Non-perturbative relation between the bare and the RGI heavy quark mass in finite-volume two-flavour QCD

ALPHA Collaboration

M. Della Morte J. Heitger H. Meyer
H. Simma R. Sommer

Patrick Fritzsch*

*Westfälische Wilhelms-Universität Münster,
Institut für Theoretische Physik

Lattice 2007, Regensburg, Jul 30 - Aug 04, 2007

Towards an accurate determination of $\kappa_\mathrm{h}(z)$

Renormalization and O(a)-improvement program using Wilson fermions

Prerequisite: mass-independent renormalization scheme, SF $(m_\mathrm{l} \equiv 0)$

renormalization of

- all bare parameters like couplings and masses

$$g_\mathrm{R}^2 = Z_\mathrm{g}(\tilde{g}_0^2, a\mu)\tilde{g}_0^2 \,, \qquad \tilde{g}_0^2 = g_0^2(1 + b_\mathrm{g} a m_\mathrm{q})$$

$$m_\mathrm{R} = Z_\mathrm{m}(\tilde{g}_0^2, a\mu)\widetilde{m}_\mathrm{q} \,, \qquad \widetilde{m}_\mathrm{q} = m_\mathrm{q}(1 + b_\mathrm{m} a m_\mathrm{q})$$

- fields $\phi_\mathrm{R}(x) = Z_\phi(\tilde{g}_0^2, a\mu)(1 + b_\phi a m_\mathrm{q})\phi_\mathrm{I}(x)$

with $b_\mathrm{X} = b_\mathrm{X}(g_0^2)$ [quenched: $b_\mathrm{g} = 0$; $am_\mathrm{q} = 0$ in $N_\mathrm{f} = 2$]

main goal of this talk: determination of *renormalization constants* and *improvement coefficients* by methods used in [Guagnelli etal;Nucl.Phys B595(2001)44] and [Heitger,Wennekers;JHEP02(2004)064] in the relevant parameter region to perform numerical simulations at several, precisely fixed values of the RGI heavy quark mass

strategy

renormalization in O(a)-improved theory $(N_\mathrm{f}=2)$ at $\mu_0 = 1/L_0$

Connection of any renormalized heavy mass to RGI mass M via

$$M = \frac{M}{\overline{m}_\mathrm{h}(\mu_0)} \times \frac{\overline{m}_\mathrm{h}(\mu_0)}{m_\mathrm{h}^\mathrm{bare}} \times m_\mathrm{h}^\mathrm{bare} \qquad , \qquad \mu_0 = \frac{1}{L_0}$$

- 1st factor *non-perturbatively known in the continuum*

$$h(L_0) \equiv \frac{M}{\overline{m}_\mathrm{h}(\mu_0)} = \frac{\overline{m}_\mathrm{h}(\mu)}{\overline{m}_\mathrm{h}(\mu_0)} \frac{M}{\overline{m}_\mathrm{h}(\mu)} \,,$$

$$\frac{M}{\overline{m}_\mathrm{h}(\mu)} = \left[2b_0\bar{g}^2\right]^{-\frac{d_0}{2b_0}} \exp\left\{-\int_0^{\bar{g}} \mathrm{d}g \left[\frac{\tau(g)}{\beta(g)} - \frac{d_0}{b_0 g}\right]\right\}$$

  using step scaling of the coupling, σ [ALPHA,2004], and of the mass, σ_P [ALPHA,2005]

- determine $\overline{m}_\mathrm{h}(\mu_0)/m_\mathrm{h}^\mathrm{bare}$ in O(a) improved theory, $(N_\mathrm{f}=2)$, accurately in dependence of g_0

strategy

renormalization in O(a)-improved theory $(N_\mathrm{f}=2)$ at $\mu_0 = 1/L_0$

Connection of any renormalized heavy mass to RGI mass M via

$$M = \; h(L_0) \; \times \frac{\overline{m}_\mathrm{h}(\mu_0)}{m_\mathrm{h}^\mathrm{bare}} \times m_\mathrm{h}^\mathrm{bare} \qquad , \qquad \mu_0 = \frac{1}{L_0}$$

