Generating Functionals for Stochastic Differential Equations (Burgers Equation)

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Navier-Stokes-Equation 1/3

Classical, newtonian, incompressible fluid to be described by:

- Navier-Stokes-Equation:

$$\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha - \nu \nabla^2 v_\alpha + \frac{1}{\rho} \partial_\alpha p = 0 \qquad (1)$$

- Condition of Incompressibility:

$$\partial_\alpha v_\alpha = 0 \qquad (2)$$

We are looking for the statistical properties of the (turbulent) 3-dimensional velocity field v.

Navier-Stokes-Equation 2/3

Why is the NSE an open problem? The equation is

- nonlinear: The term $v_\beta \partial_\beta v_\alpha$ leads to chaotic and turbulent behaviour, huge mathematical problems!
- nonlocal: Incompressibility, which enters the NSE via the pressure, renders the equation nonlocal, huge technical problems!

$\longrightarrow$ In general, the full NSE is hard (impossible) to deal with.
$\longrightarrow$ Opposing opinions concerning the nature of turbulence
$\longrightarrow$ Need for simplifications, test systems

Navier-Stokes-Equation 3/3

Possible simplications:

- Concerning Nonlinearity: Look only at systems, in which the diffusion term is much larger than the convection term,
$$R = \frac{(v\nabla)v}{\nu\nabla^2 v} \propto \frac{LV}{\nu} << 1. \tag{3}$$
$\rightarrow$ laminar regime, very realistically describes slow flows with small disturbancies.
- Concerning Nonlocality: Drop the pressure term.
$\rightarrow$ Completely compressible fluid, Burgers Equation

Burgers Equation 1/3

$$\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha - \nu \nabla^2 v_\alpha = 0 \tag{4}$$

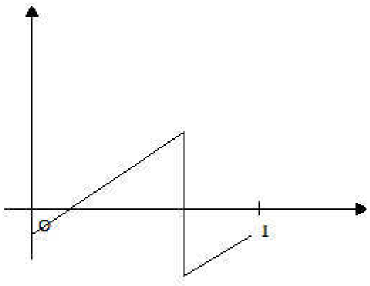
- Shock wave turbulence
- Not believed to be a good model for hydrodynamic turbulence, but can be seen as a toy model for testing methods!
- Related to KPZ-Equation (Surface Growth)

$$\partial_t \psi = \frac{1}{2} \left| \nabla \psi \right|^2 + \nu \nabla^2 \psi + F$$

- Also used in cosmology (Zel'dovich approximation), polymers,...

Burgers Equation 2/3

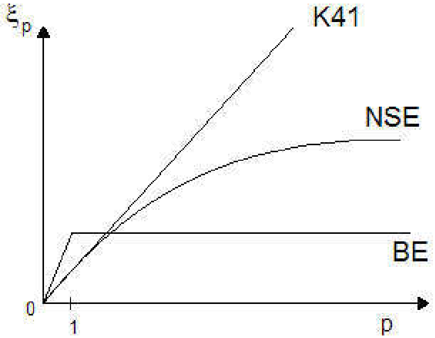
Solutions exist and are well known, in 1D for vanishing viscosity:



Kink solutions at arbitrary position and slope $\frac{1}{t}$ in periodic boundary conditions in x

Burgers Equation 3/3

Statistics are still intermittent (but in a very different way, compared to NSE):



Scaling Exponents of the structure functions

$$S_p(x) := \left\langle [(\vec{v}(\vec{r} + x\vec{l_0}) - \vec{v}(\vec{r})) * \vec{l_0}]^p \right\rangle = C_p x^{\xi_p} \qquad (5)$$

Stochastic Force

Due to viscosity, the system is nonconservative
$\longrightarrow$ Conversion of kinetic energy to heat
$\longrightarrow$ any field configuration reduces to a trivial one in the long time limit
Non-trivial, statistically static configurations only with an external source of energy
$\longrightarrow$ driving gaussian random force!

$$\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha - \nu \nabla^2 v_\alpha = f_\alpha \tag{6}$$

write that as:

$$\mathbf{N}_\alpha[v] = f_\alpha . \tag{7}$$

Sum of States 1/4

Counting the solutions of the stochastically driven Burgers Equation, we get the corresponding sum of states Z:

$$Z \propto \left\langle \int Dv\delta(v - \mathbf{N}^{-1}[f]) \right\rangle_f \tag{8}$$

$$\propto \int Df\, Dv\, \delta(v - \mathbf{N}^{-1}[f]) e^{-\frac{1}{2}\int fAf} \tag{9}$$

$$\propto \int Df\, Dv\, \delta(\mathbf{N}[v] - f) * \mathsf{DET} * e^{-\frac{1}{2}\int fAf}, \tag{10}$$

with the functional determinant

$$\mathsf{DET} = \left| \frac{\delta \mathbf{N}_\alpha[v](x)}{\delta v_\beta(y)} \right|. \tag{11}$$

