigration sequence $(\hat{X}_n^{(\hat{U}_n,\hat{T}_n)} - 1)_{n\geq 0}$. The assumption $\mathbb{P}(g'_{\Lambda_0}(1) \in \{\gamma, 0\}) < 1$ implies $\mu_{u,t} \neq \gamma$ for a $1 \leq u \leq t < \infty$ with $p_t > 0$ and $\mathbb{P}(X^{(u,t)} > 0) > 0$, and thus

$$\mu_{\hat{U}_0,\hat{T}_0} \neq \gamma \quad \text{w.p.p.} \tag{4.14}$$

Furthermore,

$$\mathbb{E}\log^+\left(\hat{X}_0^{(\hat{U}_0,\hat{T}_0)} - 1\right) \;<\; \infty$$

due to $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 < \infty$. By adapting the transformations done in (4.10), we get

$$\mathbb{E}\log \mu_{\hat{U}_0,\hat{T}_0} \;=\; \frac{\nu}{\gamma}\mathbb{E}g'_{\Lambda_0}(1)\log g'_{\Lambda_0}(1) \;\geq\; \log\gamma \;>\; 0. \tag{4.15}$$

Hence, $(\hat{Z}_{\hat{V}_n} - 1)_{n\geq 0}$ has a supercritical behavior (see Subsection 2.3), and by Theorem 3.5(a), there exists an almost surely finite random variable Z_∞ such that

$$\lim_{n\to\infty} \frac{\hat{Z}_{\hat{V}_n} - 1}{\prod_{i=0}^{n-1} \mu_{\hat{U}_i,\hat{T}_i}} \;=\; Z_\infty \quad \text{a.s.} \tag{4.16}$$

Moreover, Theorem 3.5(a) provides that Z_∞ is positive almost surely since

$$\begin{aligned}\mathbb{E}\left(\frac{X^{(\hat{U}_0,\hat{T}_0)}}{\mu_{\hat{U}_0,\hat{T}_0}}\log^+ X^{(\hat{U}_0,\hat{T}_0)}\right) &= \sum_{1\leq u\leq t<\infty} \mathbb{P}\left(\hat{U}_0 = u, \hat{T}_0 = t\right)\mathbb{E}\left(\frac{X^{(u,t)}}{\mu_{u,t}}\log^+ X^{(u,t)}\right) \\ &= \frac{1}{\gamma}\sum_{1\leq u\leq t<\infty} p_t\mathbb{E}\left(X^{(u,t)}\log^+ X^{(u,t)}\right) \\ &\leq \frac{1}{\gamma}\sum_{1\leq u\leq t<\infty} p_t\mathbb{E}\left(X^{(u,t)}\log^+ \sum_{u=1}^{t} X^{(u,t)}\right) \\ &= \frac{1}{\gamma}\mathbb{E}\mathcal{Z}_1\log^+\mathcal{Z}_1 \;<\; \infty,\end{aligned}$$

where in the second equation Lemma 2.1(*b*) was used. Thus, from (4.13), (4.14), (4.15), (4.16) and the fact that the $\mu_{\hat{U}_i,\hat{T}_i}$, $i \in \mathbb{N}_0$, are i.i.d., we get

$$\hat{W} \;=\; \limsup_{n\to\infty} \hat{W}_n \;\geq\; Z_\infty \limsup_{n\to\infty} \frac{\prod_{i=0}^{n-1} \mu_{\hat{U}_i,\hat{T}_i}}{\gamma^n} \;=\; Z_\infty \exp\left(\limsup_{n\to\infty} \sum_{i=0}^{n-1} \log\left(\frac{\mu_{\hat{U}_i,\hat{T}_i}}{\gamma} \right)\right) \;=\; \infty$$

a.s. by the law of large numbers. With the statements at the beginning of the proof, the theorem is proved. □

Remark 4.4. The condition $\mathbb{P}(g'_{\Lambda_0}(1) \in \{\gamma, 0\}) < 1$ was not needed to prove $W = 0$ a.s. in the above theorem in the case where $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 = \infty$. Thus, the $(Z \log Z)$-condition is necessary for uniform integrability of $(W_n)_{n\geq 0}$ without any further assumptions.

To cover all settings, the subsequent theorem considers the case when $\mathbb{P}(g'_{\Lambda_0}(1) \in \{\gamma, 0\}) = 1$. In this situation only a single cell line can be infected, which entails the number of parasites to behave as a BPRE. Consequently, the second condition can be omitted.

Theorem 4.5. *Let* $\mathbb{P}(g'_{\Lambda_0}(1) \in \{\gamma, 0\}) = 1$*. Then* $\mathbb{E}W = 1$ *if and only if* $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 < \infty$*.*

Proof. By Remark 4.4, $\mathbb{E}W = 1$ implies $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 < \infty$. To prove the converse, first note that $\mathbb{P}(g'_{\Lambda_0}(1) \in \{\gamma, 0\}) = 1$ implies $\mu_{u,t} \in \{0, \gamma\}$ for all $1 \leq u \leq t < \infty$ if $p_t > 0$, and thus

$$\gamma \;=\; \sum_{t=0}^{\infty} p_t \sum_{u=1}^{t} \mu_{u,t} \;=\; \gamma \underbrace{\sum_{t=0}^{\infty} p_t \#\{1 \leq u \leq t : \mathbb{P}(X^{(u,t)} > 0) > 0\}}_{=:c=1}.$$

Since c denotes the mean number of cells that are able to host parasites, we get $\mathbb{E}_z \mathcal{T}_1^* \leq c = 1$ for all $z \in \mathbb{N}$. Hence,

$$\mathbb{E}\mathcal{T}_{n+1}^* \;=\; \mathbb{E}\left(\sum_{v\in\mathbb{T}_n^*} \sum_{z=1}^{\infty} \mathbb{E}_z(\mathcal{T}_1^*)\, \mathbb{1}_{\{Z_v = z\}} \right) \;\leq\; \mathbb{E}\mathcal{T}_n^* \;\leq\; \ldots \;\leq\; 1.$$

As $\mathbb{P}(\mathsf{Surv}) > 0$ is assumed, Theorem 1.10 provides $\mathbb{P}_2(\mathcal{T}_1^* \geq 2) = 0$, and by Theorem 1.7, we infer $\mathcal{T}_n^* = 1$ a.s. on Surv. So $(\mathcal{Z}_n)_{n\geq 0}$ is a BPRE (see Subsection 1.1.3) where the reproduction law in each generation has mean γ. Now, the assertion follows with [85, Theorem 2]. □

We summarize all proved results in this chapter in the following theorem. Recall that Theorem 1.14(*b*) implies $\mathbb{P}(\mathsf{Ext}) = \mathbb{P}(W > 0)$ if $(W_n)_{n\geq 0}$ is uniformly integrable.

Theorem 4.6. *Let* $\mathbb{P}(\mathsf{Ext}) < 1$*. Then* $\mathbb{E}W \in \{0, 1\}$ *and* $\mathbb{P}(W = 0) = \mathbb{P}(\mathsf{Ext})$ *if* $\mathbb{E}W = 1$*. Furthermore:*

(a) If $\mathbb{P}(g'_{\Lambda_0}(1) \in \{\gamma, 0\}) < 1$*, then*

$$\mathbb{E}W = 1 \quad \textit{iff} \quad \mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 < \infty \textit{ and } \mathbb{E}\left(\frac{g'_{\Lambda_0}(1)}{\gamma} \log \frac{g'_{\Lambda_0}(1)}{\gamma} \right) < 0.$$

(b) If $\mathbb{P}(g'_{\Lambda_0}(1) \in \{\gamma, 0\}) = 1$, *then*

$$\mathbb{E}W = 1 \quad \textit{iff} \quad \mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 < \infty.$$

Remark 4.7. If only a single daughter cell might be infected, i.e. $\mathbb{P}_2(\mathcal{T}_1^* \geq 2) = 0$, the parasite process $(\mathcal{Z}_n)_{n\geq 0}$ forms a branching process in an i.i.d. random environment, and w.o.l.g., we assume that the possible contaminated daughter cell is the first one (see Subsection 1.1.3). Hence,

$$\mathbb{E}\left(\frac{g'_{\Lambda_0}(1)}{\gamma} \log \frac{g'_{\Lambda_0}(1)}{\gamma}\right) = \frac{1}{\nu}\sum_{t\geq 1} p_t \frac{\mu_{1,t}}{\gamma} \log \frac{\mu_{1,t}}{\gamma} \geq \frac{1}{\nu}\sum_{t\geq 1} p_t \frac{\mu_{1,t}}{\gamma} \log \sum_{t\geq 1} p_t \frac{\mu_{1,t}}{\gamma} = 0$$

by Jensen's inequality, and thus $W = 0$ a.s. if $\mathbb{P}(g'_{\Lambda_0}(1) \in \{\gamma, 0\}) < 1$. This is consistent with the known results for branching processes in random environment (see [12,85]) since the martingale norming sequence $(\prod_{i=0}^n g'_{\Lambda_i}(1))_{n\geq 0}$ grows slower than γ, as $\mathbb{E}g'_{\Lambda_0}(1) = \gamma/\nu < \gamma$.

4.2 Growth rates and the problem of finding a Heyde-Seneta norming ...

In Theorem 4.6, we have seen that $W = 0$ a.s. if the $(Z \log Z)$-condition is violated. A question naturally arising is: *What is the actual growth rate of* $(\mathcal{Z}_n)_{n\geq 0}$ *in this case?* To completely answer this question, one has to construct a sequence $(c_n)_{n\geq 0}$ in $\mathbb{R}$ such that $\lim_{n\to\infty} c_n^{-1}\mathcal{Z}_n$ exists almost surely and is positive on the set of survival Surv. Such a normalization sequence is called *Heyde-Seneta norming*, and their existence was first proved for the simple GWP in [45,76] by the eponymous authors. Efforts were made to construct similar norming sequences for other branching processes with success in [25,26,32,33,84] under the usage of diverse techniques.

However, the construction of a Heyde-Seneta norming is not the easiest task, especially in our model. Since BwBPes starting with a different number of parasites are not identically distributed, no recursive representation with i.i.d. copies can be given for $\mathcal{Z}_n$, which was fundamental for the proofs in the above mentioned articles. Consequently, the techniques used there are not easily transferable to our model. Because of this reason, we are unfortunately not proving the existence of a Heyde-Seneta norming for $(\mathcal{Z}_n)_{n\geq 0}$, but the author is optimistic that this can be done, possibly under some additional assumptions. However, if the parasite multiplication is high, a suitable normalization sequence can be constructed for $(\mathcal{T}_n^*)_{n\geq 0}$. We further determine the exponential factor of the growths rates for both processes, $(\mathcal{T}_n^*)_{n\geq 0}$ and $(\mathcal{Z}_n)_{n\geq 0}$. This weaker partial result gives a first idea how the Heyde-Seneta norming should look like since it indicates that an appropriate norming sequence is the product of this exponential factor and some strictly slower increasing function.

4.2.1 ... for the process of contaminated cells

By Theorem 1.8, the process $(\nu^{-n}\mathcal{T}_n^*)_{n\geq 0}$ forms a supermartingale. Furthermore, by Proposition 1.4 the normalization can be written as $\nu^n = \mathbb{E}\mathcal{T}_n^*/\mathbb{P}(Z'_n > 0)$, $n \in \mathbb{N}_0$, and so $\mathcal{T}_n^*$ behaves like its mean for $n \to \infty$ apart from some regulation depending on the ABPRE. Since the environmental sequence of the ABPRE takes values in a countable space, [59, Theorem 1.1] states

$$\lim_{n\to\infty} \mathbb{P}(Z'_n > 0)^{1/n} \;=\; \inf_{0\le\theta\le 1} \mathbb{E} g'_{\Lambda_0}(1)^{\theta} \;=:\; \rho \tag{4.17}$$

with $\rho = 1$ if $\mathbb{E}\log g'_{\Lambda_0}(1) \ge 0$, $\rho = \nu^{-1}\gamma$ if $\mathbb{E}\log g'_{\Lambda_0}(1) < 0$ and $\mathbb{E} g'_{\Lambda_0}(1)\log g'_{\Lambda_0}(1) \le 0$, and $\rho < \min\{1, \nu^{-1}\gamma\}$ otherwise. Hence, it is justifiable to assume that the number of contaminated cells grows with the geometric rate $(\nu\rho)^n$, and thus a proper Heyde-Seneta norming should not differ much from this sequence. This is stated in the next theorem.

Theorem 4.8. *Let* $\mathbb{P}(\mathsf{Surv}) > 0$ *(and thus particularly* $\nu > 1$*). Then* $\lim_{n\to\infty} \frac{1}{n} \log \mathcal{T}^*_n = \log \nu\rho$ $\mathbb{P}_z$*-a.s. on* Surv *for all* $z \in \mathbb{N}$.

Proof. First, let us consider the case when $\mathbb{P}_2(\mathcal{T}^*_n \ge 2) = 0$. Then on Surv, there exists exactly one contaminated cell in each generation (see Theorem 1.7), and $(\mathcal{Z}_n)_{n\ge 0}$ forms a BPRE (see Subsection 1.1.3). W.o.l.g., we can assume that the contaminated cell is always the first daughter cell. Since parasites survive w.p.p., the process along the infected cell line is supercritical and thus

$$\rho \;=\; \inf_{0\le\theta\le 1} \mathbb{E} g'_{\Lambda_0}(1)^{\theta} \;=\; \frac{1}{\nu} \inf_{0\le\theta\le 1} \sum_{t=1}^{\infty} p_t \mu_{1,t}^{\theta} \;=\; \frac{1}{\nu},$$

where the last equality can be deduced with the help of Jensen's inequality. Consequently, the assertion follows in this case since $\log \mathcal{T}^*_n = 0 = \log \nu\rho$ for all $n \in \mathbb{N}_0$.

So, from now on, assume that $\mathbb{P}_2(\mathcal{T}^*_n \ge 2) > 0$. For each $\varepsilon > 0$, the Markov inequality provides

$$\sum_{n=0}^{\infty} \mathbb{P}_z \left(\left(\frac{\mathcal{T}^*_n}{\mathbb{E}_z \mathcal{T}^*_n} \right)^{1/n} \ge 1 + \varepsilon \right) \;\le\; \sum_{n=0}^{\infty} \frac{1}{(1+\varepsilon)^n} \;<\; \infty,$$

whence by the Borel-Cantelli lemma

$$\limsup_{n\to\infty} \left(\frac{\mathcal{T}^*_n}{\mathbb{E}_z \mathcal{T}^*_n} \right)^{1/n} \;\le\; 1 \quad \mathbb{P}_z\text{-a.s.}$$

But from (4.17), Proposition 1.4 and Jensen's inequality, we infer for $n \to \infty$

$$(\mathbb{E}_z \mathcal{T}^*_n)^{1/n} \;=\; \nu \mathbb{P}_z(Z'_n > 0)^{1/n} \;\le\; \nu \left(\sum_{i=1}^{z} \mathbb{P}(Z'_n > 0) \right)^{1/n} \;\le\; \nu (z \mathbb{P}(Z'_n > 0))^{1/n} \;\to\; \nu\rho,$$

and thus

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathcal{T}^*_n \;\le\; \log \nu\rho \quad \mathbb{P}_z\text{-a.s.}$$

Concerning the lower bound, assume that the result has been already shown for $z = 1$. Then for all $z \in \mathbb{N}$, we can write $\mathsf{Surv} = \bigcup_{i=1}^{z} \{\mathcal{T}^*_{n,i} \to \infty\}$, where $\mathcal{T}^*_{n,i}$ denotes the number of contaminated cells containing a descendant of the ancestor parasite i. Since $\mathcal{T}^*_{n,i}$ is distributed as $\mathcal{T}^*_n$ when starting with a single parasite, we obtain

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathcal{T}^*_n \;\ge\; \liminf_{n\to\infty} \frac{1}{n} \log \mathcal{T}^*_{n,i} \;=\; \log \nu\rho \quad \mathbb{P}_z\text{-a.s. on } \{\mathcal{T}^*_{n,i} \to \infty\}$$

for each $1 \le i \le z$ and assertion follows. So, it is left to verify the lower bound for $z = 1$.

Theorem 1.7 entails $\mathcal{T}_n^* \to \infty$ a.s. on Surv for $n \to \infty$ and applying Fatou's lemma gives

$$\infty \;=\; \mathbb{E} \liminf_{n\to\infty} \mathcal{T}_n^* \;\le\; \liminf_{n\to\infty} \mathbb{E}\mathcal{T}_n^*. \tag{4.18}$$

In the following, we describe the construction of a sequence $(\mathbb{T}_{n,m}^*)_{n\ge 0}$ of sets of contaminated cells for each $m \in \mathbb{N}$. Step one, put $\mathbb{T}_{0,m}^* := \{\varnothing\}$ and assume the root cell to host one parasite. Next, put $\mathbb{T}_{1,m}^* := \mathbb{T}_m^*$ as the set of contaminated cell in generation m. Now, consider a cell in $\mathbb{T}_{1,m}^*$, choose from its parasites an arbitrary one and identify all contaminated cells in generation $2m$ which contain descendants of this parasite. Repeat this procedure with all cells in $\mathbb{T}_{1,m}^*$ and denote the union of all thus identified cells in generation $2m$ by $\mathbb{T}_{2,m}^*$. Use the same procedure to construct $\mathbb{T}_{n+1,m}^*$ from $\mathbb{T}_{n,m}^*$ for all $n \in \mathbb{N}$. This gives us a sequence $(\mathbb{T}_{n,m}^*)_{n\ge 0}$ of sets of contaminated cells in the generations $0, m, 2m, 3m, \ldots$ Let us put

$$S_{n,m} := \#\, \mathbb{T}_{n,m}^* \tag{4.19}$$

for $n \in \mathbb{N}_0$ as the cardinal number of these sets. Clearly,

$$\mathcal{T}_{nm}^* \;\ge\; S_{n,m} \quad \mathbb{P}\text{-a.s.}$$

for all $n \in \mathbb{N}_0$, and $(S_{n,m})_{n\ge 0}$ forms a simple GWP with reproduction law $\mathbb{P}(\mathcal{T}_m^* \in \cdot)$ and reproduction mean $\mathbb{E}\mathcal{T}_m^*$. (4.18) ensures $\mathbb{E}\mathcal{T}_m^* > 1$ and thus supercriticality of $(S_{n,m})_{n\ge 0}$ for all large m. For $m \in \mathbb{N}$ denote by Surv_m the set of non-extinction of $(S_{n,m})_{n\ge 0}$, Obviously, $\mathsf{Surv}_m \subseteq \mathsf{Surv}$ for all $m \in \mathbb{N}$. Fix m_0 such that $\mathbb{P}(\mathsf{Surv}_{m_0}) > 0$ and note that $\mathsf{Surv}_{m_0} \subseteq \mathsf{Surv}_{2m_0} \subseteq \cdots \subseteq \mathsf{Surv}$ a.s. because a GWP considered only at the points in time $l\,\mathbb{N}_0$ for a $l \in \mathbb{N}$ is also a GWP and survives if the original one does. Using these inclusions and the branching property of a GWP, we obtain

$$\mathbb{P}(\mathsf{Surv}_{km_0}) \;=\; \sum_{t=1}^{\infty} \mathbb{P}(\mathcal{T}_{km_0}^* = t)(1 - \mathbb{P}(\mathsf{Surv}_{km_0}^c)^t) \;\ge\; (1 - \mathbb{P}(\mathsf{Surv}_{m_0}^c)^s)\mathbb{P}(\mathcal{T}_{km_0}^* \ge s)$$

for all $s \in \mathbb{N}$ and $k \in \mathbb{N}$. Hence,

$$\mathbb{P}\left(\bigcup_{k\ge 0} \mathsf{Surv}_{km_0}\right) \;=\; \lim_{k\to\infty} \mathbb{P}(\mathsf{Surv}_{km_0}) \;\ge\; (1 - \mathbb{P}(\mathsf{Surv}_{m_0}^c)^s)\mathbb{P}(\mathsf{Surv})$$

for all $s \in \mathbb{N}$, and since $\mathbb{P}(\mathsf{Surv}_{m_0}) > 0$ is assumed, we get

$$\bigcup_{k\ge 0} \mathsf{Surv}_{km_0} = \mathsf{Surv} \quad \mathbb{P}\text{-a.s.} \tag{4.20}$$

by letting s tend to infinity in the above inequality.

Let now $m \in m_0\,\mathbb{N}$. For each $n \in \mathbb{N}$, we can find $k_n \in \mathbb{N}$ and $l_n \in \{0, \ldots, m-1\}$ such that $n = k_n m + l_n$ and thus on Surv_m

$$\mathcal{T}_n^* \;\ge\; \sum_{v \in \mathbb{T}_{n,m}^*} \mathcal{T}_{l_n,v}^* \quad \mathbb{P}\text{-a.s.},$$

where $\mathcal{T}^*_{l_n,v}$ denotes the number of contaminated cells in generation n rooted in cell v. So by Jensen's inequality, this yields on Surv_m

$$\log \mathcal{T}^*_n \;=\; \log^+ \mathcal{T}^*_n \;\geq\; \frac{1}{S_{k_n,m}} \sum_{v\in\mathbb{T}^*_{n,m}} \log^+ \mathcal{T}^*_{l_n,v} + \log^+ S_{k_n,m} \;\geq\; \log^+ S_{k_n,m} \quad \mathbb{P}\text{-a.s.},$$

and the classical theory of GWPes (see for example the Heyde-Seneta theorem [10, Theorem 5.1 in Chapter II]) provides

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathcal{T}^*_n \;\geq\; \liminf_{n\to\infty} \frac{1}{n} \log^+ S_{k_n,m} \;=\; \frac{1}{m} \log \mathbb{E}\mathcal{T}^*_m \;=\; \log\nu + \frac{1}{m}\log \mathbb{P}(Z'_m > 0) \quad \mathbb{P}\text{-a.s.}$$

on Surv_m, where in the last equation Proposition 1.4 has been used. As $m = km_0$ for arbitrary $k \in \mathbb{N}$, (4.17) now gives

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathcal{T}^*_n \;\geq\; \log\nu + \lim_{k\to\infty} \frac{1}{km_0} \log \mathbb{P}(Z'_{km_0} > 0) \;=\; \log\nu\rho \quad \mathbb{P}_z\text{-a.s.}$$

on Surv, by recalling (4.20). This proves the theorem. ☐

If the ABPRE survives w.p.p., $(\mathcal{T}^*_n)_{n\geq 0}$ has nearly the same growth rate as the GWP $(\mathcal{T}_n)_{n\geq 0}$ (see Theorem 1.8). Hence, the Heyde-Seneta norming of $(\mathcal{T}_n)_{n\geq 0}$ gives the right normalization for the process of contaminated cells in this case.

Theorem 4.9. *Let* $\mathbb{P}(\mathsf{Surv}) > 0$ *(and thus particularly* $\nu > 1$*) and* $z \in \mathbb{N}$*. If* $\mathbb{E}\log g'_{\Lambda_0}(1) > 0$ *and* $\mathbb{E}\log^-(1 - g_{\Lambda_0}(0)) < \infty$*, then there exists a sequence* $(c_n)_{n\geq 0}$ *in* $(0,\infty)$ *such that* $c_{n+1}/c_n \to \nu$ *and* $(c_n^{-1}\mathcal{T}^*_n)_{n\geq 0}$ *converges* $\mathbb{P}_z$*-a.s. as* $n \to \infty$ *to a finite random variable* $\tilde{L}$ *which satisfies* $\mathbb{P}_z(\tilde{L} = 0) = \mathbb{P}_z(\mathsf{Ext})$*.*

Proof. Fix $z \in \mathbb{N}$. First, note that $\mathbb{E}\log^-(1 - g_{\Lambda_0}(0)) < \infty$ ensures $\mu_{u,t} > 0$ if $p_t > 0$, and so $\mathbb{P}_2(\mathcal{T}^*_1 \geq 2) > 0$, due to $\nu > 1$. W.o.l.g. assume that $\mathbb{E}T\log T = \infty$, as otherwise $(\nu^n)_{n\geq 0}$ is a suitable norming sequence by Theorem 1.8. For each $a > 0$ such that $\mathbb{E}T\,\mathbb{1}_{\{T\leq a\}} > 1$, we define

$$c_0(a) := a \qquad \text{and} \qquad c_{n+1}(a) \;:=\; c_n(a)\mathbb{E}\left(T\,\mathbb{1}_{\{T\leq c_n(a)\}}\right), \quad n \in \mathbb{N}_0, \tag{4.21}$$

and let $(c_n)_{n\geq 0}$ be such a sequence for a fixed a. Notice that every sequence $(c_n(a))_{n\geq 0}$ is determined by the choice of a and recall that $(\mathcal{T}_n)_{n\geq 0}$ is a supercritical GWP with reproduction law $\mathcal{L}(T)$ and mean ν. Then the classical theory of GWPes (see e.g. [10, Chapter II]) provides that each thus defined sequence $(c_n(a))_{n\geq 0}$ forms a suitable Heyde-Seneta norming for $(\mathcal{T}_n)_{n\geq 0}$ with

$$\lim_{n\to\infty} \frac{c_{n+1}(a)}{c_n(a)} = \nu \qquad \text{and} \qquad \lim_{a\to\infty} \frac{1}{a}\sum_{n=0}^{\infty} c_n(a)\mathbb{P}(T > c_n(a)) = 0. \tag{4.22}$$

Furthermore, for each sequence $(c_n(a))_{n\geq 0}$ there exists a constant $y(a) \in (0,\infty)$ such that $c_n(a)/c_n \to y(a)$ as $n \to \infty$.

For $n \in \mathbb{N}_0$ and $v \in \mathbb{V}$ with $|v| = n$, put

$$T_v(a) := T_v\,\mathbb{1}_{\{T_v\leq c_n(a)\}}, \tag{4.23}$$

and let $\mathbb{T}_n^*(a)$, $\mathcal{T}_n^*(a)$ and $\mathcal{T}_n(a)$ denote the obvious in a BwBP with an underlying cell process given by $(T_v(a))_{v\in\mathbb{V}}$. It is well-known from the classical theory that the process $(c_n^{-1}(a)\mathcal{T}_n(a))_{n\geq 0}$, is a L^2-bounded martingale (see e.g. the proof of [10, Theorem 5.6 in Chapter II]). As in the proof of Theorem 1.8, we calculate for $n\in\mathbb{N}_0$

$$\begin{aligned}\mathbb{E}_z\left(\mathcal{T}_{n+1}^*(a)\mid\mathcal{F}_n\right) &= \sum_{v\in\mathbb{T}_n^*(a)}\mathbb{E}_z\left(\sum_{u=1}^{T_v(a)}\mathbb{1}_{\{Z_{vu}>0\}}\mid\mathcal{F}_n\right)\\ &\leq \mathbb{E}(T\,\mathbb{1}_{\{T\leq c_n(a)\}})\mathcal{T}_n^*(a) = \frac{c_{n+1}(a)}{c_n(a)}\mathcal{T}_n^*(a)\quad\mathbb{P}_z\text{-a.s.}\end{aligned}$$

Hence, $(c_n^{-1}(a)\mathcal{T}_n^*(a))_{n\geq 0}$ forms a positive supermartingale with $\mathbb{E}_z\mathcal{T}_n^*(a)\leq c_n(a)/a$, and since the obvious majorant $(c_n^{-1}(a)\mathcal{T}_n(a))_{n\geq 0}$ is L^2-bounded, we find an almost surely finite random variable $L(a)$ such that

$$\frac{\mathcal{T}_n^*(a)}{c_n(a)}\to L(a)\quad\mathbb{P}_z\text{-a.s. and in }L^1\tag{4.24}$$

as $n\to\infty$. The rest of the proof is split into several parts.

CONVERGENCE OF $\mathcal{T}_n^*/c_n$: With calculations as in the proof of [25, Proposition 1], we get

$$\begin{aligned}\mathbb{P}_z(\mathcal{T}_n^*(a)\neq\mathcal{T}_n^*\text{ for some }n\in\mathbb{N}_0) &= \sum_{n=1}^{\infty}\mathbb{P}_z(\mathcal{T}_1^*(a)=\mathcal{T}_1^*,\ldots,\mathcal{T}_{n-1}^*(a)=\mathcal{T}_{n-1}^*,\mathcal{T}_n^*(a)\neq\mathcal{T}_n^*)\\ &\leq \sum_{n=1}^{\infty}\sum_{x=0}^{\infty}\mathbb{P}_z(\mathcal{T}_{n-1}^*(a)=\mathcal{T}_{n-1}^*=x,\mathcal{T}_n^*(a)\neq\mathcal{T}_n^*)\\ &\leq \sum_{n=1}^{\infty}\sum_{x=0}^{\infty}\mathbb{P}_z(\mathcal{T}_{n-1}^*(a)=x)x\mathbb{P}(T>c_{n-1}(a))\\ &\leq \sum_{n=1}^{\infty}\mathbb{E}_z\mathcal{T}_{n-1}^*(a)\mathbb{P}(T>c_{n-1}(a))\\ &\leq \frac{1}{a}\sum_{n=1}^{\infty}c_{n-1}(a)\mathbb{P}(T>c_{n-1}(a))\;\to\;0\quad\text{as }a\to\infty,\end{aligned}$$

where the convergence follows from (4.22). Hence, by (4.24) we get for almost every $\omega\in\Omega$ the existence of an a_0 such that for all $a\geq a_0$

$$\frac{\mathcal{T}_n^*(\omega)}{c_n} = \frac{c_n(a)}{c_n}\frac{\mathcal{T}_n^*(a)(\omega)}{c_n(a)}\;\to\; y(a)L(a)(\omega)\tag{4.25}$$

for an appropriate constant $y(a)\in(0,\infty)$. Hence, $(c_n^{-1}\mathcal{T}_n^*)_{n\geq 0}$ converges $\mathbb{P}_z$-a.s. to a random variable $\tilde{L}$.

$\tilde{L}$ IS POSITIVE W.P.P.: By (4.25), it is enough to show $\mathbb{P}(L(a)>0)>0$ for some $a>0$. Let $\Lambda(a)=(\Lambda_n(a))_{n\geq 0}$ be a sequence of independent random variables taking values in the set of probability measures on $\mathbb{N}_0$ such that

$$\mathbb{P}\left(\Lambda_n(a)=\mathcal{L}(X^{(u,t)})\right) = \frac{p_t}{\mathbb{E}\left(T\,\mathbb{1}_{\{T\leq c_n(a)\}}\right)} = \frac{c_n(a)}{c_{n+1}(a)}p_t\tag{4.26}$$

for all $n \in \mathbb{N}_0$ and $1 \le u \le t \le c_n(a)$. Let further $(Z'_n(a))_{n\ge 0}$ be a branching process with environmental sequence $\Lambda(a)$, and let $g_{\Lambda_n(a)}(s)$ denote the random generating function of the individuals in the n^{th} generation. Recall that $(Z'_n)_{n\ge 0}$ is the ABPRE with environmental sequence Λ (see Subsection 1.2.1). Clearly,

$$\mathbb{P}_z(Z'_n(a) > 0|\Lambda(a) = \lambda) \;=\; \mathbb{P}_z(Z'_n > 0|\Lambda = \lambda)$$

as well as

$$\mathbb{P}(\Lambda_0(a) = \lambda_0, \ldots, \Lambda_n(a) = \lambda_n) \;=\; \frac{1}{c_n(a)} \prod_{k=0}^{n} p_{t_k}$$

for a sequence of probability measures $\lambda = (\lambda_k)_{k\ge 0}$ with $\lambda_k = \mathcal{L}(X^{(u_k,t_k)})$ and $u_k \le t_k \le c_k(a)$ for each $k \in \mathbb{N}_0$. Hence, by merely adjusting the summations in the proof of Proposition 1.4, we obtain for each $n \in \mathbb{N}_0$

$$\mathbb{P}(Z'_n(a) > 0) \;=\; c_n^{-1}(a)\mathbb{E}\mathcal{T}_n^*(a), \tag{4.27}$$

and as $(c_n^{-1}(a)\mathcal{T}_n^*(a))_{n\ge 0} \to L(a)$ in mean, we get

$$\mathbb{E}L(a) \;=\; \lim_{n\to\infty} \mathbb{P}(Z'_n(a) > 0).$$

For $\lambda = \mathcal{L}(X^{(u,t)})$ and $K > 0$ let

$$g_{\lambda,K}(s) \;=\; \sum_{k=0}^{K-1} s^k \mathbb{P}(X^{(u,t)} = k) \;+\; s^K \mathbb{P}(X^{(u,t)} \ge K)$$

be the generating function of the truncated random variable $X^{(u,t)} \wedge K$. As truncation reduces the reproduction, obviously

$$\mathbb{E}L(a) \;=\; \lim_{n\to\infty} \mathbb{P}(Z'_n(a) > 0) \;\ge\; \lim_{n\to\infty} \mathbb{P}(Z'_{n,K}(a) > 0),$$

where $(Z'_{n,K}(a))_{n\ge 0}$ is the branching process with environmental sequence $\Lambda(a)$ and truncated reproduction laws. The truncation further guarantees $\sup_{n\ge 0} g''_{\Lambda_n(a),K}(1)/g'_{\Lambda_n(a),K}(1) < \infty$ a.s. and hence

$$\lim_{n\to\infty} \mathbb{P}(Z'_{n,K}(a) > 0) > 0 \qquad \text{if} \qquad \sum_{n=0}^{\infty} \left(\prod_{i=0}^{n+1} g'_{\Lambda_i(a),K}(1) \right)^{-1} < \infty \quad \mathbb{P}\text{-a.s.} \tag{4.28}$$

by Agresti [3, Theorem 1].

Due to the assumptions in the theorem, [4, Theorem 2.1] gives the existence of a constant $K > 0$ such that

$$0 < \mathbb{E} \log g'_{\Lambda_0,K}(1) < \infty.$$

A look at (4.26) shows that

$$\mathbb{P}\left(\Lambda_n(a) = \mathcal{L}(X^{(u,t)})\right) \;=\; \frac{c_n(a)}{c_{n+1}(a)} p_t \;\to\; \frac{p_t}{\nu} \;=\; \mathbb{P}\left(\Lambda_0 = \mathcal{L}(X^{(u,t)})\right) \quad \text{as } n \to \infty,$$

and by an appeal to (4.22) thus

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\log g'_{\Lambda_n(a),K}(1) &= \lim_{n\to\infty}\sum_{1\le u\le t\le c_n(a)}\frac{c_n(a)}{c_{n+1}(a)}p_t\log\mathbb{E}(X^{(u,t)}\wedge K)\\ &= \sum_{1\le u\le t<\infty}\frac{p_t}{\nu}\log\mathbb{E}(X^{(u,t)}\wedge K) \;=\; \mathbb{E}\log g'_{\Lambda_0,K}(1).\end{aligned}$$

Furthermore, we obtain for all $x>0$

$$\begin{aligned}\mathbb{P}\left(\log^{\pm} g'_{\Lambda_n(a),K}(1)>x\right) &= \sum_{\substack{1\le u\le t\le c_n(a),\\ \log^{\pm}\mu_{u,t}>x}}\frac{c_n(a)}{c_{n+1}(a)}p_t\\ &\le \frac{a\nu}{c_1(a)}\sum_{\substack{1\le u\le t<\infty,\\ \log^{\pm}\mu_{u,t}>x}}\frac{p_t}{\nu} \;=\; \frac{a\nu}{c_1(a)}\mathbb{P}\left(\log^{\pm} g'_{\Lambda_0,K}(1)>x\right)\end{aligned}$$

and therefore the law of large numbers in Theorem B.1 ensures the existence of an almost surely finite random variable G such that

$$\frac{1}{n}\sum_{k=0}^{n-1}\log g'_{\Lambda_k(a),K}(1) \;\ge\; \frac{1}{2}\mathbb{E}\log g'_{\Lambda_0,K}(1) \;>\; 0 \qquad \text{for all } n\ge G.$$

But from this, we deduce

$$\begin{aligned}\sum_{n=0}^{\infty}\prod_{i=0}^{n+1}\frac{1}{g'_{\Lambda_i(a),K}(1)} &= \sum_{n=0}^{G-1}\prod_{i=0}^{n+1}\frac{1}{g'_{\Lambda_i(a),K}(1)} \;+\; \sum_{n=G}^{\infty}\exp\left(-\sum_{i=0}^{n+1}\log g'_{\Lambda_i(a),K}(1)\right)\\ &\le \sum_{n=0}^{G-1}\prod_{i=0}^{n+1}\frac{1}{g'_{\Lambda_i(a),K}(1)} \;+\; \sum_{n=G}^{\infty}\exp\left(-\frac{1}{2}\mathbb{E}\log g'_{\Lambda_0,K}(1)\right)^{n+1}\\ &< \infty \quad \text{.a.s.},\end{aligned}$$

and hence, $\mathbb{E}L(a)>0$ by an appeal to (4.28).

$\tilde{L}$ VANISHES ONLY ON Ext: Adapting the proof of Theorem 1.8(b), we set $\tau_n := \inf\{m\in\mathbb{N}_0\,|\,\mathcal{T}^*_m\ge n\}$ for each $n\in\mathbb{N}$. Then

$$\begin{aligned}\mathbb{P}_z(\tilde{L}=0) &\le \mathbb{P}_z(\tilde{L}=0|\tau_n<\infty)+\mathbb{P}_z(\tau_n=\infty)\\ &= \mathbb{P}_z\left(\lim_{m\to\infty}\frac{c_m}{c_{m+\tau_n}}\sum_{v\in\mathbb{T}^*_{\tau_n}}c_m^{-1}t^*_m(\boldsymbol{BT}^{(v)})=0\;\middle|\;\tau_n<\infty\right)+\mathbb{P}_z(\tau_n=\infty)\\ &\le \mathbb{P}\left(\bigcap_{k=1}^{n}\{\mathcal{T}^*_{m,k}/c_m\to 0\}\right)+\mathbb{P}_z(\tau_n=\infty)\\ &\le \mathbb{P}(\tilde{L}=0)^n+\mathbb{P}_z(\tau_n=\infty),\end{aligned}$$

where in the penultimate inequality we used (4.22). As $\mathbb{P}(\tilde{L}=0)<1$, letting n tend to infinity completes the proof of the theorem by an appeal to Theorem 1.7. □

4.2.2 ... for the process of parasites

Turning now to the process of parasites, we show that the norming sequence can not differ much from $(\gamma^n)_{n\geq 0}$, viz. $n^{-1}\log \mathcal{Z}_n \to \log\gamma$ a.s. for $n\to\infty$ on Surv. In the case $\mathbb{P}_2(\mathcal{T}_1^* \geq 2) = 0$, $(\mathcal{Z}_n)_{n\geq 0}$ forms a BPRE (see Subsection 1.1.3). A suitable norming was already found in [84, Theorem 1]. This case particularly comprises $P(g'_{\Lambda_0}(1) \in \{\gamma, 0\}) = 1$ as seen in the proof of Theorem 4.5. Hence, we can restrict our analysis to the case when two contaminated daughter cells may occur. As in the previous section, it is sufficient to consider only the standard starting configuration (one cell with one parasite).

Theorem 4.10. *If* $\mathbb{P}(\mathsf{Ext}) < 1$, $\mathbb{P}_2(\mathcal{T}_1^* \geq 2) > 0$ *and* $\mathbb{E}\left(\frac{g'_{\Lambda_0}(1)}{\gamma}\log\frac{g'_{\Lambda_0}(1)}{\gamma}\right) < 0$, *then* $W_n^{1/n} \to 1$ *a.s. on* Surv *as* $n\to\infty$.

Proof. Since $(W_n)_{n\geq 0}$ converges almost surely to a finite random variable by Proposition 1.12, it immediately follows that $\limsup_{n\to\infty} W_n^{1/n} \leq 1$ a.s.

To derive the other direction, we divide the proof into two cases and follow the truncation argumentation given in [25]. Note that $\mathbb{P}_2(\mathcal{T}_1^* \geq 2) > 0$ implies $P(g'_{\Lambda_0}(1) \in \{\gamma, 0\}) < 1$ (see the discussion above the theorem).

Case I: Let T be bounded, i.e. $T \leq c$ a.s. for a finite constant $c > 0$. For $a > 0$, we define for each $1 \leq u \leq t \leq c$

$$X^{(u,t)}(a) := X^{(u,t)}\,\mathbb{1}_{\{X^{(u,t)}\leq a\}}$$

and let $(\mathcal{Z}_n(a))_{n\geq 0}$ be the process of parasites and $(\mathcal{T}_n^*(a))_{n\geq 0}$ the process of contaminated cells having the truncated reproductions laws. Let further be $\gamma(a) = \mathbb{E}\mathcal{Z}_1(a)$ as well as $g_{\Lambda_0,a}(s)$ and $W_n(a)$ the obvious random variables. Since $\mathcal{T}_n^*(a) \uparrow \mathcal{T}_n^*$ as $a\to\infty$ for each $n\in\mathbb{N}$, we get for large $a > 0$ that $\mathbb{P}_2(\mathcal{T}_1^*(a) \geq 2) > 0$ as well as $\sup_{n\geq 0}\mathbb{E}\mathcal{T}_n^*(a) > 1$. Thus, $\mathbb{P}(\mathcal{Z}_n(a) \to 0) < 1$ by Theorem 1.10. Moreover,

$$\begin{aligned}
\mathbb{E}\left(g'_{\Lambda_0,a}(1)\log\frac{g'_{\Lambda_0,a}(1)}{\gamma(a)}\right) &= \mathbb{E}\left(g'_{\Lambda_0,a}(1)\log g'_{\Lambda_0,a}(1)\right) - \mathbb{E}\left(g'_{\Lambda_0,a}(1)\log\gamma(a)\right) \\
&\leq \mathbb{E}\left(g'_{\Lambda_0}(1)\log g'_{\Lambda_0}(1)\right) - \mathbb{E}\left(g'_{\Lambda_0,a}(1)\log\gamma(a)\right) \\
&\searrow \mathbb{E}\left(g'_{\Lambda_0}(1)\log g'_{\Lambda_0}(1)\right) - \mathbb{E}\left(g'_{\Lambda_0}(1)\log\gamma\right) \qquad \text{as } a\to\infty \\
&= \mathbb{E}\left(g'_{\Lambda_0}(1)\log\frac{g'_{\Lambda_0}(1)}{\gamma}\right) \; < \; 0,
\end{aligned}$$

since $\gamma(a)$ is isotone in a. Hence, by assumption there exists an $a_0 > 0$ such that for all $a \geq a_0$

$$\mathbb{E}\left(g'_{\Lambda_0,a}(1)\log\frac{g'_{\Lambda_0,a}(1)}{\gamma(a)}\right) \; < \; 0.$$

As $\mathbb{E}\mathcal{Z}_1(a)\log\mathcal{Z}_1(a) \leq ac\log ac$, Theorem 4.6 implies the existence of a finite random variable $W(a)$ such that $W_n(a) \to W(a)$ in L^1 as $n\to\infty$. In particular, $\mathbb{P}(W(a) > 0) > 0$.

