SM Exercises 1

To be handed in on 22.04.14

1. (25%) Evaluate the following structure constants of $SU(3)$: f^{123}, f^{126}, f^{147}, f^{148}, f^{156}, f^{157}, f^{246}, f^{257}, f^{345}, f^{346}, f^{367}, f^{456}, f^{458}, f^{678}. Take into account that the standard basis for the fundamental representation of $SU(3)$ is

$$t^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$t^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad t^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad t^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$t^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

2. (30%) This exercise shows how to construct all the irreducible representations of $SU(2)$. The generators of $SU(2)$ satisfy the commutation relation $[J_a, J_b] = i\varepsilon_{abc}J_c$, where ε_{abc} is the totally antisymmetric Levi-Civita symbol and $\varepsilon_{123} = 1$.

   - Define the raising and lowering operators as $J_\pm \equiv J_1 \pm iJ_2$. Obtain the commutation relations between $J_\pm$ and J_3 and write J^2 in terms of them.
   - Consider an eigenstate of J^2 and J_3 with eigenvalues λ and m: $J^2|\lambda, m\rangle = \lambda|\lambda, m\rangle$, $J_3|\lambda, m\rangle = m|\lambda, m\rangle$. Show that the states $J_\pm|\lambda, m\rangle$ are also eigenstates of J_3 but with eigenvalues $m \pm 1$ and the same λ. One can then write $J_\pm|\lambda, m\rangle = C_\pm(\lambda, m)|\lambda, m \pm 1\rangle$, where the $C_\pm(\lambda, m)$s are constants to be determined.
   - Explain why for a given λ the values of m are bounded by $\lambda - m^2 \geq 0$. Let j and j' be respectively the largest and the smallest value of m. Obtain the relation between j, j', λ. Justify why $2j$ must be an integer.
   - Find $C_\pm(\lambda, m)$ in terms of j, m.

   The states $|j, m\rangle$ with $m = j, j-1, \cdots -j$ form the basis of an $SU(2)$ irreducible representation, characterized by j which is either an integer or half-integer. Thus, the dimension of the representation is $2j + 1$.

3. (20%) Calculate the generators J_a for the $SU(2)$ representation whose basis is given by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(|1/2, 1/2\rangle + |1/2, -1/2\rangle \right) \tag{1}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(|1/2, 1/2\rangle - |1/2, -1/2\rangle \right). \tag{2}$$

4. (25%) Calculate the generators J_a for the $SU(2)$ representation whose basis is given by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |1, 1\rangle, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |1, 0\rangle, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = |1, -1\rangle. \tag{3}$$

Check that they indeed satisfy the commutation relations.