Thermodynamics of Lattice Gauge Theory

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Thermodynamics of gauge theory

Why do we study gauge theory on lattice?

The thermodynamics of color theory is:

- based on a theory with gauge symmetry over the non abelian group $SU(N_c)$,
- "asymptotic freedom", the gluons behave as a gas of free particles at high energy,
- "confinement" at low energy, $\alpha_s(q^2)$ increases to larger and larger values and a single isolated color charge cannot exist.

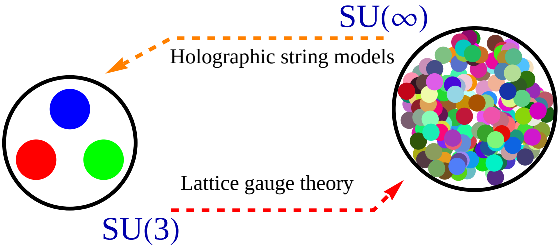
Failure of perturbative series

Due to the confinement and $\alpha_s \sim 1$, any thermodynamic quantity must be studied with non perturbative methods.

Thermodynamics of gauge theory
Why do we study gauge theory on lattice?

How does the quantum chromodynamic reach the Stefan-Boltzmann limit?

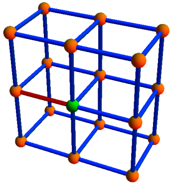
1. Recent results have shown that there isn't a gas of free gluons also when $T \sim 3T_c$.[Datta, Gupta: ArXiv:1006.0938, 12/2010]
2. Perturbative analytical previsions are difficult.
3. Non perturbative string models assume that $N_c = 3$ is "similar" to $N_c = \infty$; this hypothesis can be verified by *lattice gauge theories*.



Quantum theory on lattice

Lattice regularization and gauge invariance

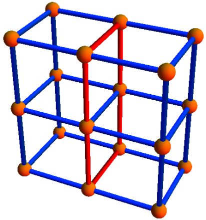
The lattice spacing $a \neq 0$ regularizes the gauge theory.



On lattice, bosonic fields are site variable; gauge fields are link variables:

- $\phi(x) \to \phi_i$
- $A_\mu(x) \to \exp\left(ig_s a A_\mu(x)\right) \equiv U_\mu(x)$

The Wilson Loop: $$W = \mathrm{Tr}\left(\prod_{x\in\mathcal{C}} U_\mu(x)\right)$$



is invariant under gauge transformations $\Omega(x)$ of the field $U_\mu(x)$:

$$U_\mu(x) \to \Omega(x) U_\mu(x) \Omega^\dagger(x+\mu)$$

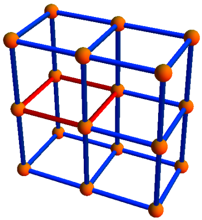
Wilson Action

Yang-Mills action on lattice

The action can be written in the simplest way using only the "plaquette", the smallest Wilson loop:

$$S_W = -\frac{\beta}{N} \sum_{x,\mu\neq\nu} \mathrm{Re}\mathrm{Tr}\left(U_\mu(x) U_\nu(x+\mu) U^\dagger_\mu(x+\nu) U_{-\nu}(x+\nu) \right)$$

defining the Wilson action [Physical Review D 10: 2445, 1974] .



When $a \to 0$, in the "naive" continuum limit, S_W tends to the usual Yang-Mills action:

$$S_W \to \left\{ -\frac{1}{4} \int d^4 x_E F^{\mu\nu} F_{\mu\nu} \right\} (1 + O(a^2))$$

if $\beta \equiv \frac{2N}{g_s^2}$.

Thermodynamics on Lattice

Study of a Yang-Mills theory at finite temperature

The partition function Z of a generic quantum system with Hamiltonian $\hat{H}$ is:

$$Z = \mathrm{Tr}\left\{ \exp\left(-\frac{\hat{H}}{T} \right) \right\} = \sum_i \exp\left(-\frac{E_i}{T} \right)$$

The partition function is equivalent to the Feynman functional, but with the temporal direction compactified:

$$Z = \int d\phi \exp\left(-\beta \int_0^{1/T} dt d^3x \mathcal{L}(\phi) \right)$$

Warning!

In the exponential β is related to g_s, the temperature is related to the length of the temporal direction $L_t = 1/T$!