Let now be $\varepsilon > 0$ and fix $a \geq a_0$ such that

$$\gamma(a) \; \geq \; (1-\varepsilon)\gamma,$$

which yields

$$\mathbb{E}\mathcal{Z}_n(a) = \gamma(a)^n \geq (1-\varepsilon)^n\gamma^n$$

for all $n \in \mathbb{N}_0$. Let $(\mathcal{Z}_{n,k}(a))_{n\geq 0}$ be the parasite process, where parasites in the first k generations beget offspring according to the original reproduction laws and from generation $k+1$ on with the truncated laws. By the above established lower bound of the means, we get

$$\mathbb{E}\mathcal{Z}_{n,k}(a) = \gamma^k \mathbb{E}\mathcal{Z}_{n-k}(a) \geq (1-\varepsilon)^n\gamma^n$$

for all $k \leq n$, $k, n \in \mathbb{N}_0$. Additionally, we find that

$$\frac{\mathcal{Z}_n}{(1-\varepsilon)^n\gamma^n} \;\geq\; \frac{\mathcal{Z}_{n,k}(a)}{(1-\varepsilon)^n\gamma^n} \;\geq\; \frac{\mathcal{Z}_{n,k}(a)}{\mathbb{E}\mathcal{Z}_{n,k}(a)} \;\geq\; \frac{1}{\gamma^k}\sum_{v\in\mathbb{T}_k^*} \frac{\mathcal{Z}_{n-k}^{(v)}(a)}{\mathbb{E}\mathcal{Z}_{n-k}(a)}$$

a.s., where $\mathcal{Z}_{n-k}^{(v)}(a)$, $v \in \mathbb{T}_n^*$, are i.i.d. random variables having the same law as $\mathcal{Z}_{n-k}(a)$ when starting with a single parasite. Because of our choice of a, taking the limit in the above inequality yields

$$\liminf_{n\to\infty} \frac{\mathcal{Z}_n}{(1-\varepsilon)^n\gamma^n} \;\geq\; \liminf_{n\to\infty} \frac{1}{\gamma^k}\sum_{v\in\mathbb{T}_k^*} \frac{\mathcal{Z}_{n-k}^{(v)}(a)}{\mathbb{E}\mathcal{Z}_{n-k}(a)} \;=\; \frac{1}{\gamma^k}\sum_{v\in\mathbb{T}_k^*} W^{(v)}(a),$$

where $W^{(v)}(a)$, $v \in \mathbb{T}_n^*$, are independent and distributed as $W(a)$ (under $\mathbb{P}$). Recalling that $\mathcal{F}_k$ is the σ-algebra of the k-past, we get from this inequality

$$\begin{aligned}\mathbb{P}\left(\liminf_{n\to\infty} \frac{\mathcal{Z}_n}{(1-\varepsilon)^n\gamma^n} > 0 \;\middle|\; \mathcal{F}_k\right) \;&\geq\; \mathbb{P}\left(\frac{1}{\gamma^k}\sum_{v\in\mathbb{T}_k^*} W^{(v)}(a) > 0 \;\middle|\; \mathcal{F}_k\right)\\ &=\; 1 - \mathbb{P}(W(a)=0)^{\mathcal{T}_k^*} \quad \text{a.s.}\end{aligned}$$

Since $\mathbb{P}(W(a) > 0) > 0$, we conclude by recalling Theorem 1.7 and letting k tend to infinity

$$\mathsf{Surv} = \{\mathcal{T}_n^* \to \infty\} \subseteq \left\{\liminf_{n\to\infty} \frac{\mathcal{Z}_n}{(1-\varepsilon)^n\gamma^n} > 0\right\} \quad \text{a.s.,}$$

and we finally get

$$\liminf_{n\to\infty} W_n^{1/n} \;\geq\; 1-\varepsilon \quad \text{a.s.}$$

on the survival set Surv. Hence, the theorem is proved in the first case.

CASE II: Let T be unbounded. We reduce this case to considerations of a bounded T by truncation and use the results of CASE I. For $b > 0$, we define

$$T(b) := T\,\mathbb{1}_{\{T\leq b\}}\,.$$

Let $(\mathcal{Z}_n(b))_{n\geq 0}$ be the process of parasites and $(\mathcal{T}_n^*(b))_{n\geq 0}$ the process of contaminated cells having the truncated reproductions law for the cells. Additionally, let $\gamma(b) = \mathbb{E}\mathcal{Z}_1(b)$, $\nu(b) = \mathbb{E}\mathcal{T}_1(b)$ and $g_{\Lambda_0,b}(s)$ be the generating function of the ABPRE of the truncated BwBP. For the truncated process, we get

$$\mathbb{E}\left(g'_{\Lambda_0,b}(1)\log g'_{\Lambda_0,b}(1)\right) \;=\; \sum_{1\leq u\leq t\leq b} \frac{p_t}{\nu(b)}\mu_{u,t}\log\mu_{u,t} \;\xrightarrow{b\to\infty}\; \mathbb{E}\left(g'_{\Lambda_0}(1)\log g'_{\Lambda_0}(1)\right)$$

as well as

$$\mathbb{E}\left(g'_{\Lambda_0,b}(1)\log\gamma(b)\right) \;=\; \frac{\nu}{\nu(b)}\frac{\gamma(b)}{\nu}\log\gamma(b) \;\xrightarrow{b\to\infty}\; \frac{\gamma}{\nu}\log\gamma \;=\; \mathbb{E}g'_{\Lambda_0}(1)\log\gamma \;\in(0,\infty).$$

Putting these two equations together and using $\gamma(b)\uparrow\gamma$ as $b\to\infty$, we obtain

$$\begin{aligned}\mathbb{E}\left(g'_{\Lambda_0,b}(1)\log\frac{g'_{\Lambda_0,b}(1)}{\gamma(b)}\right) &= \mathbb{E}\left(g'_{\Lambda_0,b}(1)\log g'_{\Lambda_0,b}(1)\right)-\mathbb{E}\left(g'_{\Lambda_0,b}(1)\log\gamma(b)\right)\\ &\xrightarrow{b\to\infty}\; \mathbb{E}\left(g'_{\Lambda_0}(1)\log g'_{\Lambda_0}(1)\right)-\mathbb{E}\left(g'_{\Lambda_0}(1)\log\gamma\right)\\ &= \mathbb{E}\left(g'_{\Lambda_0}(1)\log\frac{g'_{\Lambda_0}(1)}{\gamma}\right) \;<\; 0.\end{aligned}$$

Hence, as in the first case, for each $\varepsilon>0$, we can fix $b>0$ such that $\gamma(b)\geq(1-\varepsilon)\gamma$, $\mathbb{P}_2(\mathcal{T}_1^*(b)\geq 2)>0$, $\mathbb{P}(\mathcal{Z}_n(b)\to 0)<1$ and $\mathbb{E}\left(g'_{\Lambda_0,b}(1)\log\frac{g'_{\Lambda_0,b}(1)}{\gamma(b)}\right)<0$. Thus all conditions for Case I are fulfilled which then implies

$$\liminf_{n\to\infty}W_n^{1/n} \;\geq\; (1-\varepsilon)\liminf_{n\to\infty}\left(\frac{\mathcal{Z}_n(b)}{\gamma(b)^n}\right)^{1/n} \;\geq\; 1-\varepsilon \quad \text{a.s.}$$

This completes the proof. □

4.3 Relative proportions of contaminated cells

In this section, we are concerned with the long-run behavior of the relative proportions of contaminated cells containing a given number of parasites, viz.

$$F_n(k) := \frac{\#\{v\in\mathbb{T}_n^*\,|Z_v=k\}}{\mathcal{T}_n^*}$$

for $k\in\mathbb{N}$ and $n\to\infty$. We consider $F_n(k)$ under the event of parasite survival. Recall that $\mathbb{P}_z^*$, $z\in\mathbb{N}$, denotes the measure $\mathbb{P}_z$ conditioned under Surv. *We assume throughout this section that*

$$\mathbb{P}_2(\mathcal{T}_1^*\geq 2)>0,$$

as otherwise $\mathcal{T}_n^*=1$ $\mathbb{P}_z^*$-a.s. for all $z\in\mathbb{N}$ and $F_n(k)=\mathbb{1}_{\{\mathcal{Z}_n=k\}}$. Since $(\mathcal{Z}_n)_{n\geq 0}$ is BPRE in this situation, the known theory gives $\mathcal{Z}_n\to\infty$ on Surv and thus $F_n(k)\to 0$ a.s. (see [13]).

The asymptotic behavior of $F_n(k)$ is highly dependent on the long-term behavior of the parasite number along a cell line and thus of the ABPRE. Because of this reason, we need to consider different cases. Theorem 4.11 deals with the situation when parasites multiply at a high rate, viz. the ABPRE is supercritical. In this case the number of parasites in a cell line converges to infinity and thus every cell is hosting a large number of parasites. This entails convergence in probability of $F_n(k)$ to 0 for all $k\in\mathbb{N}$. If, on the other hand, the multiplication of parasites in low, that is when the ABPRE is strongly subcritical ($\mathbb{E}g'_{\Lambda_0}(1)\log g'_{\Lambda_0}(1)<0$), $F_n(k)$ stabilizes and converges to a deterministic limit as $n\to\infty$. This is shown in Theorem 4.12. These results are generalizations of [15, Theorem 5.1 and Theorem 5.2], where the underlying cell tree was assumed to be binary.

Before stating our results in detail, we recall that $\mathbb{P}(\mathsf{Surv})>0$ and thus $\nu>1$ is assumed.

Theorem 4.11. *Let* $\mathbb{E}\log g'_{\Lambda_0}(1) > 0$ *and* $\mathbb{E}\log^-(1 - g_{\Lambda_0}(0)) < \infty$*. Then for all* $z, k \in \mathbb{N}$ *and* $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}^*_z\left(F_n(k) > 1-\varepsilon\right) \;=\; 0.$$

Proof. Let $z \in \mathbb{N}$ and $\varepsilon, \eta > 0$. Due to the assumptions of the theorem and Theorem 4.9, we can find a sequence of positive numbers $(c_n)_{n\ge0}$ and a finite random variable $\tilde{L}$ such that $\mathbb{P}^*_z(\tilde{L} = 0) = 0$ and $\mathcal{T}^*_n \ge c_n\tilde{L}$ $\mathbb{P}_z$-a.s. for all $n \in \mathbb{N}_0$. So, we infer

$$\begin{aligned}\mathbb{P}^*_z\left(F_n(k) > 1-\varepsilon\right) \;&\le\; \mathbb{P}^*_z\left(F_n(k) > 1-\varepsilon, \tilde{L} \ge \eta\right) + \mathbb{P}^*_z(0 < \tilde{L} < \eta)\\ &\le\; \mathbb{P}^*_z\left(c_n^{-1}\#\{v \in \mathbb{T}^*_n \,|Z_v = k\} > (1-\varepsilon)\eta\right) + \mathbb{P}^*_z(0 < \tilde{L} < \eta).\end{aligned} \tag{4.29}$$

Recall the construction of the Heyde-Seneta normalization in the proof of Theorem 4.9. For each large $a > 0$, the sequence $(c_n(a))_{n\ge0}$ as defined in (4.21) is a suitable norming sequence and

$$\lim_{n\to\infty} c_n(a)/c_n = y(a) \tag{4.30}$$

for a constant $y(a) \in (0,\infty)$. Furthermore, the number of contaminated cells $\mathcal{T}^*_n(a)$ of the BwBP with the underlying cell tree generated by the truncated reproduction laws defined in (4.23) fulfills

$$\mathbb{P}_z(\mathcal{T}^*_n(a) \neq \mathcal{T}^*_n \text{ for some } n \in \mathbb{N}_0) \;\le\; \eta$$

for all large $a > 0$ (see the part CONVERGENCE OF $\mathcal{T}^*_n/c_n$). With the same argumentation to prove relation (4.27), we establish for all $k \in \mathbb{N}$

$$\mathbb{P}(Z'_n(a) = k) \;=\; c_n^{-1}(a)\mathbb{E}\#\{v \in \mathbb{T}^*_n(a)|Z_v = k\},$$

where $(Z'_n(a))_{n\ge}$ is the ABPRE of the truncated BwBP with environmental sequence $(\Lambda_n(a))_{n\ge0}$ given by (4.26) and generating functions $g_{\Lambda_n(a)}(s)$. Taking all these observations into account, we infer from (4.29) by using the Markov inequality

$$\begin{aligned}\mathbb{P}^*_z\left(F_n(k) > 1-\varepsilon\right) \;&\le\; \mathbb{P}^*_z\left(c_n^{-1}\#\{v \in \mathbb{T}^*_n \,|Z_v = k\} > (1-\varepsilon)\eta\right) + \mathbb{P}^*_z(0 < \tilde{L} < \eta)\\ &\le\; \frac{\mathbb{P}_z\left(c_n^{-1}\#\{v \in \mathbb{T}^*_n(a)|Z_v = k\} > (1-\varepsilon)\eta\right)}{\mathbb{P}_z(\mathsf{Surv})} + \eta + \mathbb{P}^*_z(0 < \tilde{L} < \eta)\\ &\le\; \frac{\mathbb{E}_z\left(c_n^{-1}\#\{v \in \mathbb{T}^*_n(a)|Z_v = k\}\right)}{\mathbb{P}_z(\mathsf{Surv})(1-\varepsilon)\eta} + \eta + \mathbb{P}^*_z(0 < \tilde{L} < \eta)\\ &\le\; \frac{c_n(a)}{c_n}\,\frac{\mathbb{P}_z(Z'_n(a) = k)}{\mathbb{P}_z(\mathsf{Surv})(1-\varepsilon)\eta} + \eta + \mathbb{P}^*_z(0 < \tilde{L} < \eta).\end{aligned}$$

Since $\mathbb{E}\log g'_{\Lambda_0}(1) > 0$, there exists a $1 \le u \le t < \infty$ such that $p_t > 0$ and $\mathbb{P}(X^{(u,t)} \ge 2) =: \alpha > 0$. Recalling (4.21) and (4.26), this implies

$$\mathbb{P}\left(g'_{\Lambda_n(a)}(0) \le 1 - \alpha/2\right) \;\ge\; \frac{c_n(a)}{c_{n+1}(a)}p_t \;\ge\; \frac{p_t}{\nu}$$

for large $a > 0$ and all $n \in \mathbb{N}$, and the Borel-Cantelli lemma entails

$$\sum_{n=1}^{\infty}(1 - g'_{\Lambda_n(a)}(0)) \;=\; \infty \quad \text{a.s.}$$

Hence, [30, Theorem 3] ensures $\lim_{n\to\infty} \mathbb{P}(Z'_n(a) = k) = 0$ for all $k \in \mathbb{N}$ and thus

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}^*_z\left(F_n(k) > 1-\varepsilon\right) &\leq \lim_{n\to\infty} \frac{c_n(a)}{c_n} \frac{\mathbb{P}_z(Z'_n(a) = k)}{\mathbb{P}_z(\mathsf{Surv})(1-\varepsilon)\eta} + \eta + \mathbb{P}^*_z(0 < \tilde{L} < \eta) \\ &= \eta + \mathbb{P}^*_z(0 < \tilde{L} < \eta),\end{aligned}$$

where (4.30) was used. Letting η tend to 0 yields the assertion. □

Turning now to the case when the ABPRE is strongly subcritical, we show convergence in probability of $F_n(k)$ to a deterministic limit, which can be determined as the quasi-stationary distribution of the ABPRE.

Theorem 4.12. *If* $\mathbb{E}\left(g'_{\Lambda_0}(1)\log g'_{\Lambda_0}(1)\right) < 0$,

$$\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 < \infty \quad \text{and} \quad \mathbb{E}\left(\frac{g'_{\Lambda_0}(1)}{\gamma} \log \frac{g'_{\Lambda_0}(1)}{\gamma}\right) < 0,$$

then for all $z, k \in \mathbb{N}$ *and* $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}^*_z\left(|F_n(k) - q_k| > \varepsilon\right) = 0,$$

where $q_k = \lim_{n\to\infty} \mathbb{P}(Z'_n = k | Z'_n > 0)$. *Furthermore,*

$$\frac{\mathcal{T}^*_n}{\gamma^n} \xrightarrow{\mathbb{P}^*_z} \frac{W}{\sum_{k=1}^\infty k q_k} \quad \text{as } n \to \infty$$

with W *being the martingale limit defined in Proposition 1.12.*

By our assumptions, all conditions for the proof of [15, Theorem 5.2] and the subsequent corollaries are fulfilled. Hence, Theorem 4.12 follows with similar arguments. However, a bit of work is still left to do since the parasite multiplication depends on the cell tree structure, which is different in [15]. To verify the assertions of the theorem, we require some preliminaries, and the next lemma comprises analogous results to [15, Lemma 6.1, Proposition 6.3, Lemma 6.5].

Lemma 4.13. *The following assertions hold true under the assumptions in Theorem 4.12:*

(a) For all $\eta > 0$ *and* $z \in \mathbb{N}$

$$\lim_{K\to\infty} \sup_{n\geq 0} \mathbb{P}^*_z\left(\frac{\sum_{v\in\mathbb{T}^*_n} Z_v \mathbb{1}_{\{Z_v > K\}}}{\mathcal{Z}_n} \geq \eta\right) = 0.$$

(b) For every $\varepsilon > 0$ *and* $z \in \mathbb{N}$ *there exists constants* $0 < a < b < \infty$ *such that*

$$\inf_{n\geq 0} \mathbb{P}^*_z\left(\frac{\mathcal{T}^*_n}{\gamma^n} \in [a, b]\right) \geq 1 - \varepsilon.$$

(c) For all $\eta > 0$ *and* $z \in \mathbb{N}$

$$\lim_{K\to\infty} \sup_{n,m\geq 0} \mathbb{P}^*_z\left(\frac{\#\{v \in \mathbb{T}^*_{n+m} : Z_{v|n} > K\}}{\mathcal{T}^*_{n+m}} \geq \eta\right) = 0.$$

Proof. Let $\eta > 0$ and $z \in \mathbb{N}$. By Theorem 4.6 there exists an almost surly finite random variable C such that $C \le \gamma^{-n}\mathcal{Z}_n$ $\mathbb{P}_z$-a.s. for all $n \in \mathbb{N}_0$ and $\mathbb{P}_z^*(C = 0) = 0$.

(a) For $K, n \in \mathbb{N}$ define

$$A_n(K,\eta) := \left\{ \frac{\sum_{v\in\mathbb{T}_n^*} Z_v \,\mathbb{1}_{\{Z_v>K\}}}{\mathcal{Z}_n} \ge \eta \right\} \cap \mathsf{Surv}\,.$$

So with C given above

$$\mathbb{E}_z\left(\sum_{v\in\mathbb{T}_n^*} Z_v \mathbb{1}_{\{Z_v>K\}}\right) \;\ge\; \eta\mathbb{E}_z\left(\mathcal{Z}_n\,\mathbb{1}_{A_n(K,\eta)}\right) \;\ge\; \gamma^n\eta\mathbb{E}_z\left(C\mathbb{1}_{A_n(K,\eta)}\right)$$

and thus by using Proposition 1.4

$$\eta\mathbb{E}_z\left(C\mathbb{1}_{A_n(K,\eta)}\right) \;\le\; \frac{1}{\gamma^n}\sum_{k>K} k\mathbb{E}_z\left(\#\{v\in\mathbb{T}_n^* : Z_v = k\}\right) \;=\; \left(\frac{\nu}{\gamma}\right)^n \mathbb{E}_z\left(Z_n'\mathbb{1}_{\{Z_n'>K\}}\right).$$

Since the ABPRE is strongly supercritical with $\mathbb{E}Z_1'\log Z_1' < \infty$, [2, Theorem 1.1 and Corollary 2.3] together with [17, Theorem 2] yield

$$\lim_{K\to\infty}\sup_{n\ge 0}\mathbb{E}_z\left(C\mathbb{1}_{A_n(K,\eta)}\right) \;=\; 0,$$

and as C is positive on Surv, the assertion follows.

(b) For each $K, n \in \mathbb{N}$ obtain

$$\frac{\mathcal{Z}_n}{\gamma^n} \;\ge\; \frac{\mathcal{T}_n^*}{\gamma^n} \;\ge\; \frac{\sum_{v\in\mathbb{T}_n^*} Z_v\,\mathbb{1}_{\{Z_v\le K\}}}{K\gamma^n} \;\ge\; \frac{C}{K}\frac{\sum_{v\in\mathbb{T}_n^*} Z_v\,\mathbb{1}_{\{Z_v\le K\}}}{\mathcal{Z}_n} \qquad \mathbb{P}_z\text{-a.s.},$$

and the assertion follows from Theorem 4.6 and (a).

(c) For $\varepsilon > 0$ fix $a > 0$ as in (b). Then for all $n, m \in \mathbb{N}$ on $\{\mathcal{T}_{n+m}^* \ge \gamma^{n+m}a\}$

$$\frac{\#\{v\in\mathbb{T}_{n+m}^* : Z_{v|n} > K\}}{\mathcal{T}_{n+m}^*} \;\le\; \frac{\#\{v\in\mathbb{T}_{n+m}^* : Z_{v|n} > K\}}{a\gamma^{n+m}} \;\le\; \frac{\sum_{v\in\mathbb{T}_{n+m}^*} \mathcal{Z}_m^{(v)}\,\mathbb{1}_{\{Z_v>K\}}}{a\gamma^{n+m}} \qquad \mathbb{P}_z\text{-a.s.},$$

where $\mathcal{Z}_m^{(v)}$ denotes the number of parasites in generation m of the subtree rooted in v. Thus,

$$\begin{aligned}\mathbb{P}_z^*\left(\frac{\#\{v\in\mathbb{T}_{n+m}^* : Z_{v|n} > K\}}{\mathcal{T}_{n+m}^*} \ge \eta\right) \;&\le\; \mathbb{P}_z^*\left(\frac{\sum_{v\in\mathbb{T}_n^*} \mathcal{Z}_m^{(v)}\,\mathbb{1}_{\{Z_v>K\}}}{a\gamma^{n+m}} \ge \eta\right) + \varepsilon \\ &\le\; \frac{1}{\eta a\gamma^n}\mathbb{E}_z^*\left(\sum_{v\in\mathbb{T}_n^*} Z_v\,\mathbb{1}_{\{Z_v>K\}}\right) + \varepsilon,\end{aligned}$$

for all $n, m \in \mathbb{N}_0$. For the last inequality, we recall that $\mathbb{E}_x\mathcal{Z}_m = x\gamma^m$ for all $x \in \mathbb{N}$. As seen in the proof of (a), the last term converges uniformly in n to 0 for $K \to \infty$, and the lemma is proved. □

Proof of Theorem 4.12: Let $\varepsilon, \eta > 0$ and $k, z \in \mathbb{N}$. First, note that

$$F_{n+m}(k) \;=\; \frac{\#\{v \in \mathbb{T}^*_{n+m} : Z_v = k, Z_{v|n} \le K\}}{\mathcal{T}^*_{n+m}} + \frac{\#\{v \in \mathbb{T}^*_{n+m} : Z_v = k, Z_{v|n} > K\}}{\mathcal{T}^*_{n+m}} \quad \mathbb{P}_z\text{-a.s.}$$

for all $n, m \in \mathbb{N}_0$ and $K > 0$. By Lemma 4.13(*c*), we can find a $K > 0$ large enough such that

$$\begin{aligned}
\mathbb{P}^*_z\left(|F_{n+m}(k) - q_k| \ge 4\eta\right) \;&\le\; \mathbb{P}^*_z\left(\left|\frac{\#\{v \in \mathbb{T}^*_{n+m} : Z_v = k, Z_{v|n} \le K\}}{\mathcal{T}^*_{n+m}} - q_k\right| \ge 3\eta\right) + \varepsilon \\
&\le\; \mathbb{P}^*_z\left(\frac{\#\{v \in \mathbb{T}^*_{n+m} : Z_{v|n} \le K\}}{\mathcal{T}^*_{n+m}}\left|\frac{\#\{v \in \mathbb{T}^*_{n+m} : Z_v = k, Z_{v|n} \le K\}}{\#\{v \in \mathbb{T}^*_{n+m} : Z_{v|n} \le K\}} - q_k\right| \ge 2\eta\right) \\
&\qquad + \mathbb{P}^*_z\left(\frac{\#\{v \in \mathbb{T}^*_{n+m} : Z_{v|n} > K\}}{\mathcal{T}^*_{n+m}} q_k \ge \eta\right) + \varepsilon \\
&\le\; \mathbb{P}^*_z\left(\left|\frac{\#\{v \in \mathbb{T}^*_{n+m} : Z_v = k, Z_{v|n} \le K\}}{\#\{v \in \mathbb{T}^*_{n+m} : Z_{v|n} \le K\}} - q_k\right| \ge 2\eta\right) + 2\varepsilon
\end{aligned} \tag{4.31}$$

for all $n, m \in \mathbb{N}_0$. We estimate the probability in (4.31) in the following two steps.

STEP 1: We show that there exists a $m_0 \in \mathbb{N}$ such that

$$\left|\frac{\sum_{v \in \mathbb{T}^*_n} \mathbb{1}_{\{Z_v \le K\}}\, \mathbb{E}_{Z_v} \#\{u \in \mathbb{T}^*_{m_0} : Z_u = k\}}{\sum_{v \in \mathbb{T}^*_n} \mathbb{1}_{\{Z_v \le K\}}\, \mathbb{E}_{Z_v} \mathcal{T}^*_{m_0}} - q_k\right| \;\le\; \eta \quad \mathbb{P}_z\text{a.s.}$$

for all $n \in \mathbb{N}$.

For that purpose, let us put

$$\mathcal{T}_m(k) := \#\{v \in \mathbb{T}_m : Z_v = k\}, \quad \text{for } m, k \in \mathbb{N}\,.$$

First, observe that

$$\frac{\mathbb{E}\mathcal{T}_m(k)}{\mathbb{E}\mathcal{T}^*_m} \;=\; \mathbb{P}(Z'_m = k | Z'_m > 0) \to q_k \quad \text{as } m \to \infty \tag{4.32}$$

by Proposition 1.4 and [40, Theorem 1.1]. Furthermore, by [17, Theorem 2 and Theorem 7] and once again Proposition 1.4, we get

$$\frac{\mathbb{E}_x\mathcal{T}_m(k)}{\mathbb{E}\mathcal{T}_m(k)} = \frac{\mathbb{P}_x(Z'_m = k | Z'_m > 0)}{\mathbb{P}(Z'_m = k | Z'_m > 0)} \frac{\mathbb{P}_x(Z'_m > 0)}{\mathbb{P}(Z'_m > 0)} \;\to\; x \qquad \text{and} \qquad \frac{\mathbb{E}_x\mathcal{T}^*_m}{\mathbb{E}\mathcal{T}^*_m} = \frac{\mathbb{P}_x(Z'_m > 0)}{\mathbb{P}(Z'_m > 0)} \;\to\; x$$

for all $x \in \mathbb{N}$ as $m \to \infty$. Hence,

$$\left|\frac{\mathbb{E}_x\mathcal{T}_m(k)}{\mathbb{E}\mathcal{T}_m(k)} - x\right| \;\le\; \frac{\eta}{4} \quad \text{and} \quad \left|\frac{\mathbb{E}_x\mathcal{T}^*_m}{\mathbb{E}\mathcal{T}^*_m} - x\right| \;\le\; \frac{\eta}{4}$$

and thus

$$|\mathbb{E}_x\mathcal{T}_m(k)\mathbb{E}\mathcal{T}^*_m - \mathbb{E}\mathcal{T}_m(k)\mathbb{E}_x\mathcal{T}^*_m| \;\le\; \frac{\eta}{2}\mathbb{E}\mathcal{T}_m(k)\mathbb{E}\mathcal{T}^*_m$$

for all $1 \le x \le K$ and all large m. From this, we infer

$$\left|\frac{\sum_{v \in \mathbb{T}^*_n} \mathbb{1}_{\{Z_v \le K\}}\, \mathbb{E}_{Z_v}\mathcal{T}_m(k)}{\sum_{v \in \mathbb{T}^*_n} \mathbb{1}_{\{Z_v \le K\}}\, \mathbb{E}_{Z_v}\mathcal{T}^*_m} - \frac{\mathbb{E}\mathcal{T}_m(k)}{\mathbb{E}\mathcal{T}^*_m}\right| \;\le\; \frac{\sum_{v \in \mathbb{T}^*_n} \mathbb{1}_{\{Z_v \le K\}}\, |\mathbb{E}_{Z_v}\mathcal{T}_m(k)\mathbb{E}\mathcal{T}^*_m - \mathbb{E}\mathcal{T}_m(k)\mathbb{E}_{Z_v}\mathcal{T}^*_m|}{\mathbb{E}\mathcal{T}^*_m \sum_{v \in \mathbb{T}^*_n} \mathbb{1}_{\{Z_v \le K\}}\, \mathbb{E}_{Z_v}\mathcal{T}^*_m}$$

$$\le\ \frac{\eta}{2}\,\frac{\sum_{v\in\mathbb{T}_n^*}\mathbb{1}_{\{Z_v\le K\}}\,\mathbb{E}\mathcal{T}_m(k)}{\sum_{v\in\mathbb{T}_n^*}\mathbb{1}_{\{Z_v\le K\}}\,\mathbb{E}_{Z_v}\mathcal{T}_m^*}\ \le\ \frac{\eta}{2}\quad \mathbb{P}_z\text{-a.s.}$$

for all large m. By an appeal to (4.32), Step 1 is proved.

Step 2: Fix $m_0\in\mathbb{N}$ according to Step 1. By an appeal to (4.31), the first convergence statement of the theorem follows if the probability

$$\mathbb{P}_z^*\left(\left|\frac{\#\{v\in\mathbb{T}_{n+m_0}^*: Z_v=k, Z_{v|n}\le K\}}{\#\{v\in\mathbb{T}_{n+m_0}^*: Z_{v|n}\le K\}}-\frac{\sum_{v\in\mathbb{T}_n^*}\mathbb{1}_{\{Z_v\le K\}}\,\mathbb{E}_{Z_v}\mathcal{T}_{m_0}(k)}{\sum_{v\in\mathbb{T}_n^*}\mathbb{1}_{\{Z_v\le K\}}\,\mathbb{E}_{Z_v}\mathcal{T}_{m_0}^*}\right|\ge\eta\right)$$

becomes small for large n.

Indeed, it is

$$\#\{v\in\mathbb{T}_{n+m_0}^*: Z_v=k, Z_{v|n}\le K\}\ =\ \sum_{x=1}^{K}\sum_{v\in\mathbb{T}_n^*}\mathbb{1}_{\{Z_v=x\}}\,\mathcal{T}_{m_0}^{(v)}(k)$$

for each $n\in\mathbb{N}_0$, where $\mathcal{T}_{m_0}^{(v)}(k)=\#\{u\in\mathbb{T}_{m_0}^{(v)}: Z_u=k\}$ and $\mathbb{T}_{m_0}^{(v)}$ denotes the set of cells in generation m_0 of the subtree rooted in v. Obviously, for each $x\in\{1,\dots,K\}$, all $\mathcal{T}_{m_0}^{(v)}(k)$ with $Z_v=x$ are i.i.d. conditioned upon $\{\mathcal{Z}_n>0\}$. Put $\mathbb{P}_z^n:=\mathbb{P}_z(\cdot|\mathcal{Z}_n>0)$ for each $n\in\mathbb{N}$ and observe that

$$\sup_{A\in\mathfrak{F}}|\mathbb{P}_z^n(A)-\mathbb{P}_z^*(A)|\ \to\ 0\quad\text{as } n\to\infty \tag{4.33}$$

by easy calculations. Then, the law of large numbers ensures for each $\eta'>0$ the existence of a $N_0>0$ such that

$$\mathbb{P}_z^n\left(\left|\frac{\sum_{v\in\mathbb{T}_n^*}\mathbb{1}_{\{Z_v=x\}}\,\mathcal{T}_{m_0}^{(v)}(k)-\mathbb{E}_x\mathcal{T}_{m_0}^{(v)}(k)}{\#\{v\in\mathbb{T}_n^*: Z_n=x\}}\right|\ge\frac{\eta'}{K},\ \#\{v\in\mathbb{T}_n^*: Z_n=x\}\ge N_0\right)\ \le\ \frac{\varepsilon}{2K}$$

for all $1\le x\le K$ and $n\in\mathbb{N}$, and by (4.33), we can find a $n_0\in\mathbb{N}$ such that

$$\mathbb{P}_z^*\left(\left|\frac{\sum_{v\in\mathbb{T}_n^*}\mathbb{1}_{\{Z_v=x\}}\,\mathcal{T}_{m_0}^{(v)}(k)-\mathbb{E}_x\mathcal{T}_{m_0}^{(v)}(k)}{\#\{v\in\mathbb{T}_n^*: Z_n=x\}}\right|\ge\frac{\eta'}{K},\ \#\{v\in\mathbb{T}_n^*: Z_n=x\}\ge N_0\right)\ \le\ \frac{\varepsilon}{K} \tag{4.34}$$

for all $1\le x\le K$ and $n\ge n_0$. Furthermore, the law of large numbers gives the existence of a finite random variable C such that

$$\mathbb{P}_z^n\left(\left|\frac{\sum_{v\in\mathbb{T}_n^*}\mathbb{1}_{\{Z_v=x\}}\,\mathcal{T}_{m_0}^{(v)}(k)-\mathbb{E}_x\mathcal{T}_{m_0}^{(v)}(k)}{\#\{v\in\mathbb{T}_n^*: Z_n=x\}}\right|\ge y\right)\ \le\ \mathbb{P}\,(C\ge y)$$

for all $1\le x\le K$, $n\in\mathbb{N}$ and $y>0$. Hence, there exists a $y_0>0$ such that

$$\mathbb{P}_z^*\left(\left|\frac{\sum_{v\in\mathbb{T}_n^*}\mathbb{1}_{\{Z_v=x\}}\,\mathcal{T}_{m_0}^{(v)}(k)-\mathbb{E}_x\mathcal{T}_{m_0}^{(v)}(k)}{\#\{v\in\mathbb{T}_n^*: Z_n=x\}}\right|\ge y_0\right)\ \le\ \frac{\varepsilon}{K} \tag{4.35}$$

for all $1\le x\le K$ and $n\in\mathbb{N}$ by using agian (4.33). Fix $N>N_0Ky_0/\eta'$. Since $\mathcal{T}_n^*\to\infty$ $\mathbb{P}_z^*$-a.s. for $n\to\infty$, Lemma 4.13(c) gives the existence of a $n(\eta')>n_0$ such that

$$\mathbb{P}_z^*\,(\#\{v\in\mathbb{T}_n^*: Z_n\le K\}\ge N)\ \ge\ 1-\varepsilon \tag{4.36}$$

for all $n \geq n(\eta')$. Using the estimations (4.34), (4.35) and (4.36), we infer

$$\begin{aligned}
&\mathbb{P}_z^*\left(\left|\frac{\#\{v \in \mathbb{T}_{n+m_0}^* : Z_v = k, Z_{v|n} \leq K\} - \sum_{v\in\mathbb{T}_n^*} \mathbb{1}_{\{Z_v\leq K\}}\, \mathbb{E}_{Z_v}\mathcal{T}_{m_0}(k)}{\#\{v \in \mathbb{T}_n^* : Z_v \leq K\}}\right| \geq \eta'\right)\\
\leq\ & \varepsilon + \mathbb{P}_z^*\left(\left|\frac{\sum_{v\in\mathbb{T}_n^*} \mathbb{1}_{\{Z_v\leq K\}}(\mathcal{T}_{m_0}^{(v)}(k) - \mathbb{E}_{Z_v}\mathcal{T}_{m_0}(k))}{\#\{v \in \mathbb{T}_n^* : Z_v \leq K\}}\right| \geq \eta',\ \#\{v \in \mathbb{T}_n^* : Z_v \leq K\} \geq N\right)\\
\leq\ & \varepsilon + \sum_{x=1}^{K}\mathbb{P}_z^*\left(\left|\frac{\sum_{v\in\mathbb{T}_n^*} \mathbb{1}_{\{Z_v=x\}}(\mathcal{T}_{m_0}^{(v)}(k) - \mathbb{E}_{Z_v}\mathcal{T}_{m_0}(k))}{\#\{v \in \mathbb{T}_n^* : Z_v \leq K\}}\right| \geq \frac{\eta'}{K},\ \#\{v \in \mathbb{T}_n^* : Z_v \leq K\} \geq N\right)\\
\leq\ & 2\varepsilon + \sum_{x=1}^{K}\mathbb{P}_z^*\left(\left|\frac{\sum_{v\in\mathbb{T}_n^*} \mathbb{1}_{\{Z_v=x\}}(\mathcal{T}_{m_0}^{(v)}(k) - \mathbb{E}_{Z_v}\mathcal{T}_{m_0}(k))}{\#\{v \in \mathbb{T}_n^* : Z_v \leq K\}}\right| \geq \frac{\eta'}{K},\ \frac{\#\{v \in \mathbb{T}_n^* : Z_v = x\}}{\#\{v \in \mathbb{T}_n^* : Z_v \leq K\}} \leq \frac{N_0}{N}\right)\\
\leq\ & 3\varepsilon + \sum_{x=1}^{K}\mathbb{P}_z^*\left(\frac{\eta'}{Ky_0} \leq \frac{\#\{v \in \mathbb{T}_n^* : Z_v = x\}}{\#\{v \in \mathbb{T}_n^* : Z_v \leq K\}} \leq \frac{N_0}{N}\right)\\
=\ & 3\varepsilon
\end{aligned}$$

for all $\eta' > 0$ and $n \geq n(\eta')$. With the same arguments, we derive for all $\eta'' > 0$ the existence of a $n(\eta'') > 0$ such that

$$\mathbb{P}_z^*\left(\left|\frac{\#\{v \in \mathbb{T}_{n+m_0}^* : Z_{v|n} \leq K\} - \sum_{v\in\mathbb{T}_n^*} \mathbb{1}_{\{Z_v\leq K\}}\, \mathbb{E}_{Z_v}\mathcal{T}_{m_0}^*}{\#\{v \in \mathbb{T}_n^* : Z_v \leq K\}}\right| \geq \eta'\right) \ \leq\ \varepsilon$$

for all $n \geq n(\eta'')$. So, with the right choice of $\eta' > 0$ and $\eta'' > 0$, we get

$$\mathbb{P}_z^*\left(\left|\frac{\#\{v \in \mathbb{T}_{n+m_0}^* : Z_v = k, Z_{v|n} \leq K\}}{\#\{v \in \mathbb{T}_{n+m_0}^* : Z_{v|n} \leq K\}} - \frac{\sum_{v\in\mathbb{T}_n^*} \mathbb{1}_{\{Z_v\leq K\}}\, \mathbb{E}_{Z_v}\mathcal{T}_{m_0}(k)}{\sum_{v\in\mathbb{T}_n^*} \mathbb{1}_{\{Z_v\leq K\}}\, \mathbb{E}_{Z_v}\mathcal{T}_{m_0}^*}\right| \geq \eta\right) \ \leq\ 5\varepsilon$$

for all large n. Hence, the second step and thus the first convergence statement of the theorem is proved.

For the second limit statement, first recall that the mean of $(q_k)_{k\geq1}$ is finite by [40, Theorem 1.1]. Moreover, note that for every $K \in \mathbb{N}$ and $n \in \mathbb{N}_0$

$$\mathcal{T}_n^* \ =\ \frac{\sum_{v\in\mathbb{T}_n^*} Z_v\, \mathbb{1}_{\{Z_v\leq K\}}}{\sum_{k=1}^{K} kF_n(k)}$$

and thus

$$\left|\frac{\mathcal{T}_n^*}{\mathcal{Z}_n} - \frac{1}{\sum_{k=1}^{\infty} kq_k}\right| \ \leq\ \left|\frac{1}{\sum_{k=1}^{K} kF_n(k)} - \frac{1}{\sum_{k=1}^{\infty} kq_k}\right| + \frac{1}{\sum_{k=1}^{\infty} kq_k}\left|\frac{\sum_{v\in\mathbb{T}_n^*} Z_v\, \mathbb{1}_{\{Z_v\leq K\}}}{\mathcal{Z}_n} - 1\right| \ \mathbb{P}_z\text{-a.s.}$$

So, applying Lemma 4.13(a), Theorem 4.6 and the convergence in probability of $F_n(k)$ to q_k for each $k \in \mathbb{N}$ yields the assertion. □

Chapter 5

Limit theorems for the BwBP in the case $\mathbb{P}(\mathsf{Surv}) = 0$

This chapter is devoted to the case when almost sure extinction of parasites holds true, namely $\mathbb{P}(\mathsf{Ext}) = 1$*, which is a standing assumption unless stated otherwise.* First, we look at the asymptotic behavior of the survival probability and determine its decay rate. Afterwards, finer convergence results are established while analyzing the BwBP conditioned under non-extinction of parasites at present time. This is done with the help of the size-biased construction (see Chapter 2) and will lead to a theorem similar to the classical one of Yaglom (see [14, Chapter I.8, Corollary 1]).

5.1 Convergence rate of the survival probability

In the simple Galton-Watson setting, the problem of finding the speed of extinction is completely solved and known under Kolmogorov's theorem (see the standard literature [14, 46]). More precisely, this result says that in the subcritical regime the survival probability decreases as fast as the population means if the $(Z \log Z)$-condition is valid. This result has been expanded to more complex branching processes like the BPRE in [35, 40].

Here, we prove an analogous theorem confirming that the survival probability has the rate γ^n if the ABPREI is positive recurrent, which particularly holds true if the ABPRE is strongly subcritical, and an additional integrability assumption is satisfied (see Theorem 5.5). But first, we prove a slightly weaker result that determines the exponential decay rates of $\mathbb{P}(\mathcal{Z}_n > 0)$ in all cases. Recall that S^* is the set of configurations of contaminated cells and the parasites they contain, see (1.15).

Theorem 5.1. *Let* $\mathbb{P}(\mathsf{Ext}) = 1$ *and* $\rho = \inf_{0 \le \theta \le 1} \mathbb{E} g'_{\Lambda_0}(1)^\theta$ *as in* (4.17). *Then for each* $(s, z) \in S^*$

$$\lim_{n\to\infty} \mathbb{P}_{(s,z)}(\mathcal{Z}_n > 0)^{1/n} \;=\; \lim_{n\to\infty} \mathbb{P}_{(s,z)}(\mathcal{T}_n^* > 0)^{1/n} \;=\; \nu\rho$$

with $\rho = 1$ *if* $\mathbb{E} \log g'_{\Lambda_0}(1) \ge 0$, $\rho = \nu^{-1}\gamma$ *if* $\mathbb{E} \log g'_{\Lambda_0}(1) < 0$ *and* $\mathbb{E} g'_{\Lambda_0}(1) \log g'_{\Lambda_0}(1) \le 0$, *and* $\rho < \min\{1, \nu^{-1}\gamma\}$ *otherwise.*

Proof. Let $(s,z) \in S^*$ and let $\mathcal{Z}_{n,i}$ denote the progeny number in the n^{th} generation stemming from parasite $i \in \{1,\dots,\sum_{j=1}^s z_j\}$. Since $\mathcal{Z}_{n,i}$ has the distribution $\mathbb{P}(\mathcal{Z}_n \in \cdot)$, the subadditivity of measures gives

$$\mathbb{P}(\mathcal{Z}_n > 0) \ \le\ \mathbb{P}_{(s,z)}(\mathcal{Z}_n > 0) \ =\ \mathbb{P}_{(s,z)}\left(\bigcup_{i=1}^{\sum_{j=1}^s z_j} \{\mathcal{Z}_{n,i} > 0\}\right) \ \le\ \mathbb{P}(\mathcal{Z}_n > 0)\sum_{j=1}^s z_j, \tag{5.1}$$

and the theorem is proved if the statement holds true under $\mathbb{P}$.

If $\mathbb{P}_2(\mathcal{T}_1^* \ge 2) = 0$, then $(\mathcal{Z}_n)_{n\ge 0}$ is a branching process in i.i.d. random environment having countable state space (see Subsection 1.1.3). Hence, the assertion follows since

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{Z}_n > 0)^{1/n} \ =\ \inf_{0\le\theta\le 1} \sum_{1\le u\le t<\infty} p_t \mu_{u,t}^{\theta} \ =\ \nu\rho$$

by [59, Theorem 1.1].