- 1st factor *non-perturbatively known in the continuum*

$$h(L_0) \equiv \frac{M}{\overline{m}_\mathrm{h}(\mu_0)} = \frac{\overline{m}_\mathrm{h}(\mu)}{\overline{m}_\mathrm{h}(\mu_0)} [2b_0\bar{g}^2]^{-\frac{d_0}{2b_0}} \exp\left\{-\int_0^{\bar{g}} \mathrm{d}g \left[\frac{\tau(g)}{\beta(g)} - \frac{d_0}{b_0 g}\right]\right\}$$

$$\frac{\overline{m}_\mathrm{h}(\mu)}{\overline{m}_\mathrm{h}(\mu_0)} = \frac{Z_\mathrm{P}(L_0)}{Z_\mathrm{P}(2^{-n}L_0)} = \left[\prod_{k=1}^{n} \sigma_\mathrm{P}(u_i)\right]^{-1}, \; u_i = \bar{g}^2(\mu_i)$$

  using step scaling of the coupling, σ [ALPHA,2004], and of the mass, σ_P [ALPHA,2005]

- determine $\overline{m}_\mathrm{h}(\mu_0)/m_\mathrm{h}^\mathrm{bare}$ in O(a) improved theory, $(N_\mathrm{f} = 2)$, accurately in dependence of g_0

ALPHA Collaboration

strategy

renormalization in O(a)-improved theory $(N_\mathrm{f}=2)$ at $\mu_0 = 1/L_0$

Connection of any renormalized heavy mass to RGI mass M via

$$M = h(L_0) \times \frac{\overline{m}_\mathrm{h}(\mu_0)}{m_\mathrm{h}^\mathrm{bare}} \times m_\mathrm{h}^\mathrm{bare} \quad , \quad \mu_0 = \frac{1}{L_0}$$

- 1st factor *non-perturbatively known in the continuum*

$$h(L_0) \equiv \frac{M}{\overline{m}_\mathrm{h}(\mu_0)} = \frac{\overline{m}_\mathrm{h}(\mu)}{\overline{m}_\mathrm{h}(\mu_0)} \left[2b_0\bar{g}^2\right]^{-\frac{d_0}{2b_0}} \exp\left\{-\int_0^{\bar{g}} \mathrm{d}g \left[\frac{\tau(g)}{\beta(g)} - \frac{d_0}{b_0 g}\right]\right\}$$

$$= 1.521(14)$$

  using step scaling of the coupling, σ [ALPHA,2004], and of the mass, σ_P [ALPHA,2005] $[N_\mathrm{f}=0:\ h(L_0)=1.544(14)]$

- determine $\overline{m}_\mathrm{h}(\mu_0)/m_\mathrm{h}^\mathrm{bare}$ in O(a) improved theory, $(N_\mathrm{f}=2)$, accurately in dependence of g_0

ALPHA Collaboration

strategy

renormalization in O(a)-improved theory $(N_\mathrm{f}=2)$ at $\mu_0 = 1/L_0$

Connection of any renormalized heavy mass to RGI mass M via

$$M = \; h(L_0) \; \times \frac{\overline{m}_\mathrm{h}(\mu_0)}{m_\mathrm{h}^\mathrm{bare}} \times m_\mathrm{h}^\mathrm{bare} + O(a^2) \;, \qquad \mu_0 = \frac{1}{L_0}$$

usually 2 versions of a renormalized mass:

(1) by current renormalization in PCAC relation, $\partial \mathrm{A} = 2 m_\mathrm{h} \mathrm{P}$,