Monte-Carlo Method
"Importance sampling"

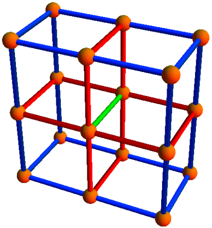
The expectation value of any observable $\mathcal{O}$:

$$\langle \mathcal{O} \rangle = \frac{\int \prod_{\mu,x} dU_\mu(x) \{\mathcal{O} \exp{(-S)}\}}{\int \prod_{\mu,x} dU_\mu(x) \{\exp{(-S)}\}}$$

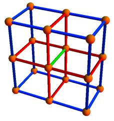
can be easily computed on a set of gauge configuration Φ_i generated with the Monte-Carlo sampling:

$$\langle \mathcal{O} \rangle = \frac{1}{N_{\mathrm{conf}}} \sum \mathcal{O}(\Phi_i)$$

and a Markov chain can be easily defined due to the locality of the pure gauge action.



[N. Metropolis et al., J. Chem. Phys. 21: 1087-1092 (1953);
M. Creutz, Physical Review D21: 2308 (1980);
N. Cabibbo, E. Marinari, Phys. Lett. B119: 387-390 (1982)]

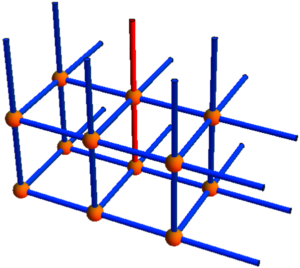


The Polyakov Loop

The order parameter of the transition of deconfinement

In a finite temperature lattice a Wilson line can be wrapped around the compactified temporal direction:

$$L(\vec{x}) = \mathrm{Tr}\left(\prod_{t=0}^{N_T-1} U_0(\vec{x}, t)\right)$$



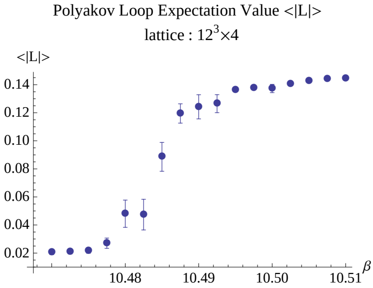
This operator is called Polyakov Loop [Phys.Lett B72: 477,1978] and it is the order parameter of the deconfinement transition.

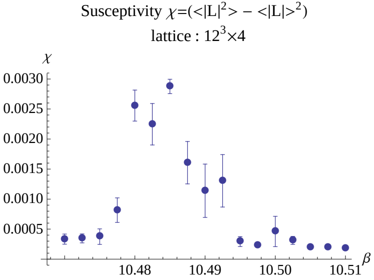
Confinement [L. G. Yaffe, B. Svetitsky, Physical Review D 26:963, 1982]

If $\langle |L| \rangle \neq 0$, then the lattice is in the deconfined phase, otherwise in those confined.

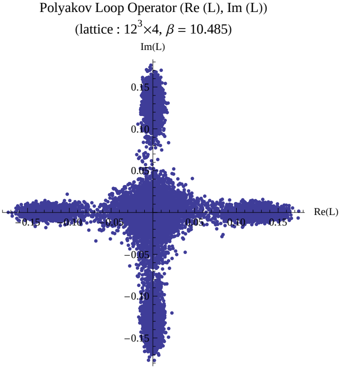
The deconfinement transition can be found on the peak of the susceptibility of the Polyakov Loop χ_L:

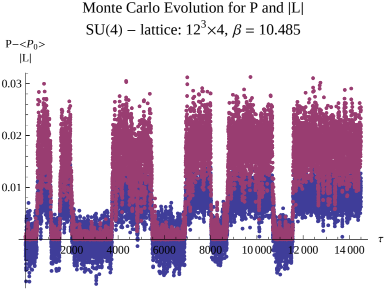
$$\frac{\chi_L}{N_s^3} = \left(\langle |L|^2 \rangle - \langle |L| \rangle^2\right)$$





The deconfinement transition for a gauge group $SU(4)$ in $4D$ is of the first order, coexistence of many phases:

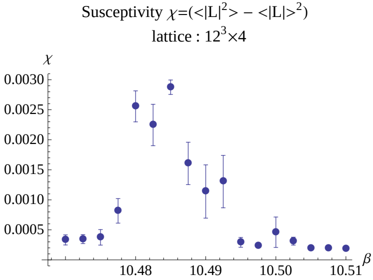


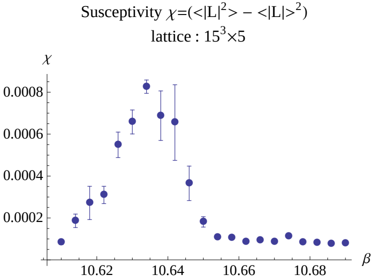


Latent heat differs from zero: $L_h/T_c^4 \sim 7.6$!

The continuum limit
Scale changes

When N_t increases, the peak of χ_L changes position:





Why? How is "T_C" defined?