Let us from now on assume that $\mathbb{P}_2(\mathcal{T}_1^* \ge 2) > 0$. By Proposition 1.4, we get

$$\mathbb{P}(\mathcal{T}_n^* > 0) \ \le\ \mathbb{E}\mathcal{T}_n^* \ =\ \nu^n \mathbb{P}(Z_n' > 0)$$

with $(Z_n')_{n\ge 0}$ being the ABPRE, and from (4.17) we deduce

$$\limsup_{n\to\infty} \mathbb{P}(\mathcal{T}_n^* > 0)^{1/n} \ \le\ \nu \limsup_{n\to\infty} \mathbb{P}(Z_n' > 0)^{1/n} \ =\ \nu\rho.$$

For the lower bound, we recall that $\sup_{n\ge 1} \mathbb{E}\mathcal{T}_n^* \le 1$ by Theorem 1.10 under the given assumptions. Furthermore, from the construction in (4.19), we get for each $m \ge 1$ a simple GWP $(S_{n,m})_{n\ge 0}$ with a single ancestor and reproduction mean $\mathbb{E}S_{1,m} = \mathbb{E}\mathcal{T}_m^* \le 1$ which satisfies

$$\mathbb{P}(\mathcal{T}_{nm}^* > 0) \ \ge\ \mathbb{P}(S_{n,m} > 0) \tag{5.2}$$

for all $n \in \mathbb{N}_0$. Let $(k_n)_{n\ge 0}$ be the sequence in $\mathbb{N}_0$ such that $k_n m < n \le (k_n+1)m$ for all $n \ge 0$. In particular, $k_n \to \infty$ as $n \to \infty$, and we deduce from (5.2) and the known theory (see e.g. [61] or [10, Theorem 1.6 in Chapter III])

$$\begin{aligned}
\liminf_{n\to\infty} \mathbb{P}(\mathcal{T}_n^* > 0)^{1/n} \ &\ge\ \liminf_{n\to\infty} \mathbb{P}(\mathcal{T}_{(k_n+1)m}^* > 0)^{1/(k_n m)} \\
&\ge\ \left(\liminf_{n\to\infty} \mathbb{P}(S_{k_n+1,m} > 0)^{1/k_n}\right)^{1/m} \\
&=\ \left(\liminf_{n\to\infty} \mathbb{P}(S_{k_n+1,m} > 0)^{(1+1/k_n)/(k_n+1)}\right)^{1/m} \\
&=\ (\mathbb{E}S_{1,m})^{1/m} \ =\ (\mathbb{E}\mathcal{T}_m^*)^{1/m}
\end{aligned}$$

for all $m \in \mathbb{N}$. Letting m tend to infinity, we get by Proposition 1.4 and once again (4.17)

$$\liminf_{n\to\infty} \mathbb{P}(\mathcal{T}_n^* > 0)^{1/n} \ \ge\ \lim_{m\to\infty} (\mathbb{E}\mathcal{T}_m^*)^{1/m} \ =\ \nu \lim_{m\to\infty} \mathbb{P}(Z_m' > 0)^{1/m} \ =\ \nu\rho,$$

and the theorem is proved. □

The just proved theorem particularly indicates that under the assumption of almost certain extinction of parasites, which by Theorem 1.10 is basically the case when $\mathbb{E}\log g'_{\Lambda_0}(1) < 0$, the decay rate of the survival probability is nearly γ^n if the ABPRE is strongly and intermediate subcritical, i.e. $\mathbb{E}g'_{\Lambda_0}(1)\log g'_{\Lambda_0}(1) \leq 0$. However, if otherwise the ABPRE is weakly subcritical, the survival probability decreases faster than γ^n. The rest of this section is dedicated to the formulation of conditions under which γ^n is the right asymptotic rate for the survival probability, viz.

$$\mathbb{P}(\mathcal{Z}_n > 0) \;\simeq\; c\gamma^n \quad \text{as } n \to \infty$$

for a constant $c > 0$. Before formulating the exact statements in the different cases, we have to ensure the convergence of $\gamma^{-n}\mathbb{P}(\mathcal{Z}_n > 0)$ for $n \to \infty$.

Theorem 5.2. *The sequence* $(\gamma^{-n}\mathbb{P}(\mathcal{Z}_n > 0))_{n\geq 0}$ *decreases for* $n \to \infty$. *In particular,*

$$\lim_{n\to\infty} \mathbb{P}(\mathcal{Z}_n > 0)/\gamma^n \;=:\; c \tag{5.3}$$

for a constant $c \in [0,\infty)$, *which is* 0 *if either* $\gamma \geq \min\{1,\nu\}$ *or* $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 = \infty$.

Proof. Let σ_n denote the leftmost cell in the first generation which has a contaminated descendant cell in generation n. We further write $\mathcal{Z}_{n-1}^{(\sigma_n)}$ for the number of parasites in these descendant cells. Since $\mathcal{Z}_n \geq \mathcal{Z}_{n-1}^{(\sigma_n)}$ a.s. and $\mathbb{E}_y\mathcal{Z}_n = y\mathbb{E}\mathcal{Z}_n$ for all $n \in \mathbb{N}$ and $y \in \mathbb{N}_0$, we calculate for each $n \in \mathbb{N}$

$$\begin{aligned}
\mathbb{E}(\mathcal{Z}_n \mid \mathcal{Z}_n > 0) &= \sum_{z\geq 1} \mathbb{P}(\mathcal{Z}_n \geq z \mid \mathcal{Z}_n > 0) \\
&\geq \sum_{z\geq 1} \mathbb{P}(\mathcal{Z}_{n-1}^{(\sigma_n)} \geq z \mid \mathcal{Z}_n > 0) \\
&= \sum_{z\geq 1}\sum_{y\geq 1} \mathbb{P}(\mathcal{Z}_{n-1}^{(\sigma_n)} \geq z,\; Z_{\sigma_n} = y \mid \mathcal{Z}_n > 0) \\
&= \sum_{z\geq 1}\sum_{y\geq 1} \mathbb{P}(Z_{\sigma_n} = y \mid \mathcal{Z}_n > 0)\mathbb{P}_y(\mathcal{Z}_{n-1} \geq z) \\
&= \sum_{y\geq 1} \mathbb{P}(Z_{\sigma_n} = y \mid \mathcal{Z}_n > 0)\mathbb{E}_y\mathcal{Z}_{n-1} \\
&= \mathbb{E}(\mathcal{Z}_{n-1} \mid \mathcal{Z}_{n-1} > 0)\sum_{y\geq 1} \mathbb{P}(Z_{\sigma_n} = y \mid \mathcal{Z}_n > 0)y\mathbb{P}(\mathcal{Z}_{n-1} > 0) \\
&\geq \mathbb{E}(\mathcal{Z}_{n-1} \mid \mathcal{Z}_{n-1} > 0)\sum_{y\geq 1} \mathbb{P}(Z_{\sigma_n} = y \mid \mathcal{Z}_n > 0)\mathbb{P}_y(\mathcal{Z}_{n-1} > 0) \\
&= \mathbb{E}(\mathcal{Z}_{n-1} \mid \mathcal{Z}_{n-1} > 0),
\end{aligned}$$

where in the last inequality (5.1) was used. Hence, the sequence $(\mathbb{E}(\mathcal{Z}_n \mid \mathcal{Z}_n > 0))_{n\geq 0}$ is increasing and consequently

$$\frac{\mathbb{P}(\mathcal{Z}_n > 0)}{\gamma^n} \;=\; \mathbb{E}(\mathcal{Z}_n \mid \mathcal{Z}_n > 0)^{-1} \;\searrow\; c \quad \text{as } n \to \infty$$

for a constant $c \in [0,\infty)$.

It is trivial that $c = 0$ if $\gamma > 1$. If otherwise $\gamma = 1$, Theorem 1.10 implies almost sure extinction of parasites and thus $c = 0$ is valid in this case. Proposition 1.4 and the Markov inequality yield

$$\mathbb{P}(\mathcal{Z}_n > 0) \ = \ \mathbb{P}(\mathcal{T}_n^* > 0) \ \leq \ \mathbb{E}\mathcal{T}_n^* \ = \ \nu^n \mathbb{P}(Z_n' > 0) \tag{5.4}$$

and thus $c = 0$ if $\gamma > \nu$. If $\gamma = \nu$, then Jensen's inequality ensures that the ABPRE is critical or subcritical. Hence, $c = 0$ by (5.4).

Let now $\gamma < \min\{1, \nu\}$ and $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 = \infty$, and we recall that $(\mathcal{F}_n)_{n\geq 0}$ is the canonical filtration of the BwBP. Denote by φ the generating function of $\mathcal{Z}_1$. Then using Jensen's inequality, we get for all $n \in \mathbb{N}$ and $s \in [0,1]$ the lower bound

$$\begin{aligned}
\mathbb{E}\left(s^{\mathcal{Z}_{n+1}} \mid \mathcal{F}_n\right) \ &= \ \prod_{v\in\mathbb{T}_n} \mathbb{E}\left(s^{\sum_{i=1}^{Z_v}\sum_{u=1}^{T_v} X_{i,v}^{(u,T_v)}} \mid Z_v\right) \\
&= \ \prod_{v\in\mathbb{T}_n} \sum_{t=0}^{\infty} p_t \mathbb{E}\left(s^{\sum_{i=1}^{Z_v}\sum_{u=1}^{t} X_{i,v}^{(u,t)}} \mid Z_v\right) \\
&= \ \prod_{v\in\mathbb{T}_n} \sum_{t=0}^{\infty} p_t \mathbb{E}\left(s^{\sum_{u=1}^{t} X^{(u,t)}}\right)^{Z_v} \\
&\geq \ \prod_{v\in\mathbb{T}_n} \left(\sum_{t=0}^{\infty} p_t \mathbb{E}\left(s^{\sum_{u=1}^{t} X^{(u,t)}}\right)\right)^{Z_v} \\
&= \ \prod_{v\in\mathbb{T}_n} \varphi(s)^{Z_v} \\
&= \ \varphi(s)^{\mathcal{Z}_n} \quad \text{a.s.}
\end{aligned}$$

Thus, iterating the above inequality yields

$$\mathbb{E}(s^{\mathcal{Z}_n}) \ \geq \ \underbrace{\varphi \circ \cdots \circ \varphi(s)}_{n\text{-times}}$$

for all $n \in \mathbb{N}$. Let $(S_n)_{n\geq 0}$ be a simple GWP with reproduction law given by φ. Then, we get from the above inequality

$$\mathbb{P}(\mathcal{Z}_n > 0) \ \leq \ 1 - \underbrace{\varphi \circ \cdots \circ \varphi(0)}_{n-\text{times}} \ = \ \mathbb{P}(S_n > 0)$$

for all $n \in \mathbb{N}$. Since $\varphi'(1) = \gamma < 1$ and $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 = \infty$ is assumed, Kolmogorov's theorem for the standard GWP (see e.g. [46, Theorem (2.6.1)]) yields $\lim_{n\to\infty} \gamma^{-n}\mathbb{P}(S_n > 0) = 0$ and thus $c = 0$. □

To identify sufficient conditions under which the limit c in (5.3) is positive, we use the spinal BwBP constructed in Chapter 2 and state an equivalent characterization for $c = 0$ in terms of this process in Lemma 5.4. This approach is similar to the one for the standard GWP (see [61]), and we also need the following measure-theoretical lemma.

Lemma 5.3 (Lemma 5.1 in [61])**.** *Let $(P_n)_{n\geq 0}$ be a sequence of probability measures on the positive integers with finite means m_n and for $n \in \mathbb{N}_0$ let $\hat{P}_n$ be the size-biased measure of P_n. If $(\hat{P}_n)_{n\geq 0}$ is tight, then $\sup_{n\geq 0} m_n < \infty$, while if $\hat{P}_n(\{z\}) \to 0$ for each $z \in \mathbb{N}$, then $\sup_{n\geq 0} m_n = \infty$.*

Proof. For each $z \in \mathbb{N}$, we get

$$\begin{aligned} \frac{1}{\sup_{n\geq 0} m_n} - \frac{1}{z} \;\leq\; \frac{1}{m_n} - \frac{1}{z} \;\leq\; \frac{P_n(\{1,\dots,z\})}{m_n} \;&\leq\; \hat{P}_n(\{1,\dots,z\}) \\ = \frac{\sum_{x=1}^{z} x P_n(\{x\})}{m_n} \;&\leq\; \frac{z P_n(\{1,\dots,z\})}{m_n} \;\leq\; \frac{z}{m_n}. \end{aligned}$$

So, if $(\hat{P}_n)_{n\geq 0}$ is tight, there exists a $z \in \mathbb{N}$ such that

$$\frac{1}{2} \;\leq\; \hat{P}_n(\{1,\dots,z\}) \;\leq\; \frac{z}{m_n}$$

for all $n \geq 0$, hence $\sup_{n\geq 0} m_n < \infty$. If, conversely, $\sup_{n\geq 0} m_n < \infty$, there exists a $z \in \mathbb{N}$ such that

$$0 \;<\; \frac{1}{\sup_{n\geq 0} m_n} - \frac{1}{z} \;\leq\; \lim_{n\to\infty} \hat{P}_n(\{1,\dots,z\}).$$

□

Lemma 5.4. $\lim_{n\to\infty} \gamma^{-n} \mathbb{P}(\mathcal{Z}_n > 0) = \lim_{n\to\infty} \mathbb{E}(\mathcal{Z}_n | \mathcal{Z}_n > 0)^{-1} = 0$ *if and only if* $\hat{\mathcal{Z}}_n \xrightarrow{\mathbb{P}} \infty$.

Proof. For $n \in \mathbb{N}_0$, put $P_n := \mathbb{P}(\mathcal{Z}_n \in \cdot \mid \mathcal{Z}_n > 0)$, and let $\hat{P}_n$ its corresponding size-biased distribution. Then,

$$\hat{P}_n(\{z\}) \;=\; \frac{z\mathbb{P}(\mathcal{Z}_n = z | \mathcal{Z}_n > 0)}{\mathbb{E}(\mathcal{Z}_n | \mathcal{Z}_n > 0)} \;=\; \frac{z\mathbb{P}(\mathcal{Z}_n = z)}{\mathbb{E}\mathcal{Z}_n} \;=\; \mathbb{P}(\hat{\mathcal{Z}}_n = z) \tag{5.5}$$

for all $n, z \in \mathbb{N}$. If $\hat{\mathcal{Z}}_n \xrightarrow{\mathbb{P}} \infty$, the above equation yields $\hat{P}_n(\{z\}) \to 0$ for all $z \in \mathbb{N}$, and Lemma 5.3 ensures $\sup_{n\geq 0} \mathbb{E}(\mathcal{Z}_n | \mathcal{Z}_n > 0) = \infty$. Now, suppose $\lim_{n\to\infty} \gamma^{-n} \mathbb{P}(\mathcal{Z}_n > 0) = 0$. Then,

$$0 \;=\; \lim_{n\to\infty} \mathbb{P}(\mathcal{Z}_n > 0)/\gamma^n \;\geq\; \lim_{n\to\infty} \mathbb{P}(\mathcal{Z}_n = z)/\gamma^n \;=\; \lim_{n\to\infty} \mathbb{P}(\hat{\mathcal{Z}}_n = z)/z$$

for all $z \in \mathbb{N}$, and thus $\mathbb{P}(\hat{\mathcal{Z}}_n = z) \to 0$ for all $z \in \mathbb{N}$ as $n \to \infty$. Since $\hat{\mathcal{Z}}_n \geq 1$ a.s. for all $n \in \mathbb{N}$, this implies $\hat{\mathcal{Z}}_n \to \infty$ in probability. □

We have seen in Theorem 5.2 that $\gamma < \min\{1, \nu\}$, which implies subcriticality of the ABPRE by Jensen's inequality, is necessary for the survival probability to decay like the mean number of parasites. So, we consider the three different subcritical cases $\mathbb{E} g'_{\Lambda_0}(1) \log g'_{\Lambda_0}(1) < 0, = 0$ and > 0 and give sufficient conditions under which the limit in (5.3) is positive. More precisely, if the ABPRE is weakly subcritical, then $\lim_{n\to\infty} \gamma^{-n} \mathbb{P}(\mathcal{Z}_n > 0) = 0$. In the other cases, the positivity of this limit depends on the existence of a positive recurrent state of $(\hat{Z}_{\hat{V}_n} - 1)_{n\geq 0}$ and the mean offspring number of parasites this process produces during two visits of such a positive recurrent state. Before stating the result in detail, we need some notation. Let

$$\kappa := \inf \left\{ z \in \mathbb{N}_0 \mid \mathbb{P}(X_{1,\varnothing}^{(\hat{U}_0,\hat{T}_0)} = 0, \hat{X}_0^{(\hat{U}_0,\hat{T}_0)} - 1 = z) > 0 \right\}$$

and put $\kappa = \infty$ if the set is empty. Furthermore, let

$$\tau := \inf \left\{ n \in \mathbb{N} \mid \hat{Z}_{\hat{V}_n} - 1 = \kappa \right\}$$

denote the first time the Markov chain $(\hat{Z}_{\hat{V}_n} - 1)_{n \geq 0}$ hits κ. If $\hat{Z}_{\hat{V}_0} - 1 = \kappa$, then the BwBP starts with $\kappa + 1$ parasites and thus $\mathbb{E}_{\kappa+1}\tau < \infty$ if κ is positive recurrent, whereas $\mathbb{E}_{\kappa+1}\tau = \infty$ otherwise. We further define

$$\hat{Y} := \sum_{j=0}^{\tau-1} \sum_{u \in \{1,\dots,\hat{T}_j\} \setminus \{\hat{U}_j\}} \hat{Z}_{\hat{V}_j u}$$

as the total number of parasites in non-spinal cells which have a spinal mother cell up to generation τ. According to standard convention, we set $\min_{1 \leq u \leq T} \mathbb{P}(X^{(u,T)} = 0 | T) = 1$ on $\{T = 0\}$.

Theorem 5.5. *Let* $\gamma < \min\{1, \nu\}$ *and* $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 < \infty$.

(a) If $\mathbb{E}g'_{\Lambda_0}(1) \log g'_{\Lambda_0}(1) > 0$, *then* $\lim_{n\to\infty} \gamma^{-n}\mathbb{P}(\mathcal{Z}_n > 0) = 0$.

(b) Let $\mathbb{E}g'_{\Lambda_0}(1) \log g'_{\Lambda_0}(1) \leq 0$. *Then* $\lim_{n\to\infty} \gamma^{-n}\mathbb{P}(\mathcal{Z}_n > 0) > 0$ *if one of the following conditions holds true:*

(i) $\mathbb{E}_{\kappa+1}\tau < \infty$ *and* $\mathbb{E}_{\kappa+1} \log^+ \hat{Y} < \infty$

(ii) $\mathbb{P}(X^{(u,t)} = 0) \in \{0, 1\}$ *for all* $u \leq t$ *with* $p_t > 0$.

(c) Let $\mathbb{E}g'_{\Lambda_0}(1) \log g'_{\Lambda_0}(1) \leq 0$. *Then* $\lim_{n\to\infty} \gamma^{-n}\mathbb{P}(\mathcal{Z}_n > 0) = 0$ *if one of the following conditions holds true:*

(i) $\mathbb{E}_{\kappa+1}\tau = \infty$

(ii) $\mathbb{E}_{\kappa+1} \log^+ \hat{Y} = \infty$ *and* $\mathbb{E} \log^- \left(1 - \min_{1 \leq u \leq T} \mathbb{P}(X^{(u,T)} = 0 | T)\right) < \infty$

Proof. (a) follows directly from Theorem 5.1 since in this case $\lim_{n\to\infty} \mathbb{P}(\mathcal{Z}_n > 0)^{1/n} < \min\{1, \gamma\}$.

Let from now on $\mathbb{E}g'_{\Lambda_0}(1) \log g'_{\Lambda_0}(1) \leq 0$ hold true. Recall that $(\hat{Z}_{\hat{V}_n} - 1)_{n\geq 0}$ is a BPREI with immigration components $(\hat{X}_n^{(\hat{U}_n,\hat{T}_n)} - 1)_{n\geq 0}$ and i.i.d. environmental sequence $[\hat{U}_n, \hat{T}_n]_{n\geq 0}$. Furthermore, this process is critical or subcritical due to Remark 2.5 and the conditions given in this theorem. The proof of the remaining statement (b) and (c) is divided into several parts.

DEGENERATED CASES: We first consider the case when $\mathbb{P}(X^{(u,t)} = 0) \in \{0, 1\}$ for all $1 \leq u \leq t < \infty$ with $p_t > 0$, which particularly means $\hat{X}_0^{(\hat{U}_0,\hat{T}_0)} = X_0^{(\hat{U}_0,\hat{T}_0)} = 1$ a.s. Hence, $\mathcal{Z}_n = \mathcal{T}_n^*$ a.s. for all $n \in \mathbb{N}_0$ and $(\mathcal{Z}_n)_{n\geq 0}$ forms a standard GWP. Then the classical result by Kolmogorov [46, Theorem (2.6.1)] provides $\lim_{n\to\infty} \gamma^{-n}\mathbb{P}(\mathcal{Z}_n > 0) > 0$, $(b)(ii)$ is proved.

Let now $\mathbb{P}(X^{(u,t)} = 0) \in (0, 1)$ for some $1 \leq u \leq t < \infty$ and $\mathbb{P}(X^{(u,t)} \leq 1) = 1$ for all $1 \leq u \leq t < \infty$ with $p_t > 0$. Consequently, $\mathcal{Z}_n = \mathcal{T}_n^*$ a.s. for all $n \in \mathbb{N}_0$ and $(\mathcal{Z}_n)_{n\geq 0}$ forms a standard GWP. As seen above, $\lim_{n\to\infty} \gamma^{-n}\mathbb{P}(\mathcal{Z}_n > 0) > 0$ follows. Furthermore, $\kappa = 0$ and $\tau = 1$ $\mathbb{P}$-a.s., as only the spinal parasite is in the spinal cell due to the assumptions. Consequently, $\hat{Y} \leq \hat{\mathcal{Z}}_1$ $\mathbb{P}$-a.s. and $\mathbb{E} \log^+ \hat{Y} \leq \mathbb{E} \log \hat{\mathcal{Z}}_1 < \infty$ by Remark 2.3 and the assumptions of this theorem.

ASSUMPTIONS: *Let from now on*

$$\mathbb{P}(X^{(u,t)} = 0) \in (0,1) \quad \text{for some} \quad 1 \leq u \leq t < \infty \quad \text{with} \quad p_t > 0$$

and

$$\mathbb{P}(X^{(u,t)} \geq 2) > 0 \quad \text{for some} \quad 1 \leq u \leq t < \infty \quad \text{with} \quad p_t > 0.$$

These two assumptions particularly imply $\mathbb{P}(\hat{X}^{(\hat{U},\hat{T})} > 1) > 0$ and $0 < \mathbb{P}(X^{(\hat{U},\hat{T})} = 0) < 1$. Hence, by Lemma 3.1, the Markov chain $(\hat{Z}_{\hat{V}_n} - 1)_{n\geq 0}$, independent from the starting state, eventually hits an irreducible and aperiodic set C_κ containing κ.

PROOF OF $(b)(i)$: Let $\mathbb{E}_{\kappa+1}\tau < \infty$ and $\mathbb{E}_{\kappa+1}\log^+\hat{Y} < \infty$. By Lemma 5.4, it is enough to show that $\hat{\mathcal{Z}}_n$ does not converge to infinity in probability, which follows if $(\mathbb{P}(\hat{\mathcal{Z}}_n \in \cdot))_{n\geq 0}$ is tight. Under the given assumptions, κ is positive recurrent, and thus $(\hat{Z}_{\hat{V}_n} - 1)_{n\geq 0}$ visits this state infinitely often. Let $(\sigma_i)_{i\geq 0}$ with $\sigma_0 = 0$ be the successive return times of state κ, i.e.

$$\sigma_{i+1} \;=\; \inf\{k > \sigma_i \mid \hat{Z}_{\hat{V}_k} - 1 = \kappa\} \quad \text{for } i \geq 0.$$

We further put

$$\tau_i = \sigma_i - \sigma_{i-1} \quad \text{for } i \geq 1.$$

In particular, $\tau_1 = \sigma_1 = \tau$ a.s. Recalling that $\hat{\mathcal{Z}}_n^{(v)}$ denotes the number of parasites in generation n of the subtree rooted in cell v, we obtain that

$$\hat{\mathcal{Z}}_n \;=\; \hat{\mathcal{Z}}_{n-\sigma_1}^{(\hat{V}_{\sigma_1})} + \sum_{u\in\hat{\mathbb{T}}_{\sigma_1}\setminus\{\hat{V}_{\sigma_1}\}} \hat{\mathcal{Z}}_{n-\sigma_1}^{(u)} \quad \text{a.s.} \tag{5.6}$$

for all $n \geq \sigma_1$. Since $\hat{\mathcal{Z}}_{n-\sigma_1}^{(u)}$ for $u \neq \hat{V}_{\sigma_1}$ behaves as a non-spinal process and $\mathbb{P}(\mathsf{Ext}) = 1$ is assumed, it converges to 0 a.s. Hence, the right sum converges to 0 a.s. too, as $\sigma_1 < \infty$ a.s. Consequently, it is enough to consider the process starting with $\kappa + 1$ parasites in the root cell, which is assumed from now on. For convenience, we omit the index in the probability measures and write $\mathbb{P}$ instead of $\mathbb{P}_{\kappa+1}$ in the rest of this proof.

Let $\hat{H}_k := \{1, .., \hat{T}_k\} \setminus \{\hat{U}_k\}$ for $k \geq 0$, and with $\mathcal{G}$ being the σ-algebra as defined in (4.6), we put

$$\mathcal{H} \;:=\; \sigma\left(\mathcal{G} \cup \sigma\left((\hat{Z}_{\hat{V}_n} - 1)_{n\geq 0}, (X_{i,\hat{V}_n}^{(\bullet,\hat{T}_n)})_{i\geq 1, n\geq 0}\right)\right).$$

We show the existence of an almost surely finite random variable which dominates $\mathbb{E}(\hat{\mathcal{Z}}_n|\mathcal{H})$ for all $n \geq 0$ stochastically. Indeed,

$$\hat{\mathcal{Z}}_n \;=\; \hat{Z}_{\hat{V}_n} + \sum_{k=0}^{n-1}\sum_{u\in\hat{H}_k} \hat{\mathcal{Z}}_{n-1-k}^{(\hat{V}_k u)} \;=\; \hat{Z}_{\hat{V}_n} + \sum_{k=0}^{\sigma_{r_n}-1}\sum_{u\in\hat{H}_k} \mathcal{Z}_{n-1-k}^{(\hat{V}_k u)} + \sum_{k=\sigma_{r_n}}^{n-1}\sum_{u\in\hat{H}_k} \mathcal{Z}_{n-1-k}^{(\hat{V}_k u)} \tag{5.7}$$

for each $n \geq 0$, where $r_n \in \mathbb{N}_0$ such that $\sigma_{r_n} \leq n < \sigma_{r_n+1}$, and it is recalled that host-parasite processes not containing the spine behave as an ordinary BwBP. As all σ_k, $\hat{H}_k$ and $\hat{Z}_{\hat{V}_k}u$ are $\mathcal{H}$-measurable, we get

$$\mathbb{E}\left(\hat{\mathcal{Z}}_n|\mathcal{H}\right) \;=\; \hat{Z}_{\hat{V}_n} + \sum_{k=0}^{\sigma_{r_n}-1}\sum_{u\in\hat{H}_k} \hat{Z}_{\hat{V}_k u}\gamma^{n-k-1} + \sum_{k=\sigma_{r_n}}^{n-1}\sum_{u\in\hat{H}_k} \hat{Z}_{\hat{V}_k u}\gamma^{n-k-1}$$

$$\leq \hat{Z}_{\hat{V}_n} + \underbrace{\sum_{k=0}^{\sigma_{r_n}-1} \sum_{u\in\hat{H}_k} \hat{Z}_{\hat{V}_k u}\gamma^{\sigma_{r_n}-k-1}}_{(*)_{r_n}} + \sum_{k=\sigma_{r_n}}^{\sigma_{r_n+1}} \sum_{u\in\hat{H}_k} \hat{Z}_{\hat{V}_k u} \quad \text{a.s.,} \tag{5.8}$$

where in the last inequality $\gamma < 1$ has been used. Since $(\hat{Z}_{\hat{V}_n} - 1)_{n\geq 0}$ is positive recurrent (see the discussion at the beginning of this part of the proof), it converges in distribution to a finite random variable. Furthermore, the last sum in the above inequality is almost surely finite and has the same distribution for each $n \in \mathbb{N}_0$. Hence, there exists a finite random variable C_1 dominating $\hat{Z}_{\hat{V}_n}$ and the last sum stochastically for each $n \geq 0$, i.e.

$$\mathbb{P}\Bigg(\hat{Z}_{\hat{V}_n} + \sum_{k=\sigma_{r_n}}^{\sigma_{r_n+1}} \sum_{u\in\hat{H}_k} \hat{Z}_{\hat{V}_k u} \geq m\Bigg) \;\leq\; \mathbb{P}(C_1 \geq m) \quad \text{for all } m, n \in \mathbb{N}_0. \tag{5.9}$$

Let us now consider $(*)_n$. Put

$$\hat{Y}_k \;:=\; \sum_{j=\sigma_k}^{\sigma_{k+1}-1} \sum_{u\in\hat{H}_j} \hat{Z}_{\hat{V}_j u}$$

for $k \geq 0$ and observe that

$$\begin{aligned}(*)_n \;&=\; \gamma^{\sigma_n - 1}\sum_{k=0}^{n-1}\sum_{j=0}^{\tau_{k+1}-1} \sum_{u\in\hat{H}_{\sigma_k+j}} \hat{Z}_{\hat{V}_{\sigma_k+j}u}\gamma^{-\sigma_k-j} \\ &\leq\; \gamma^{\sigma_n}\sum_{k=0}^{n-1}\gamma^{-\sigma_{k+1}} \sum_{j=0}^{\tau_{k+1}-1} \sum_{u\in\hat{H}_{\sigma_k+j}} \hat{Z}_{\hat{V}_{\sigma_k+j}u} \;=\; \gamma^{\sigma_n}\sum_{k=0}^{n-1}\gamma^{-\sigma_{k+1}}\hat{Y}_k \quad \text{a.s.}\end{aligned}$$

Thanks to the positive recurrence of κ, the segments

$$\hat{\mathbf{Z}}_i \;:=\; \left(\hat{Z}_{\hat{V}_{\sigma_i}}, \ldots, \hat{Z}_{\hat{V}_{\sigma_{i+1}-1}}\right), \quad i \geq 0, \tag{5.10}$$

are i.i.d. and thus particularly the τ_{k+1}, $k \geq 0$. Due to the reproduction mechanism of cells and parasites, this ensures that the vectors

$$\hat{\mathbf{Y}}_k := \left(\tau_{k+1}, \hat{Y}_k\right), \quad k \geq 0,$$

are i.i.d. too. In particular, $\hat{Y}_k$ is distributed as $\hat{Y}$ for each $k \geq 0$. Hence,

$$\psi\left(\hat{\mathbf{Y}}_\mathbf{0}, \ldots, \hat{\mathbf{Y}}_\mathbf{n}\right) \;\overset{d}{=}\; \psi\left(\hat{\mathbf{Y}}_\mathbf{n}, \ldots, \hat{\mathbf{Y}}_\mathbf{0}\right)$$

for all measurable functions $\psi : \mathbb{N}_0^{2(n+1)} \to \mathbb{R}$. This yields

$$\begin{aligned}\gamma^{\sigma_n}\sum_{k=0}^{n-1}\gamma^{-\sigma_{k+1}}\hat{Y}_k \;&=\; \sum_{k=0}^{n-1}\gamma^{\sum_{i=k+1}^{n-1}\tau_{i+1}}\hat{Y}_k \\ &\overset{d}{=}\; \sum_{k=0}^{n-1}\gamma^{\sum_{i=k+1}^{n-1}\tau_{n-i}}\hat{Y}_{n-k-1}\end{aligned}$$

$$\begin{aligned} &= \sum_{k=0}^{n-1} \gamma^{\sigma_{n-k-1}} \hat{Y}_{n-k-1} \\ &= \sum_{k=0}^{n-1} \gamma^{\sigma_k} \hat{Y}_k \\ &\le \sum_{k=0}^{\infty} \exp\left(\frac{1}{k+1} \log^+ \hat{Y}_k + \frac{\sigma_k}{k+1} \log \gamma\right)^{k+1} \end{aligned} \tag{5.11}$$

for all $n \ge 0$. Lemma 3.4 and the law of large numbers yield

$$\limsup_{k\to\infty} \frac{1}{k} \log^+ \hat{Y}_k \;=\; 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{\sigma_k}{k} \;=\; \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \tau_i \;=\; \mathbb{E}\tau \;<\; \infty$$

since the $(\hat{\mathbf{Y}}_k)_{k\ge 0}$ are i.i.d. and $\mathbb{E}\log^+ \hat{Y} < \infty$ is assumed. Thus, the sum in (5.11) is almost surely finite. Consequently, $(*)_n$ is stochastically bounded from above by a finite random variable C_2 uniformly in n, i.e.

$$\mathbb{P}((*)_n \ge m) \;\le\; \mathbb{P}(C_2 \ge m) \quad \text{for all } m, n \in \mathbb{N}_0.$$

Together with (5.8) and (5.9), this ensures that we find for each $\varepsilon > 0$ a constant $d > 0$ such that

$$\sup_{n\ge 0} \mathbb{P}\left(\mathbb{E}\left(\hat{\mathcal{Z}}_n | \mathcal{H}\right) \ge d\right) \;\le\; \varepsilon.$$

But from this, we infer

$$\begin{aligned} \mathbb{P}\left(\hat{\mathcal{Z}}_n \ge \varepsilon^{-1} d\right) \;&=\; \mathbb{E}\left(\mathbb{P}(\hat{\mathcal{Z}}_n \ge \varepsilon^{-1} d | \mathcal{H})\right) \\ &\le\; \varepsilon + \mathbb{E}\left(\mathbb{P}(\hat{\mathcal{Z}}_n \ge \varepsilon^{-1} d | \mathcal{H})\, \mathbb{1}_{\left\{\mathbb{P}(\hat{\mathcal{Z}}_n \ge \varepsilon^{-1} d | \mathcal{H}) \ge \varepsilon\right\}}\right) \\ &\le \varepsilon + \mathbb{P}\left(\mathbb{P}(\hat{\mathcal{Z}}_n \ge \varepsilon^{-1} d | \mathcal{H}) \ge \varepsilon\right) \\ &\le \varepsilon + \mathbb{P}\left(\mathbb{E}(\hat{\mathcal{Z}}_n | \mathcal{H}) \ge d\right) \\ &\le 2\varepsilon \end{aligned}$$

for all $n \ge 0$, which implies tightness of $(\mathbb{P}(\hat{\mathcal{Z}}_n \in \cdot))_{n\ge 0}$. Consequently, $\hat{\mathcal{Z}}_n$ cannot converge to infinity in probability, and $(b)(i)$ is proved.

Proof of $(c)(i)$: Let $\mathbb{E}_{\kappa+1} = \infty$. By the observations in the part Assumptions and Corollary 3.2, we get $\hat{Z}_{\hat{V}_n} - 1 \xrightarrow{\mathbb{P}_x} \infty$ for all $x \in \mathbb{N}_0$, and since

$$\hat{\mathcal{Z}}_n \;\ge\; \hat{Z}_{\hat{V}_n} - 1 \quad \text{a.s.}$$

by (4.13), this yields $\hat{\mathcal{Z}}_n \xrightarrow{\mathbb{P}} \infty$. Lemma 5.4 finishes the proof of $(b)(i)$.

Proof of $(c)(ii)$: Let the assumptions in $(c)(ii)$ and additionally $\mathbb{E}_{\kappa+1} < \infty$ hold true. This gives positive recurrence of $(\hat{Z}_{\hat{V}_n} - 1)_{n\ge 0}$. We use the same notation as in the part Proof of $(b)(i)$. By (5.6), it is enough to consider the process with $\kappa + 1$ parasites in the root cell and, as before, we write $\mathbb{P}$ as shorthand for $\mathbb{P}_{\kappa+1}$ from now on in this proof. Again, we use Lemma 5.4 and show that $\hat{\mathcal{Z}}_n$ converges to infinity in probability. Observe that it is enough to consider the sequence $(\hat{\mathcal{Z}}_{\sigma_n})_{n\geq 0}$ since this argumentation can be repeated for each positive recurrent state. As $(\hat{Z}_{\hat{V}_n} - 1)_{n\geq 0}$ is in one of this states in every time step, the assertion follows.

From (5.7), we get

$$\hat{\mathcal{Z}}_{\sigma_n} \;\geq\; \sum_{k=0}^{\sigma_n-1} \sum_{u\in\hat{H}_k} \mathcal{Z}_{n-1-k}^{(\hat{V}_k u)} \;=\; \sum_{k=0}^{n-1} \sum_{j=0}^{\tau_{k+1}-1} \sum_{u\in\hat{H}_{\sigma_k+j}} \mathcal{Z}_{\sigma_n-1-\sigma_k-j}^{(\hat{V}_{\sigma_k+j}u)} \quad \text{a.s.} \tag{5.12}$$

As before, the segments $\hat{\mathbf{Z}}_i$, $i \geq 0$, as defined in (5.10), are i.i.d. Now, the branching property ensures that the vectors of subtrees spawning from these segments are i.i.d. too since the number of these subtrees and the number of their root parasites are i.i.d. More precisely, the vectors

$$\mathbf{BT_i} \;:=\; \Big(\tau_{i+1}, \hat{\mathbf{Z}}_i, \boldsymbol{BT}(\sigma_i), \ldots, \boldsymbol{BT}(\sigma_{i+1}-1)\Big), \quad i \geq 0,$$

are i.i.d., where the $\boldsymbol{BT}(\sigma_i + l) := (\boldsymbol{BT}^{(\hat{V}_{(\sigma_i+l)u})})_{u\in\hat{H}_{\sigma_i+l}}$, $0 \leq l < \tau_{i+1}$, denote the subtrees spawning from the spinal cell $\hat{V}_{\sigma_i+l}$ in segment i. In particular, each $\boldsymbol{BT}^{(\hat{V}_{(\sigma_i+l)u})}$ behaves as an ordinary BwBP with one root cell and $\hat{Z}_{\hat{V}_{(\sigma_i+l)u}}$ parasites. Hence,

$$\psi\left(\mathbf{BT_0}, \ldots, \mathbf{BT_n}\right) \;\overset{d}{=}\; \psi\left(\mathbf{BT_n}, \ldots, \mathbf{BT_0}\right)$$

for all measurable functions $\psi : R \to \mathbb{R}$ on a proper space R. As $\mathcal{Z}_n = z_n(\boldsymbol{BT})$ for the measurable mapping $z_n : \mathbb{S} \to \mathbb{N}_0$ for each $n \in \mathbb{N}_0$ (see Subsection 1.1.2), we get

$$\begin{aligned}
\sum_{k=0}^{n-1} \sum_{j=0}^{\tau_{k+1}-1} \sum_{u\in\hat{H}_{\sigma_k+j}} \mathcal{Z}_{\sigma_n-1-\sigma_k-j}^{(\hat{V}_{\sigma_k+j}u)} &= \sum_{k=0}^{n-1} \sum_{j=0}^{\tau_{k+1}-1} \sum_{u\in\hat{H}_{\sigma_k+j}} \mathcal{Z}_{\sum_{i=k}^{n-1}\tau_{i+1}-1-j}^{(\hat{V}_{\sigma_k+j}u)} \\
&\overset{d}{=} \sum_{k=0}^{n-1} \sum_{j=0}^{\tau_{n-k}-1} \sum_{u\in\hat{H}_{\sigma_{n-k}+j}} \mathcal{Z}_{\sigma_{n-k}-1-j}^{(\hat{V}_{\sigma_{n-k-1}+j}u)} \\
&= \sum_{k=0}^{n-1} \sum_{j=0}^{\tau_{k+1}-1} \sum_{u\in\hat{H}_{\sigma_k+j}} \mathcal{Z}_{\sigma_{k+1}-1-j}^{(\hat{V}_{\sigma_k+j}u)}.
\end{aligned}$$

By an appeal to (5.12), we deduce that $\hat{\mathcal{Z}}_{\sigma_n}$ converges to infinity in probability if

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\tau_{k+1}-1} \sum_{u\in\hat{H}_{\sigma_k+j}} \mathcal{Z}_{\sigma_{k+1}-1-j}^{(\hat{V}_{\sigma_k+j}u)} \;=\; \infty \quad \text{a.s.,}$$

which follows if infinitely many $\mathcal{Z}_{\sigma_{k+1}-1-j}^{(\hat{V}_{\sigma_k+j}u)}$ are positive. But conditioned under $\mathcal{H}$, the host-parasite trees spawning from the spinal cell line are independent, and hence the Borel-Cantelli lemma gives that

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\tau_{k+1}-1} \sum_{u\in\hat{H}_{\sigma_k+j}} \mathbb{P}\left(\mathcal{Z}_{\sigma_{k+1}-1-j}^{(\hat{V}_{\sigma_k+j}u)} > 0 \;\middle|\; \mathcal{H}\right) \;=\; \infty \quad \text{a.s.}$$

is sufficient for $\hat{\mathcal{Z}}_{\sigma_n} \xrightarrow{\mathbb{P}} \infty$. Since the $(\tau_i)_{i\geq 1}$ are i.i.d. with finite mean $\mathbb{E}\tau$, the law of large numbers provides us with an almost surely finite random variable K such that $\sigma_k \leq 2k\lceil\mathbb{E}\tau\rceil$ for all $k \geq K$. Recalling (5.1), we infer

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{j=0}^{\tau_{k+1}-1}\sum_{u\in\hat{H}_{\sigma_k+j}} \mathbb{P}\left(\mathcal{Z}_{\sigma_{k+1}-1-j}^{(\hat{V}_{\sigma_k+j}u)} > 0 \mid \mathcal{H}\right) &\geq \sum_{k=0}^{\infty}\sum_{j=0}^{\tau_{k+1}-1}\sum_{u\in\hat{H}_{\sigma_k+j}} \mathbb{P}\left(\mathcal{Z}_{\sigma_{k+1}}^{(\hat{V}_{\sigma_k+j}u)} > 0 \mid \mathcal{H}\right)\\ &\geq \sum_{k=K}^{\infty}\sum_{j=0}^{\tau_{k+1}-1}\sum_{u\in\hat{H}_{\sigma_k+j}} \mathbb{P}\left(\mathcal{Z}_{2(k+1)\lceil\mathbb{E}\tau\rceil}^{(\hat{V}_{\sigma_k+j}u)} > 0 \mid \mathcal{H}\right)\\ &\geq \sum_{k=K}^{\infty}\mathbb{P}_{\hat{Y}_k}\left(\mathcal{Z}_{4k\lceil\mathbb{E}\tau\rceil} > 0 \mid \mathcal{H}\right) \quad \text{a.s.} \qquad (5.13)\end{aligned}$$

Since $\mathbb{E}\log^+\hat{Y} = \infty$, it is $\limsup_{n\to\infty} n^{-1}\log^+\hat{Y}_n = \infty$ a.s. and thus

$$\mathbb{P}\left(\hat{Y}_n \geq a^n \text{ infinitely often}\right) = 1$$

for each $a > 0$ by Lemma 3.4. In particular, this holds true for $a = \exp(\log(a')4\lceil\mathbb{E}\tau\rceil)$ for all $a' > 0$. Thus, it is left to prove that there exist some $a > 0$ and $b > 0$ such that

$$\mathbb{P}_{a^n}\left(\mathcal{Z}_n > 0\right) \geq b \qquad (5.14)$$

for all large n. Indeed, if this holds true, then (5.13) is almost surely infinite and thus $\hat{\mathcal{Z}}_n \xrightarrow{\mathbb{P}} \infty$ as $n \to \infty$. We prove (5.14) in the subsequent lemma. ☐

Lemma 5.6. *Let $\gamma < 1$ and $\mathbb{E}\log^-\left(1 - \min_{1\leq u\leq T}\mathbb{P}(X^{(u,T)} = 0|T)\right) < \infty$. There exists an $a > 0$ such that for all $\varepsilon > 0$ there exists a $n_0 \geq 0$ such that*

$$\mathbb{P}_z\left(\mathcal{Z}_n > 0\right) \geq \frac{1}{2}(1-\varepsilon)$$

for all $z \geq a^n$ and $n \geq n_0$.

Proof. Consider the function

$$h : [0,1] \to [0,1], \quad h(x) = 1 - x^z - (1-x)^z,$$

which is strictly concave for $z \geq 2$. Furthermore, $h(0) = h(1) = 0$ and thus $h(x) > 0$ for all $x \in (0,1)$, and h is symmetric in $1/2$, i.e. $h(x) = h(1-x)$ for all $x \in [0,1]$. The concativity and symmetry yield

$$h(x) \geq \begin{cases} \frac{h(x_0)}{x_0}x & \text{if } x \leq x_0,\\ h(x_0) & \text{if } x \in (x_0, 1-x_0),\\ \frac{h(x_0)}{x_0}(1-x) & \text{if } x \geq x_0 \end{cases}$$

for all $x_0 \in (0,1)$.