(2) by definition of mass renormalization, $m_\mathrm{q} = m_0 - m_\mathrm{c}$,

$$\overline{m}_\mathrm{h}(\mu_0) = \frac{Z_\mathrm{A}(g_0)(1 + b_\mathrm{A}\, a m_\mathrm{q,h})}{Z_\mathrm{P}(g_0, L_0)(1 + b_\mathrm{P}\, a m_\mathrm{q,h})} \times m_\mathrm{h} + O(a^2) \qquad (1)$$

$$\overline{m}_\mathrm{h}(\mu_0) = Z_\mathrm{m}(g_0, L_0)(1 + b_\mathrm{m}\, a m_\mathrm{q,h}) m_\mathrm{q,h} \quad + O(a^2) \qquad (2)$$

strategy

renormalization in O(a)-improved theory $(N_{\rm f}=2)$ at $\mu_0 = 1/L_0$

Connection of any renormalized heavy mass to RGI mass M via

$$M = h(L_0) \times \frac{\overline{m}_{\rm h}(\mu_0)}{m_{\rm h}^{\rm bare}} \times m_{\rm h}^{\rm bare} + O(a^2) \ , \qquad \mu_0 = \frac{1}{L_0}$$

usually 2 versions of a renormalized mass:

(1) by current renormalization in PCAC relation, $\partial \mathrm{A} = 2 m_{\rm h} \mathrm{P}$,

(2) by definition of mass renormalization, $m_{\rm q} = m_0 - m_{\rm c}$,

$$\overline{m}_{\rm h}(\mu_0) = \frac{Z_{\rm A}(g_0)(1 + b_{\rm A}\, a m_{\rm q,h})}{Z_{\rm P}(g_0, L_0)(1 + b_{\rm P}\, a m_{\rm q,h})} \times m_{\rm h} + O(a^2) \qquad (1)$$

$$\overline{m}_{\rm h}(\mu_0) = Z_{\rm m}(g_0, L_0)(1 + b_{\rm m}\, a m_{\rm q,h}) m_{\rm q,h} + O(a^2) \qquad (2)$$

Problem: $b_{\rm A} - b_{\rm P}$, $b_{\rm m}$, $Z_{\rm P}$, $Z_{\rm m}$ currently not NP known in the relevant β region

strategy

renormalization in O(a)-improved theory $(N_\mathrm{f}=2)$ at $\mu_0 = 1/L_0$

Connection of any renormalized heavy mass to RGI mass M via

$$M = h(L_0) \times \frac{\overline{m}_\mathrm{h}(\mu_0)}{m_\mathrm{h}^\mathrm{bare}} \times m_\mathrm{h}^\mathrm{bare} + O(a^2) \,, \qquad \mu_0 = \frac{1}{L_0}$$

usually 2 versions of a renormalized mass:

(1) by current renormalization in PCAC relation, $\partial \mathrm{A} = 2 m_\mathrm{h} \mathrm{P}$,

(2) by definition of mass renormalization, $m_\mathrm{q} = m_0 - m_\mathrm{c}$,

$$\overline{m}_\mathrm{h}(\mu_0) = \frac{Z_\mathrm{A}(g_0)(1 + b_\mathrm{A} a m_{\mathrm{q,h}})}{Z_\mathrm{P}(g_0, L_0)(1 + b_\mathrm{P} a m_{\mathrm{q,h}})} \times m_\mathrm{h} + O(a^2) \qquad (1)$$

$$\overline{m}_\mathrm{h}(\mu_0) = Z_\mathrm{m}(g_0, L_0)(1 + b_\mathrm{m} a m_{\mathrm{q,h}}) m_{\mathrm{q,h}} + O(a^2) \qquad (2)$$

Strategy: eliminate Z_m in favor of $Z \equiv Z_\mathrm{m} Z_\mathrm{P} / Z_\mathrm{A}$

[$Z_\mathrm{A}(g_0^2)$ NP known from [Della Morte etal;JHEP07(2005)007]]