The continuum limit

Extrapolate physical results from the lattice

The results of simulations with the Wilson action show that β_c is independent from N_s, but not clearly from N_t:

N_t	N_s	Number of "sweeps"	Critical value β_c
4	12	217500	10.486(5)
4	14	517500	10.4875(25)
4	16	379500	10.490(5)
5	15	144000	10.6352(3)
5	17	264000	10.6352(3)
6	18	170000	10.7816(33)
7	21	64000	10.92(2)

The continuum limit
Extrapolate physical results from the lattice

The temperature of the lattice is:

$$T = \frac{1}{aN_t}$$

The "scale fixing" is needed for knowing the lattice spacing a. Starting from the knowledge of the couples (N_t, β_c):

$$\begin{aligned} a(\beta_c) &= \frac{1}{T_c N_t} \\ T_c &\simeq 260 \text{ MeV} \end{aligned}$$

$a(\beta)$ is obtained extrapolating the global behavior with a fit.

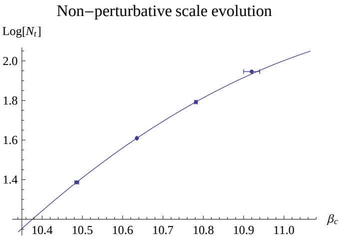
Trace anomaly [D.J. Gross, F. Wilczek, Phys. Rew. Lett. 30 (26): 1343-1346 (1973)]

The scale change when the gauge coupling change too, so the quantum fluctuations break the conformal symmetry.

The continuum limit

Extrapolate physical results from the lattice

The data are well fitted by an exponential ($\chi_R = 0.23$):



The fit gives the relation between T and β ($\bar{\beta} = \beta - 10.71$):

$$T = \frac{T_c}{N_t} \exp\left(c_0 + c_1 \bar{\beta} + c_2 \bar{\beta}^2\right)$$

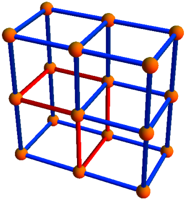
where $c_0 = 1.707397(2)$, $c_1 = 1.2491(4)$ e $c_2 = -0.81(1)$.

“Improved” actions

Reduce discretizzation errors

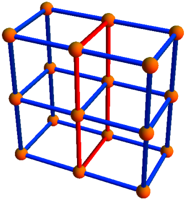
The Wilson action can be “improved” by adding irrelevant operators (loop larger than plaquette) for increasing the convergence to the continuum limit and the symmetries of operators.

The Symanzik action [Nucl. Phys. B226: 187 (1983)]:

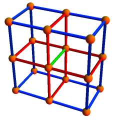


$$S_{SY} = -\frac{\beta}{N_c} \mathrm{ReTr} \sum_{\mu \neq \nu} \left(\frac{5}{3} U_{\mu\nu} - \frac{1}{12} R_{\mu\nu} \right)$$

$$\to \left\{ -\frac{1}{4} \int d^4 x_E F^{\mu\nu} F_{\mu\nu} \right\} (1 + O(a^4))$$

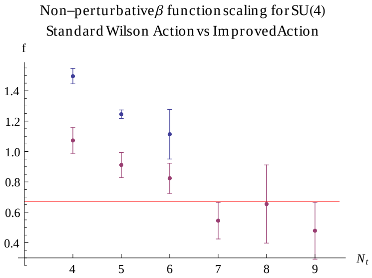


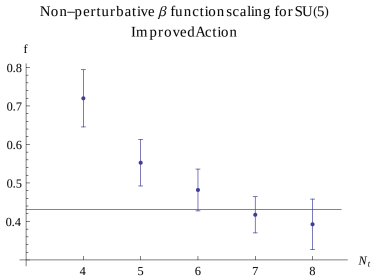
has relative discretization errors of order $O(a^4)$.



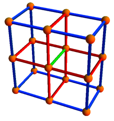
"Improved" actions
Non-perturbative scaling

The scaling for β_c with the improved Symanzik action is better and it reaches faster the perturbative value:



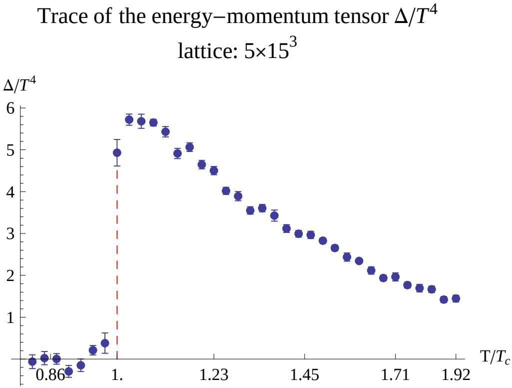


where $f(N_t) = \dfrac{\log(N_t+1) - \log(N_t)}{\beta_c(N_t+1) - \beta_c(N_t)} \overset{N_t \to \infty}{\longrightarrow} \dfrac{48\pi^2}{44 N_c^2}$