Fix $\varepsilon > 0$ and put

$$a := \exp\left(2\,\mathbb{E}\log^-\left(1 - \min_{1\le u\le T}\mathbb{P}(X^{(u,T)} = 0|T)\right)\right).$$

Observe that $a > 1$, as otherwise $X^{(u,t)} \ge 1$ a.s. for at least one $1 \le u \le t$ for each $t \ge 1$ with $p_t > 0$. But the assumption $\mathbb{E}\log^-(1 - \min_{1\le u\le T}\mathbb{P}(X^{(u,T)} = 0|T)) < \infty$ implies $p_0 = 0$ and thus $\gamma = \sum_{t=1}^{\infty} p_t \sum_{u=1}^{t} \mu_{u,t} \ge 1$, which contradicts $\gamma < 1$.

Conditioned under $\mathbb{T}$ the descendants of a root parasite behave in an i.i.d. manner and thus

$$\begin{aligned}
\mathbb{P}_z(\mathcal{Z}_n > 0) &= 1 - \mathbb{P}_z(\mathcal{Z}_n = 0)\\
&= 1 - \mathbb{E}(\mathbb{P}_z(\mathcal{Z}_n = 0|\,\mathbb{T}))\\
&= \mathbb{E}(1 - \mathbb{P}(\mathcal{Z}_n = 0|\,\mathbb{T})^z)\\
&= \mathbb{E}((1 - \mathbb{P}(\mathcal{Z}_n = 0|\,\mathbb{T}))^z) + \mathbb{E}(h(\mathbb{P}(\mathcal{Z}_n = 0|\,\mathbb{T}))\\
&\ge h(x_0)\mathbb{P}(\mathbb{P}(\mathcal{Z}_n = 0|\,\mathbb{T}) \in (x_0, 1 - x_0)) \qquad (5.15)
\end{aligned}$$

for all $x_0 \in (0,1)$, $z \ge 2$ and $n \in \mathbb{N}_0$. For $1 \le u \le t < \infty$, let $g_{(u,t)}(s)$ be the generating function of $X^{(u,t)}$. Furthermore, let $u(t)$ denote the smallest index in $\{1,\dots,t\}$ for which

$$g_{(u(t),t)}(0) \;\le\; g_{(u,t)}(0) \quad \text{for all } 1 \le u \le t.$$

We shortly write g_t for the generating function of $X^{(u(t),t)}$. So, given $(T_v)_{v\in\mathbb{V}} = (t_v)_{v\in\mathbb{V}}$ and thus $\mathbb{T} = \tau$, we obtain

$$\mathbb{P}(\mathcal{Z}_n = 0|\,\mathbb{T} = \tau) \;\le\; \mathbb{P}(Z_u = 0|\,\mathbb{T} = \tau) \;=\; g_{(u_1,t_\varnothing)} \circ g_{(u_2,t_{u_1})} \circ \dots \circ g_{(u_n,t_{u_1\dots u_{n-1}})}(0)$$

for all $n \in \mathbb{N}$ and $u = u_1 \dots u_n \in \mathbb{V}$. Note that the right hand side is 1 if $u \notin \tau$. Thus,

$$\mathbb{P}(\mathcal{Z}_n = 0|\,\mathbb{T} = \tau) \;\le\; g_{t_\varnothing} \circ g_{t_{u_1}} \circ \dots \circ g_{t_{u_1\dots u_{n-1}}}(0),$$

and the convexity of the generating functions entails

$$\mathbb{P}(\mathcal{Z}_n > 0|\,\mathbb{T}) \;\ge\; 1 - g_{T_{\mathcal{U}_0}} \circ g_{T_{\mathcal{U}_1}} \circ \dots \circ g_{T_{\mathcal{U}_{n-1}}}(0) \;\ge\; \prod_{i=0}^{n-1}(1 - g_{T_{\mathcal{U}_i}}(0)) \quad \text{a.s.}$$

with $\mathcal{U}_0 = \varnothing$ and $\mathcal{U}_{i+1} = \mathcal{U}_i u(T_{\mathcal{U}_i})$ for $i \ge 0$. Since the $g_{T_{\mathcal{U}_i}}(0)$ are i.i.d. and distributed as $\min_{1\le u\le T}\mathbb{P}(X^{(u,T)} = 0|T)$, the law of large numbers and the condition of the lemma ensure

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=0}^{n-1}\log(1 - g_{T_{\mathcal{U}_i}}(0)) \ge \log(x_0)\right) \;\ge\; 1 - \varepsilon$$

for all large n and $x_0 := a^{-1} \in (0,1)$. Hence, we find a $n_0 \in \mathbb{N}$ such that

$$\mathbb{P}(\mathbb{P}(\mathcal{Z}_n = 0|\,\mathbb{T}) \le 1 - x_0^n) \;=\; \mathbb{P}(\mathbb{P}(\mathcal{Z}_n > 0|\,\mathbb{T}) \ge x_0^n) \;\ge\; 1 - \varepsilon \qquad (5.16)$$

for all $n \ge n_0$. Furthermore,

$$\mathbb{P}(\mathcal{Z}_n = 0) \;=\; \mathbb{E}(\mathbb{P}(\mathcal{Z}_n = 0|\,\mathbb{T})) \;\le\; x_0^n + (1 - x_0^n)\mathbb{P}(\mathbb{P}(\mathcal{Z}_n = 0|\,\mathbb{T}) \ge x_0^n)$$

and thus

$$\mathbb{P}\left(\mathbb{P}\left(\mathcal{Z}_n = 0 \middle| \mathbb{T}\right) \geq x_0^n\right) \;\geq\; \frac{\mathbb{P}(\mathcal{Z}_n = 0) - x_0^n}{1 - x_0^n} \;\geq\; 1 - \varepsilon$$

for all large n, as $\mathbb{P}(\mathcal{Z}_n = 0) \to 1$ for $n \to \infty$. In combination with (5.16), this gives

$$\mathbb{P}\left(\mathbb{P}\left(\mathcal{Z}_n = 0 \middle| \mathbb{T}\right) \in (x_0^n, 1 - x_0^n)\right) \;\geq\; 1 - 2\varepsilon,$$

and thus we find an $n_1 \geq n_0$ such that

$$\mathbb{P}_z(\mathcal{Z}_n > 0) \;\geq\; h(x_0^n)(1 - 2\varepsilon) \tag{5.17}$$

for all $n \geq n_1$ and $z \geq 2$ by an appeal to (5.15).

So, the proof is complete if $h(x_0^n) \geq 1/2$ for all $z \geq a^n$ and large n. Indeed, this statement holds true if $x_0^{nz} + (1 - x_0)^{nz} \leq 2^{-1}$. Observe that

$$(1 - x_0^n)^{a^n} \;=\; \left(1 - \frac{1}{a^n}\right)^{a^n} \;\to\; e^{-1} \quad \text{as } n \to \infty$$

and thus there exists a $n_2 \geq n_1$ such that

$$(1 - x_0^n)^z \;\leq\; e^{-0.9} \;<\; \frac{1}{2}.$$

for all $n \geq n_2$ and $z \geq a^n$. Obviously $x_0^n \to 0$ for $n \to \infty$ and thus x_0^{nz} becomes small for all large n. This proves the claimed and thus the lemma. □

Remark 5.7. *(a)* The condition $\mathbb{E}_{\kappa+1}\tau = \infty$ in part (i) of the previous Theorem 5.5(b) is valid if the ABPREI is critical, i.e. $\mathbb{E}g'_{\Lambda_0}(1)\log g'_{\Lambda_0}(1) = 0$ by Remark 2.5, and the integrability assumptions $0 < \mathbb{E}\left(\log^2 \mu_{\hat{U}_0,\hat{T}_0}\right) < \infty$ and

$$\mathbb{E}\left(\left(1 + \log \mu_{\hat{U}_0,\hat{T}_0}\right) \frac{\mathbb{E}(X^{(\hat{U}_0,\hat{T}_0)}(X^{(\hat{U}_0,\hat{T}_0)} - 1)|\hat{U}_0,\hat{T}_0)}{2\mu_{\hat{U}_0,\hat{T}_0}}\right) < \infty$$

hold true. This follows directly from Proposition 3.8. Hence, $\lim_{n\to\infty} \gamma^{-n}\mathbb{P}(\mathcal{Z}_n > 0) = 0$ in this case.

(b) That the critical ABPREI may have a class of positive recurrent states, which is hit eventually, was shown by Seneta in [77], where he gave an example in the case of a constant environmental sequence. So $\mathbb{E}_{\kappa+1}\tau < \infty$ is valid under suitable integrability assumptions.

(c) If the ABPREI is strongly subcritical, the state κ is positive recurrent and thus $\mathbb{E}_{\kappa+1}\tau < \infty$. Indeed, $\mathbb{E}g'_{\Lambda_0}(1)\log g'_{\Lambda_0}(1) < 0$ yields $\mu_{u,t} \in (0,1)$ and thus $\mathbb{P}(X^{(u,t)} = 0) \in (0,1)$ for some $1 \leq u \leq t < \infty$ with $p_t > 0$. If $\mathbb{P}(X^{(u,t)} \leq 1) = 1$ for all $1 \leq u \leq t < \infty$ with $p_t > 0$, then obviously $\kappa = 0$ and $\mathbb{E}\tau = 1$. If otherwise $p_t\mathbb{P}(X^{(u,t)} \geq 2) > 0$ for some $1 \leq u \leq t < \infty$, we get $\mathbb{P}(\hat{X}_0^{(\hat{U},\hat{T}_0)} - 1 > 0) > 0$ and $\mathbb{P}(X^{(\hat{U}_0,\hat{T}_0)} = 0) > 0$. Then Lemma 3.1 states that $(\hat{Z}_{\hat{V}_n} - 1)_{n\geq 0}$ hits the irreducible and aperiodic set C_κ eventually. Furthermore, this process along the spine is subcritical due to $\mathbb{E}g'_{\Lambda_0}(1)\log g'_{\Lambda_0}(1) < 0$ and Remark 2.5. But as $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 < \infty$, it converges in distribution to a finite random variable by Theorem 3.11. Standard Markov theory then gives the positive recurrence of state κ.

(d) To verify $\mathbb{E}_{\kappa+1} \log^+ \hat{Y} < \infty$ is not the easy task. However, we conjecture that this condition is implied by $\mathbb{E}_{\kappa+1}\tau < \infty$ and $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 < \infty$ and vice versa, maybe under some further mild assumptions. This should be true for at least the strongly subcritical case.

(e) If the ABPREI is weakly or intermediate subcritical (under minor conditions, for example the ones in (a) of this remark), the decay rate of the survival probability is strictly less than γ^n. In view of Theorem 5.1 and the results in [40], we conjecture that the proper rate is of order

$$(\nu\rho)^n \cdot \begin{cases} n^{-1/2}, & \text{(intermediate subcritical case)}, \\ n^{-3/2}, & \text{(weakly subcritical case)}, \end{cases}$$

in the two different regimes.

An additional outcome of the proof of Theorem 5.5$(b)(i)$ is the tightness of the probability measures $(\mathbb{P}(\hat{\mathcal{Z}}_n \in \cdot))_{n\geq 0}$. This holds true if

$$\begin{gathered} \gamma < \min\{1, \nu\}, \quad \mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 < \infty, \quad \mathbb{E}g'_{\Lambda_0}(1) \log g'_{\Lambda_0}(1) \leq 0, \\ \mathbb{E}_{\kappa+1} < \infty \quad \text{and} \quad \mathbb{E}_{\kappa+1} \log^+ \hat{Y} < \infty \end{gathered} \tag{R}$$

and additionally $\sup_{1\leq u\leq t<\infty} p_t \mathbb{P}(X^{(u,t)} \geq 2) > 0$ are valid. In particular, the process of parasites along the spinal cells eventually hits a positive recurrent state. However, in the case where the last condition is violated, there is no immigration in the ABPREI. Hence, the number of parasites along the spine is non-decreasing, and if

$$\mathbb{P}(X^{(u,t)} = 0) \in (0,1) \quad \text{for some } 1 \leq u \leq t < \infty \text{ with } p_t > 0, \tag{R*}$$

the process $(\hat{Z}_{\hat{V}_n})_{n\geq 0}$ eventually reaches the absorbing state 1. As every parasite population in a subtree branching off the spine dies out almost surely, the process has the asymptotic behavior as starting with a single parasites and one parasite in each spinal cell. Hence, the proof of Theorem 5.5$(b)(i)$ can be adapted, and we obtain tightness in this situation too. Since (R*) is implied by $\sup_{1\leq u\leq t<\infty} p_t \mathbb{P}(X^{(u,t)} \geq 2) > 0$, as seen in the proof of Theorem 5.5(b), this condition is a relaxation of the latter one. If both (R) and (R*) are valid, we refer to them by (R1), i.e.

$$\text{(R)} \quad \text{and} \quad \text{(R*)}. \tag{R1}$$

In the case of part (ii) of Theorem 5.5(b), that is when

$$\begin{gathered} \gamma < \min\{1, \nu\}, \quad \mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 < \infty, \quad \mathbb{E}g'_{\Lambda_0}(1) \log g'_{\Lambda_0}(1) \leq 0 \quad \text{and} \\ \mathbb{P}(X^{(u,t)} = 0) \in \{0,1\} \quad \text{for all } 1 \leq u \leq t < \infty \text{ with } p_t > 0, \end{gathered} \tag{R2}$$

we are in the standard Galton-Watson setting and thus tightness of the above measures holds true under the $(Z \log Z)$-condition by the classical theory (see e.g. [61, 62]). We summarize all these observations in the next lemma.

Lemma 5.8. *Let* (R1) *or* (R2) *hold true. The measures* $(\mathbb{P}_{(s,z)}(\hat{\mathcal{Z}}_n \in \cdot))_{n\geq 0}$ *are tight for each* $(s,z) \in S^*$.

Proof. Tightness follows immediately from (5.1) and the discussion above the lemma. $\square$

The results of Theorem 5.5 can be generalized to an arbitrary number of root cells and parasites, and limits can be determined. The proof of the exact limit in the case when (R1) holds true can be done by elementary but cumbersome calculations. However, these result follows as an easy corollary from Proposition 5.12, which is why we omit the proof here and refer to Corollary 5.14 for details. We mention that the below theorem is not needed to proof Proposition 5.12.

Theorem 5.9. *Let $c \in [0,\infty)$ as defined in* (5.3). *If $c = 0$, then $\gamma^{-n}\mathbb{P}_{(s,z)}(\mathcal{Z}_n > 0) \to 0$ as $n \to \infty$ for all $(s,z) \in S^*$. Furthermore,*

$$\lim_{n\to\infty} \frac{1}{\gamma^n}\mathbb{P}_{(s,z)}(\mathcal{Z}_n > 0) \;=\; \begin{cases} c\sum_{i=1}^{s} z_i & \textit{if } \text{(R1)} \textit{ holds true,} \\ cs & \textit{if } \text{(R2)} \textit{ holds true} \end{cases}$$

for all $(s,z) \in S^*$.

Proof. If $c = 0$, the assertion follows immediately from (5.1). The case where (R1) holds true follows from Corollary 5.14.

If (R2) is valid, viz. $\mathbb{P}(X^{(u,t)} = 0) \in \{0,1\}$ for all $1 \le u \le t < \infty$ with $p_t > 0$, it follows that $\mu_{u,t} = 0$ or $\mu_{u,t} \ge 1$ for all $1 \le u \le t < \infty$ with $p_t > 0$. Since $\mathbb{E}g'_{\Lambda_0}(1)\log g'_{\Lambda_0}(1) \le 0$, this gives

$$\mathbb{P}(X^{(u,t)} = 0) = 1 \quad \text{or} \quad \mathbb{P}(X^{(u,t)} = 1) = 1$$

for all $1 \le u \le t < \infty$ with $p_t > 0$. Consequently, every infected cell contains as many parasites as the root cell, i.e. $\mathcal{Z}_n = x\mathcal{T}^*_n$ $\mathbb{P}_x$-a.s. for each $x \in \mathbb{N}$, and $(\mathcal{T}^*_n)_{n\ge 0}$ forms a standard GWP. Hence,

$$\mathbb{P}_{(s,z)}(\mathcal{Z}_n > 0) \;=\; \mathbb{P}\left(\sum_{i=1}^{s} \mathcal{T}^*_{n,i} > 0\right),$$

where $(\mathcal{T}^*_{n,i})_{n\ge 0}$, $1 \le i \le s$, are independent copies of $(\mathcal{T}^*_n)_{n\ge 0}$. So, $\mathbb{P}_{(s,z)}(\mathcal{Z}_n > 0)$ has the same asymptotic behavior as the survival probability of a standard GWP with s ancestors. By Lemma 1.9, the assertion follows, as only one of the $\mathcal{T}^*_{n,i}$ survives in the long run and each of them has the same probability to be that particular process. $\square$

Remark 5.10. In the case where (R2) holds true, c is the limit in Kolmogorov's classical theorem [46, Theorem (2.6.1)]. If otherwise (R1) holds true, c can be determined as $\sum_{k=1}^{\infty} k^{-1}\hat{\theta}(k)$, where $\hat{\theta}$ is the limit distribution of $\mathbb{P}(\hat{\mathcal{Z}}_n \in \cdot)$. This is shown in the later Corollary 5.13.

5.2 Conditional limit theorems

Considering almost sure extinction, the process of parasites $(\mathcal{Z}_n)_{n\ge 0}$ reaches the absorbing state 0 eventually. Therefore, for a more detailed description of the behavior of the BwBP, one has to study its distribution conditioned under the event $\{\mathcal{Z}_n > 0\}$. For this purpose, *we assume* *that* (R1) *or* (R2) *hold true in this section*, which ensures the limit c in (5.3) to be positive by Theorem 5.5.

Before stating and proving results in detail, we shortly recall that the homogeneous Markov chain $\boldsymbol{BPG}$ describes the number of infected cells and its containing parasites. $\boldsymbol{BPG}$ has state space $S_0^* = S^* \cup \{(0,0)\}$, transient states S^* and transition probabilities given by (1.17), i.e.

$$p((s,x),(t,z)) := \mathbb{P}_{(s,x)}(\boldsymbol{BPG}_1 = (t,z))$$

for $(s,x),(t,z) \in S_0^*$.

Since we are interested in the asymptotic behavior of the BwBP conditioned under $\{\mathcal{Z}_n > 0\}$, we introduce some further notation. We put

$$\mathbb{P}^n_{(t,z)} := \mathbb{P}_{(t,z)}(\cdot|\mathcal{Z}_n > 0)$$

for all $n \in \mathbb{N}_0$ and $(t,z) \in S_0^*$ and denote by $\mathbb{E}^n_{(t,z)}$ the corresponding expectation. As usual we write $\mathbb{P}^n_x$ if we start with one cell hosting $x \in \mathbb{N}_0$ parasites, and omit the index if the starting cell contains only one parasite.

With this notation, the opening question for the analysis of this section can be formulated as follows: Does there exist a probability distribution π on S^* such that

$$\lim_{n\to\infty} \mathbb{P}^n_{(t,z)}(\boldsymbol{BPG}_n \in \cdot) \;=\; \pi(\cdot)$$

for each $(t,z) \in S^*$? Such a limiting distribution is called *quasi-stationary distribution* or *Yaglom-limit*. This section gives a positive answer to this question. We even prove that the above convergence holds true in total variation distance d_{TV}. Moreover, we study the distribution of $\boldsymbol{BPG}_n$ conditioned under $\{\mathcal{Z}_{n+k} > 0\}$ for $k \in \mathbb{N}$ arbitrary and determine its limit for $n \to \infty$. Similar results can be looked up in [2, 14] for the GWP and the BPRE. Before analyzing the situation in the general model, we shortly look at the simple case in which the number of parasites can be expressed via a weighted sum of standard GWPes.

5.2.1 A simple Galton-Watson case

We assume throughout this subsection that (R2) *holds true.* This particularly means that

$$\mathbb{P}(X^{(u,t)} = 0) = 1 \quad \text{or} \quad \mathbb{P}(X^{(u,t)} = 1) = 1 \qquad \text{for all } 1 \le u \le t < \infty \text{ with } p_t > 0$$

as seen in the proof of Theorem 5.9. Hence, every contaminated cell contains exactly as many parasites as the root cell. So, starting with $s \in \mathbb{N}$ root cells each containing a single parasite, denoted by $(s,1) \in S^*$, provides $\mathcal{Z}_n = \mathcal{T}_n^*$ for all $n \in \mathbb{N}$. Moreover, $(\mathcal{Z}_n)_{n\ge 0}$ forms a simple GWP starting with s ancestors. In this case, the introductory question of this section has been entirely answered by the classical theory (see e.g. [14, 46, 61]).

- *(Yaglom's theorem)* For all $k \in \mathbb{N}_0$ there exists a probability distribution π_k on $\mathbb{N}$ such that for all $(s,1) \in S^*$
$$\lim_{n\to\infty} \mathbb{P}^{n+k}_{(s,1)}(\mathcal{Z}_n \in \cdot) \;=\; \pi_k(\cdot). \tag{5.18}$$
Furthermore, the mean of π_1 is c^{-1}, and if $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 < \infty$, then $\lim_{k\to\infty} \pi_k(z) = cz\pi_1(z)$ for every $z \in \mathbb{N}$. In particular, $(\mathcal{Z}_n|\mathcal{Z}_n > 0)_{n\ge 0}$ is uniformly integrable.

- *(Q-process)* For all $z \in \mathbb{N}$ and $(s,1) \in S^*$
$$\lim_{k\to\infty} \mathbb{P}^{n+k}_{(s,1)}(\mathcal{Z}_n = z) \;=\; \frac{z}{s\gamma^n}\mathbb{P}_{(s,1)}(\mathcal{Z}_n = z) \;=\; \frac{z}{s\gamma^n}\mathbb{P}_{(s,1)}(\boldsymbol{BPG}_n = (z,1)), \tag{5.19}$$
and if additionally $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 < \infty$,
$$\lim_{n\to\infty}\lim_{k\to\infty} \mathbb{P}^{n+k}_{(s,1)}(\mathcal{Z}_n = z) \;=\; cz\pi_1(z).$$
A Markov chain with transition probabilities given by (5.19) is called a *Q-process* and converges to its stationary distribution $(cz\pi_1(z))_{z\geq 1}$ if $\mathbb{E}\mathcal{Z}_1 \log \mathcal{Z}_1 < \infty$ (see [14, Chapter I.14]).

If cells are able to host more than one parasite, the situation becomes a bit more complex. For $(s,z) \in S^*$ with $z = (z_1,\dots,z_s)$ let $\boldsymbol{BT}^{(1)},\dots,\boldsymbol{BT}^{(s)}$ denote the independent BwBPes with ancestor parasites $z_1,\dots,z_s$. The number of parasites can be expressed as
$$\mathcal{Z}_n \;=\; \sum_{i=1}^{s} z_i t^*_n(\boldsymbol{BT}^{(i)}), \quad n \in \mathbb{N}_0,$$
with $t^*_n(\boldsymbol{BT}^{(1)}),\dots,t^*_n(\boldsymbol{BT}^{(s)})$ denoting the number of contaminated cells in the separate cell trees. These processes of infected cells are i.i.d and each forms a standard GWP. By Lemma 1.9, only one subtree survives and the surviving tree is picked uniformly due to the i.i.d. property of the spawning trees. Thus the limiting distribution of $\boldsymbol{BPG}$ can be expressed as a convex combination of Yaglom distributions of simple GWPes with a single ancestor. More precisely, Yaglom's theorem yields for every $k \in \mathbb{N}_0$
$$\lim_{n\to\infty} \mathbb{P}^{n+k}_{(s,z)}(\boldsymbol{BPG}_n \in \cdot) \;=\; \sum_{i=1}^{s} \frac{1}{s}\pi^i_k(\cdot), \tag{5.20}$$
where $\pi_k(x) = \pi^i_k((x,(z_i)))$ for all $1 \leq i \leq s$ and π_k as in Equation (5.18). Thereby, $(x,(z_i)) \in S^*$ means that each of the x cells contains z_i parasites. In particular, the limiting distribution depends on the starting number of cells and parasites. Letting k tend to infinity, Yaglom's theorem once again gives for $x \in \mathbb{N}$
$$\lim_{k\to\infty} \sum_{i=1}^{s} \frac{1}{s}\pi^i_k((x,(z_i))) \;=\; \sum_{i=1}^{s} \frac{xc}{sz_i}\pi^i_1((x,(z_i))),$$
with $(xc\pi^i_1((x,(z_i)))/z_i)_{x\in\mathbb{N}}$ being the size-biased distribution of π^i_1.

If we first let k tend to infinity for a fixed n, we get
$$\begin{aligned}\mathbb{P}^{n+k}_{(s,x)}(\boldsymbol{BPG}_n = (t,z)) \;&=\; \frac{\mathbb{P}_{(t,z)}(\mathcal{Z}_k > 0)\mathbb{P}_{(s,x)}(\boldsymbol{BPG}_n = (t,z))}{\mathbb{P}_{(s,x)}(\mathcal{Z}_{n+k} > 0)} \\ &\to\; \frac{t}{s\gamma^n}\mathbb{P}_{(s,x)}(\boldsymbol{BPG}_n = (t,z)) \quad \text{as } k \to \infty\end{aligned}$$
for each $(s,x),(t,z) \in S^*$ by using Theorem 5.9. Letting now n go to infinity yields
$$\lim_{n\to\infty}\lim_{k\to\infty} \mathbb{P}^{n+k}_{(s,x)}(\boldsymbol{BPG}_n = (t,z)) \;=\; \lim_{n\to\infty} \frac{t}{s\gamma^n}\mathbb{P}_{(s,x)}(\boldsymbol{BPG}_n = (t,z))$$

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{t\mathbb{P}_{(s,x)}(\mathcal{Z}_n > 0)}{s\gamma^n}\mathbb{P}^n_{(s,x)}(\boldsymbol{BPG}_n = (t,z)) \\
&= \sum_{i=1}^{s} \frac{tc}{sx_i}\pi_1^i((t,z))
\end{aligned}$$

by (5.20) and Theorem 5.9. So, Yaglom's theorem as well as the limit of the Q-process are transfered to the general setting of an arbitrary starting configuration in this special scenario.

5.2.2 The general branching within branching case

After the short interlude about the simple case in the previous subsection, we consider the general setting and *assume from now on that* (R1) *holds true.* In particular, under the latter assumption $(\mathcal{Z}_n)_{n\geq 0}$ loses its Galton-Watson properties and the classical theorems do not apply any more. However, the convergence results mentioned in the introduction of this section are still true and later proved. The methods of proof in this subsection are orientated on the ones given in [2] for the BPRE.

The proofs of the afore mentioned results use a process $\widehat{\boldsymbol{BPG}}$ living on S^* and describing the number of contaminated cells and its parasites in the generations of the size-biased BwBP $\widehat{\boldsymbol{BP}}$. The process $\widehat{\boldsymbol{BPG}}$ is the analog to $\boldsymbol{BPG}$ of the ordinary BwBP. Thus, $\widehat{\boldsymbol{BPG}} = (\widehat{\boldsymbol{BPG}}_n)_{n\geq 0}$ is defined by

$$\widehat{\boldsymbol{BPG}}_n := \chi_n(\widehat{\boldsymbol{BP}}_n), \quad n \in \mathbb{N}_0,$$

with the mapping χ_n given in (1.16). For $(s,z) = (s,(z_1,\ldots,z_s)) \in S^*_0$, we introduce the notation $\bar{z}$ for the sum of all elements in the vector z, i.e.

$$\bar{z} := \sum_{i=1}^{s} z_i.$$

By Remark 2.3, this yields for all $(t,z) \in S^*$ with $z = (z_1,\ldots,z_t)$ and $n \in \mathbb{N}_0$

$$\mathbb{P}_{(t,z)}\left((\widehat{\boldsymbol{BPG}}_k)_{k\leq n} \in \cdot\right) = \frac{1}{\bar{z}}\mathbb{E}_{(t,z)}\left(W_n \mathbb{1}_{\{(\boldsymbol{BPG}_k)_{k\leq n}\in\cdot\}}\right). \tag{5.21}$$

As a result, we get for each $n \in \mathbb{N}_0$ and $(s_0,x_0),\ldots,(s_n,x_n) \in S^*$

$$\begin{aligned}
\mathbb{P}_{(s_0,x_0)}\left((\widehat{\boldsymbol{BPG}}_k)_{k\leq n} = (s_k,x_k)_{k\leq n}\right) &= \frac{\bar{x}_n}{\bar{x}_0\gamma^n}\mathbb{P}_{(s_0,x_0)}\left((\boldsymbol{BPG}_k)_{k\leq n} = (s_k,x_k)_{k\leq n}\right) \\
&= \frac{\bar{x}_n}{\bar{x}_0\gamma^n}\prod_{k=0}^{n-1} p((s_k,x_k),(s_{k+1},x_{k+1})) \\
&= \prod_{k=0}^{n-1} \frac{\bar{x}_{k+1}}{\bar{x}_k\gamma}p((s_k,x_k),(s_{k+1},x_{k+1})) \\
&= \prod_{k=0}^{n-1} \mathbb{P}_{(s_k,x_k)}\left(\widehat{\boldsymbol{BPG}}_1 = (s_{k+1},x_{k+1})\right),
\end{aligned}$$

where we used the Markov property of $\boldsymbol{BPG}$ from Proposition 1.5. Hence, $\widehat{\boldsymbol{BPG}}$ is a homogeneous Markov chain, and we summarize the obtained results in the following proposition.

Proposition 5.11. *$\widehat{\boldsymbol{BPG}}$ is a homogeneous Markov chain with state space S^* and transition probabilities*

$$\hat{p}((s,x),(t,z)) := \frac{\bar{z}}{\bar{x}\gamma} p((s,x),(t,z)) \tag{5.22}$$

for $(s,x),(t,z) \in S^*$.

For $n \in \mathbb{N}$ and $(s,x),(t,z) \in S^*$, let us define the n-step transition probabilities of $\boldsymbol{BPG}$ resp. $\widehat{\boldsymbol{BPG}}$ by

$$p_n((s,x),(t,z)) := \mathbb{P}_{(s,x)}\left(\boldsymbol{BPG}_n = (t,z)\right)$$

and

$$\hat{p}_n((s,x),(t,z)) := \mathbb{P}_{(s,x)}\left(\widehat{\boldsymbol{BPG}}_n = (t,z)\right) = \frac{\bar{z}}{\bar{x}\gamma^n} p_n((s,x),(t,z)). \tag{5.23}$$

Proposition 5.12. *Given the Assumptions* (R1)*, the Markov chain $\widehat{\boldsymbol{BPG}}$ has a unique recurrence class $\hat{R}$ which is aperiodic and positive recurrent. Furthermore, the chain hits $\hat{R}$ with probability one from every initial state.*

Proof. First, we prove that there are states in S^* which can be reached from every other state. In particular, (R*) yields the existence of a tuple $(u,t) \in \mathbb{N}^2$ such that $p_t > 0$ and

$$\mathbb{P}(X^{(u,t)} = \breve{x})\mathbb{P}(X^{(u,t)} = 0) > 0$$

for some $\breve{x} \in \mathbb{N}$. Let $(x_1,\dots,x_t) \in \mathbb{N}_0^t$ and $(y_1,\dots,y_t) \in \mathbb{N}_0^t$ such that $x_u = \breve{x} > 0$, $y_u = 0$ and

$$\mathbb{P}(X^{(1,t)} = x_1,\dots,X^{(t,t)} = x_t) > 0 \quad \text{and} \quad \mathbb{P}(X^{(1,t)} = y_1,\dots,X^{(t,t)} = y_t) > 0. \tag{5.24}$$

For $1 \le i \le t$ define $x'_i := x_i + \sum_{j=1}^{\breve{x}-1} y_i = x_i + (\breve{x}-1)y_i$ and $t' := \#\{1 \le i \le t : x'_i > 0\}$. Let further denote x' the increasingly ordered vector of the strictly positive x'_i. Thus, x' is of length t' and $(t',x') \in S^*$. By the definition of (t',x'), this yields

$$p((1,\breve{x}),(t',x')) > 0, \tag{5.25}$$

and we show that this constructed state can be reached from every other state. To see that, put

$$A(\breve{x}) := \{(s,(z_1,\dots,z_s)) \in S^* \ : \ \breve{x} \in \{z_1,\dots,z_s\}\}$$

as the set of host-parasite configurations in which at least one cell contains $\breve{x}$ parasites. Obviously, $\mathbb{P}_y(\mathcal{Z}_1 = 0) \ge \mathbb{P}_{y+1}(\mathcal{Z}_1 = 0) > 0$ for all $y \in \mathbb{N}$, and utilizing this, the branching property, (5.21) and (5.25), we deduce for all $(s',z') \in A(\breve{x})$ with $z' = (z'_1,\dots,z'_s)$

$$\hat{p}((s',z'),(t',x')) \ = \ \frac{\bar{x}'}{\bar{z}'\gamma} p((s',z'),(t',x')) \ \ge \ \frac{\bar{x}'}{\bar{z}'\gamma} p((1,\breve{x}),(t',x'))\mathbb{P}_{z'_s}(\mathcal{Z}_1 = 0)^{s-1} \ > \ 0. \tag{5.26}$$

By using the same arguments, we further obtain for each $(s,z) \in S^*$ with $z = (z_1,\dots,z_s)$

$$\begin{aligned} \mathbb{P}_{(s,z)}\left(\widehat{\boldsymbol{BPG}}_1 \in A(\breve{x})\right) \ &\ge \ \mathbb{P}_{z_1}\left(\widehat{\boldsymbol{BPG}}_1 \in A(\breve{x})\right)\mathbb{P}_{z_s}(\mathcal{Z}_1 = 0)^{s-1} \\ &\ge \ p_t \mathbb{P}(X^{(1,t)} = x_1,\dots,X^{(t,t)} = x_t)\mathbb{P}(X^{(1,t)} = y_1,\dots,X^{(t,t)} = y_t)^{z_1-1}\mathbb{P}_{z_s}(\mathcal{Z}_1 = 0)^{s-1} \ > \ 0. \end{aligned}$$

Putting this together with (5.26) gives for all $(s, z) \in S^*$

$$\hat{p}_2((s,z),(t',x') \ \geq \sum_{(s',z')\in A(\breve{x})} \hat{p}((s,x),(s',z'))\hat{p}((s',z'),(t',x')) \ > \ 0.$$

Hence, (t', x') can be reached from every other state in S^* in at least two time steps, and so there exists at most one recurrence class $\hat{R}$. As $(t', x') \in A(\breve{x})$ by construction, (5.26) provides the aperiodicity of this state. Thus, $\hat{R}$ is aperiodic, if it exists.

Next, we show that $\hat{R}$ exists and is positive recurrent. Indeed, Lemma 5.8 ensures for all $(s, x) \in S^*$ and $\varepsilon > 0$ the existence of a $z \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \mathbb{P}_{(s,x)}\left(\widehat{\boldsymbol{BPG}}_n \in B(z)\right) \ = \ \lim_{n\to\infty} \mathbb{P}_{(s,x)}(\hat{\mathcal{Z}}_n \leq z) \ \geq \ 1-\varepsilon \ > \ 0 \tag{5.27}$$

with $B(z) := \{(t, y) \in S^* : \bar{y} \leq z\}$. Since $B(z)$ is finite, there is at least one positive recurrent state contained in $B(z)$.

Since transient states are visited only finitely often by a Markov chain and $B(z)$ is a finite set for each $z \in \mathbb{N}$, we get for each $(s, x) \in S^*$

$$\lim_{n\to\infty} \mathbb{P}_{(s,x)}\left(\widehat{\boldsymbol{BPG}}_n \in B(z)\cap \hat{R}^c\right) \ = \ 0.$$

Thus, we infer from (5.27) for each $(s, x) \in S^*$

$$\lim_{n\to\infty} \mathbb{P}_{(s,x)}\left(\widehat{\boldsymbol{BPG}}_n \in \hat{R}\right) \geq \lim_{n\to\infty} \mathbb{P}_{(s,x)}\left(\widehat{\boldsymbol{BPG}}_n \in B(z)\cap\hat{R}\right) = \lim_{n\to\infty} \mathbb{P}_{(s,x)}(\hat{\mathcal{Z}}_n \leq z) \geq 1-\varepsilon.$$

Letting ε tend to 0 yields that $\widehat{\boldsymbol{BPG}}$ hits $\hat{R}$ a.s. from every initial state. This proves the assertion. ☐

As an immediate consequence of Proposition 5.12 and the ergodic theory for Markov chains, we get the convergence of $\widehat{\boldsymbol{BPG}}$ to a stationary distribution in total variation distance d_{TV}. We briefly recall that the total variation distance for two probability measures P_1 and P_2 on a countable space $\mathcal{X}$ is defined by

$$d_{TV}\left(P_1, P_2\right) \ := \ \frac{1}{2}\sum_{x\in\mathcal{X}} |P_1(\{x\}) - P_2(\{x\})|.$$

Corollary 5.13. *Let* (R1) *hold true. There exists a distribution* $\hat{\pi} = (\hat{\pi}((t,z)))_{(t,z)\in S^*}$ *on* S^* *such that*

$$\lim_{n\to\infty} d_{TV}\left(\mathbb{P}_{(s,x)}\left(\widehat{\boldsymbol{BPG}}_n \in \cdot\right),\ \hat{\pi}(\cdot)\right) \ = \ 0$$

for all $(s, x) \in S^*$. *In particular, there exists a probability distribution* $\hat{\theta} = (\hat{\theta}(k))_{k\geq 1}$ *on the positive integers such that*

$$\lim_{n\to\infty} \mathbb{P}_{(s,z)}(\hat{\mathcal{Z}}_n \in \cdot) \ = \ \hat{\theta}(\cdot)$$

for all $(s, z) \in S^*$. *Furthermore,* $c = \sum_{k=1} k^{-1}\hat{\theta}(k)$ *for* c *as in Theorem 5.9.*

Proof. The limit statements follow immediately from Proposition 5.12 and the ergodic theory for Markov chains. Recalling the size-biased construction and particular Lemma 2.2, we get

$$\sum_{k=1}^{l} \frac{1}{k}\mathbb{P}(\hat{\mathcal{Z}}_n = k) \;\leq\; \frac{1}{\gamma^n}\mathbb{P}(\mathcal{Z}_n > 0) \;= \sum_{k=1}^{\infty} \frac{1}{k}\mathbb{P}(\hat{\mathcal{Z}}_n = k) \;\leq\; \sum_{k=1}^{l} \frac{1}{k}\mathbb{P}(\hat{\mathcal{Z}}_n = k) + \frac{1}{l} \tag{5.28}$$

for all $l \geq 1$. Hence,

$$\sum_{k=1}^{l} \frac{1}{k}\theta(k) \;\leq\; c \;\leq\; \sum_{k=1}^{l} \frac{1}{k}\theta(k) + \frac{1}{l}$$

by letting n tend to infinity. Finally $l \to \infty$ yields the assertion. □

As an consequence of the above corollary, we get the part of Theorem 5.9, where (R1) holds true.

Corollary 5.14. *Let* (R1) *hold true, and let* $c \in [0, \infty)$ *be the constant as defined in* (5.3)*. Then*

$$\lim_{n\to\infty} \frac{1}{\gamma^n}\mathbb{P}_{(s,z)}(\mathcal{Z}_n > 0) \;=\; c\bar{z}$$

for all $(s, z) \in S^*$*.*

Proof. Let $(s, z) \in S^*$. As in (5.28), we get

$$\bar{z}\sum_{k=1}^{l} \frac{1}{k}\mathbb{P}_{(s,z)}(\hat{\mathcal{Z}}_n = k) \;\leq\; \frac{1}{\gamma^n}\mathbb{P}_{(s,z)}(\mathcal{Z}_n > 0) \;\leq\; \bar{z}\sum_{k=1}^{l} \frac{1}{k}\mathbb{P}_{(s,z)}(\hat{\mathcal{Z}}_n = k) + \frac{\bar{z}}{l}$$

for all $n \in \mathbb{N}_0$ and $l \in \mathbb{N}$. Letting n tend to infinity provides

$$\bar{z}\sum_{k=1}^{l} \frac{1}{k}\theta(k) \;\leq\; \liminf_{n\to\infty} \frac{1}{\gamma^n}\mathbb{P}_{(s,z)}(\mathcal{Z}_n > 0) \;\leq\; \limsup_{n\to\infty} \frac{1}{\gamma^n}\mathbb{P}_{(s,z)}(\mathcal{Z}_n > 0) \;\leq\; \bar{z}\sum_{k=1}^{l} \frac{1}{k}\theta(k) + \frac{\bar{z}}{l}$$

under utilizing Corollary 5.13. For $l \to \infty$, the assertion follows by an other appeal to the above corollary. □

The next two additional results follow directly from Theorem 5.9 and Corollary 5.13. They say that the process $(\mathcal{Z}_n|\mathcal{Z}_n > 0)_{n\geq 0}$ is uniformly integrable and only descendants of one ancestor parasite survive conditioned upon survival up to the present time.

Proposition 5.15. *Let* (R1) *hold true.*

(a) *For all* $(s, x) \in S^*$

$$\lim_{z\to\infty} \sup_{n\geq 0} \mathbb{E}^n_{(s,x)}\left(\mathcal{Z}_n \mathbb{1}_{\{\mathcal{Z}_n > z\}}\right) \;=\; 0.$$

(b) *For all* $x \geq 2$

$$\lim_{n\to\infty} \mathbb{P}^n_x\left(\exists\ 1 \leq i < j \leq x \text{ s.t. } \mathcal{Z}_{n,i} \wedge \mathcal{Z}_{n,j} > 0\right) \;=\; 0,$$

where $\mathcal{Z}_{n,i}$ *denotes the number of descendants in generation* n *stemming from the* i^{th} *initial parasite.*

Proof. (*a*) Theorem 5.9, Corollary 5.13 and Lemma 2.2 provide for $(s,x) \in S^*$ with $x = (x_1, \ldots, x_s)$

$$\begin{aligned}\mathbb{E}_{(s,x)}\left(\mathcal{Z}_n \mathbb{1}_{\{\mathcal{Z}_n > z\}} \,|\mathcal{Z}_n > 0\right) &= \frac{1}{\mathbb{P}_{(s,x)}(\mathcal{Z}_n > 0)} \sum_{y>z} y \mathbb{P}_{(s,x)}(\mathcal{Z}_n = y) \\ &= \frac{\gamma^n \sum_{i=1}^s x_i}{\mathbb{P}_{(s,x)}(\mathcal{Z}_n > 0)} \mathbb{P}_{(s,x)}(\hat{\mathcal{Z}}_n > z) \\ &\xrightarrow{n\to\infty} c \sum_{y>z} \hat{\theta}(y) \\ &\xrightarrow{z\to\infty} 0.\end{aligned}$$

(*b*) Let $x \geq 2$. Since $\mathcal{Z}_{n,i}$ is distributed as $\mathcal{Z}_n$ when starting with a single parasite for each $1 \leq i \leq x$, we obtain

$$\begin{aligned}\mathbb{P}_x^n\left(\text{ex. } 1 \leq i < j \leq x \text{ s.t. } \mathcal{Z}_{n,i} > 0 \text{ and } \mathcal{Z}_{n,j} > 0\right) &\leq \binom{x}{2} \frac{\mathbb{P}_2(\mathcal{Z}_{n,1} > 0, \mathcal{Z}_{n,2} > 0)}{\mathbb{P}_x(\mathcal{Z}_n > 0)} \\ &\leq \binom{x}{2} \mathbb{P}_2^n(\mathcal{Z}_{n,1} > 0, \mathcal{Z}_{n,2} > 0),\end{aligned}$$

by an appeal to (1.20). So, it is enough to consider the case $x = 2$. But for each $n \in \mathbb{N}$, we get

$$\begin{aligned}1 \;=\; \mathbb{P}_2^n(\mathcal{Z}_{n,1} + \mathcal{Z}_{n,2} > 0) &= \mathbb{P}_2^n(\mathcal{Z}_{n,1} > 0) + \mathbb{P}_2^n(\mathcal{Z}_{n,1} = 0, \mathcal{Z}_{n,2} > 0) \\ &= \frac{\mathbb{P}(\mathcal{Z}_n > 0)}{\mathbb{P}_2(\mathcal{Z}_n > 0)} + \mathbb{P}_2^n(\mathcal{Z}_{n,1} = 0, \mathcal{Z}_{n,2} > 0),\end{aligned}$$

and from Theorem 5.9, we deduce

$$\lim_{n\to\infty} \mathbb{P}_2^n(\mathcal{Z}_{n,1} = 0, \mathcal{Z}_{n,2} > 0) \;=\; 1 - \lim_{n\to\infty} \frac{\mathbb{P}(\mathcal{Z}_n > 0)}{\mathbb{P}_2(\mathcal{Z}_n > 0)} \;=\; \frac{1}{2}.$$

By symmetry, the assertion follows. ☐

After having established distributional convergence of $\widehat{\boldsymbol{BPG}}$, we can now use these results to derive limits for $\boldsymbol{BPG}$. For $(t,z) \in S^*$ and $k \in \mathbb{N}_0$ let

$$\pi((t,z)) \;:=\; \frac{1}{c\bar{z}} \hat{\pi}((t,z)) \quad \text{and} \quad \pi_k((t,z)) \;:=\; \frac{\mathbb{P}_{(t,z)}(\mathcal{Z}_k > 0)}{\gamma^k} \pi((t,z)) \tag{5.29}$$

with $\hat{\pi}$ being the probability distribution given in Corollary 5.13 and c as defined in (5.3). In particular, $\pi_0 = \pi$ and π_k is a proper probability distribution on S^*. Moreover, π_k is the distributional limit of $\boldsymbol{BPG}_n$ conditioned under $\{\mathcal{Z}_{n+k} > 0\}$ for each $k \in \mathbb{N}_0$. This is an analogous result to the one of Yaglom in the classical Galton-Watson setting and stated in detail in the subsequent theorem.