ALPHA Collaboration

definitions and conventions

- match both definitions to get dependency $m_{\mathrm{h}} = m_{\mathrm{h}}(m_{\mathrm{q,h}})$

$$m_{\mathrm{h}} = Z \times \frac{(1 + b_{\mathrm{P}}\, a m_{\mathrm{q,h}})(1 + b_{\mathrm{m}}\, a m_{\mathrm{q,h}})}{(1 + b_{\mathrm{A}}\, a m_{\mathrm{q,h}})} \times m_{\mathrm{q,h}} + O(a^2)$$

  with in the SF O(a) improved PCAC mass

$$m_{ij}\big(x_0; \{L/a, T/L, \theta\}\big) = \frac{\tilde{\partial}_0 f_{\mathrm{A}}^{ij}(x_0) + a c_{\mathrm{A}} \partial_0^* \partial_0 f_{\mathrm{P}}^{ij}(x_0)}{2 f_{\mathrm{P}}^{ij}(x_0)}$$

  defined through $\partial_\mu A_\mu = (m_i + m_j)P$,

$$m_1 = m_{\mathrm{l}}\,, \qquad m_2 = m_{\mathrm{h}}\,, \qquad m_3 = (m_1 + m_2)/2\,.$$

- use std. & improved derivatives:

$$\tilde{\partial}_0 \to \tilde{\partial}_0\left(1 - \tfrac{1}{6} a^2 \partial_0^* \partial_0\right)\,, \quad \partial_0^* \partial_0 \to \partial_0^* \partial_0\left(1 - \tfrac{1}{12} a^2 \partial_0^* \partial_0\right)$$

  to reduce errors to $O(g_0^2 a^2, a^4)$

mass ratios

How to extract improvement coefficients?

Definition

$$R_{\rm AP} = \frac{2(2m_{12}-m_{11}-m_{22})}{(m_{11}-m_{22})(am_{\rm q,1}-am_{\rm q,2})} = b_{\rm A}-b_{\rm P} + O(am_{\rm q,1}+am_{\rm q,2})$$

$$R_{\rm m} = \frac{4(m_{12}-m_{33})}{(m_{11}-m_{22})(am_{\rm q,1}-am_{\rm q,2})} = b_{\rm m} \qquad + O(am_{\rm q,1}+am_{\rm q,2})$$

$$R_Z = \frac{m_{11}-m_{22}}{m_{\rm q,1}-m_{\rm q,2}} + [b_{\rm A}-b_{\rm P}-b_{\rm m}](am_{11}+am_{22}) = Z + O(a^2)$$

with a priori local ratios $R_{\rm X} = R_{\rm X}(x_0)$ like masses

setting up the **constant physics condition** by

$$\bar{g}^2(L_1/2) = 2.989\,, \qquad m_{\rm l} = 0\,, \qquad z = LM = const.$$

defines any improvement coefficient exactly

ALPHA Collaboration

mass ratios

How to extract improvement coefficients?

Definition

$$R_{\rm AP} = \frac{2(2m_{12} - m_{11} - m_{22})}{(m_{11} - m_{22})(am_{\rm q,1} - am_{\rm q,2})} = b_{\rm A} - b_{\rm P} + O(am_{\rm q,1} + am_{\rm q,2})$$

$$R_{\rm m} = \frac{4(m_{12} - m_{33})}{(m_{11} - m_{22})(am_{\rm q,1} - am_{\rm q,2})} = b_{\rm m} \qquad + O(am_{\rm q,1} + am_{\rm q,2})$$

$$R_Z = \frac{m_{11} - m_{22}}{m_{\rm q,1} - m_{\rm q,2}} + [\ R_{\rm AP} - R_{\rm m}\ ](am_{11} + am_{22}) = Z + O(a^2)$$

with a priori local ratios $R_{\rm X} = R_{\rm X}(x_0)$ like masses

computed for 2 choices of **constant physics condition**, $z = \text{const.}$,

$$z = 0.5\,, \qquad\qquad z = 2.5$$

referred to as 'set1' and 'set2' respectively (to check that intrinsic O(a) ambiguities vanishes as $a \to 0$)

simulation parameters

small volume ($L = L_0$) lattice QCD

L_0/a	β	$\bar{g}^2$	κ_c	Z_P	am	$\tilde{\kappa}_c$
20	6.6380	2.989(43)	0.135163	0.5962(22)	+0.00091(10)	0.1351937
16	6.5113	2.989(28)	0.135441	0.6016(24)	−0.00056(16)	0.1354220
12	6.3158	2.989(28)	0.135793	0.6087(10)	−0.00062(17)	0.1357721
10	6.1906	2.989(21)	0.136016	0.6111(6)	−0.00055(8)	0.1359972