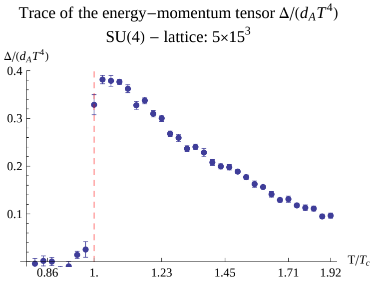


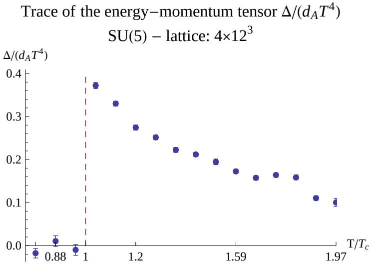
The trace of the energy-momentum tensor
The Stefan-Boltzmann limit

The trace of the energy-momentum tensor $\Delta = (\epsilon - 3p)/T^4$ has a peak near the critical point ($\simeq 1.04\,T_c$) and it slowly goes to zero:



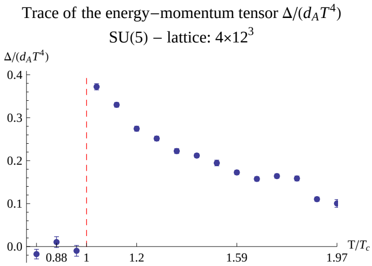
The trace of the energy-momentum tensor has small differences between $SU(4)$ and $SU(5)$:

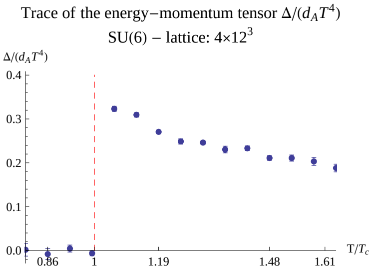




The trace of the energy-momentum tensor
The $N_c \to \infty$ limit

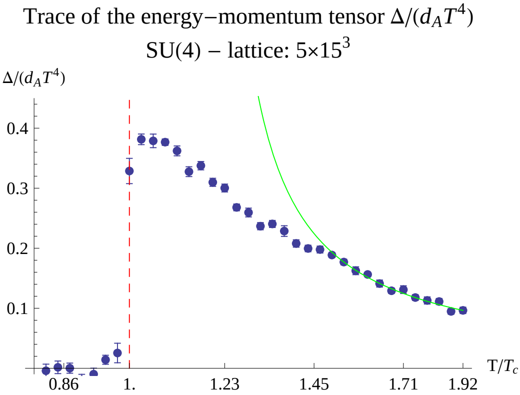
The trace of the energy-momentum tensor has small differences between $SU(5)$ and $SU(6)$:





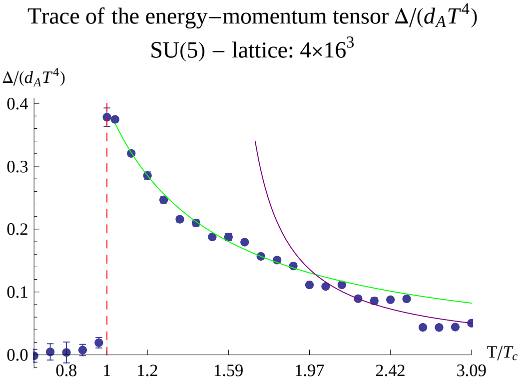
The limit $N_c \to \infty$ exists!

Δ has many deviation from perturbative estimations. Why?

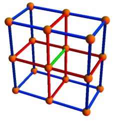


$\sim \log(T/T_c)$ fails when $T > T_c$.[J. O. Andersen et al., hep-ph:1106-0514 (2011)]

Δ has many deviation from perturbative estimations. Why?



T^2 contribution when $T > T_c$. [R. D. Pisarski, hep-ph:0612191 (2006)]



Conclusions

Thermodynamics of lattice gauge theory

1. Lattice simulations are an important non-perturbative tool
2. Lattice discretization errors can be “easily” reduced with improved actions
3. Answers:
   1. Phase transition of the same order when $N_c \geq 3$
   2. No differences in thermodynamic variables for $T > T_c$
   3. Small linear differences for $T \sim T_c$
   4. The limit $N_c \to 3^+$ exists!
4. Questions:
   1. How does a color theory reach the Stephan-Boltzmann limit?
   2. Non perturbative effects are still present also after the deconfinement transition
   3. Where does the related mass scale come from?