Theorem 5.16. *Let* (R1) *hold true.* π_k *as defined in* (5.29) *is a probability distribution on* S^* *for every* $k \in \mathbb{N}_0$ *and for all* $(s,x) \in S^*$

$$\lim_{n\to\infty} d_{TV}\left(\mathbb{P}_{(s,x)}^{n+k}\left(\boldsymbol{BPG}_n \in \cdot\right),\; \pi_k(\cdot)\right) \;=\; 0.$$

Furthermore, $c^{-1} = \sum_{(t,z)\in S^*} \bar{z}\pi((t,z))$ *as well as* $\lim_{k\to\infty} \pi_k = \hat{\pi}$ *in total variation distance.*

Proof. Let $k \in \mathbb{N}_0$ and $(t,z) \in S^*$. Then by definition of the size-biased process (Lemma 2.2) and π

$$\begin{aligned} \pi_k((t,z)) &= \frac{\mathbb{P}_{(t,z)}(\mathcal{Z}_k > 0)}{\gamma^k}\pi((t,z)) = \sum_{x\geq 1}\frac{\bar{z}}{x}\mathbb{P}_{(t,z)}(\hat{\mathcal{Z}}_k = x)\pi((t,z)) \\ &= \frac{1}{c}\sum_{x\geq 1}\frac{1}{x}\mathbb{P}_{(t,z)}(\hat{\mathcal{Z}}_k = x)\hat{\pi}((t,z)). \end{aligned}$$

Summation over all $(t,z) \in S^*$ and the stationarity of $\hat{\pi}$ (see Corollary 5.13) yield

$$\sum_{(t,z)\in S^*}\pi_k((t,z)) = \frac{1}{c}\sum_{x\geq 1}\frac{1}{x}\sum_{(t,z)\in S^*}\mathbb{P}_{(t,z)}(\hat{\mathcal{Z}}_k = x)\hat{\pi}((t,z)) = \frac{1}{c}\sum_{x\geq 1}\frac{1}{x}\hat{\theta}(x) = 1,$$

while in the last equation again Corollary 5.13 was used. By definition, it follows directly

$$\sum_{(t,z)\in S^*}\bar{z}\pi((t,z)) = \frac{1}{c}\sum_{(t,z)\in S^*}\hat{\pi}((t,z)) = \frac{1}{c}$$

and

$$d_{TV}\left(\pi_k, \hat{\pi}\right) = \frac{1}{2}\sum_{(s,x)\in S^*}\hat{\pi}((s,x))\left|\frac{\mathbb{P}_{(s,x)}(\mathcal{Z}_k > 0)}{c\gamma^k\bar{x}} - 1\right| \to 0, \qquad \text{for } k \to \infty,$$

by utilizing dominated convergence with the fact that $\mathbb{P}_{(s,x)}(\mathcal{Z}_k > 0) \leq \mathbb{E}_{(s,x)}\mathcal{Z}_k = \bar{x}\gamma^k$ for each $k \in \mathbb{N}$ and $(s,x) \in S^*$.

Let us first prove the convergence in total variation distance for $k = 0$. Using (5.21), the definition of π, Theorem 5.9 and Corollary 5.13, we obtain for every $(s,x) \in S^*$

$$\begin{aligned} & d_{TV}\left(\mathbb{P}^n_{(s,x)}\left(\boldsymbol{BPG}_n \in \cdot\right),\ \pi(\cdot)\right) \\ &= \frac{1}{2}\sum_{(t,z)\in S^*}\left|\mathbb{P}^n_{(s,x)}\left(\boldsymbol{BPG}_n = (t,z)\right) - \pi((t,z))\right| \\ &= \frac{1}{2}\sum_{(t,z)\in S^*}\left|\frac{\bar{x}\gamma^n}{\mathbb{P}_{(s,x)}(\mathcal{Z}_n > 0)}\frac{1}{\bar{z}}\mathbb{P}_{(s,x)}\left(\widehat{\boldsymbol{BPG}}_n = (t,z)\right) - \frac{1}{c\bar{z}}\hat{\pi}((t,z))\right| \\ &\leq \frac{\bar{x}\gamma^n}{\mathbb{P}_{(s,x)}(\mathcal{Z}_n > 0)}d_{TV}\left(\mathbb{P}_{(s,x)}\left(\widehat{\boldsymbol{BPG}}_n \in \cdot\right), \hat{\pi}(\cdot)\right) + \frac{1}{2}\left|\frac{\bar{x}\gamma^n}{\mathbb{P}_{(s,x)}(\mathcal{Z}_n > 0)} - \frac{1}{c}\right| \\ &\xrightarrow{n\to\infty} 0. \end{aligned}$$

Let $k \in \mathbb{N}_0$ be arbitrary. Once again, using (5.21), the definition of π_k and the Markov property, we obtain

$$\begin{aligned} & d_{TV}\left(\mathbb{P}^{n+k}_{(s,x)}\left(\boldsymbol{BPG}_n \in \cdot\right),\ \pi_k(\cdot)\right) \\ &= \frac{1}{2}\sum_{(t,z)\in S^*}\left|\mathbb{P}^{n+k}_{(s,x)}\left(\boldsymbol{BPG}_n = (t,z)\right) - \pi_k((t,z))\right| \\ &= \frac{1}{2}\sum_{(t,z)\in S^*}\left|\frac{\mathbb{P}_{(s,x)}\left(\boldsymbol{BPG}_n = (t,z), \mathcal{Z}_{n+k} > 0\right)}{\mathbb{P}_{(s,x)}(\mathcal{Z}_{n+k} > 0)} - \pi_k((t,z))\right| \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{(t,z) \in S^*} \mathbb{P}_{(t,z)}(\mathcal{Z}_k > 0) \left| \frac{\mathbb{P}_{(s,x)}(\mathcal{Z}_n > 0)}{\mathbb{P}_{(s,x)}(\mathcal{Z}_{n+k} > 0)} \mathbb{P}^n_{(s,x)}\left(\boldsymbol{BPG}_n = (t,z)\right) - \frac{1}{\gamma^k} \pi((t,z)) \right| \\
&\leq \frac{\mathbb{P}_{(s,x)}(\mathcal{Z}_n > 0)}{\mathbb{P}_{(s,x)}(\mathcal{Z}_{n+k} > 0)} d_{TV}\left(\mathbb{P}^n_{(s,x)}\left(\boldsymbol{BPG}_n \in \cdot\right), \pi(\cdot) \right) + \frac{1}{2} \left| \frac{\mathbb{P}_{(s,x)}(\mathcal{Z}_n > 0)}{\mathbb{P}_{(s,x)}(\mathcal{Z}_{n+k} > 0)} - \frac{1}{\gamma^k} \right|.
\end{aligned}$$

Theorem 5.9 and the already established convergence for $k = 0$ finishes the proof when letting n tend to infinity. □

In the following corollary, we state the distributional convergence of the important processes $(\mathcal{T}_n^*)_{n \geq 0}$ and $(\mathcal{Z}_n)_{n \geq 0}$ conditioned upon survival at present time. These special cases follow directly from the above theorem.

Corollary 5.17. *Let* (R1) *hold true. Then there exist probability distributions θ and ϑ on $\mathbb{N}$ with finite mean, such that for all $(s,x) \in S^*$*

$$\lim_{n \to \infty} \mathbb{P}^n_{(s,x)}\left(\mathcal{Z}_n \in \cdot\right) \;=\; \theta(\cdot) \quad \textit{and} \quad \lim_{n \to \infty} \mathbb{P}^n_{(s,x)}\left(\mathcal{T}_n^* \in \cdot\right) \;=\; \vartheta(\cdot)$$

in total variation distance. Furthermore, $\hat{\theta}$ (from Corollary 5.13) is the size-biased distribution of θ and

$$\lim_{n \to \infty} \mathbb{E}^n_{(s,x)} \mathcal{Z}_n \;=\; \sum_{k=1}^{\infty} k\theta(k) \ (= c^{-1}) \quad \textit{as well as} \quad \lim_{n \to \infty} \mathbb{E}^n_{(s,x)} \mathcal{T}_n^* \;=\; \sum_{k=1}^{\infty} k\vartheta(k).$$

Proof. The convergences in total variation distance follow immediately from Theorem 5.16. By Proposition 5.15, the process $(\mathcal{Z}_n | \mathcal{Z}_n > 0)_{n \geq 0}$ is uniformly integrable and thus $(\mathcal{T}_n^* | \mathcal{Z}_n > 0)_{n \geq 0}$ is uniformly integrable too. Hence, the convergence of the means is ensured. That θ is the size-biased distribution of θ is derived from (5.5) and Corollary 5.13 as

$$\hat{\theta}(k) \;=\; \lim_{n \to \infty} \mathbb{P}(\hat{\mathcal{Z}}_n = k) \;=\; \lim_{n \to \infty} \frac{k}{\mathbb{E}^n \mathcal{Z}_n} \mathbb{P}^n(\mathcal{Z}_n = k) \;=\; ck\theta(k)$$

for each $k \in \mathbb{N}$. □

Consider again the distribution of $\boldsymbol{BPG}_n$ conditioned under $\{\mathcal{Z}_{n+k} > 0\}$. Instead of n, we let k tend to infinity first. This can be thought of looking at the process conditioned under non-extinction of parasites in the far future, but on certain extinction in the even more distant future. As in the standard Galton-Watson setting the considered distribution converges in total variation distance to a distribution generated by a positive recurrent Markov chain. This is described in more detail in the next theorem.

Theorem 5.18. *Let* (R1) *hold true and let $(k_n)_{n \geq 0}$ be a sequence of natural numbers such that $k_n \leq n$ for all $n \in \mathbb{N}$ and $n - k_n \to \infty$ as $n \to \infty$. Then for all $(s,x) \in S^*$*

$$\lim_{n \to \infty} d_{TV}\left(\mathbb{P}^n_{(s,x)}\left((\boldsymbol{BPG}_k)_{k \leq k_n} \in \cdot\right),\ \mathbb{P}_{(s,x)}\left((\widehat{\boldsymbol{BPG}}_k)_{k \leq k_n} \in \cdot\right) \right) \;=\; 0.$$

Proof. Fix $(s,x)\in S^*$. For all $l\le n$ and $(t_1,z_1),\dots,(t_l,z_l)\in S^*$, we get, using the definition of the transition probabilities of $\boldsymbol{BPG}$ and $\widehat{\boldsymbol{BPG}}$ (see (5.22) and (5.23)),

$$\begin{aligned}
&\mathbb{P}^n_{(s,x)}\left(\boldsymbol{BPG}_1=(t_1,z_1),\dots,\boldsymbol{BPG}_l=(t_l,z_l)\right)\\
&\quad=\sum_{(t',z')\in S^*}\frac{\mathbb{P}_{(s,x)}\left(\boldsymbol{BPG}_1=(t_1,z_1),\dots,\boldsymbol{BPG}_l=(t_l,z_l),\boldsymbol{BPG}_n=(t',z')\right)}{\mathbb{P}_{(s,x)}(\mathcal{Z}_n>0)}\\
&\quad=\sum_{(t',z')\in S^*}\frac{p((s,x),(t_1,z_1))}{\mathbb{P}_{(s,x)}(\mathcal{Z}_n>0)}\left(\prod_{i=1}^{l-1}p((t_i,z_i),(t_{i+1},z_{i+1}))\right)p_{n-l}((t_l,z_l),(t',z'))\\
&\quad=\sum_{(t',z')\in S^*}\frac{\bar{x}\gamma^l}{\bar{z}'_l}\frac{\hat{p}((s,x),(t_1,z_1))}{\mathbb{P}_{(s,x)}(\mathcal{Z}_n>0)}\left(\prod_{i=1}^{l-1}\hat{p}((t_i,z_i),(t_{i+1},z_{i+1}))\right)p_{n-l}((t_l,z_l),(t',z'))\\
&\quad=\sum_{(t',z')\in S^*}\mathbb{P}_{(s,x)}\left(\widehat{\boldsymbol{BPG}}_1=(t_1,z_1),\dots,\widehat{\boldsymbol{BPG}}_l=(t_l,z_l)\right)\frac{\bar{x}\gamma^l}{\bar{z}'_l}\frac{p_{n-l}((t_l,z_l),(t',z'))}{\mathbb{P}_{(s,x)}(\mathcal{Z}_n>0)}\\
&\quad=\mathbb{P}_{(s,x)}\left(\widehat{\boldsymbol{BPG}}_1=(t_1,z_1),\dots,\widehat{\boldsymbol{BPG}}_l=(t_l,z_l)\right)\frac{\bar{x}\gamma^l}{\bar{z}_l}\frac{\mathbb{P}_{(t_l,z_l)}(\mathcal{Z}_{n-l}>0)}{\mathbb{P}_{(s,x)}(\mathcal{Z}_n>0)}.
\end{aligned}$$

For $l,n\in\mathbb{N}_0$ with $l\le n$ and $(t,z)\in S^*$, let us define

$$h(l,n,(t,z)):=\frac{\bar{x}\gamma^l}{\bar{z}}\frac{\mathbb{P}_{(t,z)}(\mathcal{Z}_{n-l}>0)}{\mathbb{P}_{(s,x)}(\mathcal{Z}_n>0)}.$$

Theorem 5.9 implies

$$h(k_n,n,(t,z))=\frac{\bar{x}\gamma^{k_n}}{\bar{z}}\frac{\mathbb{P}_{(t,z)}(\mathcal{Z}_{n-k_n}>0)}{\mathbb{P}_{(s,x)}(\mathcal{Z}_n>0)}\to 1\quad\text{as } n\to\infty$$

if $n-k_n\to\infty$, and the Markov inequality and once again Theorem 5.9 ensure the existence of a constant $c^*>0$ such that for all $(t,z)\in S^*$ and $l,n\in\mathbb{N}_0$ with $l\le n$

$$h(l,n,(t,z))\;=\;\frac{\bar{x}\gamma^l}{\bar{z}}\frac{\mathbb{P}_{(t,z)}(\mathcal{Z}_{n-l}>0)}{\mathbb{P}_{(s,x)}(\mathcal{Z}_n>0)}\;\le\;\frac{\bar{x}\gamma^l}{\bar{z}}\frac{\mathbb{E}_{(t,z)}\mathcal{Z}_{n-l}}{\mathbb{P}_{(s,x)}(\mathcal{Z}_n>0)}\;=\;\frac{\bar{x}\gamma^n}{\mathbb{P}_{(s,x)}(\mathcal{Z}_n>0)}\;\le\;c^*.$$

Taking all these established properties into account, we get by dominated convergence

$$\begin{aligned}
&d_{TV}\left(\mathbb{P}^n_{(s,x)}\left((\boldsymbol{BPG}_k)_{k\le k_n}\in\cdot\right),\ \mathbb{P}_{(s,x)}\left((\widehat{\boldsymbol{BPG}}_k)_{k\le k_n}\in\cdot\right)\right)\\
&\quad=\frac{1}{2}\sum_{(t_1,z_1),\dots,(t_{k_n},z_{k_n})\in S^*}\mathbb{P}_{(s,x)}\left((\widehat{\boldsymbol{BPG}}_k)_{k\le k_n}=((t_k,z_k))_{k\le k_n}\right)\left|h(k_n,n,(t_{k_n},z_{k_n}))-1\right|\\
&\quad=\frac{1}{2}\sum_{(t,z)\in S^*}\mathbb{P}_{(s,x)}\left(\widehat{\boldsymbol{BPG}}_{k_n}=(t,z)\right)\left|h(k_n,n,(t,z))-1\right|\\
&\quad\xrightarrow{n\to\infty}0
\end{aligned}$$

when $n-k_n\to\infty$. □

Chapter 6

A host-parasite model for a two-type cell population

This chapter studies a host-parasite branching model with two types of cells (the hosts), here called A and B, and proliferating parasites colonizing the cells. In this particular model unilateral cell type heredity is assumed, meaning that B-cells can only split into cells with the same type, whereas type-A cells are able to split into both types. The model grew out of a discussion with biologists in an attempt to provide a first very simple setup that allows to study coevolutionary adaptations, here due to the presence of two different cell types.

The reciprocal, adaptive genetic change of two antagonists (e.g. different species or genes) through reciprocal, selective pressures is known as host-parasite coevolution. It may be observed even in real-time under both, field and laboratory conditions, if reciprocal adaptations occur rapidly and generation times are short. For more information see e.g. [57, 89].

In the first section, the model is introduced in detail and a connection to the BwBP is established. We then focus on the case of non-extinction of contaminated A-cells. Results on the number of contaminated cells of the various types, including the growth rate of contaminated cells with a certain type, are shown in Section 6.2. These will be partly instrumental for the proofs of our results on the asymptotic behavior of the relative proportion of contaminated cells with k parasites within the population of all contaminated cells. The analysis of the model makes use of the previously established branching within branching theory. Most of the presented results have been published in [6].

6.1 Description of the model

Consider a cell population where each cell is of one of the two types A or B, and proliferating parasites living in these cells. All cells behave independently and split into two daughter cells after one unit of time. The daughters of a type-B cell are thereby again of type B whereas type-A cells divide into cells of both types according to a random mechanism. Parasites in a cell multiply in an i.i.d. manner to produce a random number of offspring with a distribution which depends on the type of this cell as well as on those of its daughter cells. The same holds true for the random mechanism by which the offspring is shared into these daughter cells. This model is described in more detail in the following.

We use the notation of [6, 15], which is slightly different to the one of the previous chapters. Making the usual assumption of starting from one ancestor cell, denoted by $\varnothing$, we put $\mathbb{G}_0 := \{\varnothing\}$, $\mathbb{G}_n := \{0,1\}^n$ for $n \in \mathbb{N}$, and let

$$\mathbb{V}_2 := \bigcup_{n\in\mathbb{N}_0} \mathbb{G}_n \quad \text{with} \quad \mathbb{G}_n := \{0,1\}^n$$

be the binary Ulam-Harris tree rooted at $\varnothing$, which describes the cell tree. For any cell $v \in \mathbb{V}_2$, let $\mathsf{T}_v \in \{\mathsf{A}, \mathsf{B}\}$ denote its type and Z_v the number of parasites it contains. *Unless stated otherwise, the ancestor cell is assumed to be of type* A *and to contain one parasite, i.e.*

$$\mathsf{T}_\varnothing = \mathsf{A} \quad \text{and} \quad Z_\varnothing = 1. \tag{SA1}$$

Then, for $\mathsf{t} \in \{\mathsf{A}, \mathsf{B}\}$ and $n \in \mathbb{N}_0$ define

$$\mathbb{G}_n(\mathsf{t}) := \{v \in \mathbb{G}_n : \mathsf{T}_v = \mathsf{t}\} \quad \text{and} \quad \mathbb{G}_n^*(\mathsf{t}) := \{v \in \mathbb{G}_n(\mathsf{t}) : Z_v > 0\}$$

as the sets of type-t cells and contaminated type-t cells in generation n, respectively. The set of all contaminated cells in generation n is denoted by $\mathbb{G}_n^* := \mathbb{G}_n^*(\mathsf{A}) \cup \mathbb{G}_n^*(\mathsf{B})$.

The process $(\mathsf{T}_v)_{v\in\mathbb{V}_2}$ is a Markov process indexed by the tree $\mathbb{V}_2$ with transition probabilities

$$\mathbb{P}(\mathsf{T}_{v0} = \mathsf{x}, \mathsf{T}_{v1} = \mathsf{y} \,|\, \mathsf{T}_v = \mathsf{A}) = p_{\mathsf{xy}}, \quad (\mathsf{x}, \mathsf{y}) \in \{(\mathsf{A},\mathsf{A}), (\mathsf{A},\mathsf{B}), (\mathsf{B},\mathsf{B})\},$$
$$\mathbb{P}(\mathsf{T}_{v0} = \mathsf{B}, \mathsf{T}_{v1} = \mathsf{B} \,|\, \mathsf{T}_v = \mathsf{B}) = 1.$$

For information on tree-indexed Markov chains with independent and symmetric transitions see [20]. We further denote by

$$\mathsf{p}_0 := p_{\mathsf{AA}} + p_{\mathsf{AB}} = 1 - p_{\mathsf{BB}} \quad \text{and} \quad \mathsf{p}_1 := p_{\mathsf{AA}}$$

the probabilities that the first and the second daughter cell are of type A, respectively. In order to rule out total segregation of type-A and type-B cells, which would just lead back to the model studied in [15], it will be assumed throughout that

$$p_{\mathsf{AA}} < 1. \tag{SA2}$$

The family $(\mathsf{T}_v)_{v\in\mathbb{V}_2}$ indicates which cells are of type A, and by the transition probabilities, $(\mathbb{G}_n(\mathsf{A}))_{n\geq 0}$ forms a Galton-Watson cell tree with a cell giving rise to at most two daughter cells and reproduction mean

$$\nu := \mathsf{p}_0 + \mathsf{p}_1 .$$

Then the classical theory (see e.g. [14]) provides

$$\#\,\mathbb{G}_n(\mathsf{A}) \to 0 \quad \text{a.s.} \qquad \text{iff} \qquad \nu \leq 1 \quad \text{and} \quad p_{\mathsf{AB}} < 1. \tag{6.1}$$

To describe the multiplication of parasites, let Z_v denote the number of parasites in cell v and let

$$\left\{\left(X_{k,v}^{(0)}(\mathsf{t},\mathsf{s}), X_{k,v}^{(1)}(\mathsf{t},\mathsf{s})\right) \mid k \in \mathbb{N}, v \in \mathbb{V}_2\right\}, \quad \mathsf{t} \in \{\mathsf{A},\mathsf{B}\},\ \mathsf{s} \in \{\mathsf{AA},\mathsf{AB},\mathsf{BB}\}$$

be independent families of i.i.d. $\mathbb{N}_0^2$-valued random vectors with respective generic copies of $(X^{(0)}(\mathsf{t},\mathsf{s}), X^{(1)}(\mathsf{t},\mathsf{s}))$. If v is of type t and their daughter cells are of type x and y, then $X_{k,v}^{(i)}(\mathsf{t},\mathsf{xy})$ gives the offspring number of the k^{th} parasite in cell v that is shared into the daughter cell vi of v. Since type-B cells can only produce daughter cells of the same type, we will write $(X_{k,v}^{(0)}(\mathsf{B}), X_{k,v}^{(1)}(\mathsf{B}))$ as shorthand for $(X_{k,v}^{(0)}(\mathsf{B},\mathsf{BB}), X_{k,v}^{(1)}(\mathsf{B},\mathsf{BB}))$.

Next, observe that

$$(Z_{v0}, Z_{v1}) = \sum_{\mathsf{t}\in\{\mathsf{A},\mathsf{B}\}} \mathbb{1}_{\{\mathsf{T}_v=\mathsf{t}\}} \sum_{\mathsf{s}\in\{\mathsf{AA},\mathsf{AB},\mathsf{BB}\}} \mathbb{1}_{\{(\mathsf{T}_{v0},\mathsf{T}_{v1})=\mathsf{s}\}} \sum_{k=1}^{Z_v} (X_{k,v}^{(0)}(\mathsf{t},\mathsf{s}), X_{k,v}^{(1)}(\mathsf{t},\mathsf{s})),$$

and concentrating on the process of A-cells by ignoring cells of type B, gives a BwBP by definition (see Section 1.1.1). Thus, *all results established in the previous chapters are applicable to the process of* A-*cells and the parasites it contains*, which is why we mainly focus on the behavior of B-cells and its proportion to A-cells in the current chapter. This observation is summarized in the following proposition.

Proposition 6.1. *The process of type-*A *cells and the parasites it contains forms a BwBP.*

We put $\mu_{i,\mathsf{t}}(\mathsf{s}) := \mathbb{E}X^{(i)}(\mathsf{t},\mathsf{s})$ for $i \in \{0,1\}$ and t,s as before, write $\mu_{i,\mathsf{B}}$ as shorthand for $\mu_{i,\mathsf{B}}(\mathsf{BB})$ and assume throughout that $\mu_{i,\mathsf{t}}(\mathsf{s})$ are finite and

$$\mu_{0,\mathsf{A}}(\mathsf{AA}),\ \mu_{1,\mathsf{A}}(\mathsf{AA}),\ \mu_{0,\mathsf{B}},\ \mu_{1,\mathsf{B}}\ >\ 0. \tag{SA3}$$

Furthermore, we allow cell infection of both types from a type-A cell, viz.

$$\mathbb{P}\left(\mathbb{G}_1^*(\mathsf{A}) \neq \emptyset | (\mathsf{T}_\varnothing, Z_\varnothing) = (\mathsf{A},1)\right) > 0 \ \text{ and } \ \mathbb{P}\left(\mathbb{G}_1^*(\mathsf{B}) \neq \emptyset | (\mathsf{T}_\varnothing, Z_\varnothing) = (\mathsf{A},1)\right) > 0, \tag{SA4}$$

as otherwise, parasites are concentrated in a one type cell process. It is noted that $\mu_{0,\mathsf{A}}(\mathsf{AB})$ and $\mu_{1,\mathsf{A}}(\mathsf{AB})$ might be unequal, which then indicates the preference of parasites of one cell type over the other. In extreme cases, one or both of the means may even be zero. To avoid further trivialities and exceptions, it is always assumed hereafter that

$$\mathbb{P}\left(X^{(0)}(\mathsf{B}) \leq 1,\ X^{(1)}(\mathsf{B}) \leq 1\right) < 1. \tag{SA5}$$

The total number of parasites in cells of type $\mathsf{t} \in \{\mathsf{A},\mathsf{B}\}$ at generation n is denoted by

$$\mathcal{Z}_n(\mathsf{t}) := \sum_{v\in\mathbb{G}_n(\mathsf{t})} Z_v,$$

and we put $\mathcal{Z}_n := \mathcal{Z}_n(\mathsf{A}) + \mathcal{Z}_n(\mathsf{B})$, namely the total number of all parasites at generation n. To enforce actual growth of the type-A parasite number, we further assume

$$\mathbb{P}(\mathcal{Z}_1(\mathsf{A}) = 1) < 1. \tag{SA6}$$

Under this assumption, both, $(\mathcal{Z}_n)_{n\geq 0}$ and $(\mathcal{Z}_n(\mathsf{A}))_{n\geq 0}$, are transient Markov chains with absorbing state 0. That this holds true for $(\mathcal{Z}_n(\mathsf{A}))_{n\geq 0}$, follows directly from Proposition 6.1 and Corollary 1.6, and if this Markov chain reaches 0, then all remaining parasites are in B-cells and form a standard GWP. Hence, transience is ensured. According to our notation of the BwBP the extinction events are defined as

$$\mathsf{Ext} := \{\mathcal{Z}_n \to 0\} \quad \text{and} \quad \mathsf{Ext}(\mathsf{t}) := \{\mathcal{Z}_n(\mathsf{t}) \to 0\}, \quad \mathsf{t} \in \{\mathsf{A},\mathsf{B}\},$$

and their complements by Surv and $\mathsf{Surv}(\mathsf{t})$, respectively.

As in [15] and the chapters about the BwBP, we are interested in the stochastic properties of an infinite *random cell line*. But since the process concentrated on the A-cells is a BwBP, a random cell line picked from those lines consisting of A-cells only, the so-called *random* A-*cell line*, behaves like a BPRE denoted by $(Z_n(\mathsf{A}))_{n\geq 0}$ in an i.i.d. environmental sequence (see Subsection 1.2.1). The environment of this process is denoted by $\Lambda := (\Lambda_n)_{n\geq 1}$ and takes values in $\{\mathcal{L}(X^{(0)}(\mathsf{A},\mathsf{AA})), \mathcal{L}(X^{(1)}(\mathsf{A},\mathsf{AA})), \mathcal{L}(X^{(0)}(\mathsf{A},\mathsf{AB}))\}$ with

$$\mathbb{P}\Big(\Lambda_1 = \mathcal{L}(X^{(0)}(\mathsf{A},\mathsf{AB}))\Big) = \frac{p_{\mathsf{AB}}}{\nu} \quad \text{and} \quad \mathbb{P}\Big(\Lambda_1 = \mathcal{L}(X^{(i)}(\mathsf{A},\mathsf{AA}))\Big) = \frac{p_{\mathsf{AA}}}{\nu}$$

with $i \in \{0,1\}$. Furthermore, for $s \in [0,1]$

$$g_\lambda(s) := \mathbb{E}(s^{Z_1(\mathsf{A})}|\Lambda_1 = \lambda) = \sum_{n\geq 0} \lambda_n s^n$$

for any distribution $\lambda = (\lambda_n)_{n\geq 0}$ on $\mathbb{N}_0$. Moreover, the g_{Λ_n}, $n \in \mathbb{N}$, are i.i.d. with

$$\mathbb{E}g'_{\Lambda_1}(1) = \mathbb{E}Z_1(\mathsf{A}) \ = \frac{p_{\mathsf{AA}}}{\nu}\Big(\mu_{0,\mathsf{A}}(\mathsf{AA}) + \mu_{1,\mathsf{A}}(\mathsf{AA})\Big) + \frac{p_{\mathsf{AB}}}{\nu}\mu_{0,\mathsf{A}}(\mathsf{AB}) = \frac{\gamma}{\nu},$$

where

$$\gamma := \mathbb{E}\mathcal{Z}_1(\mathsf{A}) = p_{\mathsf{AA}}\left(\mu_{0,\mathsf{A}}(\mathsf{AA}) + \mu_{1,\mathsf{A}}(\mathsf{AA})\right) + p_{\mathsf{AB}}\mu_{0,\mathsf{A}}(\mathsf{AB})$$

denotes the expected total number of parasites in cells of type A in the first generation (recall from (SA1) that $Z_\varnothing = Z_\varnothing(\mathsf{A}) = 1$).

Looking now on a random cell line trough the whole tree $\mathbb{V}_2$ leads to another BPRE but without i.i.d. environment since multiplication of parasites depend on their hosting cell. However, one of the cells in such a cell line is of type B eventually due to $p_{\mathsf{AA}} < 1$ and the Borel-Cantelli lemma. As B-cells produce only daughter cells of the same type, this process starting from such a cell now behaves in an i.i.d. environmental manner. Hence, we are in the same situation as [15] and properties of a random B-cell line can be looked up there.

For a stringent definition of a random cell line, let $U = (U_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence of symmetric Bernoulli variables independent of the parasite evolution and put $V_n := U_1...U_n$. Then

$$\varnothing =: V_0 \to V_1 \to V_2 \to ... \to V_n \to ...$$

provides us with a random cell line in the binary Ulam-Haris tree, and we denote by

$$\mathsf{T}_{[n]} = \mathsf{T}_{V_n} \quad \text{and} \quad Z_{[n]} = Z_{V_n}, \quad n \in \mathbb{N}_0,$$

the cell types and the number of parasites along that random cell line. A random A-cell line up to generation n is obtained when $\mathsf{T}_{[n]} = \mathsf{A}$, for then $\mathsf{T}_{[k]} = \mathsf{A}$ for any $k = 0, ..., n-1$ as well. As will be shown in Proposition 6.2, the conditional law of $(Z_{[0]}, ..., Z_{[n]})$ given $\mathsf{T}_{[n]} = \mathsf{A}$ equals the law of $(Z_n(\mathsf{A}))_{n\geq 0}$ up to generation n for each $n \in \mathbb{N}$. As mentioned before, this cannot be generally true for the unconditional law of $(Z_{[0]}, ..., Z_{[n]})$ due to the multi-type structure of the cell population.

Proposition 6.2. *The law of* $(Z_{[0]}, ..., Z_{[n]})$ *conditioned under* $\mathsf{T}_{[n]} = \mathsf{A}$ *equals the law of* $(Z_0(\mathsf{A}), ..., Z_n(\mathsf{A}))$, *for each* $n \in \mathbb{N}_0$.

Proof. We use induction over n and begin by noting that nothing has to be shown for $n = 0$. For $n \geq 1$ and $(z_0, ..., z_n) \in \mathbb{N}_0^{n+1}$, we introduce the notation

$$C_{z_0,...,z_n} := \{(Z_{[0]}, ..., Z_{[n]}) = (z_0, ..., z_n)\} \quad \text{and} \quad C^{\mathsf{A}}_{z_0,...,z_n} := C_{z_0,...,z_n} \cap \{\mathsf{T}_{[n]} = \mathsf{A}\}$$

and note that

$$\mathbb{P}\left(\mathsf{T}_{[n]} = \mathsf{A}\right) = 2^{-n}\,\mathbb{E}\left(\sum_{v\in\mathbb{G}_n} \mathbb{1}_{\{\mathsf{T}_v=\mathsf{A}\}}\right) = \left(\frac{\nu}{2}\right)^n, \tag{6.2}$$

for each $n \in \mathbb{N}$, in particluar

$$\mathbb{P}(\mathsf{T}_{[n]} = \mathsf{A}\,|\,\mathsf{T}_{[n-1]} = \mathsf{A}) = \frac{\mathbb{P}(\mathsf{T}_{[n]} = \mathsf{A})}{\mathbb{P}(\mathsf{T}_{[n-1]} = \mathsf{A})} = \frac{\nu}{2}.$$

Assuming the assertion holds for $n-1$ (inductive hypothesis), thus

$$\mathbb{P}(C_{z_0,...,z_{n-1}}|\,\mathsf{T}_{[n-1]} = \mathsf{A}) = \mathbb{P}\left(Z_0(\mathsf{A}) = z_0, ..., Z_{n-1}(\mathsf{A}) = z_{n-1}\right)$$

for any $(z_0, ..., z_{n-1}) \in \mathbb{N}_0^n$, we infer with the help of the Markov property that

$$\begin{aligned}
&\mathbb{P}\left((Z_{[0]}, ..., Z_{[n]}) = (z_0, ..., z_n)|\,\mathsf{T}_{[n]} = \mathsf{A}\right)\\
&= \frac{\mathbb{P}(C^{\mathsf{A}}_{z_0,...,z_n})}{\mathbb{P}(\mathsf{T}_{[n]} = \mathsf{A})}\\
&= \mathbb{P}\left(C_{z_0,...,z_{n-1}}|\,\mathsf{T}_{[n-1]} = \mathsf{A}\right)\,\mathbb{P}(Z_{[n]} = z_n, \mathsf{T}_{[n]} = \mathsf{A}\,|C^{\mathsf{A}}_{z_0,...,z_{n-1}})\,\frac{\mathbb{P}(\mathsf{T}_{[n-1]} = \mathsf{A})}{\mathbb{P}(\mathsf{T}_{[n]} = \mathsf{A})}\\
&= \mathbb{P}\left(Z_0(\mathsf{A}) = z_0, ..., Z_{n-1}(\mathsf{A}) = z_{n-1}\right)\,\frac{\mathbb{P}(Z_{[1]} = z_n, \mathsf{T}_{[1]} = \mathsf{A}\,|Z_{[0]} = z_{n-1}, \mathsf{T}_{[0]} = \mathsf{A})}{\mathbb{P}(\mathsf{T}_{[n]} = \mathsf{A}\,|\,\mathsf{T}_{[n-1]} = \mathsf{A})}\\
&= \mathbb{P}\left(Z_0(\mathsf{A}) = z_0, ..., Z_{n-1}(\mathsf{A}) = z_{n-1}\right)\,\frac{2\mathbb{P}(\mathsf{T}_{[1]} = \mathsf{A})}{\nu}\,\frac{\mathbb{P}(Z_{[0]} = z_{n-1}, Z_{[1]} = z_n|\,\mathsf{T}_{[1]} = \mathsf{A})}{\mathbb{P}(Z_{[0]} = z_{n-1}, \mathsf{T}_{[0]} = \mathsf{A})}\\
&= \mathbb{P}\left(Z_0(\mathsf{A}) = z_0, ..., Z_{n-1}(\mathsf{A}) = z_{n-1}\right)\,\mathbb{P}\left(Z_n(\mathsf{A}) = z_n|Z_{n-1}(\mathsf{A}) = z_{n-1}\right)\\
&= \mathbb{P}\left(Z_0(\mathsf{A}) = z_0, ..., Z_n(\mathsf{A}) = z_n\right).
\end{aligned}$$

This proves the assertion. □

As in the BwBP, the process along a random cell line has a closely relation to the number of contaminated cells.

Lemma 6.3. *For every* $k, n \in \mathbb{N}$

$$\mathbb{E}\#\{v \in \mathbb{G}_n^* \,|Z_v = k\} \;=\; 2^n \mathbb{P}(Z_{[n]} = k) \quad \textit{and} \quad \mathbb{E}\#\,\mathbb{G}_n^* = 2^n \mathbb{P}(Z_{[n]} > 0)$$

as well as

$$\mathbb{E}\#\{v \in \mathbb{G}_n^*(\mathsf{A})|Z_v = k\} \;=\; \nu^n \mathbb{P}(Z_n(\mathsf{A}) = k) \quad \textit{and} \quad \mathbb{E}\#\,\mathbb{G}_n^*(\mathsf{A}) = \nu^n \mathbb{P}(Z_n(\mathsf{A}) > 0).$$

Proof. The statements for the type-A cells follow directly from Proposition 1.4. The assertion for the random cell line process through the whole tree is ensured by the equation

$$2^{-n}\mathbb{E}\#\{v \in \mathbb{G}_n^* \,|Z_v = k\} \;=\; 2^{-n} \sum_{v \in \mathbb{G}_n} \mathbb{P}(Z_v = k) \;=\; \mathbb{P}(Z_{[n]} = k)$$

for $k, n \in \mathbb{N}$. □

Aiming at a study of host-parasite coevolution in the framework of a multi-type host population, our model may be viewed as the simplest possible alternative. There are only two types of host cells and reproduction is unilateral in the sense that cells of type A may give birth to both, A- and B-cells, but those of type B will never produce cells of the other type. The basic idea behind this restriction is that of irreversible mutations that generate new types of cells but never lead back to already existing ones. Observe that the setup could readily be generalized without changing much the mathematical structure by allowing the occurrence of further irreversible mutations from cells of type B to cells of type C, and so on.

For convenience, we define for the rest of this chapter

$$\mathbb{P}_{\mathsf{t},z} := \mathbb{P}(\cdot|Z_\varnothing = z, \mathsf{T}_\varnothing = \mathsf{t}), \quad z \in \mathbb{N},\ \mathsf{t} \in \{\mathsf{A}, \mathsf{B}\},$$

and use $\mathbb{E}_{\mathsf{t},z}$ for expectation under $\mathbb{P}_{\mathsf{t},z}$. Recalling that $\mathbb{P}$ stands for $\mathbb{P}_{\mathsf{A},1}$, we put $\mathbb{P}^* := \mathbb{P}(\cdot|\,\mathsf{Surv}(\mathsf{A}))$ and, furthermore,

$$\mathbb{P}^*_{\mathsf{t},z} := \mathbb{P}_{\mathsf{t},z}(\cdot|\,\mathsf{Surv}(\mathsf{A})) \quad \text{and} \quad \mathbb{P}^n_{\mathsf{t},z} := \mathbb{P}_{\mathsf{t},z}(\cdot|\mathcal{Z}_n(\mathsf{A}) > 0)$$

for $z \in \mathbb{N}$ and $\mathsf{t} \in \{\mathsf{A}, \mathsf{B}\}$.

6.2 Properties of $\#\,\mathbb{G}_n^*(\mathsf{t})$

This section is dedicated to the analysis of the asymptotic behavior of $\mathbb{G}_n^*(\mathsf{A})$ and $\mathbb{G}_n^*$ conditioned under $\mathsf{Surv}(\mathsf{A})$ and Surv, respectively. We begin by ratifying the extinction-explosion principle for these two processes, saying that contaminated cells tend to infinity if the parasite population explodes. But before, note that the conditions

$$p_{\mathsf{AA}} > 0 \quad \text{and} \quad \mathbb{P}(X^{(0)}(\mathsf{A}, \mathsf{AA}) = X^{(1)}(\mathsf{A}, \mathsf{AA}) = X^{(0)}(\mathsf{A}, \mathsf{AB}) = 1) < 1 \qquad \text{(AsBP)}$$

in addition with the standard assumptions (SA1)-(SA6) ensure the validation of (A1)-(A3) of the BwBP as well as $\mathbb{P}_{\mathsf{A},2}(\#\,\mathbb{G}_1^*(\mathsf{A}) \geq 2) > 0$. Hence, under these premises all results of the BwBP in all previous chapters can be applied without further concern.

Theorem 6.4.

(a) If $\mathbb{P}(\mathsf{Surv}(\mathsf{A})) > 0$ *and* $p_{\mathsf{AA}} > 0$*, then* $\mathbb{P}(\#\,\mathbb{G}_n^*(\mathsf{A}) \to \infty |\,\mathsf{Surv}(\mathsf{A})) = 1$.

(b) If $\mathbb{P}(\mathsf{Surv}) > 0$*, then* $\mathbb{P}(\#\,\mathbb{G}_n^*(\mathsf{B}) \to \infty |\,\mathsf{Surv}) = 1$.

In particular, $\mathsf{Surv}(\mathsf{A}) = \{\mathcal{Z}_n(\mathsf{A}) \to \infty\} \subseteq \mathsf{Surv}(\mathsf{B}) = \{\mathcal{Z}_n(\mathsf{B}) \to \infty\}$ *a.s.*

Proof. (a) If $X^{(0)}(\mathsf{A},\mathsf{AA}) = X^{(1)}(\mathsf{A},\mathsf{AA}) = X^{(0)}(\mathsf{A},\mathsf{AB}) = 1$ a.s., then $\mathcal{Z}_n(\mathsf{A}) = \#\,\mathbb{G}_n^*(\mathsf{A})$ for all $n \in \mathbb{N}_0$ and the assertion follows as $\mathcal{Z}_n(\mathsf{A}) \to \infty$ on $\mathsf{Surv}(\mathsf{A})$. If otherwise (AsBP) holds true, then Proposition 6.1 and Theorem 1.7 finishes the proof of (a).

(b) First note that, given Surv, a contaminated B-cell is eventually created with probability one and then spawns a single-type cell process (as $\mathbb{E}\mathcal{Z}_1(\mathsf{B}) > 0$ by (SA4)). Hence, the assertion follows from [15, Theorem 4.1] and the Borel-Cantelli lemma if $\mu_{\mathsf{B}} = \mu_{0,\mathsf{B}} + \mu_{1,\mathsf{B}} > 1$.