- Z_P computed on L_0^4 SF $N_f = 2$ config.s at $x_0 = T/2$
- PCAC mass $am(x_0)$ computed on $(L_0, T = \frac{3}{2} L_0)$ SF $N_f = 2$ configurations at $x_0 = T/2$
- $\tilde{\kappa}_c \equiv \kappa_c[L_1]$ estimated by shifting w.r.t.

$$\tilde{\kappa}_c = \kappa_c + am \cdot 2\kappa_c^2 / Z \,, \qquad Z(g_0^2) = 1 + 0.0905 \cdot g_0^2$$

setup of simulations

Algorithms etc.

- $O(a)$ improved Schrödinger Functional setup with $\mathcal{BF}=0$ and $\theta=0.5$
- HMC with multiple timescale integration with $T=3L_0/2$ for $L_0 \in \{10, 12, 16, 20\}$ and $T=L_1$ for $L_1=2L_0 \in \{20, 24, 32, 40\}$
- even-odd preconditioning
- mass preconditioning, [Hasenbusch,2001], with optimized $\rho = \{\langle\lambda_{\min}\rangle\langle\lambda_{\max}\rangle\}^{1/4}$

H.Meyer etal;'Exploring the HMC trajectory-length dependence of autocorrelation times in lattice QCD',Comput.Phys.Commun.176:91-97(2007)

setting up the condition of constant physics

small volume ($L = L_0$) lattice QCD

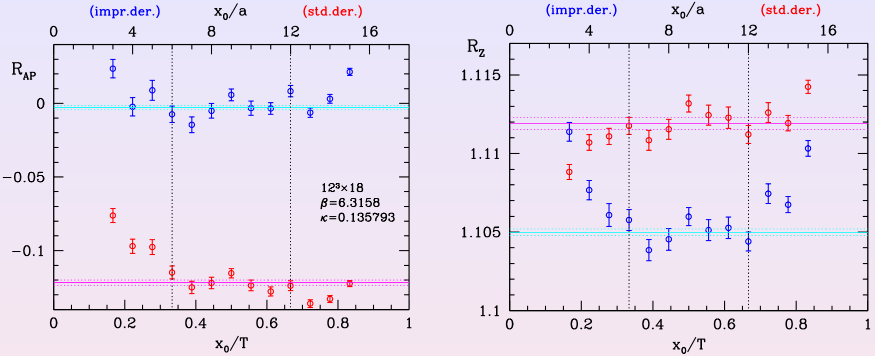
L_0/a	β	$\tilde{z} = Lm_2$		$z = Lm_2$		#meas*#rep
		set1	set2	set1	set2	
20	6.6380	0.5602(13)	2.6314(12)	0.5019(12)	2.5030(14)	20*8
16	6.5113	0.5398(23)	2.5983(21)	0.4949(13)	2.4955(13)	200*1
12	6.3158	0.5375(13)	2.5858(15)	0.50082(90)	2.50071(97)	74*8
10	6.1906	0.5379(10)	2.5832(11)	0.50045(61)	2.50096(71)	30*64

- plateau averaged dimensionless PCAC masses

$$z = N \cdot \sum_{x_0=T/3}^{2T/3} Lm(x_0)$$

- to estimate κ corresponding to $z = 0.5$, interpolate between $Lm_2(\kappa_2)$ and $Lm_3(\kappa_3)$ coming from a first computation with low statistic