Left with the case $\mu_{\mathsf{B}} \leq 1$, it follows that

$$\mathbb{P}(\mathsf{Surv}(\mathsf{A})|\,\mathsf{Surv}) = 1,$$

for otherwise, given Surv, only B-parasites would eventually be left w.p.p. which however would die out almost surely, as they then form a standard GWP. Next, $p_{\mathsf{AA}} > 0$ implies $\#\,\mathbb{G}_n^*(\mathsf{A}) \to \infty$ almost surly by (a), and so the same holds true for the number of contaminated B cells since every contaminated A cell has a positive probability to beget an infected B daughter cell by (SA4). Hence, the law of large numbers entails

$$\begin{aligned}\liminf_{n\to\infty} \#\,\mathbb{G}_{n+1}^*(\mathsf{B}) \;&\geq\; \liminf_{n\to\infty} \sum_{v\in\mathbb{G}_n^*(\mathsf{A})} \mathbb{1}_{\{Z_{v0}>0,\mathsf{T}_{v0}=\mathsf{B}\}} + \mathbb{1}_{\{Z_{v1}>0,\mathsf{T}_{v1}=\mathsf{B}\}} \\ &\geq\; \mathbb{P}(\mathcal{Z}_1(\mathsf{B}) > 0) \liminf_{n\to\infty} \#\,\mathbb{G}_n^*(\mathsf{A}) \;=\; \infty \quad \text{a.s.}\end{aligned}$$

on Surv. It remains to consider the situation when $p_{\mathsf{AA}} = 0$ and thus $p_{\mathsf{AB}} = 1$, as otherwise $\mathcal{Z}_n(\mathsf{A}) = 0$ eventually. In this case there is a single line of A-cells, namely $\varnothing \to 0 \to 00 \to ...$, and $(\mathcal{Z}_n(\mathsf{A}))_{n\geq 0}$ is an ordinary GWP tending $\mathbb{P}(\cdot|\,\mathsf{Surv}(\mathsf{A}))$-a.s. to infinity. For $n, k \in \mathbb{N}$, let

$$\mathcal{Z}_k(n,\mathsf{B}) := \sum_{v\in\mathbb{G}_{n+k+1}(\mathsf{B}):v|n+1=0^n1} Z_v$$

denote the number of B-parasites at generation k sitting in cells of the subpopulation stemming from the cell 0^n1, where $0^n := 0...0$ (n-times). Using $p_{\mathsf{AB}} = 1$, (SA3) and (SA4), notably $\mu_{1,\mathsf{A}}(\mathsf{AB}) > 0, \mu_{0,\mathsf{B}} > 0$ and $\mu_{1,\mathsf{B}} > 0$, it is readily seen that

$$\mathbb{P}^*\left(\lim_{n\to\infty} \mathcal{Z}_0(n-k,\mathsf{B}) = \infty\right) = 1$$

and thus

$$\mathbb{P}^*\left(\lim_{n\to\infty} \mathcal{Z}_K(n-k,\mathsf{B}) = 0\right) = 0$$

for all $K \in \mathbb{N}$ and $k \leq K$. Consequently,

$$\mathbb{P}^*\left(\liminf_{n\to\infty} \#\,\mathbb{G}_n^*(\mathsf{B}) \leq K\right) \;\leq\; \mathbb{P}^*\left(\lim_{n\to\infty} \min_{0\leq k\leq K} \mathcal{Z}_k(n-k,\mathsf{B}) = 0\right)$$

$$\leq \sum_{k=0}^{K} \mathbb{P}^*\left(\lim_{n\to\infty} \mathcal{Z}_K(n-k,\mathsf{B}) = 0\right) \;=\; 0$$

for all $K \in \mathbb{N}$ □

The next result provides us with the geometric rate at which the number of contaminated cells tends to infinity, which is basically a restatement of Theorem 1.8 in this special situation.

Theorem 6.5. *The process* $(\nu^{-n}\#\,\mathbb{G}_n^*(\mathsf{A}))_{n\geq 0}$ *is a non-negative supermartingale and therefore a.s. convergent to a random variable* $L(\mathsf{A})$ *as* $n\to\infty$. *Furthermore,*

- *(a)* $L(\mathsf{A}) = 0$ *a.s. iff* $\mathbb{E}\log g'_{\Lambda_1}(1) \leq 0$ *or* $p_{\mathsf{AB}} < \nu \leq 1$
- *(b)* $\mathbb{P}(L(\mathsf{A}) = 0) < 1$ *implies* $\{L(\mathsf{A}) = 0\} = \mathsf{Ext}(\mathsf{A})$ *a.s.*

Proof. If $p_{\mathsf{AA}} = 0$ and thus $p_{\mathsf{AB}} = \nu$, then $\mathbb{E}(\#\,\mathbb{G}_{n+1}^*(\mathsf{A})|\#\,\mathbb{G}_n^*(\mathsf{A})) \leq \#\,\mathbb{G}_n^*(\mathsf{A})p_{\mathsf{AB}}$ for all $n\in\mathbb{N}_0$, and the supermartingale property follows. Furthermore, if $p_{\mathsf{AB}} < 1$, then $\#\,\mathbb{G}_n^*(\mathsf{A}) \leq \#\,\mathbb{G}_n(\mathsf{A}) = 0$ eventually by (6.1), and if otherwise $p_{\mathsf{AB}} = 1$, $(\mathcal{Z}_n(\mathsf{A}))_{n\geq 0}$ is an ordinary GWP and hence dies out almost surely if and only if its reproduction mean $\mu_{0,\mathsf{A}}(\mathsf{AB})$ is less or equal to 1. (b) is clear in this case.

Let now be $p_{\mathsf{AA}} > 0$. If additionally $X^{(0)}(\mathsf{A},\mathsf{AA}) = X^{(1)}(\mathsf{A},\mathsf{AA}) = X^{(0)}(\mathsf{A},\mathsf{AB}) = 1$ a.s., then $\mathcal{Z}_n(\mathsf{A}) = \#\,\mathbb{G}_n^*(\mathsf{A}) = \#\,\mathbb{G}_n(\mathsf{A})$ for all $n\in\mathbb{N}_0$. Hence, $(\#\,\mathbb{G}_n^*(\mathsf{A}))_{n\geq 0}$ forms a GWP with reproduction mean ν, and the classical theory yields the assertions (see e.g. [14]). So let now (AsBP) hold true, but then the theorem follows from Proposition 6.1 and Theorem 1.8. □

Since $\nu < 2$ by (SA2) and $(\nu^{-n}\#\,\mathbb{G}_n(\mathsf{A}))_{n\geq 0}$ is a non-negative, a.s. convergent martingale, we see that $2^{-n}\#\,\mathbb{G}_n^*(\mathsf{A}) \leq 2^{-n}\#\,\mathbb{G}_n(\mathsf{A}) \to 0$ a.s. and therefore

$$\frac{\#\,\mathbb{G}_n^*}{2^n} \simeq \frac{\#\,\mathbb{G}_n^*(\mathsf{B})}{2^n} \quad \text{as } n\to\infty.$$

That is, the asymptotic proportion of all contaminated cells is the same as the asymptotic proportion of contaminated B-cells. Note also that

$$\mathbb{P}(\mathsf{T}_{[n]} = \mathsf{A}) = \mathbb{E}\left(\frac{\#\,\mathbb{G}_n(\mathsf{A})}{2^n}\right) \to 0 \quad \text{as } n\to\infty. \tag{6.3}$$

Further information is provided by the next result.

Theorem 6.6. *There exists a random variable* $L\in[0,1]$ *such that* $\#\,\mathbb{G}_n^*/2^n \to L$ *a.s. Moreover,*

- *(a)* $L = 0$ *a.s. iff* $\mu_{0,\mathsf{B}}\mu_{1,\mathsf{B}} \leq 1$.
- *(b) If* $\mathbb{P}(L=0) < 1$, *then* $\{L = 0\} = \mathsf{Ext}$ *a.s.*

Proof. The existence of L follows because $2^{-n}\#\,\mathbb{G}_n^*$ is obviously decreasing. As for (a), suppose first that $\mu_{0,\mathsf{B}}\mu_{1,\mathsf{B}} \leq 1$ and note that this is equivalent to almost certain extinction of a random B-cell line, i.e.

$$\lim_{n\to\infty} \mathbb{P}(Z_{[n]} > 0 | Z_\varnothing = k, \mathsf{T}_{[0]} = \mathsf{B}) = 0$$

for any $k \in \mathbb{N}$. This follows because, starting from a B-cell, we are in the one-type model studied in [15]. There it is stated that $(Z_{[n]})_{n\geq 0}$ forms a BPRE which dies out a.s. iff $\mu_{0,\mathsf{B}}\mu_{1,\mathsf{B}} \leq 1$ (see [15, Proposition 2.1]). Now, fix any $\varepsilon > 0$ and choose $m \in \mathbb{N}$ so large that $\mathbb{P}(\mathsf{T}_{[m]} = \mathsf{A}) \leq \varepsilon$, which is possible by (6.3). Then, by the monotone convergence theorem, Lemma 6.3 and the Markov property of a BPRE, we find that for sufficiently large $K \in \mathbb{N}$

$$\begin{aligned}
\mathbb{E}L &= \lim_{n\to\infty} \mathbb{P}(Z_{[n+m]} > 0) \\
&\leq \lim_{n\to\infty} \mathbb{P}(Z_{[n+m]} > 0, \mathsf{T}_{[m]} = \mathsf{B}) + \varepsilon \\
&= \lim_{n\to\infty} \sum_{k=0}^{\infty} \mathbb{P}(Z_{[n+m]} > 0, Z_{[m]} = k, \mathsf{T}_{[m]} = \mathsf{B}) + \varepsilon \\
&\leq \lim_{n\to\infty} \sum_{k=0}^{K} \mathbb{P}(Z_{[n]} > 0 | Z_{[0]} = k, \mathsf{T}_{[0]} = \mathsf{B}) + 2\varepsilon \\
&\leq 2\varepsilon
\end{aligned}$$

and thus $\mathbb{E}L = 0$. For the converse, note that

$$\begin{aligned}
0 &= \mathbb{E}L \\
&= \lim_{n\to\infty} \mathbb{P}(Z_{[n+1]} > 0) \\
&\geq \lim_{n\to\infty} \mathbb{P}(Z_{[1]} > 0, \mathsf{T}_{[1]} = \mathsf{B})\mathbb{P}(Z_{[n]} > 0 | \, \mathsf{T}_{[0]} = \mathsf{B})
\end{aligned}$$

implies $0 = \lim_{n\to\infty} \mathbb{P}(Z_{[n]} > 0 | \, \mathsf{T}_{[0]} = \mathsf{B})$ and thus $\mu_{0,\mathsf{B}}\mu_{1,\mathsf{B}} \leq 1$ as well.

The proof of (b) follows along similar lines as Theorem 1.8(b). If $\mathbb{P}(L = 0) < 1$, (a) implies $\mu_{0,\mathsf{B}}\mu_{1,\mathsf{B}} \leq 1$ and [15, Theorem 3.1] then $\mathbb{P}_{\mathsf{B},1}(L = 0) < 1$. Hence, there exists a constant $\varrho < 1$ such that

$$\max\{\mathbb{P}_{\mathsf{A},1}(L = 0), \mathbb{P}_{\mathsf{B},1}(L = 0)\} \leq \varrho < 1.$$

Defining $\tau_n = \inf\{m \in \mathbb{N} : \# \, \mathbb{G}^*_m \geq n\}$, we find that

$$\begin{aligned}
\mathbb{P}(L = 0) &\leq \mathbb{P}(L = 0 | \tau_n < \infty) + \mathbb{P}(\tau_n = \infty) \\
&\leq \mathbb{P}\left(\bigcap_{v\in\mathbb{G}^*_{\tau_n}} \{\# \, \mathbb{G}^*_{m,v} / 2^m \to 0\} \middle| \tau_n < \infty \right) + \mathbb{P}(\tau_n = \infty) \\
&\leq \max\{\mathbb{P}_{\mathsf{A},1}(L = 0), \mathbb{P}_{\mathsf{B},1}(L = 0)\}^n + \mathbb{P}(\tau_n = \infty) \\
&\leq \varrho^n + \mathbb{P}(\tau_n = \infty)
\end{aligned}$$

for all $n \in \mathbb{N}$, where the $\# \, \mathbb{G}^*_{m,v}$, $v \in \mathbb{G}^*_{\tau_n}$, are independent processes of contaminated cells each starting with a single parasite in a cell of type T_v. Since $\varrho < 1$, Theorem 6.4 implies

$$\mathbb{P}(L = 0) \leq \lim_{n\to\infty} \mathbb{P}(\tau_n = \infty) = \mathbb{P}\left(\sup_{n\geq 1} \# \, \mathbb{G}^*_n < \infty\right) = \mathbb{P}(\mathsf{Ext}),$$

which in combination with $\mathsf{Ext} \subseteq \{L = 0\}$ a.s. proves the assertion. □

Remark 6.7. At the end of this section, we shortly comment on the case when Assumption (SA6) fails. In this situation, $\mathcal{Z}_n(\mathsf{A}) = 1$ a.s. for each $n \in \mathbb{N}_0$ and the only type-A cell spawns type-B parasites in each generation due to (SA4). Since this reproduction mechanism is of an i.i.d. manner, the number of B-parasites $\mathcal{Z}_n(\mathsf{B})$ forms a branching process with immigration, and the results in the standard literature [43, 44, 78, 80] yield

$$\mathcal{Z}_n(\mathsf{B}) \begin{cases} \xrightarrow{a.s.} \infty \\ \xrightarrow{d} \mathcal{Z}_\infty \end{cases} \qquad \text{if} \quad \mu_{\mathsf{B}} := \mu_{0,\mathsf{B}} + \mu_{1,\mathsf{B}} \begin{cases} > 1 \\ \leq 1, \end{cases}$$

where $\mathcal{Z}_\infty$ is almost surely finite if $\mu_{\mathsf{B}} < 1$, whereas infinite if $\mu_{\mathsf{B}} = 1$ and additionally $\mathcal{Z}_1(\mathsf{B})$ square integrable under $\mathbb{P}_{\mathsf{B},1}$.

With little effort, a similar result can be established for the number contaminated B cells $(\#\,\mathbb{G}^*_n(\mathsf{B}))_{n\geq 0}$. More precisely,

$$\#\,\mathbb{G}^*_n(\mathsf{B}) \begin{cases} \xrightarrow{a.s.} \infty \\ \xrightarrow{d} G_\infty \end{cases} \qquad \text{if} \quad \mu_{\mathsf{B}} := \mu_{0,\mathsf{B}} + \mu_{1,\mathsf{B}} \begin{cases} > 1 \\ \leq 1, \end{cases}$$

where G_∞ is almost surely finite if $\mu_{\mathsf{B}} < 1$, whereas infinite if $\mu_{\mathsf{B}} = 1$ and $\mathbb{E}_{\mathsf{B},1}\mathcal{Z}_1^2 < \infty$. To see that, note that when $\mathcal{Z}_n(\mathsf{B}) \to \infty$ a.s., there exist at least one type-B subtree in which parasites survive. Hence, [15, Theorem 4.1] states $\#\,\mathbb{G}^*_n(\mathsf{B}) \to \infty$ a.s. if $\mu_{\mathsf{B}} > 1$. Since the immigration of new type-B cells per generation is of the i.i.d. kind and B-parasites multiply in an i.i.d. manner, we get

$$\#\,\mathbb{G}_n(\mathsf{B}) \overset{d}{=} \sum_{i=0}^{n-1} \#\,\mathbb{G}_i(\mathsf{B}, 0^i 1)\,\mathbb{1}_{\{T_{0^i} = \mathsf{AB}\}} \to \sum_{i=0}^{\infty} \#\,\mathbb{G}_i(\mathsf{B}, 0^i 1)\,\mathbb{1}_{\{T_{0^i} = \mathsf{AB}\}} =: G_\infty \quad \text{for } n \to \infty,$$

where the $\#\,\mathbb{G}_i(\mathsf{B}, 0^i 1)$, $i \in \mathbb{N}_0$, are independent, and $\#\,\mathbb{G}_i(\mathsf{B}, 0^i 1)$ is distributed as $\#\,\mathbb{G}^*_i(\mathsf{B})$ with $Z_{0^i 1}$ parasites in the root cell. Obviously, $\mathcal{Z}_\infty$ dominates G_∞ stochastically and thus $G_\infty < \infty$ a.s. if $\mu_{\mathsf{B}} < 1$. However, [15, Corollary 5.6] yields in the case when $\mu_{\mathsf{B}} = 1$ and $\mathbb{E}_{\mathsf{B},1}\mathcal{Z}_1^2 < \infty$

$$\mathbb{P}\left(\#\,\mathbb{G}_n(\mathsf{B}, 0^n 1)\,\mathbb{1}_{\{T_{0^n} = \mathsf{AB}\}} \geq n\right) \geq \frac{p_{\mathsf{AB}}}{2}\mathbb{P}(\mathcal{Z}_1(\mathsf{B}) > 0)\mathbb{P}_{\mathsf{B},1}(\#\,\mathbb{G}^*_n > 0)\mathbb{P}(\mathcal{E} \geq c) > 0, \tag{6.4}$$

for all $n \geq n_0$, a suitable $n_0 \in \mathbb{N}$ and constant $c > 0$, and an exponential distributed random variable $\mathcal{E}$. Thus,

$$\sum_{n=0}^{\infty} \mathbb{P}\left(\#\,\mathbb{G}_n(\mathsf{B}, 0^n 1)\,\mathbb{1}_{\{T_{0^n} = \mathsf{AB}\}} \geq n\right) \geq \frac{p_{\mathsf{AB}}}{2}\mathbb{P}(\mathcal{Z}_1(\mathsf{B}) > 0)\mathbb{P}(\mathcal{E} \geq c) \sum_{n=n_0}^{\infty} \mathbb{P}_{\mathsf{B},1}(\mathcal{Z}_n > 0) = \infty,$$

by [14, Theorem 9.1 in Chapter I], as $(\mathcal{Z}_n(\mathsf{B}))_{n\geq 0}$ starting in a B cell is a critical GWP. Hence, applying the Borel-Cantelli lemma gives $G_\infty = \infty$ a.s.

Theorem 6.5 is trivial in the case when (SA6) fails, and the latter assumption was not used to prove Theorem 6.6, which is why it stays valid in this situation.

6.3 Relative proportions of contaminated cells

We now turn to statements that are concerned with the long-run behavior of relative proportions of contaminated cells containing a given number of parasites, viz.

$$F_n(k) := \frac{\#\{v \in \mathbb{G}_n^* \,|Z_v = k\}}{\# \, \mathbb{G}_n^*}$$

for $k \in \mathbb{N}$ and $n \to \infty$, and of the corresponding quantities when restricting to contaminated cells of a given type t, viz.

$$F_n(k, \mathsf{t}) := \frac{\#\{v \in \mathbb{G}_n^*(\mathsf{t})|Z_v = k\}}{\# \, \mathbb{G}_n^*(\mathsf{t})}$$

for $\mathsf{t} \in \{\mathsf{A}, \mathsf{B}\}$. Note that

$$F_n(k) = F_n(k, \mathsf{A}) \, \frac{\# \, \mathbb{G}_n^*(\mathsf{A})}{\# \, \mathbb{G}_n^*} + F_n(k, \mathsf{B}) \, \frac{\# \, \mathbb{G}_n^*(\mathsf{B})}{\# \, \mathbb{G}_n^*}.$$

The limit behaviors of $F_k(n, \mathsf{A})$, $\# \, \mathbb{G}_n^*(\mathsf{A})/\# \, \mathbb{G}_n^*$ and $F_k(n, \mathsf{B})$ depend on that of $\mathcal{Z}_n(\mathsf{A})$ and $Z_n(\mathsf{A})$ in a crucial way. In the following, we consider two different cases: $(Z_n(\mathsf{A}))_{n\geq 0}$ is supercritical

$$\mathbb{E} \log g'_{\Lambda_1}(1) > 0, \tag{SupC}$$

that is when parasites along an infinite A-cell line may tend to infinity, and the case where $(Z_n(\mathsf{A}))_{n\geq 0}$ is strongly subcritical but type-A parasites survive w.p.p. and their number growths like its means, viz.

$$\mathbb{E}\mathcal{Z}_1(\mathsf{A}) \log \mathcal{Z}_1(\mathsf{A}) < \infty, \ \mathbb{E} \left(\frac{g'_{\Lambda_1}(1)}{\gamma} \log \frac{g'_{\Lambda_1}(1)}{\gamma} \right) < 0 \quad \text{and} \quad \mathbb{E} g'_{\Lambda_1}(1) \log g'_{\Lambda_1}(1) < 0. \tag{SubC}$$

We assume from now on that $\mathbb{P}(\mathsf{Surv}(\mathsf{A})) > 0$ and furthermore *that* (AsBP) *holds true* to avoid too many exceptions. Under the given assumptions, the asymptotic of $F_n(k, \mathsf{A})$ was already detected in Theorem 4.11 and Theorem 4.12. We restate these results for the present studied bifurcating host-parasite model in the subsequent corollary.

Corollary 6.8. *Let* (AsBP) *hold true.*

(a) If (SupC)*, then* $F_n(k, \mathsf{A})$ *converges to* 0 *in probability conditioned under* $\mathsf{Surv}(\mathsf{A})$ *as* $n \to \infty$ *for each* $k \in \mathbb{N}$.

(b) If (SubC)*, then* $(F_n(k, \mathsf{A}))_{k\geq 1}$ *converges in probability conditioned under* $\mathsf{Surv}(\mathsf{A})$ *as* $n \to \infty$ *to a probability distribution* $(q_k)_{k\geq 1}$ *on* $\mathbb{N}$ *with*

$$q_k = \lim_{n\to\infty} \mathbb{P}(Z_n(\mathsf{A}) = k|Z_n(\mathsf{A}) > 0) \quad \textit{for } k \in \mathbb{N}\,. \tag{6.5}$$

Furthermore, $\# \, \mathbb{G}_n^*(\mathsf{A})/\gamma^n \to W'$ *in probability, where* W' *is almost surely finite and strictly positive on* $\mathsf{Surv}(\mathsf{A})$.

6.3.1 Statement of the results

The first theorem deals with the situation when B-parasites multiply at a high rate, viz.

$$\mu_{0,\mathsf{B}}\mu_{1,\mathsf{B}} > 1.$$

In essence, it asserts that among all contaminated cells in generation n those of type B prevail as $n \to \infty$. This may be surprising at first glance because multiplication of A-parasites can also be high (or even higher), namely if

$$\mu_{0,\mathsf{A}}(\mathsf{AA})^{p_{\mathsf{AA}}}\mu_{1,\mathsf{A}}(\mathsf{AA})^{p_{\mathsf{AA}}}\mu_{0,\mathsf{A}}(\mathsf{AB})^{p_{\mathsf{AB}}} > 1,$$

which is equivalent to (SupC). On the other hand, it should be recalled that the subpopulation of A-cells grows at rate $\nu < 2$ only (recall (SA2)), whereas the growth rate of B-cells is 2. Hence, prevalence of B-cells in the subpopulation of all contaminated cells is observed whenever $\#\,\mathbb{G}_n^*(\mathsf{B})/\#\,\mathbb{G}_n(\mathsf{B})$, the relative proportion of contaminated cells within the n^{th} generation of all B-cells, is asymptotically positive as $n \to \infty$.

Theorem 6.9. *Assuming $\mu_{0,\mathsf{B}}\mu_{1,\mathsf{B}} > 1$, the following assertions hold true:*

(a) As $n \to \infty$

$$\frac{\#\,\mathbb{G}_n^*(\mathsf{A})}{\#\,\mathbb{G}_n^*} \to 0 \quad \mathbb{P}^*\text{-a.s.}$$

(b) Conditioned upon survival of A-cells, $F_n(k,\mathsf{B})$ converges to 0 in probability for any $k \in \mathbb{N}$.

Limits of $\#\,\mathbb{G}_n^*(\mathsf{A})/\#\,\mathbb{G}_n^*$ and $F_k(n,\mathsf{B})$ in the case (SubC) are given in Theorem 6.10. By Corollary 6.8 the partition of contaminated A-cells stabilizes, and hence, the number of parasites entering the type-B cell population is drawn by this limit distribution in distant generations. This immigration dynamic allows us to control the B-cells and the parasites within. We are able to show convergence in probability to a deterministic limit for the proportion of these B-cells with a given number of parasites to all contaminated B-cells. This limit highly depends on the relation of the parasite multiplication rates of both types. Roughly speaking, given a higher multiplication rate of B-parasites, i.e. $\mu_{\mathsf{B}} \geq \gamma$, B-cells and parasites dominate the cell tree. Thus, contaminated B-cells prevail in the long-run and $F_n(k,\mathsf{B})$ behaves as in the one-type model as given in [15]. For these results to be true, we need to consider additional integrability assumptions for the reproduction law of B-parasites, namely

$$\mathbb{E}_{\mathsf{B},1}\mathcal{Z}_1^2 < \infty,\ \mu_{\mathsf{B}} > 1 \quad \text{and} \quad \mu_{0,\mathsf{B}}\log\mu_{0,\mathsf{B}} + \mu_{1,\mathsf{B}}\log\mu_{1,\mathsf{B}} < 0. \tag{B}$$

If, on the other hand, $\mu_{\mathsf{B}} < \gamma$, the proportion of infected A-cells to all contaminated cells converges to a positive constant. Moreover, $(F_n(k,\mathsf{B}))_{k\geq 1}$ converges to a distribution as well.

Theorem 6.10. *Let* (AsBP) *and* (SubC) *hold true.*

(a) If $\mu_{\mathsf{B}} \geq \gamma$ and (B)*, then*

$$\frac{\#\,\mathbb{G}_n^*(\mathsf{A})}{\#\,\mathbb{G}_n^*} \xrightarrow{\mathbb{P}^*} 0 \quad \textit{and} \quad F_n(k,\mathsf{B}) \xrightarrow{\mathbb{P}^*} q_k(\mathsf{B})$$

for each $k \in \mathbb{N}$ as $n \to \infty$, where $q_k(\mathsf{B}) = \lim_{n\to\infty}\mathbb{P}_{\mathsf{B},1}(Z_{[n]} = k|Z_{[n]} > 0)$.

(b) If $\mu_{\mathsf{B}} < \gamma$*, then*

$$\frac{\#\,\mathbb{G}_n^*(\mathsf{A})}{\#\,\mathbb{G}_n^*} \xrightarrow{\mathbb{P}^*} \frac{1}{1+\rho} > 0 \quad \textit{and} \quad F_n(k,\mathsf{B}) \xrightarrow{\mathbb{P}^*} \frac{\rho_k}{\rho}$$

for each $k \in \mathbb{N}$ *and* $n \to \infty$*, where*

$$\rho_k := \sum_{n=0}^{\infty} (2/\gamma)^{n+1} \sum_{z=1}^{\infty} q_z \mathbb{P}_{\mathsf{A},z}(Z_{[n+1]} = k, \mathsf{T}_{[1]} = \mathsf{B}) \quad \textit{and} \quad \rho = \sum_{k=1}^{\infty} \rho_k$$

with the probability distribution $(q_z)_{z\geq 1}$ *as given in* (6.5).

Finally, we consider (SupC) and thus a high multiplication rate of A-parasites. Given that, contaminated B-cells still prevail in the long-run because, roughly speaking, highly infected A-cells eventually produce highly infected B-cells whose offspring m generations later for any fixed m are all contaminated (thus 2^m in number). However, as in Theorem 6.11, $(F_n(k,\mathsf{B}))_{k\geq 1}$ behaves as in the one-type model if the multiplication of type-B parasites is higher than the one of type-A parasites.

Theorem 6.11. *Let* (AsBP) *and* (SupC) *hold true, then*

$$\frac{\#\,\mathbb{G}_n^*(\mathsf{A})}{\#\,\mathbb{G}_n^*} \xrightarrow{\mathbb{P}^*} 0, \quad n \to \infty.$$

Furthermore, if $\mu_{\mathsf{B}} > \gamma$ *and* (B)*, then*

$$F_n(k,\mathsf{B}) \xrightarrow{\mathbb{P}^*} q_k(\mathsf{B})$$

for each $k \in \mathbb{N}$ *as* $n \to \infty$*, where* $q_k(\mathsf{B}) = \lim_{n\to\infty} \mathbb{P}_{\mathsf{B},1}(Z_{[n]} = k | Z_{[n]} > 0)$.

6.3.2 Proofs

Here, we present the proofs of the theorems stated in the previous subsection.

Proof of Theorem 6.9

(a) By Theorem 6.6, $2^{-n}\#\,\mathbb{G}_n^* \to L$ $\mathbb{P}^*$-a.s. and $\mathbb{P}^*(L > 0) = 1$, while Theorem 6.5 shows that $\nu^{-n}\#\,\mathbb{G}_n^*(\mathsf{A}) \to L(\mathsf{A})$ $\mathbb{P}$-a.s. for an a.s. finite random variable $L(\mathsf{A})$. Consequently,

$$\frac{\#\,\mathbb{G}_n^*(\mathsf{A})}{\#\,\mathbb{G}_n^*} = \left(\frac{\nu}{2}\right)^n \left(\frac{2^n}{\#\,\mathbb{G}_n^*}\right) \left(\frac{\#\,\mathbb{G}_n^*(\mathsf{A})}{\nu^n}\right) \simeq \frac{1}{L}\left(\frac{\nu}{2}\right)^n \frac{\#\,\mathbb{G}_n^*(\mathsf{A})}{\nu^n} \to 0 \quad \mathbb{P}^*\text{-a.s.}$$

as $n \to \infty$, for $\nu < 2$.

(b) Fix arbitrary $\varepsilon, \delta > 0$ and $K \in \mathbb{N}$ and define

$$D_n := \left\{ \sum_{k=1}^{K} F_n(k,\mathsf{B}) > \delta \right\} \cap \mathsf{Surv}(\mathsf{A}).$$

By another appeal to Theorem 6.6, $\#\,\mathbb{G}_n^*(\mathsf{B}) \geq 2^n L$ $\mathbb{P}^*$-a.s. for all $n \in \mathbb{N}$ and L as above. It follows that

$$\#\{v \in \mathbb{G}_n(\mathsf{B}) : 0 < Z_v \leq K\} \;\geq\; \delta\,\#\,\mathbb{G}_n^*(\mathsf{B})\,\mathbb{1}_{D_n} \;\geq\; \delta\,2^n\,L\,\mathbb{1}_{D_n},$$

and by taking the expectation, we obtain for $m \le n$

$$\begin{aligned}
\delta\,\mathbb{E}\left(L\,\mathbb{1}_{D_n}\right) &\le \frac{1}{2^n}\,\mathbb{E}\left(\sum_{v\in\mathbb{G}_n}\mathbb{1}_{\{0<Z_v\le K,\mathsf{T}_v=\mathsf{B}\}}\right)\\
&\le \frac{1}{2^n}\,\mathbb{E}\left(\#\big\{v\in\mathbb{G}_n : \mathsf{T}_{v|m}=\mathsf{A},\mathsf{T}_v=\mathsf{B}\big\}+\sum_{v\in\mathbb{G}_n}\mathbb{1}_{\{0<Z_v\le K,\mathsf{T}_{v|m}=\mathsf{B}\}}\right)\\
&\le \frac{1}{2^m}\,\mathbb{E}\#\,\mathbb{G}_m(\mathsf{A})+\frac{1}{2^n}\sum_{v\in\mathbb{G}_n}\mathbb{P}\left(0<Z_v\le K,\mathsf{T}_{v|m}=\mathsf{B}\right)\\
&= \left(\frac{\nu}{2}\right)^m+\frac{1}{2^n}\sum_{z=1}^{\infty}\sum_{v\in\mathbb{G}_n}\mathbb{P}\left(0<Z_v\le K,Z_{v|m}=z,\mathsf{T}_{v|m}=\mathsf{B}\right)\\
&= \left(\frac{\nu}{2}\right)^m+\sum_{z=1}^{\infty}\left(\sum_{u\in\mathbb{G}_m}\frac{\mathbb{P}(Z_u=z,\mathsf{T}_u=\mathsf{B})}{2^m}\right)\left(\sum_{u\in\mathbb{G}_{n-m}}\frac{\mathbb{P}_{\mathsf{B},z}(0<Z_v\le K)}{2^{n-m}}\right)\\
&= \left(\frac{\nu}{2}\right)^m+\sum_{z=1}^{\infty}\mathbb{P}(Z_{[m]}=z,\mathsf{T}_{[m]}=\mathsf{B})\,\mathbb{P}_{\mathsf{B},z}\left(0<Z_{[n-m]}\le K\right).
\end{aligned}$$

Since $\nu<2$, we can fix $m\in\mathbb{N}$ such that $(\nu/2)^m\le\varepsilon$. Also fix $z_0\in\mathbb{N}$ such that

$$\mathbb{P}(Z_{[m]}>z_0)\le\varepsilon.$$

Then

$$\begin{aligned}
\delta\,\mathbb{E}\left(L\,\mathbb{1}_{D_n}\right) &\le \sum_{z=1}^{\infty}\mathbb{P}(Z_{[m]}=z,\mathsf{T}_{[m]}=\mathsf{B})\,\mathbb{P}_{\mathsf{B},z}\left(0<Z_{[n-m]}\le K\right)+\left(\frac{\nu}{2}\right)^m\\
&\le \sum_{z=1}^{z_0}\mathbb{P}_{\mathsf{B},z}\left(0<Z_{[n-m]}\le K\right)+2\varepsilon.
\end{aligned}$$

But the last sum converges to zero as $n\to\infty$ because, under $\mathbb{P}_{\mathsf{B},z}$, $(Z_{[n]})_{n\ge0}$ is a single-type BPRE (see [15]) and thus satisfies the extinction-explosion principle. So we have shown that $\mathbb{E}L\,\mathbb{1}_{D_n}\to0$ implying $\mathbb{P}(D_n)\to0$ because $L>0$ on Surv. Hence, the theorem is proved.

Proof of Theorem 6.10

Firs, we note that given the additional assumptions (B), the process of a random cell line starting from a B-cell is strongly subcritical but descendants of a B-parasite survive w.p.p. Thus, Theorem 5.2 and the following corollaries in [15] are applicable in this case. Furthermore, $\mathbb{P}^n$ converges to $\mathbb{P}^*$ in total variation distance ($\mathbb{P}^n \xrightarrow{d_{TV}} \mathbb{P}^*$), which can be shown with easy calculations.

Before turning to the proof of the theorem, we first give some auxiliary lemmata after the following notations: For $v\in\mathbb{G}_n$ and $k\in\mathbb{N}$, let

$$\mathbb{G}_k^*(\mathsf{t},v):=\{u\in\mathbb{G}_{n+k}^*(\mathsf{t}) : v<u\}$$

denote the set of all infected t-cells in generation $n+k$ stemming from v. Let further be

$$\mathbb{G}_n^*(\mathsf{A},\mathsf{B}):=\{u\in\mathbb{G}_{n+1}^*(\mathsf{B}) : \mathsf{T}_{u|n}=\mathsf{A}\},$$

which is the set of all infected B-cells in generation $n+1$ whose mother cells are of type A.

Lemma 6.12. *Assuming* (AsBP) *and* (SubC), *then for all* $z \in \mathbb{N}$

$$\frac{\#\{v \in \mathbb{G}_n^*(\mathsf{A},\mathsf{B}) | Z_v = z\}}{\#\,\mathbb{G}_n^*(\mathsf{A})} \xrightarrow{\mathbb{P}^*} c(z) \quad \text{and} \quad \frac{\#\,\mathbb{G}_n^*(\mathsf{A},\mathsf{B})}{\#\,\mathbb{G}_n^*(\mathsf{A})} \xrightarrow{\mathbb{P}^*} \sum_{z=1}^{\infty} c(z),$$

where $c(z) := 2\sum_{x=1}^{\infty} q_x \mathbb{P}_{\mathsf{A},x}(Z_{[1]} = z, \mathsf{T}_{[1]} = \mathsf{B})$ *and* $(q_x)_{x\geq 1}$ *as in* (6.5).

Proof. Let $z \in \mathbb{N}$. Then

$$\frac{\#\{v \in \mathbb{G}_n^*(\mathsf{A},\mathsf{B}) | Z_v = z\}}{\#\mathbb{G}_n^*(\mathsf{A})} = \sum_{x=1}^{\infty} F_n(x,\mathsf{A}) \frac{\#\{v \in \mathbb{G}_n^*(\mathsf{A},\mathsf{B}) | Z_{v|n-1} = x, Z_v = z\}}{\#\{v \in \mathbb{G}_n^*(\mathsf{A}) | Z_v = x\}}$$

for all $n \in \mathbb{N}_0$. By Corollary 6.8(b), $F_n(x,\mathsf{A}) \to q_x$ for each $x \in \mathbb{N}$ and $n \to \infty$. Since $\mathbb{P}^n \xrightarrow{d_{TV}} \mathbb{P}^*$, the law of large numbers yields

$$\begin{aligned} \frac{\#\{v \in \mathbb{G}_n^*(\mathsf{A},\mathsf{B}) | Z_{v|n-1} = x, Z_v = z\}}{\#\{v \in \mathbb{G}_n^*(\mathsf{A}) | Z_v = x\}} &\xrightarrow{\mathbb{P}^*} \mathbb{E}_{\mathsf{A},x} \#\{v \in \mathbb{G}_1^*(\mathsf{B}) | Z_v = z\} \\ &= 2\mathbb{P}_{\mathsf{A},x}(Z_{[1]} = z, \mathsf{T}_{[1]} = \mathsf{B}) \end{aligned}$$

for $n \to \infty$ on $\{\#\{v \in \mathbb{G}_n^*(\mathsf{A}) | Z_v = x\} \to \infty\}$. Thus,

$$\sum_{x=1}^{K} F_n(x,\mathsf{A}) \frac{\#\{v \in \mathbb{G}_n^*(\mathsf{A},\mathsf{B}) | Z_{v|n-1} = x, Z_v = z\}}{\#\{v \in \mathbb{G}_n^*(\mathsf{A}) | Z_v = x\}} \xrightarrow{\mathbb{P}^*} 2\sum_{x=1}^{K} q_x \mathbb{P}_{\mathsf{A},x}(Z_{[1]} = z, \mathsf{T}_{[1]} = \mathsf{B}) \tag{6.6}$$

for all $K \in \mathbb{N}$. Finally, for each $\varepsilon > 0$, we can choose K large enough such that

$$\sum_{x>K} q_x \mathbb{P}_{\mathsf{A},x}(Z_{[1]} = z, \mathsf{T}_{[1]} = \mathsf{B}) \;\leq\; \sum_{x>K} q_x \leq \frac{\varepsilon}{4},$$

and thus

$$\begin{aligned} \mathbb{P}^*\left(\sum_{x>K} F_n(x,\mathsf{A}) \frac{\#\{v \in \mathbb{G}_n^*(\mathsf{A},\mathsf{B}) | Z_{v|n-1} = x, Z_v = z\}}{\#\{v \in \mathbb{G}_n^*(\mathsf{A}) | Z_v = x\}} \geq \varepsilon\right) \;&\leq\; \mathbb{P}^*\left(\sum_{x>K} F_n(x,\mathsf{A}) \geq \frac{\varepsilon}{2}\right) \\ = \; \mathbb{P}^*\left(\sum_{x=1}^{K} F_n(x,\mathsf{A}) \leq 1 - \frac{\varepsilon}{2}\right) \;&\to\; 0 \end{aligned}$$

for $n \to \infty$ by an appeal to Corollary 6.8(b). In combination with (6.6) this finishes the proof. □

Lemma 6.13. *Let* (AsBP), (SubC), (B) *and* $\mu_{\mathsf{B}} \geq \gamma$ *hold true. For all* $\varepsilon > 0$ *there exist a constant* $\eta > 0$ *such that*

$$\inf_{n\geq 0} \mathbb{P}^*\left(\frac{\#\,\mathbb{G}_n^*(\mathsf{B})}{\#\,\mathbb{G}_n^*(\mathsf{A})} \geq n\eta\right) \geq 1 - \varepsilon.$$

Proof. Fix $\varepsilon > 0$ and note that $\{\#\,\mathbb{G}_n^*(\mathsf{A},\mathsf{B}) \to \infty\} = \mathsf{Surv}(\mathsf{A})$ a.s. by the Borel-Cantelli lemma. By Theorem 4.6 and the assumptions of the lemma, there exists a finite random variable D which is strictly positive on $\mathsf{Surv}(A)$ with $\gamma^{-n}\mathcal{Z}_n(\mathsf{A}) \leq D$ a.s. for all $n \in \mathbb{N}_0$. From this, we get

$$\frac{\#\,\mathbb{G}_n^*(\mathsf{B})}{\#\,\mathbb{G}_n^*(\mathsf{A})} \;=\; \frac{1}{\#\,\mathbb{G}_n^*(\mathsf{A})} \sum_{m=0}^{n-1} \sum_{v\in\mathbb{G}_m^*(\mathsf{A},\mathsf{B})} \#\,\mathbb{G}_{n-m-1}^*(\mathsf{B},v)$$

$$
\begin{aligned}
&\geq \frac{1}{\mathcal{Z}(\mathsf{A})} \sum_{m=0}^{n-1} \sum_{v\in\mathbb{G}_m^*(\mathsf{A},\mathsf{B})} \#\,\mathbb{G}_{n-m-1}^*(\mathsf{B},v,1) \\
&\geq \frac{1}{D} \sum_{m=0}^{n-1} \sum_{v\in\mathbb{G}_m^*(\mathsf{A},\mathsf{B})} \frac{1}{\gamma^n}\#\,\mathbb{G}_{n-m-1}^*(\mathsf{B},v,1) \\
&\geq \frac{\eta_1}{D} \sum_{m=0}^{n-1} \sum_{v\in\mathbb{G}_m^*(\mathsf{A},\mathsf{B})} \frac{1}{\gamma^{m+1}} \mathbb{1}_{\left\{\#\,\mathbb{G}_{n-m-1}^*(\mathsf{B},v,1)\geq\eta_1\gamma^{n-m-1}\right\}} \quad \text{a.s.}
\end{aligned}
$$

for all $n \in \mathbb{N}$ and $\eta_1 > 0$, where $\#\,\mathbb{G}_{n-m-1}^*(\mathsf{B},v,1)$ gives the number of cells in generation n infected with a parasite stemming from the first parasite in cell v. Furthermore, conditioned upon $\{\#\,\mathbb{G}_n^*(\mathsf{A},\mathsf{B}) \to \infty\}(=\mathsf{Surv}(\mathsf{A}))$ these random variables are independent to each other as well as to the type-A cells. By choosing η_1 small enough, [15, Corollary 5.3] provides

$$
\mathbb{P}_{\mathsf{B},1}\left(\#\,\mathbb{G}_n^*(\mathsf{B}) \geq \eta_1\gamma^n\right) \;\leq\; \mathbb{P}_{\mathsf{B},1}\left(\#\,\mathbb{G}_n^*(\mathsf{B}) \geq \eta_1\mu_{\mathsf{B}}^n\right) \;\geq\; \delta \;>\; 0
$$

for a $\delta > 0$ and all $n \in \mathbb{N}$ by recalling that $\mu_{\mathsf{B}} \leq \gamma$. Hence,

$$
\mathbb{P}^*\left(\frac{\#\,\mathbb{G}_n^*(\mathsf{B})}{\#\,\mathbb{G}_n^*(\mathsf{A})} \geq t\right) \;\geq\; \mathbb{P}^*\left(\frac{\eta_1}{D}\sum_{m=0}^{n-1}\sum_{v\in\mathbb{G}_m^*(\mathsf{A},\mathsf{B})} \frac{\beta_v}{\gamma^{m+1}} \geq t\right) \tag{6.7}
$$

for all $t \geq 0$ and $n \in \mathbb{N}_0$, where $(\beta_v)_{v\in\mathbb{V}_2}$ are i.i.d., independent of the A-cells an A-parasites and Bernoulli distributed with success probability δ. Since

$$
\begin{aligned}
\sum_{m=0}^{\infty}\sum_{v\in\mathbb{G}_m^*(\mathsf{A},\mathsf{B})} \frac{1}{(m+1)^2}\mathbb{V}ar\left(\frac{\beta_v}{\gamma^{m+1}}\right) \;&\leq\; \sum_{m=0}^{\infty} \frac{\#\,\mathbb{G}_m^*(\mathsf{A},\mathsf{B})}{\gamma^{2(m+1)}} \\
&\leq\; 2\sum_{m=0}^{\infty}\frac{\mathcal{Z}_m(\mathsf{A})}{\gamma^{2(m+1)}} \;\leq\; \frac{2D}{\gamma^2}\sum_{m=0}^{\infty}\frac{1}{\gamma^m} \;<\; \infty \quad \text{a.s.,}
\end{aligned}
$$

the law of large numbers ensures

$$
\lim_{n\to\infty}\frac{1}{n}\sum_{m=0}^{n-1}\sum_{v\in\mathbb{G}_m^*(\mathsf{A},\mathsf{B})}\frac{\beta_v-\delta}{\gamma^{m+1}} \;=\; 0 \quad \mathbb{P}^*\text{-a.s.} \tag{6.8}
$$

Furthermore,

$$
\frac{1}{n}\sum_{m=0}^{n-1}\sum_{v\in\mathbb{G}_m^*(\mathsf{A},\mathsf{B})}\frac{\delta}{\gamma^{m+1}} \;=\; \frac{\delta}{n}\sum_{m=0}^{n-1}\frac{\#\,\mathbb{G}_m^*(\mathsf{A},\mathsf{B})}{\gamma^{m+1}} \;\geq\; \frac{\delta\eta_2}{n}\sum_{m=0}^{n-1}\mathbb{1}_{\{\#\,\mathbb{G}_m^*(\mathsf{A},\mathsf{B})\geq\eta_2\gamma^{m+1}\}} \quad \mathbb{P}^*\text{-a.s.}
$$

for every $\eta_2 > 0$ and $n \to \infty$. Using Corollary 6.8 and Lemma 6.12, we can choose η_2 small enough such that

$$
\mathbb{P}^*\left(\#\,\mathbb{G}_m^*(\mathsf{A},\mathsf{B}) \geq \eta_2\gamma^{m+1}\right) \;\geq\; 1-\varepsilon
$$

for all $m \in \mathbb{N}$, and hence

$$
1-\varepsilon \;\leq\; \mathbb{E}^*\left(\frac{1}{n}\sum_{m=0}^{n-1}\mathbb{1}_{\{\#\,\mathbb{G}_m^*(\mathsf{A},\mathsf{B})\geq\eta_2\gamma^{m+1}\}}\right) \;=\; \int_0^1 \mathbb{P}^*\left(\frac{1}{n}\sum_{m=0}^{n-1}\mathbb{1}_{\{\#\,\mathbb{G}_m^*(\mathsf{A},\mathsf{B})\geq\eta_2\gamma^{m+1}\}} > t\right) dt
$$

$$\leq\ \varepsilon + (1-\varepsilon)\mathbb{P}^*\left(\frac{1}{n}\sum_{m=0}^{n-1}\mathbb{1}_{\{\#\,\mathbb{G}_m^*(\mathsf{A},\mathsf{B})\geq \eta_2\gamma^{m+1}\}} > \varepsilon\right).$$

This implies

$$\mathbb{P}^*\left(\frac{1}{n}\sum_{m=0}^{n-1}\sum_{v\in\mathbb{G}_m^*(\mathsf{A},\mathsf{B})}\frac{\delta}{\gamma^{m+1}} > \delta\eta_2\varepsilon\right)\ \geq\ \mathbb{P}^*\left(\frac{1}{n}\sum_{m=0}^{n-1}\mathbb{1}_{\{\#\,\mathbb{G}_m^*(\mathsf{A},\mathsf{B})\geq\eta_2\gamma^{m+1}\}} > \varepsilon\right)\ \geq\ \frac{1-2\varepsilon}{1-\varepsilon}$$

for all $n\in\mathbb{N}$. This together with the right choice of t in (6.7) and (6.8) yields the assertion. □

The next lemma ensures that the contribution of highly infected type-B cells is negligible to the total number of B-cells if the multiplication of type-B parasites is higher then the one of type-A parasites, i.e. $\mu_{\mathsf{B}}\geq\gamma$.