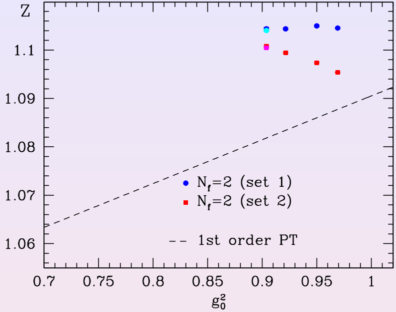
Results

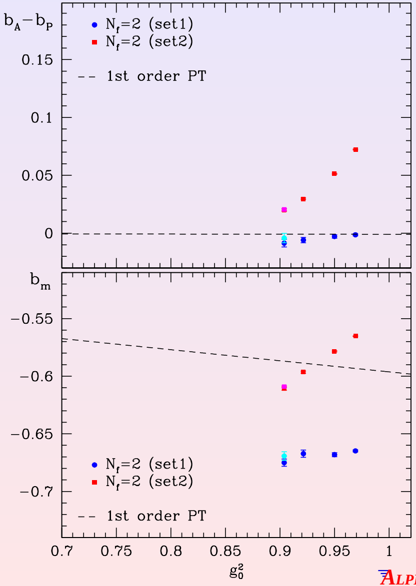


L_0/a	β	$b_A - b_P$	b_m	Z	$b_A - b_P - b_m$
20	6.6380	−0.0042(40)	−0.6693(37)	+1.10404(24)	+0.6651(32)
16	6.5113	−0.0059(23)	−0.6672(31)	+1.10438(22)	+0.6614(22)
12	6.3158	−0.0028(15)	−0.6681(17)	+1.10499(20)	+0.6653(13)
10	6.1906	−0.0006(9)	−0.6643(9)	+1.10455(13)	+0.6637(8)

Results $(N_\mathrm{f} = 2)$

... for Z, $b_\mathrm{A} - b_\mathrm{P}$, b_m





ren. cond. of $N_\mathrm{f} = 2$ simul. imposed:

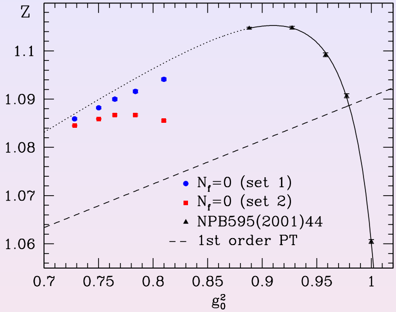
$$\bar{g}^2(L_0) = 2.989 \text{ fixed}, \quad L_0 = L_1/2$$

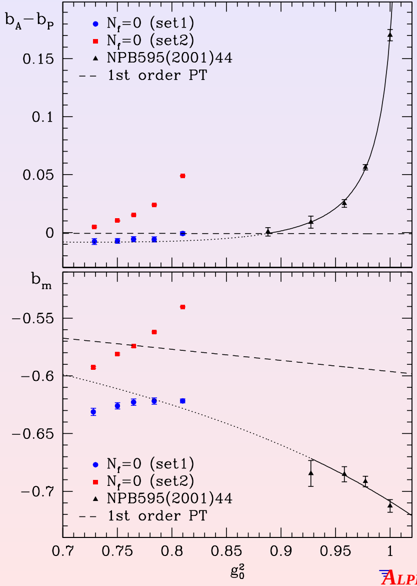
corresponding to matching volume

$$L_1 \approx 0.4 - 0.5\,\mathrm{fm}$$

Results ($N_{\mathrm{f}} = 0$)

... for Z, $b_{\mathrm{A}} - b_{\mathrm{P}}$, b_{m}





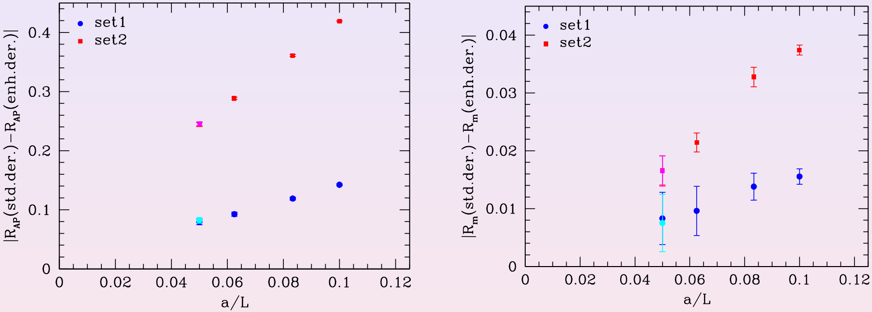
compared to renorm. condition with

$$\bar{g}^2 = 1.8811 \text{ fixed}$$

in quenched simulations

Results

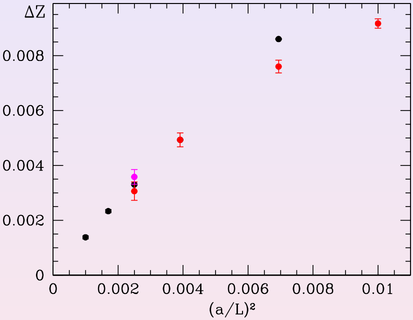
O(a) ambiguities in $b_{\rm A} - b_{\rm P}$ and $b_{\rm m}$

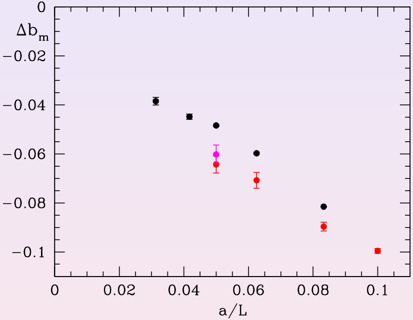


$$\Delta R_{\rm X}(a/L) = |R_{\rm X}(\text{std.der.}) - R_{\rm X}(\text{impr.der.})|$$

Results

O(a) ambiguities in Z and $b_{\rm m}$





$$\Delta Z(g_0^2) = Z(g_0^2)\Big|_{\rm set1} - Z(g_0^2)\Big|_{\rm set2} \qquad \Delta b_{\rm m}(g_0^2) = b_{\rm m}(g_0^2)\Big|_{\rm set1} - b_{\rm m}(g_0^2)\Big|_{\rm set2}$$

Results for $\kappa_{\rm h}$

for given values of

$$z = L_1 M = L_1 Z_{\rm M} \widetilde{m}_{\rm q,h} = L_1 h(L_0) Z_{\rm m} \widetilde{m}_{\rm q,h} \quad \Leftrightarrow \quad a\widetilde{m}_{\rm q,h} = \frac{a}{L_0} \frac{z}{Z_{\rm M}}$$

in the b-quark region use

$$am_{\rm q,h} = (\kappa_{\rm h}^{-1} - \kappa_{\rm c}^{-1})/2 \,, \qquad a\widetilde{m}_{\rm q,h} = am_{\rm q,h}(1 + b_{\rm m} am_{\rm q,h})$$

and invert for $\kappa_{\rm h}(z)$:

L_1/a	β	$\kappa_{\rm h}(z=10.0)$	$\kappa_{\rm h}(z=12.0)$	$\kappa_{\rm h}(z=14.0)$
20	6.1906	0.1208197	0.1209418	0.1210709
24	6.3158	0.1205529	0.1206457	0.1207425
32	6.5113	0.1201557	0.1202201	0.1202864
40	6.6380	0.1198801	0.1199283	0.1199776

error estimation not yet done, but will be of order $\sim 1\%$ because it is dominated by $h(L_0)$ (as in the quenched case)

Status & Outlook

so far so good

TODO:

- produce L_1^4 configurations (on the way) and ...
- compute heavy-light meson correlation functions in LQCD ...
- to do the matching in small volume as explained by J.Heitger

future plans:

- computation of $b_{\rm A} - b_{\rm P}$, $b_{\rm m}$, Z also for the β range relevant for $N_{\rm f} = 2$ simulations in physically large volume (needed e.g. for the computation of the mass of the charm quark)
- test of HQET in finite volume for $N_{\rm f} = 2$, similar to earlier quenched works of the ALPHA-Coll.

ALPHA Collaboration