Lemma 6.14. *Assuming* (AsBP), (B) *and* $\mu_{\mathsf{B}}>\gamma$, *then for all* $\varepsilon>0$

$$\lim_{z\to\infty}\sup_{n\geq 0}\mathbb{P}^*\left(\frac{\#\{v\in\mathbb{G}_n^*(\mathsf{B})\mid Z_v>z\}}{\#\,\mathbb{G}_n^*(\mathsf{B})}\geq\varepsilon\right)\ =\ 0.$$

If additionally (SubC), *the above limit even holds true for* $\mu_{\mathsf{B}}=\gamma$.

Proof. First, we note that $(Z_{[n]})_{n\geq 0}$ is a BPRE with mean $\mu_{\mathsf{B}}/2$ when starting with a B-cell hosting one parasite (see [15]). Second, we get by (6.2) and Proposition 6.2

$$\mathbb{E}Z_{[n]}\,\mathbb{1}_{\{\mathsf{T}_{[n]}=\mathsf{A}\}}\ =\ \mathbb{P}(\mathsf{T}_{[n]}=\mathsf{A})\,\mathbb{E}Z_n(\mathsf{A})\ =\ \left(\frac{\gamma}{2}\right)^n$$

and thus

$$\begin{aligned}\mathbb{E}Z_{[n]}\ &=\ \mathbb{E}Z_{[n]}\,\mathbb{1}_{\{\mathsf{T}_{[n]}=\mathsf{A}\}}+\sum_{m=0}^{n-1}\mathbb{E}Z_{[n]}\,\mathbb{1}_{\{\mathsf{T}_{[m]}=\mathsf{A},\mathsf{T}_{[m+1]}=\mathsf{B}\}}\\ &=\ \left(\frac{\gamma}{2}\right)^n+\sum_{m=0}^{n-1}\mathbb{E}Z_{[m]}\,\mathbb{1}_{\{\mathsf{T}_{[m]}=\mathsf{A}\}}\ \mathbb{E}_{\mathsf{A},1}Z_{[1]}\,\mathbb{1}_{\{\mathsf{T}_{[1]}=\mathsf{B}\}}\ \mathbb{E}_{\mathsf{B},1}Z_{[n-m-1]}\\ &=\ \left(\frac{\gamma}{2}\right)^n+\eta\sum_{m=0}^{n-1}\left(\frac{\gamma}{2}\right)^m\left(\frac{\mu_{\mathsf{B}}}{2}\right)^{n-m-1}\end{aligned}\tag{6.9}$$

for all $n\in\mathbb{N}$ where $\eta:=\mathbb{E}_{\mathsf{A},1}Z_{[1]}\,\mathbb{1}_{\{\mathsf{T}_{[1]}=\mathsf{B}\}}$.

Fix $\varepsilon>0$ and let $\mu_{\mathsf{B}}>\gamma$. By [15, Proposition 6.3] there exists a constant $a>0$ such that

$$\mathbb{P}_{\mathsf{B},1}\left(\frac{\#\,\mathbb{G}_n^*(\mathsf{B})}{\mu_{\mathsf{B}}^n}\geq a\ \middle|\ \mathcal{Z}_n(\mathsf{B})\to\infty\right)\geq 1-\varepsilon$$

for all $n\in\mathbb{N}_0$, and thus

$$\mathbb{P}\left(\sum_{k=1}^{K}\frac{\#\,\mathbb{G}_{n,k}^*(\mathsf{B})}{\mu_{\mathsf{B}}^n}\geq a\right)\geq 1-2\varepsilon\tag{6.10}$$

for large $K\in\mathbb{N}$, where $(\#\,\mathbb{G}_{n,k}^*(\mathsf{B}))_{k\geq 1}$ are i.i.d. copies of $\#\,\mathbb{G}_n^*(\mathsf{B})$ starting in a B-cell with a single parasite. By Theorem 6.4, we can find for each $K\in\mathbb{N}$ a $n_0\in\mathbb{N}$ such that

$$\mathbb{P}^*(\#\,\mathbb{G}_n^*(\mathsf{B})\geq K)\geq 1-\varepsilon$$

for all $n \geq n_0$, and with (6.10), this implies

$$\begin{aligned}\inf_{n\geq 0} \mathbb{P}^*\left(\frac{\#\,\mathbb{G}_n^*(\mathsf{B})}{\mu_{\mathsf{B}}^n} \geq c\right) \;&\geq\; (1-3\varepsilon) \wedge \inf_{n\geq n_0} \mathbb{P}^*\left(\sum_{k=1}^{K} \frac{\#\,\mathbb{G}_{n-n_0,k}^*(\mathsf{B})}{\mu_{\mathsf{B}}^{n-n_0}} \geq c\mu_{\mathsf{B}}^{n_0},\ \#\,\mathbb{G}_{n_0}^*(\mathsf{B}) \geq K\right) \\ &\geq\; 1-3\varepsilon\end{aligned}$$

for a suitable small $c > 0$. Fix such a c and define

$$E_n(z) := \left\{\frac{\#\{v \in \mathbb{G}_n^*(\mathsf{B}) \mid Z_v > z\}}{\#\,\mathbb{G}_n^*(\mathsf{B})} \geq \varepsilon\right\} \cap \left\{\frac{\#\,\mathbb{G}_n^*(\mathsf{B})}{\mu_{\mathsf{B}}^n} \geq c\right\}$$

for $z \in \mathbb{N}$. Then

$$\#\{v \in \mathbb{G}_n^* \mid Z_v > z\} \;\geq\; \#\{v \in \mathbb{G}_n^*(\mathsf{B}) \mid Z_v > z\} \;\geq\; \varepsilon c \mu_{\mathsf{B}}^n \mathbb{1}_{E_n(z)} \qquad \text{a.s.}$$

for all $n \in \mathbb{N}$, and consequently, by using Lemma 6.3 and (6.9),

$$\begin{aligned}\varepsilon\mathbb{P}(E_n(z)) \;&\leq\; \frac{1}{\mu_{\mathsf{B}}^n}\mathbb{E}\#\{v \in \mathbb{G}_n^* \mid Z_v > z\} \;=\; \left(\frac{2}{\mu_{\mathsf{B}}}\right)^n \mathbb{P}(Z_{[n]} > z) \\ &\leq\; \frac{1}{z}\left(\frac{2}{\mu_{\mathsf{B}}}\right)^n \mathbb{E}Z_{[n]} \;\leq\; \frac{1}{z}\,\tilde{c}\sum_{m=0}^{\infty}\left(\frac{\gamma}{\mu_{\mathsf{B}}}\right)^m \;<\; \infty\end{aligned}$$

for some $\tilde{c} < \infty$. Letting z tend to infinity proves the first statement of this lemma.

Let now (SubC) and $\mu_{\mathsf{B}} = \gamma$. Then Lemma 6.13 and Corollary 6.8 ensure the existence of a constant $c > 0$ such that

$$\inf_{n\geq 0} \mathbb{P}^*\left(\frac{\#\,\mathbb{G}_n^*(\mathsf{B})}{\mu_{\mathsf{B}}^n} \geq nc\right) \;\geq\; 1-\varepsilon,$$

and defining

$$E_n'(z) := \left\{\frac{\#\{v \in \mathbb{G}_n^*(\mathsf{B}) \mid Z_v > z\}}{\#\,\mathbb{G}_n^*(\mathsf{B})} \geq \varepsilon\right\} \cap \left\{\frac{\#\,\mathbb{G}_n^*(\mathsf{B})}{\mu_{\mathsf{B}}^n} \geq nc\right\}$$

for $n \geq 0$, we find by an analogous argumentation as above that

$$\varepsilon\mathbb{P}(E_n(z)) \;\leq\; \frac{1}{n\mu_{\mathsf{B}}^n}\mathbb{E}\#\{v \in \mathbb{G}_n^*(\mathsf{B}) \mid Z_v > z\} \;\leq\; \frac{1}{nz}\left(\frac{2}{\mu_{\mathsf{B}}}\right)^n \mathbb{E}Z_{[n]} \;\leq\; \frac{\tilde{c}}{z}$$

for a constant $\tilde{c} < \infty$. Hence, the assertion follows for $z \to \infty$. □

Having verified the above lemmata, we are now able to prove Theorem 6.10.

Proof of Theorem 6.10: (*a*) Lemma 6.13 provides $\#\,\mathbb{G}_n^*(\mathsf{B})/\#\,\mathbb{G}_n^*(\mathsf{A}) \xrightarrow{\mathbb{P}^*} \infty$, and we infer

$$\frac{\#\,\mathbb{G}_n^*(\mathsf{A})}{\#\,\mathbb{G}_n^*} \;=\; \frac{1}{1+\#\,\mathbb{G}_n^*(\mathsf{B})/\#\,\mathbb{G}_n^*(\mathsf{A})} \;\xrightarrow{\mathbb{P}^*}\; 0 \quad \text{as } n \to \infty. \tag{6.11}$$

Let $k \in \mathbb{N}$ and $m \in \mathbb{N}$ be large. For $n \in \mathbb{N}$, (6.11) and the fact that $\#\,\mathbb{G}_n^*(\mathsf{B}) \to \infty$ $\mathbb{P}^*$-a.s. (Theorem 6.4) ensure

$$\frac{\#\{v \in \mathbb{G}_{n+m}^*(\mathsf{B}) | \,\mathsf{T}_{v|n} = \mathsf{A}\}}{\#\,\mathbb{G}_{n+m}^*(\mathsf{B})} \;\leq\; 2^m \frac{\#\,\mathbb{G}_n^*(\mathsf{A})}{\#\,\mathbb{G}_n^*(\mathsf{B})}\,\frac{\#\,\mathbb{G}_n^*(\mathsf{B})}{\#\,\mathbb{G}_{n+m}^*(\mathsf{B})} \;\xrightarrow{\mathbb{P}^*}\; 0$$

and thus

$$\frac{\#\{v \in \mathbb{G}^*_{n+m}(\mathsf{B}) |\, \mathsf{T}_{v|n} = \mathsf{B}\}}{\#\, \mathbb{G}^*_{n+m}(\mathsf{B})} \overset{\mathbb{P}^*}{\longrightarrow} 1 \quad \text{as } n \to \infty.$$

Hence,

$$\begin{aligned} F_{n+m}(k, \mathsf{B}) &\overset{\mathbb{P}^*}{\simeq} \frac{\#\{v \in \mathbb{G}^*_{n+m}(\mathsf{B}) | Z_v = k, \mathsf{T}_{v|n} = \mathsf{B}\}}{\#\{v \in \mathbb{G}^*_{n+m}(\mathsf{B}) |\, \mathsf{T}_{v|n} = \mathsf{B}\}} \\ &= \frac{\sum_{u \in \mathbb{G}^*_n(\mathsf{B})} \#\{v \in \mathbb{G}^*_m(\mathsf{B}, u) | Z_v = k\}}{\sum_{u \in \mathbb{G}^*_n(\mathsf{B})} \#\, \mathbb{G}^*_m(\mathsf{B}, u)} \quad \text{for } n \to \infty, \end{aligned} \tag{6.12}$$

where $a_n \overset{\mathbb{P}^*}{\simeq} b_n$ means that $\lim_{n\to\infty} \mathbb{P}^*(|a_n - b_n| \geq \eta) = 0$ for all $\eta > 0$. Let us set

$$\mathbb{G}^*_m(\mathsf{B}, u, k) := \{v \in \mathbb{G}^*_m(\mathsf{B}, u) | Z_v = k\}$$

for $m \in \mathbb{N}$ and $u \in \mathbb{V}_2$, and let $\varepsilon, \delta > 0$. Lemma 6.14 yields

$$\mathbb{P}^* \left(\#\{v \in \mathbb{G}^*_n(\mathsf{B}) : Z_v \leq z\} \geq (1 - \eta) \#\, \mathbb{G}^*_n(\mathsf{B})\right) \geq 1 - \varepsilon \tag{6.13}$$

for all $\eta > 0$, $n \in \mathbb{N}_0$ and suitable large $z \in \mathbb{N}$, and thus

$$\mathbb{P}^* \left(\left| \frac{\sum_{u \in \mathbb{G}^*_n(\mathsf{B})} \#\, \mathbb{G}^*_m(\mathsf{B}, u, k)}{\sum_{u \in \mathbb{G}^*_n(\mathsf{B})} \#\, \mathbb{G}^*_m(\mathsf{B}, u)} - \frac{\sum_{u \in \{v \in \mathbb{G}^*_n(\mathsf{B}) : Z_v \leq z\}} \#\, \mathbb{G}^*_m(\mathsf{B}, u, k)}{\sum_{u \in \{v \in \mathbb{G}^*_n(\mathsf{B}) : Z_v \leq z\}} \#\, \mathbb{G}^*_m(\mathsf{B}, u)} \right| \geq \delta \right) \leq \varepsilon \tag{6.14}$$

for all $n \in \mathbb{N}_0$ and a large $z \in \mathbb{N}$. Further note that by definition $\{v \in \mathbb{G}^*_n(\mathsf{B}) : Z_v = x\} = \mathbb{G}^*_n(\mathsf{B}, \varnothing, x)$ for every $x \in \{1, \ldots, z\}$. As the random variables $\#\, \mathbb{G}^*_m(\mathsf{B}, u)$, $u \in \mathbb{G}^*_n(\mathsf{B}, \varnothing, x)$, are i.i.d. conditioned upon $\{\mathcal{Z}_n(\mathsf{B}) > 0\}$, we can find a $K_x \in \mathbb{N}$ such that

$$\mathbb{P} \left(\left| \frac{\sum_{u \in \mathbb{G}^*_n(\mathsf{B}, \varnothing, x)} \#\, \mathbb{G}^*_m(\mathsf{B}, u)}{\#\, \mathbb{G}^*_n(\mathsf{B}, \varnothing, x)} - \mathbb{E}_{\mathsf{B},x} \#\, \mathbb{G}^*_m \right| \geq \delta,\ \#\, \mathbb{G}^*_n(\mathsf{B}, \varnothing, x) \geq K_x \,\middle|\, \mathcal{Z}_n(\mathsf{B}) > 0 \right) \leq \varepsilon$$

for all $n \in \mathbb{N}$ by the law of large numbers. Since $\mathbb{P}(\cdot | \mathcal{Z}_n(\mathsf{B}) > 0) \to \mathbb{P}(\cdot |\, \mathsf{Surv})$ in total variation distance as $n \to \infty$, and $\mathsf{Surv}(\mathsf{A}) \subseteq \mathsf{Surv}$ by Theorem 6.4, there exists a $n_1 \in \mathbb{N}$ such that

$$\mathbb{P}^* \left(\left| \frac{\sum_{u \in \mathbb{G}^*_n(\mathsf{B}, \varnothing, x)} \#\, \mathbb{G}^*_m(\mathsf{B}, u)}{\#\, \mathbb{G}^*_n(\mathsf{B}, \varnothing, x)} - \mathbb{E}_{\mathsf{B},x} \#\, \mathbb{G}^*_m \right| \geq \delta,\ \#\, \mathbb{G}^*_n(\mathsf{B}, \varnothing, x) \geq K_x \right) \leq 2\varepsilon$$

for all $n \geq n_1$. From (6.13), we get that $\mathbb{P}^*(\#\{v \in \mathbb{G}^*_n(\mathsf{B}) : Z_v \leq z\} \to \infty) \geq 1 - \varepsilon$, which provides us with a $n_2 \geq n_1$ such that

$$\begin{aligned} &\mathbb{P}^* \left(\left| \frac{\sum_{u \in \mathbb{G}^*_n(\mathsf{B}, \varnothing, x)} \#\, \mathbb{G}^*_m(\mathsf{B}, u)}{\#\{v \in \mathbb{G}^*_n(\mathsf{B}) : Z_v \leq z\}} - \frac{\#\, \mathbb{G}^*_n(\mathsf{B}, \varnothing, x)}{\#\{v \in \mathbb{G}^*_n(\mathsf{B}) : Z_v \leq z\}} \mathbb{E}_{\mathsf{B},x} \#\, \mathbb{G}^*_m \right| \geq \delta \right) \\ &\quad = \mathbb{P}^* \left(\frac{\#\, \mathbb{G}^*_n(\mathsf{B}, \varnothing, x)}{\#\{v \in \mathbb{G}^*_n(\mathsf{B}) : Z_v \leq z\}} \left| \frac{\sum_{u \in \mathbb{G}^*_n(\mathsf{B}, \varnothing, x)} \#\, \mathbb{G}^*_m(\mathsf{B}, u)}{\#\, \mathbb{G}^*_n(\mathsf{B}, \varnothing, x)} - \mathbb{E}_{\mathsf{B},x} \#\, \mathbb{G}^*_m \right| \geq \delta \right) \\ &\quad \leq 2\varepsilon + \mathbb{P}^* \left(\frac{K_x 2^{m+1}}{\#\{v \in \mathbb{G}^*_n(\mathsf{B}) : Z_v \leq z\}} \geq \delta,\ \#\, \mathbb{G}^*_n(\mathsf{B}, \varnothing, x) < K_x \right) \\ &\quad \leq 3\varepsilon \end{aligned}$$

for all $n \geq n_2$, where we used that $\#\,\mathbb{G}_m^* \leq 2^m$ a.s. for all $m \in \mathbb{N}$. This estimation can be done for each $x \in \{1, \ldots, z\}$, and thus we find a $n_3 \geq n_2$ such that

$$\mathbb{P}^*\left(\left|\frac{\sum_{u\in\{v\in\mathbb{G}_n^*(\mathsf{B}):Z_v\leq z\}}\#\,\mathbb{G}_m^*(\mathsf{B},u)}{\#\{v\in\mathbb{G}_n^*(\mathsf{B}):Z_v\leq z\}} - \frac{\sum_{x=1}^{z}\#\,\mathbb{G}_n^*(\mathsf{B},\varnothing,x)\mathbb{E}_{\mathsf{B},x}\#\,\mathbb{G}_m^*}{\#\{v\in\mathbb{G}_n^*(\mathsf{B}):Z_v\leq z\}}\right| \geq 2\delta\right) \;\leq\; 4\varepsilon$$

for all $n \geq n_3$. Repeating the same argumentation for the random variables $\#\,\mathbb{G}_m^*(\mathsf{B}, u, k)$ now gives for all $n \geq n_4$ with $n_4 \geq n_3$ large enough

$$\mathbb{P}^*\left(G_n(z,k) \geq 3\delta\right) \;\leq\; 5\varepsilon, \tag{6.15}$$

for the random variable

$$G_n(z,k) := \left|\frac{\sum_{u\in\{v\in\mathbb{G}_n^*(\mathsf{B}):Z_v\leq z\}}\#\,\mathbb{G}_m^*(\mathsf{B},u,k)}{\sum_{u\in\{v\in\mathbb{G}_n^*(\mathsf{B}):Z_v\leq z\}}\#\,\mathbb{G}_m^*(\mathsf{B},u)} - \frac{\sum_{x=1}^{z}\#\,\mathbb{G}_n^*(\mathsf{B},\varnothing,x)\mathbb{E}_{\mathsf{B},x}\#\,\mathbb{G}_m^*(\mathsf{B},\varnothing,k)}{\sum_{x=1}^{z}\#\,\mathbb{G}_n^*(\mathsf{B},\varnothing,x)\mathbb{E}_{\mathsf{B},x}\#\,\mathbb{G}_m^*}\right|.$$

Finally, Lemma 6.3 yields

$$\frac{\sum_{x=1}^{z}\#\,\mathbb{G}_n^*(\mathsf{B},\varnothing,x)\mathbb{E}_{\mathsf{B},x}\#\,\mathbb{G}_m^*(\mathsf{B},\varnothing,k)}{\sum_{x=1}^{z}\#\,\mathbb{G}_n^*(\mathsf{B},\varnothing,x)\mathbb{E}_{\mathsf{B},x}\#\,\mathbb{G}_m^*} \;=\; \frac{\sum_{x=1}^{z}\#\,\mathbb{G}_n^*(\mathsf{B},\varnothing,x)\mathbb{P}_{\mathsf{B},x}(Z_{[m]}=k)}{\sum_{x=1}^{z}\#\,\mathbb{G}_n^*(\mathsf{B},\varnothing,x)\mathbb{P}_{\mathsf{B},x}(Z_{[m]}>0)}$$

for $n \in \mathbb{N}$, and since [17, Theorem 7] states

$$\lim_{m\to\infty}\mathbb{P}_{\mathsf{B},x}(Z_{[m]}=k|Z_{[m]}>0) \;=\; q_k(\mathsf{B}),$$

for all $1 \leq x \leq z$, we infer

$$\left|\frac{\sum_{x=1}^{z}\#\,\mathbb{G}_n^*(\mathsf{B},\varnothing,x)\mathbb{E}_{\mathsf{B},x}\#\,\mathbb{G}_m^*(\mathsf{B},\varnothing,k)}{\sum_{x=1}^{z}\#\,\mathbb{G}_n^*(\mathsf{B},\varnothing,x)\mathbb{E}_{\mathsf{B},x}\#\,\mathbb{G}_m^*} - q_k(\mathsf{B})\right| \;\leq\; \delta$$

for all $n \in \mathbb{N}$ by having m chosen large enough at the beginning of the proof. Putting this, (6.12), (6.14) and (6.15) together finishes the proof of (*a*).

(*b*) Note that $\mathbb{E}_{\mathsf{B},x}Z_{[n]} = x(\mu_{\mathsf{B}}/2)^n$ and $\mathbb{E}_{\mathsf{A},x}Z_{[1]} = x\mathbb{E}_{\mathsf{A},1}Z_{[1]}$ for all $n \in \mathbb{N}$ and $x \in \mathbb{N}$, which entails

$$\begin{aligned}
\rho \;&=\; \sum_{n=0}^{\infty}(2/\gamma)^{n+1}\sum_{z=1}^{\infty}q_z\mathbb{P}_{\mathsf{A},z}(Z_{[n+1]}>0,\mathsf{T}_{[1]}=\mathsf{B}) \\
&\leq\; \sum_{n=0}^{\infty}(2/\gamma)^{n+1}\sum_{z=1}^{\infty}q_z\sum_{x=1}^{\infty}\mathbb{P}_{\mathsf{A},z}(Z_{[1]}=x,\mathsf{T}_{[1]}=\mathsf{B})\mathbb{E}_{\mathsf{B},x}Z_{[n]} \\
&\leq\; \frac{1}{\gamma}\sum_{n=0}^{\infty}(\mu_{\mathsf{B}}/\gamma)^n\sum_{z=1}^{\infty}q_z\sum_{x=1}^{\infty}x\mathbb{P}_{\mathsf{A},z}(Z_{[1]}=x) \\
&\leq\; \frac{\mathbb{E}_{\mathsf{A},1}Z_{[1]}}{\gamma}\sum_{n=0}^{\infty}(\mu_{\mathsf{B}}/\gamma)^n\sum_{z=1}^{\infty}zq_z \;<\; \infty,
\end{aligned}$$

as $\mu_{\mathsf{B}} < \gamma$ and $\sum_{z=1}^{\infty} zq_z < \infty$ by [40, Theorem 1.1]. Furthermore,

$$\frac{\#\,\mathbb{G}_n^*(\mathsf{A})}{\mathbb{G}_n^*} = \frac{1}{1+\#\,\mathbb{G}_n^*(\mathsf{B})/\,\mathbb{G}_n^*(\mathsf{A})} \quad\text{and}\quad F_n(k,\mathsf{B}) = \frac{\#\,\mathbb{G}_n^*(\mathsf{A})}{\#\,\mathbb{G}_n^*(\mathsf{B})}\cdot\frac{\#\{v\in\mathbb{G}_n^*(\mathsf{B})|Z_v=k\}}{\#\,\mathbb{G}_n^*(\mathsf{A})},$$

for all $n, k \in \mathbb{N}$, and so (b) follows if

$$\frac{\#\,\mathbb{G}_n^*(\mathsf{B})}{\#\,\mathbb{G}_n^*(\mathsf{A})} \xrightarrow{\mathbb{P}^*} \rho \quad \text{and} \quad \frac{\#\{v \in \mathbb{G}_n^*(\mathsf{B}) | Z_v = k\}}{\#\,\mathbb{G}_n^*(\mathsf{A})} \xrightarrow{\mathbb{P}^*} \rho_k.$$

We use the following representation of ρ

$$\rho \;=\; \frac{1}{\gamma}\sum_{n=0}^{\infty}(2/\gamma)^n \sum_{z=1}^{\infty} c(z)\mathbb{P}_{\mathsf{B},z}(Z_{[n]} > 0) \;=\; \sum_{n=0}^{\infty}\sum_{z=1}^{\infty}\frac{c(z)}{\gamma^{n+1}}\mathbb{E}_{\mathsf{B},z}\#\,\mathbb{G}_n^*,$$

where we recall the definition of $c(z)$ in Lemma 6.12. The triangular inequality yields

$$\left|\frac{\#\,\mathbb{G}_n^*(\mathsf{B})}{\#\,\mathbb{G}_n^*(\mathsf{A})} - \rho\right| \;\leq\; (*) + (**) + (***)$$

with

$$(*) \quad = \left|\frac{1}{\#\,\mathbb{G}_n^*(\mathsf{A})}\sum_{m=1}^{n-N-1}\sum_{u \in \mathbb{G}_{m-1}^*(\mathsf{A},\mathsf{B})}\#\,\mathbb{G}_{n-m}^*(\mathsf{B},u) - \sum_{m=N+1}^{\infty}\sum_{z=1}^{\infty}\frac{c(z)}{\gamma^{m+1}}\mathbb{E}_{\mathsf{B},z}\#\,\mathbb{G}_m^*\right|$$

$$(**) \quad = \left|\frac{1}{\#\,\mathbb{G}_n^*(\mathsf{A})}\sum_{m=n-N}^{n}\sum_{u \in \{v \in \mathbb{G}_{m-1}^*(\mathsf{A},\mathsf{B}) : Z_v > z_0\}}\#\,\mathbb{G}_{n-m}^*(\mathsf{B},u) - \sum_{m=0}^{N}\sum_{z > z_0}\frac{c(z)}{\gamma^{m+1}}\mathbb{E}_{\mathsf{B},z}\#\,\mathbb{G}_m^*\right|$$

$$(***) \quad = \left|\frac{1}{\#\,\mathbb{G}_n^*(\mathsf{A})}\sum_{m=n-N}^{n}\sum_{u \in \{v \in \mathbb{G}_{m-1}^*(\mathsf{A},\mathsf{B}) : Z_v \leq z_0\}}\#\,\mathbb{G}_{n-m}^*(\mathsf{B},u) - \sum_{m=0}^{N}\sum_{z=1}^{z_0}\frac{c(z)}{\gamma^{m+1}}\mathbb{E}_{\mathsf{B},z}\#\,\mathbb{G}_m^*\right|$$

for each $n, N \in \mathbb{N}$ with $N \leq n$ and $z_0 \in \mathbb{N}$. We show that these three summands become sufficiently small for the right choice of N and z_0.

Estimation of $(*)$: Since $\rho < \infty$, pick N so large that

$$\sum_{m=N+1}^{\infty}\sum_{z=1}^{\infty}\frac{c(z)}{\gamma^{m+1}}\mathbb{E}_{\mathsf{B},z}\#\,\mathbb{G}_m^* \;\leq\; \delta,$$

and by Corollary 6.8, we can find a constant $c > 0$ such that for all $n \in \mathbb{N}$

$$\mathbb{P}^*\left(\frac{\gamma^n}{\#\,\mathbb{G}_n^*(\mathsf{A})} > c\right) \;\leq\; \varepsilon. \tag{6.16}$$

Using these two estimations, $\mathbb{E}\mathcal{Z}_m(\mathsf{A}) = \gamma^m$ and $\mathbb{E}_{\mathsf{B},x}\mathcal{Z}_m = x\mu_{\mathsf{B}}^m$ for each $x, m \in \mathbb{N}$, we get

$$\begin{aligned}
\mathbb{P}^*\left((***) \geq 2\delta\right) \;&\leq\; \mathbb{P}^*\left(\frac{1}{\#\,\mathbb{G}_n^*(\mathsf{A})}\sum_{m=1}^{n-N-1}\sum_{u \in \mathbb{G}_{m-1}^*(\mathsf{A},\mathsf{B})}\#\,\mathbb{G}_{n-m}^*(\mathsf{B},u) \geq \delta\right) \\
&\leq\; \mathbb{P}^*\left(\frac{1}{\gamma^n}\sum_{m=1}^{n-N-1}\sum_{u \in \mathbb{G}_{m-1}^*(\mathsf{A},\mathsf{B})}\#\,\mathbb{G}_{n-m}^*(\mathsf{B},u) \geq \frac{\delta}{c}\right) + \varepsilon \\
&\leq\; \frac{c}{\delta\gamma^n}\sum_{m=1}^{n-N-1}\mathbb{E}^*\left(\sum_{u \in \mathbb{G}_{m-1}^*(\mathsf{A},\mathsf{B})}\#\,\mathbb{G}_{n-m}^*(\mathsf{B},u)\right) + \varepsilon
\end{aligned}$$

$$\begin{aligned}
&\le \frac{c}{\delta\gamma^n}\sum_{m=1}^{n-N-1}\mathbb{E}^*\left(\sum_{u\in\mathbb{G}^*_{m-1}(\mathsf{A},\mathsf{B})}\mathbb{E}_{\mathsf{B},Z_u}\mathcal{Z}_{n-m}\right)+\varepsilon\\
&= \frac{c}{\delta\gamma^n}\sum_{m=1}^{n-N-1}\mu_{\mathsf{B}}^{n-m}\mathbb{E}^*\left(\sum_{u\in\mathbb{G}^*_{m-1}(\mathsf{A},\mathsf{B})}Z_u\right)+\varepsilon\\
&= \frac{c}{\delta\mathbb{P}(\mathsf{Surv}(\mathsf{A}))\gamma^n}\sum_{m=1}^{n-N-1}\mu_{\mathsf{B}}^{n-m}\mathbb{E}\mathcal{Z}_{m-1}(\mathsf{A})\mathbb{E}\mathcal{Z}_1(\mathsf{B})+\varepsilon\\
&\le \frac{c\mathbb{E}\mathcal{Z}_1(\mathsf{B})}{\delta\mathbb{P}(\mathsf{Surv}(\mathsf{A}))\gamma}\sum_{m=N-1}^{\infty}\left(\frac{\mu_{\mathsf{B}}}{\gamma}\right)^m+\varepsilon\\
&\le 2\varepsilon,
\end{aligned}$$

possibly after enlarging N. Recall that $\mu_{\mathsf{B}}<\gamma$ is assumed.

Estimation of $(**)$: Let N be as chosen in the estimation of $(*)$ and fix $z_0\in\mathbb{N}$ large enough such that

$$\sum_{m=0}^{N}\sum_{z>z_0}\frac{c(z)}{\gamma^{m+1}}\mathbb{E}_{\mathsf{B},z}\#\,\mathbb{G}^*_m\ \le\ \delta\quad\text{and}\quad\sum_{z>z_0}c(z)\sum_{m=0}^{N}\frac{1}{\gamma^{m+1}}\le\frac{\delta}{2^{N+1}}.$$

From Lemma 6.12 and Corollary 6.8(b), we deduce

$$\sum_{m=0}^{N}\frac{\#\{v\in\mathbb{G}^*_{n-m-1}(\mathsf{A},\mathsf{B}):Z_v>z_0\}}{\#\,\mathbb{G}^*_n(\mathsf{A})}\ \xrightarrow{\mathbb{P}^*}\ \sum_{z>z_0}c(z)\sum_{m=0}^{N}\frac{1}{\gamma^{m+1}}\quad\text{as } n\to\infty,$$

and since $\#\,\mathbb{G}^*_{n-m}(\mathsf{B},u)\le 2^{n-m}$ a.s. for all $m\le n$, we infer

$$\begin{aligned}
\mathbb{P}^*\left((**)\ge 2\delta\right)\ &\le\ \mathbb{P}^*\left(\frac{1}{\#\,\mathbb{G}^*_n(\mathsf{A})}\sum_{m=n-N}^{n}\ \sum_{u\in\{v\in\mathbb{G}^*_{m-1}(\mathsf{A},\mathsf{B}):Z_v>z_0\}}\#\,\mathbb{G}^*_{n-m}(\mathsf{B},u)\ge\delta\right)\\
&\le\ \mathbb{P}^*\left(\sum_{m=n-N}^{n}\frac{\#\{v\in\mathbb{G}^*_{m-1}(\mathsf{A},\mathsf{B}):Z_v>z_0\}}{\#\,\mathbb{G}^*_n(\mathsf{A})}\ge\frac{\delta}{2^N}\right)\\
&=\ \mathbb{P}^*\left(\sum_{m=0}^{N}\frac{\#\{v\in\mathbb{G}^*_{n-m-1}(\mathsf{A},\mathsf{B}):Z_v>z_0\}}{\#\,\mathbb{G}^*_n(\mathsf{A})}\ge\frac{\delta}{2^N}\right)\\
&\to\ 0,\qquad n\to\infty.
\end{aligned}$$

Estimation of $(***)$: Let N and z_0 chosen according to the estimations of $(*)$ and $(**)$. Once again, by Lemma 6.12 and Corollary 6.8(b), we obtain

$$\frac{\#\{v\in\mathbb{G}^*_{n-m}(\mathsf{A},\mathsf{B})|Z_v=z\}}{\#\,\mathbb{G}^*_n(\mathsf{A})}\ \xrightarrow{\mathbb{P}^*}\ \frac{c(z)}{\gamma^m}\quad\text{as } n\to\infty \tag{6.17}$$

for each $z\in\mathbb{N}$ and $m\in\mathbb{N}$. Let $z\in\mathbb{N}$ with $c(z)>0$. For each $m\in\mathbb{N}$, the random variables $\#\,\mathbb{G}^*_m(\mathsf{B},u)$ with $u\in\{v\in\mathbb{G}^*_{n-m-1}(\mathsf{A},\mathsf{B})|Z_v=z\}$ are conditioned upon $\{\mathcal{Z}_{n-m}>0\}$ ($\supseteq$

$\{\mathcal{Z}_{n-m}(\mathsf{A})>0\}$) i.i.d., and thus the law of large numbers with (6.17) gives for each $\eta_1,\eta_2>0$ a $n_0\in\mathbb{N}$ such that

$$\mathbb{P}\left(\left|\frac{\sum_{u\in\{v\in\mathbb{G}^*_{n-m-1}(\mathsf{A},\mathsf{B})|Z_v=z\}}\#\,\mathbb{G}^*_m(\mathsf{B},u)}{\#\{v\in\mathbb{G}^*_{n-m-1}(\mathsf{A},\mathsf{B})|Z_v=z\}}-\mathbb{E}_{\mathsf{B},z}\#\,\mathbb{G}^*_m\right|\geq\eta_1\;\middle|\;\mathcal{Z}_{n-m}>0\right)\;\leq\;\eta_2$$

for all $n\geq n_0$. Since $\mathbb{P}(\cdot|\mathcal{Z}_n(\mathsf{B})>0)\to\mathbb{P}(\cdot|\,\mathsf{Surv})$ in total variation distance as $n\to\infty$, and $\mathsf{Surv}(\mathsf{A})\subseteq\mathsf{Surv}$ by Theorem 6.4, this leads to

$$\frac{1}{\#\,\mathbb{G}^*_n(\mathsf{A})}\sum_{u\in\{v\in\mathbb{G}^*_{n-m-1}(\mathsf{A},\mathsf{B})|Z_v=z\}}\#\,\mathbb{G}^*_m(\mathsf{B},u)\;\xrightarrow{\mathbb{P}^*}\;\frac{c(z)}{\gamma^{m+1}}\mathbb{E}_{\mathsf{B},z}\#\,\mathbb{G}^*_m$$

for $n\to\infty$ by an appeal to (6.17). But this implies

$$\frac{1}{\#\,\mathbb{G}^*_n(\mathsf{A})}\sum_{m=n-N}^{n}\;\sum_{u\in\{v\in\mathbb{G}^*_{m-1}(\mathsf{A},\mathsf{B}):Z_v\leq z_0\}}\#\,\mathbb{G}^*_{n-m}(\mathsf{B},u)$$

$$=\sum_{m=0}^{N}\sum_{z=1}^{z_0}\frac{1}{\#\,\mathbb{G}^*_n(\mathsf{A})}\sum_{u\in\{v\in\mathbb{G}^*_{n-m-1}(\mathsf{A},\mathsf{B})|Z_v=z\}}\#\,\mathbb{G}^*_m(\mathsf{B},u)\;\xrightarrow{\mathbb{P}^*}\;\sum_{m=0}^{N}\sum_{z=1}^{z_0}\frac{c(z)}{\gamma^{m+1}}\mathbb{E}_{\mathsf{B},z}\#\,\mathbb{G}^*_m$$

for $n\to\infty$, completing the proof of the theorem. □

Proof of Theorem 6.11

As in the proof of Theorem 6.10, we first show an auxiliary lemma which provides us with a cell type change rate.

Lemma 6.15. *Let* (AsBP) *and* (SupC) *hold true. Then for all* $z\in\mathbb{N}$

$$\frac{\#\{v\in\mathbb{G}^*_n(\mathsf{A},\mathsf{B})|Z_v\geq z\}}{\#\mathbb{G}^*_n(\mathsf{A})}\;\xrightarrow{\mathbb{P}^*}\;\beta\;>\;0\quad\textit{as } n\to\infty,$$

where $\beta:=\lim_{z\to\infty}\mathbb{E}_{\mathsf{A},z}\#\,\mathbb{G}^*_1(\mathsf{B})$. *In particular,* $\#\,\mathbb{G}^*_n(\mathsf{A},\mathsf{B})/\#\mathbb{G}^*_n(\mathsf{A})$ *conditioned upon* $\mathsf{Surv}(\mathsf{A})$ *converges in probability to* β.

Proof. Since $z\mapsto\mathbb{E}_{\mathsf{A},z}\#\,\mathbb{G}^*_1(\mathsf{B})$ is increasing and $\mathbb{E}_{\mathsf{A},1}\#\,\mathbb{G}^*_1(\mathsf{B})>0$ by our standing assumption (SA4), we see that β must be positive. Moreover, $\sum_{k\geq z_0}F_n(k,\mathsf{A})\xrightarrow{\mathbb{P}^*}1$ by Corollary 6.8(a), whence

$$\frac{\#\{v\in\mathbb{G}^*_n(\mathsf{A},\mathsf{B}):Z_v\geq z_0\}}{\#\,\mathbb{G}^*_n(\mathsf{A},\mathsf{B})}\xrightarrow{\mathbb{P}^*}1.$$

Thus, it is enough to prove the result with $\#\,\mathbb{G}^*_n(\mathsf{A},\mathsf{B})$ as numerator. Next, observe that

$$\#\,\mathbb{G}^*_n(\mathsf{A},\mathsf{B})=\sum_{v\in\mathbb{G}^*_{n-1}(\mathsf{A})}\#\,\mathbb{G}^*_1(\mathsf{B},v)$$

for each $n\in\mathbb{N}$, where the $\#\,\mathbb{G}^*_1(\mathsf{B},v)$ are conditionally independent given $\{\mathcal{Z}_n(\mathsf{A})>0\}$. Since $\#\,\mathbb{G}^*_n(\mathsf{A})\to\infty$ $\mathbb{P}^*$-a.s. (Theorem 6.4) and $\mathbb{P}^n\xrightarrow{d_{TV}}\mathbb{P}^*$, it is not difficult to infer with the help of the law of large numbers that

$$\frac{\#\mathbb{G}^*_n(\mathsf{A},\mathsf{B})}{\#\mathbb{G}^*_n(\mathsf{A})}-\frac{1}{\#\,\mathbb{G}^*_n(\mathsf{A})}\sum_{v\in\mathbb{G}^*_n(\mathsf{A})}\mathbb{E}_{\mathsf{A},Z_v}\#\,\mathbb{G}^*_1(\mathsf{B})\;\xrightarrow{\mathbb{P}^*}\;0\quad\text{as } n\to\infty.$$

Now, use $\mathbb{E}_{\mathsf{A},z}\#\,\mathbb{G}_1^*(\mathsf{B}) \uparrow \beta$ to infer the existence of a $z_0 \in \mathbb{N}$ such that

$$\mathbb{E}_{\mathsf{A},z}\#\,\mathbb{G}_1^*(\mathsf{B}) \geq \beta(1-\varepsilon)$$

for all $z \geq z_0$. After these observations, we finally obtain by an appeal to Corollary 6.8(a) that

$$\begin{aligned}
\beta &\geq \frac{1}{\#\,\mathbb{G}_n^*(\mathsf{A})} \sum_{v\in\mathbb{G}_n^*(\mathsf{A})} \mathbb{E}_{\mathsf{A},Z_v}\#\,\mathbb{G}_1^*(\mathsf{B}) \\
&\geq \sum_{z\geq z_0} \frac{F_n(z,\mathsf{A})}{\#\{v\in\mathbb{G}_n^*(\mathsf{A})|Z_v\geq z_0\}} \sum_{v\in\{u\in\mathbb{G}_n^*(\mathsf{A})|Z_u\geq z_0\}} \mathbb{E}_{\mathsf{A},Z_v}\#\,\mathbb{G}_1^*(\mathsf{B}) \\
&\geq \beta(1-\varepsilon)\sum_{z\geq z_0} F_n(z,\mathsf{A}) \\
&\to \beta(1-\varepsilon), \quad n\to\infty.
\end{aligned}$$

□

Proof of Theorem 6.11: If $\mu_{\mathsf{B}} > \gamma$, Lemma 6.14 holds true and the correctness of the limit statement of $F_n(k,\mathsf{B})$ for $n\to\infty$ follows with the exact same argumentation as in the proof of Theorem 6.10(a), which is why we omit it here.

For the other assertion, let $\varepsilon > 0$ and $N\in\mathbb{N}$. Then

$$\begin{aligned}
\#\,\mathbb{G}_n^*(\mathsf{B}) &= \sum_{k=0}^{n-1} \sum_{v\in\mathbb{G}_k^*(\mathsf{A},\mathsf{B})} \#\mathbb{G}_{n-k-1}^*(\mathsf{B},v) \\
&\geq \sum_{k=0}^{n-1} \sum_{v\in\{u\in\mathbb{G}_k^*(\mathsf{A},\mathsf{B})|Z_u\geq z\}} \#\mathbb{G}_{n-k-1}^*(\mathsf{B},v) \\
&\geq \sum_{v\in\{u\in\mathbb{G}_{n-1-m}^*(\mathsf{A},\mathsf{B})|Z_u\geq z\}} \#\mathbb{G}_m^*(\mathsf{B},v) \quad \text{a.s.}
\end{aligned}$$

for all $n > m \geq 1$ and $z\in\mathbb{N}$, hence

$$\begin{aligned}
\mathbb{P}^*\left(\frac{\#\mathbb{G}_n^*(\mathsf{A})}{\#\mathbb{G}_n^*} > \frac{1}{N+1}\right) &= \mathbb{P}^*\left(N\,\#\mathbb{G}_n^*(\mathsf{A}) > \#\mathbb{G}_n^*(\mathsf{B})\right) \\
&\leq \mathbb{P}^*\left(N\,\#\mathbb{G}_n^*(\mathsf{A}) > \sum_{v\in\{u\in\mathbb{G}_{n-1-m}^*(\mathsf{A},\mathsf{B})|Z_u\geq z\}} \#\mathbb{G}_m^*(\mathsf{B},v)\right).
\end{aligned} \tag{6.18}$$

Fix m so large that

$$2^m(1-\varepsilon) > \frac{4N}{\beta}.$$

Then, since

$$\lim_{z\to\infty} \mathbb{P}_{\mathsf{B},z}(\#\mathbb{G}_m^* = 2^m) = 1,$$

there exists a $z_0\in\mathbb{N}$ such that

$$\mathbb{P}_{\mathsf{B},z}(\#\mathbb{G}_m^* = 2^m) \geq 1-\varepsilon$$

and therefore

$$\mathbb{E}_{\mathsf{B},z}\#\mathbb{G}_m^* \;\geq\; (1-\varepsilon)2^m \;>\; \frac{4N}{\beta} \tag{6.19}$$

for all $z \geq z_0$. Moreover, Lemma 6.15 yields

$$\frac{\#\{v \in \mathbb{G}_n^*(\mathsf{A},\mathsf{B}) : Z_v \geq z_0\}}{\#\,\mathbb{G}_n^*(\mathsf{A})} \xrightarrow{\mathbb{P}^*} \beta$$

and thereupon

$$\mathbb{P}^*\left(\frac{\#\{v \in \mathbb{G}_n^*(\mathsf{A},\mathsf{B}) : Z_v \geq z_0\}}{\#\,\mathbb{G}_n^*(\mathsf{A})} \geq \frac{\beta}{2}\right) \geq 1-\varepsilon \tag{6.20}$$

for all $n \geq n_0$ and some $n_0 \in \mathbb{N}$. By combining (6.18) and (6.20), we now infer

$$\begin{aligned}
&\mathbb{P}^*\left(\frac{\#\mathbb{G}_n^*(\mathsf{A})}{\#\mathbb{G}_n^*} > \frac{1}{N+1}\right)\\
&\leq\; \mathbb{P}^*\left(N\,\#\mathbb{G}_n^*(\mathsf{A}) > \sum_{v\in\{u\in\mathbb{G}_{n-1-m}^*(\mathsf{A},\mathsf{B}):Z_u\geq z_0\}} \#\mathbb{G}_m^*(\mathsf{B},v)\right)\\
&\leq\; \mathbb{P}^*\left(\frac{2N}{\beta} > \frac{\sum_{v\in\{u\in\mathbb{G}_{n-1-m}^*(\mathsf{A},\mathsf{B}):Z_u\geq z_0\}}\#\mathbb{G}_m^*(\mathsf{B},v)}{\#\{u\in\mathbb{G}_{n-1-m}^*(\mathsf{A},\mathsf{B}) : Z_u \geq z_0\}}\right) + \varepsilon\\
&\leq\; \mathbb{P}^{n-m}\left(\frac{2N}{\beta} > \frac{\sum_{v\in\{u\in\mathbb{G}_{n-1-m}^*(\mathsf{A},\mathsf{B}):Z_u\geq z_0\}}\#\mathbb{G}_m^*(\mathsf{B},v)}{\#\{u\in\mathbb{G}_{n-1-m}^*(\mathsf{A},\mathsf{B}) : Z_u \geq z_0\}}\right)\frac{\mathbb{P}(\mathcal{Z}_{n-m}(\mathsf{A})>0)}{\mathbb{P}(\mathsf{Surv}(\mathsf{A}))} + \varepsilon\\
&\leq\; \mathbb{P}^{n-m}\left(\frac{2N}{\beta} > \frac{\sum_{i=1}^{\#\{u\in\mathbb{G}_{n-1-m}^*(\mathsf{A},\mathsf{B}):Z_u\geq z_0\}}\mathcal{G}_{i,m}(z_0)}{\#\{u\in\mathbb{G}_{n-1-m}^*(\mathsf{A},\mathsf{B}) : Z_u \geq z_0\}}\right)\frac{\mathbb{P}(\mathcal{Z}_{n-m}(\mathsf{A})>0)}{\mathbb{P}(\mathsf{Surv}(\mathsf{A}))} + \varepsilon
\end{aligned}$$

for all $n \geq n_0 + m$, where the $\mathcal{G}_{i,m}(z_0)$ are i.i.d. with the same law as $\#\{v \in \mathbb{G}_m^*(\mathsf{B}) : Z_\varnothing = z_0, \mathsf{T}_\varnothing = \mathsf{B}\}$. The law of large numbers, together with Lemma 6.15 and $\mathbb{P}^n \xrightarrow{d_{TV}} \mathbb{P}^*$, provides us with a $n_1 \geq n_0 + m$ such that

$$\mathbb{P}^{n-m}\left(\frac{\sum_{i=1}^{\#\{u\in\mathbb{G}_{n-1-m}^*(\mathsf{A},\mathsf{B}):Z_u\geq z\}}\mathcal{G}_{i,m}(z_0)}{\#\{u\in\mathbb{G}_{n-1-m}^*(\mathsf{A},\mathsf{B}) : Z_u \geq z\}} \geq \mathbb{E}\mathcal{G}_{i,m}(z_0)/2\right) \geq 1-\varepsilon$$

for all $n \geq n_1$. By combining this with (6.19), we can further estimate in the above inequality

$$\begin{aligned}
\mathbb{P}^*\left(\frac{\#\mathbb{G}_n^*(\mathsf{A})}{\#\mathbb{G}_n^*} > \frac{1}{N+1}\right) \;&\leq\; \left(\mathbb{P}^{n-m}\left(\frac{2N}{\beta} > \frac{\mathbb{E}\mathcal{G}_{i,m}(z_0)}{2}\right) + \varepsilon\right)\frac{\mathbb{P}(\mathcal{Z}_{n-m}(\mathsf{A})>0)}{\mathbb{P}(\mathsf{Surv}(\mathsf{A}))} + \varepsilon\\
&=\; \left(\frac{\mathbb{P}(\mathcal{Z}_{n-m}(\mathsf{A})>0)}{\mathbb{P}(\mathsf{Surv}(\mathsf{A}))} + 1\right)\varepsilon \;\xrightarrow{n\to\infty}\; 2\varepsilon.
\end{aligned}$$

This completes the proof. □

Remark 6.16. We briefly remark on the cases when (AsBP) or (SA6) is violated.

(a) Let $p_{\mathsf{AA}} > 0$ and $\mathbb{P}(X^{(0)}(\mathsf{A},\mathsf{AA}) = X^{(1)}(\mathsf{A},\mathsf{AA}) = X^{(0)}(\mathsf{A},\mathsf{AB}) = 1) = 1$. Then $\mathcal{Z}_n(\mathsf{A}) = \#\,\mathbb{G}_n^*(\mathsf{A})$ a.s. for all $n \in \mathbb{N}_0$ and $\#\,\mathbb{G}_n^*(\mathsf{A}) \to \infty$ on $\mathsf{Surv}(\mathsf{A})$ by Theorem 6.4. Moreover, $(\#\,\mathbb{G}_n^*(\mathsf{A}))_{n\geq 0}$ forms a standard GWP with finite reproduction variance, and consequently,

$(\#\,\mathbb{G}_n^*(\mathsf{A})/\nu^n)_{n\geq 0}$ is an L^2-bounded martingale (see e.g. [14, Theorem 2 in Chapter I.6]). As every type-A cell contains exactly one parasite, evidently $F_n(1,\mathsf{A}) = 1$ $\mathbb{P}^*$-a.s., and the assertions in Theorem 6.10 can be analogously proved by using the just mentioned properties.

(b) Let $p_{\mathsf{AA}} = 0$ and $\mathbb{P}(\mathcal{Z}_n(\mathsf{A}) \to \infty) > 0$, which particularly entails $p_{\mathsf{AB}} = 1$ as $\mathbb{P}(\mathsf{Surv}(\mathsf{A})) > 0$. Moreover, $F_n(k,\mathsf{A}) = \mathbb{1}_{\{\mathcal{Z}_n(\mathsf{A})=k\}} \to 0$ $\mathbb{P}^*$-a.s. for all $k \in \mathbb{N}$ and so Lemma 6.14 as well as Lemma 6.15 for $\beta = 1$ can be shown in this case with analogous arguments. The statement in the latter lemma can be even expanded to almost sure convergence. By just copying the proof of Theorem 6.11, the assertions given there follow for this case, whereby the first limit result can be expanded to almost sure convergence.

(c) Let (SA6) fail. In this case $\mathcal{Z}_n(\mathsf{A}) = 1$ $\mathbb{P}^*$-a.s and therefore $F_n(1,\mathsf{A}) = 1$ $\mathbb{P}^*$-a.s. for all $n \in \mathbb{N}_0$. Remark 6.7 yields

$$\frac{\#\,\mathbb{G}_n^*(\mathsf{A})}{\#\,\mathbb{G}_n^*} = \frac{1}{1+\#\,\mathbb{G}_n^*(\mathsf{B})} \begin{cases} \xrightarrow{a.s.} 0 \\ \xrightarrow{d} (1+G_\infty)^{-1} \end{cases} \qquad \text{if} \quad \mu_{\mathsf{B}} \begin{cases} > 1 \\ \leq 1, \end{cases}$$

where, as in Remark 6.7, the random variable G_∞ is finite if $\mu_{\mathsf{B}} < 1$, whereas infinite if $\mu_{\mathsf{B}} = 1$ and $\mathbb{E}_{\mathsf{B},1}\mathcal{Z}_1^2 < \infty$.

Following the short proof in Remark 6.7, it is not hard to see that

$$\#\{v \in \mathbb{G}_n^*(\mathsf{B}) : Z_v = k\} \xrightarrow{d} G_{\infty,k} \quad \text{as } n \to \infty$$

for each $k \in \mathbb{N}$ if $\mu_{\mathsf{B}} < 1$, where $G_{\infty,k}$ is an almost surely finite random variable. Thus, for each $k \in \mathbb{N}$

$$F_n(k,\mathsf{B}) \xrightarrow{d} \frac{G_{\infty,k}}{G_\infty} \qquad \text{as } n \to \infty$$

if $\mu_{\mathsf{B}} < 1$. If $\mu_{\mathsf{B}} \leq 1$ and $\mathbb{E}_{\mathsf{B},1}\mathcal{Z}_1^2 < \infty$ in the case when equality holds, Lemma 6.14 can be easily verified under the present conditions with similar arguments given there and by the usage of (6.4). Hence, Theorem 6.10(*a*) is still valid is this case and can be shown with the same arguments done in the theorem's proof.

Appendix A

Calculation of the variance

Here, we proof the exact formula for the variance of the process of parasites $(\mathcal{Z}_n)_{n\geq 0}$.

Lemma A.1. *Let* $\sigma^2 := \mathbb{V}ar\mathcal{Z}_1 < \infty$ *and* $\tau^2 := \nu\mathbb{E}g''_{\Lambda_0}(1)$. *Then*

$$\mathbb{V}ar\mathcal{Z}_n \;=\; \sigma^2\gamma^{n-1}\sum_{k=0}^{n-1}\gamma^k + c\tau^2\frac{\gamma^{2(n-1)}}{\tilde{\gamma}}\sum_{k=0}^{n-1}\left(\tilde{\gamma}\gamma^{-2}\right)^k\;\sum_{j=0}^{k-1}\left(\frac{\gamma}{\tilde{\gamma}}\right)^j$$

for $n \in \mathbb{N}$, *where*

$$\tilde{\gamma} \;:=\; \nu\,\mathbb{E}g'_{\Lambda_0}(1)^2 \;=\; \sum_{t=1}^{\infty}p_t\sum_{u=1}^{t}\mu_{u,t}^2 \quad \textit{and} \quad c := \mathbb{C}ov\left(\sum_{u=1}^{T_\varnothing}X_{1,\varnothing}^{(u,T_\varnothing)},\sum_{u=1}^{T_\varnothing}X_{2,\varnothing}^{(u,T_\varnothing)}\right).$$

In particular

$$\mathbb{V}ar\mathcal{Z}_n \;=\; \begin{cases} \sigma^2\gamma^{n-1}\frac{\gamma^n-1}{\gamma-1} + c\tau^2\frac{1}{\gamma-\tilde{\gamma}}\left(\frac{\gamma^{n-1}(\gamma^n-1)}{\gamma-1} - \frac{\gamma^{2n}-\tilde{\gamma}^n}{\gamma^2-\tilde{\gamma}}\right) & \textit{if } \gamma\neq 1,\ \tilde{\gamma}\neq 1, \gamma\neq\tilde{\gamma}\neq\gamma^2, \\ \sigma^2\gamma^{n-1}\frac{\gamma^n-1}{\gamma-1} + c\tau^2\frac{\gamma^{n-1}}{\gamma(1-\gamma)}\left(\frac{\gamma^n-1}{\gamma-1} - \gamma^{n-1}n\right) & \textit{if } \gamma\neq 1,\ \tilde{\gamma}\neq 1, \gamma^2=\tilde{\gamma}, \\ \sigma^2\gamma^{n-1}\frac{\gamma^n-1}{\gamma-1} + c\tau^2\frac{\gamma^n}{\gamma^2(\gamma-1)}\ \left(\frac{\gamma^n-1}{\gamma-1} - n\right) & \textit{if } \gamma\neq 1,\ \tilde{\gamma}\neq 1, \gamma=\tilde{\gamma}, \\ \sigma^2\gamma^{n-1}\frac{\gamma^n-1}{\gamma-1} + c\tau^2\frac{1}{\gamma-1}\left(\frac{\gamma^{n-1}(\gamma^n-1)}{\gamma-1} - \frac{\gamma^{2n}-1}{\gamma^2-1}\right) & \textit{if } \gamma\neq 1,\ \tilde{\gamma}=1, \\ \sigma^2 n + c\tau^2\frac{1}{\tilde{\gamma}-1}\left(\frac{\tilde{\gamma}^n-1}{\tilde{\gamma}-1} - n\right) & \textit{if } \gamma=1,\ \tilde{\gamma}\neq 1, \\ \sigma^2 n + c\tau^2\frac{n(n-1)}{2} & \textit{if } \gamma=1,\ \tilde{\gamma}=1. \end{cases}$$

Proof. First, we like to point out that $\tilde{\gamma}$ and τ^2 are finite since Jensen's inequality yields

$$\tilde{\gamma} \;\leq\; \sum_{t=1}^{\infty}p_t\sum_{u=1}^{t}\mathbb{E}\left(X^{(u,t)\,2}\right) \;=\; \nu\mathbb{E}\left(Z_1'^{2}\right) \;\leq\; \sum_{t=1}^{\infty}p_t\mathbb{E}\left(\sum_{u=1}^{t}X^{(u,t)}\right)^2 \;=\; \mathbb{E}\mathcal{Z}_1^2 \;<\; \infty.$$

Furthermore, by the Cauchy-Schwarz inequality

$$c \;\leq\; \mathbb{V}ar\left(\sum_{u=1}^{T_\varnothing}X_{1,\varnothing}^{(u,T_\varnothing)}\right)^{\frac{1}{2}}\mathbb{V}ar\left(\sum_{u=1}^{T_\varnothing}X_{2,\varnothing}^{(u,T_\varnothing)}\right)^{\frac{1}{2}} \;=\; \mathbb{V}ar\mathcal{Z}_1 < \infty$$

because $\sum_{u=1}^{T_\varnothing} X_{i,\varnothing}^{(u,T_\varnothing)}$ is distributed as $\mathcal{Z}_1$ for $i \in \{1,2\}$. Another appeal to Jensen's inequality provides

$$\begin{aligned}\mathbb{E}\left(\sum_{u=1}^{T_\varnothing} X_{1,\varnothing}^{(u,T_\varnothing)} \sum_{u=1}^{T_\varnothing} X_{2,\varnothing}^{(u,T_\varnothing)}\right) &= \sum_{t=1}^{\infty} p_t \sum_{1\le u,v\le t} \mu_{u,t}\mu_{v,t} \;=\; \sum_{t=1}^{\infty} p_t \left(\sum_{1\le u\le t} \mu_{u,t}\right)^2 \\ &\ge \left(\sum_{t=1}^{\infty} p_t \sum_{1\le u\le t} \mu_{u,t}\right)^2 \;=\; (\mathbb{E}\mathcal{Z}_1)^2\end{aligned}$$

and thus non-negativity of c.

Let $n \in \mathbb{N}$. To compute the stated variance we use the formula

$$\mathbb{V}ar X = \mathbb{E}\left(\mathbb{V}ar(X|\mathcal{G})\right) + \mathbb{V}ar\left(\mathbb{E}(X|\mathcal{G})\right)$$

for a random variable X and a σ-algebra $\mathcal{G}$. Since conditioned under $\mathcal{F}_n$ the subtrees rooted in cells of generation n are independent and depend only on the number of parasites in their ancestor cell, we infer

$$\begin{aligned}\mathbb{V}ar\mathcal{Z}_{n+1} &= \mathbb{E}\left(\mathbb{V}ar(\mathcal{Z}_{n+1}|\mathcal{F}_n)\right) + \mathbb{V}ar\left(\mathbb{E}(\mathcal{Z}_{n+1}|\mathcal{F}_n)\right) \\ &= \mathbb{E}\left(\sum_{v\in\mathbb{T}_n} \mathbb{V}ar\left(\sum_{i=1}^{Z_v}\sum_{u=1}^{T_v} X_{i,v}^{(u,T_v)} \;\middle|\; Z_v\right)\right) + \gamma^2 \mathbb{V}ar\left(\mathcal{Z}_n\right). \end{aligned} \tag{A.1}$$

Recalling that each $\sum_{u=1}^{T_v} X_{i,v}^{(u,T_v)}$, $i \in \mathbb{N}$, is identically distributed as $\mathcal{Z}_1$ with mean γ, we get for the conditioned variance

$$\begin{aligned}&\mathbb{V}ar\left(\sum_{i=1}^{Z_v}\sum_{u=1}^{T_v} X_{i,v}^{(u,T_v)} \;\middle|\; Z_v\right) \\ &= \mathbb{E}\left(\left(\sum_{i=1}^{Z_v}\sum_{u=1}^{T_v} X_{i,v}^{(u,T_v)} - \mathbb{E}\left(\sum_{i=1}^{Z_v}\sum_{u=1}^{T_v} X_{i,v}^{(u,T_v)}\middle| Z_v\right)\right)^2 \;\middle|\; Z_v\right) \\ &= \mathbb{E}\left(\left(\sum_{i=1}^{Z_v}\left(\sum_{u=1}^{T_v} X_{i,v}^{(u,T_v)} - \gamma\right)\right)^2 \;\middle|\; Z_v\right) \\ &= \sum_{i=1}^{Z_v} \mathbb{V}ar\left(\sum_{u=1}^{T_v} X_{i,v}^{(u,T_v)}\right) + Z_v(Z_v-1)\mathbb{E}\left(\left(\sum_{u=1}^{T_v} X_{1,v}^{(u,T_v)} - \gamma\right)\left(\sum_{u=1}^{T_v} X_{2,v}^{(u,T_v)} - \gamma\right)\right) \\ &= Z_v\sigma^2 + Z_v(Z_v-1)c.\end{aligned}$$

By plugging this equation into (A.1), we establish the recursive formula

$$\begin{aligned}\mathbb{V}ar\mathcal{Z}_{n+1} &= \gamma^2\mathbb{V}ar\mathcal{Z}_n + \sigma^2\mathbb{E}\mathcal{Z}_n + c\left(\mathbb{E}\left(\sum_{v\in\mathbb{T}_n} Z_v^2 - \mathcal{Z}_n\right)\right) \\ &= \gamma^2\mathbb{V}ar\mathcal{Z}_n + \sigma^2\gamma^n + c\nu^n\left(\mathbb{E}Z_n'^2 - \mathbb{E}Z_n'\right) \\ &= \gamma^2\mathbb{V}ar\mathcal{Z}_n + \sigma^2\gamma^n + c\nu^n f_n''(1)\end{aligned}$$

with $f_n(1)$ being the generating function of Z'_n (see Subsection 1.2.1) and thus via iteration

$$\mathbb{V}ar\mathcal{Z}_{n+1} \;=\; \sigma^2\gamma^n\sum_{k=0}^{n}\gamma^k + c\sum_{k=0}^{n}\nu^k\gamma^{2(n-k)}f''_k(1), \tag{A.2}$$

while $f''_0(1) \equiv 0$. Consider the recursive representation for the second derivative of the generating function of a branching process in varying environments established by Fearn in [37] and use the i.i.d. property of Λ to obtain for each $k \in \mathbb{N}$

$$\begin{aligned} f''_k(1) \;&=\; \mathbb{E}\left(f''_k(1|\Lambda)\right) \;=\; \mathbb{E}\Big(f''_{k-1}(1|\Lambda)g'_{\Lambda_{k-1}}(1)^2 + f'_{k-1}(1|\Lambda)g''_{\Lambda_{k-1}}(1)\Big) \\ &=\; f''_{k-1}(1)\mathbb{E}g'_{\Lambda_0}(1)^2 + f'_{k-1}(1)\mathbb{E}g''_{\Lambda_0}(1) \\ &=\; f''_{k-1}(1)\frac{\tilde{\gamma}}{\nu} + \left(\frac{\gamma}{\nu}\right)^{k-1}\frac{\tau^2}{\nu} \\ &=\; \ldots \;=\; \frac{\tau^2}{\nu^k}\sum_{j=0}^{k-1}\tilde{\gamma}^{k-1-j}\gamma^j. \end{aligned}$$

Combined with (A.2), we get

$$\begin{aligned} \mathbb{V}ar\mathcal{Z}_{n+1} \;&=\; \sigma^2\gamma^n\sum_{k=0}^{n}\gamma^k + c\tau^2\sum_{k=0}^{n}\gamma^{2(n-k)}\sum_{j=0}^{k-1}\tilde{\gamma}^{k-j-1}\gamma^j \\ &=\; \sigma^2\gamma^n\sum_{k=0}^{n}\gamma^k + c\tau^2\frac{\gamma^{2n}}{\tilde{\gamma}}\sum_{k=0}^{n}\left(\tilde{\gamma}\gamma^{-2}\right)^k\sum_{j=0}^{k-1}\left(\frac{\gamma}{\tilde{\gamma}}\right)^j \end{aligned} \tag{A.3}$$

and thus the formula of the variance.

Based on this equation, we consider six different cases.

Case $\gamma = \tilde{\gamma} = 1$:

$$\mathbb{V}ar\mathcal{Z}_n \;=\; \sigma^2 n + c\tau^2\sum_{k=0}^{n-1}k \;=\; \sigma^2 n + c\tau^2\frac{n(n-1)}{2}.$$

Case $\gamma = 1, \tilde{\gamma} \neq 1$:

$$\begin{aligned} \mathbb{V}ar\mathcal{Z}_n \;&=\; \sigma^2 n + c\tau^2\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}\tilde{\gamma}^{k-1-j} \;=\; \sigma^2 n + c\tau^2\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}\tilde{\gamma}^j \\ &=\; \sigma^2 n + c\tau^2\sum_{k=0}^{n-1}\frac{\tilde{\gamma}^k - 1}{\tilde{\gamma} - 1} \;=\; \sigma^2 n + \frac{c\tau^2}{\tilde{\gamma} - 1}\left(\frac{\tilde{\gamma}^n - 1}{\tilde{\gamma} - 1} - n\right). \end{aligned}$$

Case $\gamma \neq 1, \tilde{\gamma} = 1$:

$$\begin{aligned} \mathbb{V}ar\mathcal{Z}_n \;&=\; \sigma^2\gamma^{n-1}\frac{\gamma^n - 1}{\gamma - 1} + c\tau^2\gamma^{2(n-1)}\sum_{k=0}^{n-1}\gamma^{-2k}\sum_{j=0}^{k-1}\gamma^j \\ &=\; \sigma^2\gamma^{n-1}\frac{\gamma^n - 1}{\gamma - 1} + c\tau^2\gamma^{2(n-1)}\sum_{k=0}^{n-1}\gamma^{-2k}\frac{\gamma^k - 1}{\gamma - 1} \end{aligned}$$

$$\begin{aligned}
&= \sigma^2\gamma^{n-1}\frac{\gamma^n-1}{\gamma-1}+c\tau^2\frac{\gamma^{2(n-1)}}{\gamma-1}\left(\sum_{k=0}^{n-1}\gamma^{-k}-\sum_{k=0}^{n-1}\gamma^{-2k}\right)\\
&= \sigma^2\gamma^{n-1}\frac{\gamma^n-1}{\gamma-1}+c\tau^2\frac{1}{\gamma-1}\left(\gamma^{n-1}\frac{\gamma^n-1}{\gamma-1}-\frac{\gamma^{2n}-1}{\gamma^2-1}\right).
\end{aligned}$$

CASE $\gamma\neq 1, \tilde{\gamma}\neq 1, \gamma=\tilde{\gamma}$:

$$\begin{aligned}
\mathbb{V}ar\mathcal{Z}_n &= \sigma^2\gamma^{n-1}\frac{\gamma^n-1}{\gamma-1}+c\tau^2\frac{\gamma^{2(n-1)}}{\gamma}\sum_{k=0}^{n-1}k\gamma^{-k}\\
&= \sigma^2\gamma^{n-1}\frac{\gamma^n-1}{\gamma-1}+c\tau^2\frac{\gamma^{2n}}{\gamma^2}\frac{d}{d\gamma}\left(-\sum_{k=0}^{n-1}\gamma^{-k}\right)\\
&= \sigma^2\gamma^{n-1}\frac{\gamma^n-1}{\gamma-1}+c\tau^2\frac{\gamma^{2n}}{\gamma^2}\frac{d}{d\gamma}\frac{1-\gamma^{-n}}{\gamma^{-1}-1}\\
&= \sigma^2\gamma^{n-1}\frac{\gamma^n-1}{\gamma-1}+c\tau^2\frac{\gamma^{2n}}{\gamma^2}\;\frac{n\gamma^{-n-1}(\gamma^{-1}-1)+(1-\gamma^{-n})\gamma^{-2}}{(\gamma^{-1}-1)^2}\\
&= \sigma^2\gamma^{n-1}\frac{\gamma^n-1}{\gamma-1}+c\tau^2\frac{\gamma^n}{\gamma^2(\gamma-1)}\;\left(\frac{\gamma^n-1}{\gamma-1}-n\right).
\end{aligned}$$

CASE $\gamma\neq 1, \tilde{\gamma}\neq 1, \gamma\neq\tilde{\gamma}, \gamma^2=\tilde{\gamma}$:

$$\begin{aligned}
\mathbb{V}ar\mathcal{Z}_n &= \sigma^2\gamma^{n-1}\frac{\gamma^n-1}{\gamma-1}+c\tau^2\frac{\gamma^{2(n-1)}}{\gamma^2}\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}\gamma^{-j}\\
&= \sigma^2\gamma^{n-1}\frac{\gamma^n-1}{\gamma-1}+c\tau^2\frac{\gamma^{2(n-1)}}{\gamma(1-\gamma)}\sum_{k=0}^{n-1}(\gamma^{-k}-1)\\
&= \sigma^2\gamma^{n-1}\frac{\gamma^n-1}{\gamma-1}+c\tau^2\frac{\gamma^{2(n-1)}}{\gamma(1-\gamma)}\left(\frac{\gamma^{-n}-1}{\gamma^{-1}-1}-n\right)\\
&= \sigma^2\gamma^{n-1}\frac{\gamma^n-1}{\gamma-1}+c\tau^2\frac{\gamma^{n-1}}{\gamma(1-\gamma)}\left(\frac{\gamma^n-1}{\gamma-1}-\gamma^{n-1}n\right).
\end{aligned}$$

CASE $\gamma\neq 1, \tilde{\gamma}\neq 1, \gamma\neq\tilde{\gamma}, \gamma^2\neq\tilde{\gamma}$:

$$\begin{aligned}
\mathbb{V}ar\mathcal{Z}_n &= \sigma^2\gamma^{n-1}\frac{\gamma^n-1}{\gamma-1}+c\tau^2\frac{\gamma^{2(n-1)}}{\tilde{\gamma}}\sum_{k=0}^{n-1}(\tilde{\gamma}\gamma^{-2})^k\frac{\left(\frac{\gamma}{\tilde{\gamma}}\right)^k-1}{\frac{\gamma}{\tilde{\gamma}}-1}\\
&= \sigma^2\gamma^{n-1}\frac{\gamma^n-1}{\gamma-1}+c\tau^2\frac{\gamma^{2(n-1)}}{\gamma-\tilde{\gamma}}\sum_{k=0}^{n-1}\left(\gamma^{-k}-\left(\frac{\tilde{\gamma}}{\gamma^2}\right)^k\right)\\
&= \sigma^2\gamma^{n-1}\frac{\gamma^n-1}{\gamma-1}+c\tau^2\frac{\gamma^{2(n-1)}}{\gamma-\tilde{\gamma}}\left(\frac{1-\gamma^{-n}}{1-\gamma^{-1}}-\frac{(\frac{\tilde{\gamma}}{\gamma^2})^n-1}{\frac{\tilde{\gamma}}{\gamma^2}-1}\right)\\
&= \sigma^2\gamma^{n-1}\frac{\gamma^n-1}{\gamma-1}+c\tau^2\frac{1}{\gamma-\tilde{\gamma}}\left(\gamma^{n-1}\frac{\gamma^n-1}{\gamma-1}-\frac{\tilde{\gamma}^n-\gamma^{2n}}{\tilde{\gamma}-\gamma^2}\right).
\end{aligned}$$

□

Appendix B

A law of large numbers for stochastically bounded random variables

In this short section, we present a law of large numbers for a sequence of independent random variables which are not assumed to have second moments. But, instead of the latter, these random variables are stochastically bounded, despite a constant factor, by another random variable. This guarantees that the sum is not dominated by only finite summands.

Theorem B.1. *Let* $(X_n)_{n\geq 1}$ *be independent random variables, and* X *an integrable random variable such that*

$$\sup_{n\geq 1} \mathbb{P}(X_n^+ > k) \ \leq \ c\mathbb{P}(X^+ > k) \quad \text{and} \quad \sup_{n\geq 1} \mathbb{P}(X_n^- > k) \ \leq \ c\mathbb{P}(X^- > k)$$

for all $k \in \mathbb{N}_0$ *and a finite constant* $c > 0$*. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (X_i - \mathbb{E}X_i) \ = \ 0 \quad a.s.$$

If additionally $\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X$*, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i \ = \ \mathbb{E}X \quad a.s.$$

Proof. First, note that X_n is integrable for each $n \in \mathbb{N}$ by our assumptions, and we set $S_n := \sum_{i=1}^{n}(X_i - \mathbb{E}X_i)$ for $n \in \mathbb{N}$. To prove the assertion, we use a truncation argument analog to the one in the proof of Etemadi for the law of large numbers (see e.g. [36, Theorem 2.4.1]). For $n \in \mathbb{N}$, define

$$Y_n \ := \ X_n \mathbb{1}_{\{|X_n| \leq n\}}$$

and set $T_n := \sum_{i=1}^{n}(Y_i - \mathbb{E}Y_i)$. Observe that

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n \neq Y_n) \ = \ \sum_{n=1}^{\infty} \mathbb{P}(|X_n| > n) \ \leq \ c \sum_{n=1}^{\infty} \mathbb{P}(|X| > n) \ \leq \ c\mathbb{E}|X| \ < \ \infty,$$

and thus the Borel-Cantelli lemma provides

$$\mathbb{P}(X_n \neq Y_n \ \infty\text{-often}) = 0.$$

Furthermore, we estimate for the difference of the means

$$|\mathbb{E}(X_n - Y_n)| \ \leq \ \mathbb{E}|X_n - Y_n| \ = \ \mathbb{E}|X_n| \, \mathbb{1}_{\{|X_n|>n\}} \ \leq \ c\mathbb{E}|X| \, \mathbb{1}_{\{|X|>n\}},$$

and thus we infer from the integrability of X

$$\begin{aligned} \frac{1}{n}\left|\sum_{i=1}^{n} \mathbb{E}(X_i - Y_i)\right| \ &\leq \ \frac{c}{n}\sum_{i=1}^{n} \mathbb{E}|X| \, \mathbb{1}_{\{|X|>i\}} \ \leq \ \frac{c}{n}\sum_{i=1}^{m} \mathbb{E}|X| \, \mathbb{1}_{\{|X|>i\}} + c\frac{n-m}{n}\mathbb{E}|X| \, \mathbb{1}_{\{|X|>m\}} \\ &\to \ c\mathbb{E}|X| \, \mathbb{1}_{\{|X|>m\}} \qquad \text{as } n \to \infty \\ &\to \ 0 \qquad \text{as } m \to \infty. \end{aligned}$$

This yields

$$\lim_{n\to\infty} \frac{1}{n}|S_n - T_n| \ = \ 0 \quad \text{a.s.}$$

Hence, it is enough to show the law of large numbers for the sequence $(Y_n)_{n\geq 0}$. To see that, first observe for each $n \in \mathbb{N}$

$$\mathbb{E}Y_n^2 \ = \ \mathbb{E}X_n^2 \, \mathbb{1}_{\{X_n \leq n\}} \ = \ \int_0^n 2t\mathbb{P}(X_n > t)dt \ \leq \ c\int_0^n 2t\mathbb{P}(X > t)dt \ = \ c\mathbb{E}X^2 \, \mathbb{1}_{\{X\leq n\}},$$

which together with [36, Lemma 2.4.3] entails

$$\sum_{n\geq 1} \frac{\mathbb{E}Y_n^2}{n^2} \ \leq \ c\sum_{n\geq 1} \frac{\mathbb{E}X^2 \, \mathbb{1}_{\{X\leq n\}}}{n^2} \ \leq \ 4c\mathbb{E}X \ < \ \infty.$$

Hence, Kolmogorov's criterion for the strong law of large numbers is satisfied, and the assertion follows (see e.g. [36, Exercise 2.5.4]). □

List of Abbreviations

ABPRE	associated branching process in random environment
ABPREI	associated branching process in random environment with immigration
BPRE	branching process in random environment
BPREI	branching process in random environment with immigration
BwBP	branching within branching process
GWP	Galton-Watson process
GWT	Galton-Watson tree

a.s.	almost surely
i.i.d.	independent and identically distributed
w.o.l.g.	without loss of generality
w.p.p.	with positive probability

List of Symbols

Symbols of the branching within branching process, Chapter 1-5

$\boldsymbol{BP}$	branching within branching process
$\boldsymbol{BP}_n$	n^{th} generation of the branching within branching process
$\boldsymbol{BPG}$	process denoting the number of contaminated cells and the parasites they contain
$\boldsymbol{BPG}_n$	number of infected cells and the parasites they contain in generation n
$\boldsymbol{BT}$	branching within branching tree
$\boldsymbol{BT}_n$	branching within branching tree up to generation n
$\boldsymbol{BT}^{(v)}$	subtree rooted in v
Δ	$= (\Delta_n)_{n\geq 0}$, environmental sequence of the ABPREI
$g_{\Delta_n}(s)$	generating function giving the n^{th} reproduction law of the ABPREI
$g_{\Lambda_n}(s)$	generating function giving the n^{th} reproduction law of the ABPRE
Λ	$= (\Lambda_n)_{n\geq 0}$, environmental sequence of the ABPRE
$\mathbb{T}$	tree of (alive) cells
$\mathbb{T}_n$	set of (alive) cells in the n^{th} generation
$\mathbb{T}^*$	set of contaminated cells
$\mathbb{T}^*_n$	set of contaminated cells in the n^{th} generation
$\mathcal{T}_n$	number of (alive) cells in the n^{th} generation
$\mathcal{T}^*_n$	number of contaminated cells in the n^{th} generation
T_v	number of daughter cells of cell v
T_v	indicator if cell v is alive ($v \in \mathbb{T}$)
$\mathbb{V}$	infinite Ulam-Harris tree
$(V_n)_{n\geq 0}$	random cell line through $\mathbb{T}$
$(\hat{V}_n)_{n\geq 0}$	spinal cells in the size-biased BwBP
W_n	mean normalized number of parasites in generation n, $:= \gamma^{-n}\mathcal{Z}_n$
W	martingale limit of $(W_n)_{n\geq 0}$
$(X^{(1,t)}_{i,v}, \dots, X^{(t,t)}_{i,v})$	offspring numbers of the i^{th} parasite in cell v provided that v has t daughter cells
$X^{(u,t)}_{i,v}$	offspring number of the i^{th} parasite in cell v which goes into the u^{th} of t daughter cells
$X^{(\bullet,t)}_{i,v}$	$:= (X^{(1,t)}_{i,v}, \dots, X^{(t,t)}_{i,v})$

$\mathcal{Z}_n$	number of parasites in the n^{th} generation
Z'_n	number of individuals of the ABPRE in generation n
$\hat{Z}'_n$	number of individuals of the ABPREI in generation n
Z_v	number of parasites in cell v
$\hat{Z}_{\hat{V}_n}$	number of parasites in the n^{th} spinal cell

γ	mean number of offspring of a parasite, $:= \mathbb{E}\mathcal{Z}_1$
$\mu_{u,t}$	mean offspring number of a parasite which goes into the u^{th} of t daughter cells, $:= \mathbb{E}X^{(u,t)}$
ν	reproduction mean of a cell, $:= \mathbb{E}\mathcal{T}_1$

$(\mathcal{F}_n)_{n\geq 0}$	canonical filtration
$(\mathbb{S}, \mathcal{S})$	space of host-parasite trees
$\mathcal{S}_n$	sub-σ-algebra of $\mathcal{S}$ generated by the projections on the first n generations of the host-parasite trees
S	set of all possible root configurations
S^*	set of configurations of contaminated cells in a generation
S^*_0	$:= S^* \cup \{(0,0)\}$
S	state space of a cell, $:= \{0,1\} \times \mathbb{N}_0$

$\theta(\cdot)$	limit distribution of $\mathbb{P}^n(\mathcal{Z}_n \in \cdot)$
$\pi(\cdot)$	Yaglom-limit of ***BPG***
$(p_k)_{k\geq 0}$	reproduction law of a cell
$\mathbb{P}_{(t,z)}$	probability measure under which the process starts with t cells containing $z = (z_1, \ldots, z_t)$ parasites
$\mathbb{P}^*_{(t,z)}$	the same as before but conditioned upon Surv
$\mathbb{P}^n_{(t,z)}$	the same as before but conditioned upon $\{\mathcal{Z}_n > 0\}$
$\mathbb{P}_z, \mathbb{P}^*_z, \mathbb{P}^n_z$	the same probability measures as before for $(1, z)$

$\hat{a}$, $\hat{A}$	variable (object) a resp. random variable A in the size-biased BwBP
Ext / Surv	event of extinction/survival of parasites
$\bar{z}$	sum of all entries in a vector $z = (z_1, \ldots, z_t)$
$\varnothing$	root cell of $\mathbb{V}$

Symbols of the two-type host parasite model, Chapter 6

$\mathbb{G}_n$	set of cells in generation n
$\mathbb{G}_n(\mathsf{t})$	set of cells of type t in generation n
$\mathbb{G}^*_n$	set of contaminated cells in generation n

$\mathbb{G}^*_n(\mathsf{t})$	set of contaminated cells of type t in generation n
$g_{\Lambda_n}(s)$	generating function giving the n-th reproduction law of the process of a random A cell line
T_v	type of cell v
$\mathbb{V}_2$	binary cell tree
$(X^{(0)}(\mathsf{A},\mathsf{s}),X^{(1)}(\mathsf{A},\mathsf{s}))$	offspring numbers of an A-parasite with daughter cells of type $\mathsf{s}\in\{\mathsf{AA},\mathsf{AB},\mathsf{BB}\}$
$(X^{(0)}(\mathsf{B}),X^{(1)}(\mathsf{B}))$	offspring numbers of a B-parasite
$\mathcal{Z}_n$	number of parasites in generation n
$\mathcal{Z}_n(\mathsf{t})$	number of parasites in t-cells in generation n
$Z_{[n]}$	number of parasites in a random cell in generation n
$Z_n(\mathsf{A})$	number of parasites of a random A-cell in generation n
Z_v	number of parasites in cell v
γ	mean number of offspring of an A-parasite which goes in an A-cell, $:=\mathbb{E}_{\mathsf{A},1}\mathcal{Z}_1(\mathsf{A})$
$\mu_{i,\mathsf{A}}(\mathsf{s})$	mean offspring number of a A-parasite which goes in daughter cell $i\in\{0,1\}$ if daughter cells are of type $\mathsf{s}\in\{\mathsf{AA},\mathsf{AB},\mathsf{BB}\}$, $:=\mathbb{E}X^{(i)}(\mathsf{A},\mathsf{s})$
$\mu_{i,\mathsf{B}}$	mean offspring number of B-parasites which goes in daughter cell $i\in\{0,1\}$, $:=\mathbb{E}X^{(i)}(\mathsf{B})$
μ_{B}	reproduction mean of a parasite in a B-cell, $:=\mu_{0,\mathsf{B}}+\mu_{1,\mathsf{B}}$
ν	mean number of type-A daughter cells of an A-cell, $:=\mathbb{E}_{\mathsf{A},1}\#\,\mathbb{G}_1(\mathsf{A})$
p_{s}	probability that the daughter cell of an A-cell is of type $\mathsf{s}\in\{\mathsf{AA},\mathsf{AB},\mathsf{BB}\}$
p_0	probability that the 1^{st} daughter cell of an A-cell is of type A
p_1	probability that the 2^{nd} daughter cell of an A-cell is of type A
$\mathbb{P}_{\mathsf{t},z}$	probability measure under which the process starts with one t-cell containing z parasites
$\mathbb{P}^*_{\mathsf{t},z}$	the same as before but conditioned upon $\mathsf{Surv}(\mathsf{A})$
$\mathbb{P}^n_{\mathsf{t},z}$	the same as before but conditioned upon survival of A-parasites in generation n
$\mathsf{Ext}\,/\,\mathsf{Surv}$	event of extinction/survival of parasites
$\mathsf{Ext}(\mathsf{t})/\,\mathsf{Surv}(\mathsf{t})$	event of extinction/survival of type-t parasites

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