Sum of States 2/4

The δ-function can be rewritten by means of a functional Fourier-transformation:

$$\delta(\mathbf{N}[v] - f) = \int Du\, e^{i \int u(\mathbf{N}[v]-f)}. \tag{12}$$

The new field u can be looked at as

- Lagrange-multiplicator forcing the Burgers Equation to be valid, or as
- book-keeping field picking up the contributions to the energy which otherwise would be converted to heat (original ansatz followed by MSR).

Sum of States 3/4

The gaussian random force is integrated out now, in 1D leading to:

$$Z \propto \int DuDv * \mathsf{DET} * e^{i \int u(\partial_t v + v \partial_x v - \nu \nabla^2 v) - \frac{1}{2} \int u A^{-1} u}. \tag{13}$$

- uu-Propagator: $-\frac{1}{2} \int u A^{-1} u$
- uv-Propagator: $i \int u(\partial_t - \nu \nabla^2) v$
  $\rightarrow$ diffusion propagator, *only* causal solution!
  $\rightarrow \propto \theta(t)$
- uvv-Vertex: $i \int uv \partial_x v$

Sum of States 4/4

We might even integrate out the field u:

$$Z \quad \propto \quad \int Dv * \mathsf{DET} * e^{-\frac{1}{2}\int(\partial_t v + v\partial_x v - \nu\nabla^2 v)A(\partial_t v + v\partial_x v - \nu\nabla^2 v)}. \tag{14}$$

- vv-Propagator: Can be read off the quadratic term, *only* Feynman-solution!
  $\rightarrow$ Pertubation theory possible in the standard way.
- various interactions

Determinant - Version 1/2

writing the determinant as

$$\text{DET} = e^{\ln \det(O)} = e^{\mathbf{Tr} \ln(O)} \tag{15}$$

permits to calculate it explicitly, we get for a general stochastic differential equation of first order in the time derivative

$$\left(\partial_t - D_0(\nabla)\right) v - F[v] = f \tag{16}$$

$$\rightarrow \text{DET} \propto e^{-\theta(0)\mathbf{Tr}\left[\frac{\delta F}{\delta v}\right]} \tag{17}$$

what is $\theta(0)$??? why this ambiguity?

Parenthesis - the $\theta(0)$-Problem 1/4

Also is known from path integral quantization of a charge in a magnetic field,

$$H = \frac{1}{2m}[p + eA(q)]^2 + V(q), \tag{18}$$

leading to the hermitic quantum hamiltonian

$$\hat{H} = \frac{1}{2m}[\hat{p}^2 + e\hat{p}\hat{A}(\hat{q}) + e\hat{A}(\hat{q})\hat{p} + e^2\hat{A}(\hat{q})^2] + V(\hat{q}). \tag{19}$$

If V is not altered, this is dictated by hermiticity. But also any non-hermitic hamiltonian will naively lead to the same continuum limit of the generating functional with the action

$$S(q) = \int \frac{1}{2}\dot{(q)}^2 + ieA(q)\dot{(q)} + V(q)!?! \tag{20}$$

Parenthesis - the $\theta(0)$-Problem 2/4

Hermiticity is invariance of S under complex conjugation and inversion of time.
Another ordering of $\hat{p}$ and $\hat{A}$ in $\hat{H}$ corresponds to a non-symmetric definition of the time derivation, and is compensated by a term proportional to the commutator:

$$-\frac{1}{2} i e \epsilon(0) \hbar \nabla A(q) \tag{21}$$

This term exactly cancels the direct $\dot{q}q$-loops!
The choice $\epsilon(0) = 0$ lets time differentiation and averaging commute.

Parenthesis - the $\theta(0)$-Problem 3/4

Analogy between quantization of a charge in a magnetic field and our problem with Burgers Equation:

- DET exactly cancels the direct uv loops, and
- corresponds to different discretizations of the time derivative.

But - why can there be an operator ordering problem in a classical theory?

Parenthesis - the $\theta(0)$-Problem 3/4

Analogy between quantization of a charge in a magnetic field and our problem with Burgers Equation:

- DET exactly cancels the direct uv loops, and
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But - why can there be an operator ordering problem in a classical theory?

(later!)

Parenthesis - the $\theta(0)$-Problem 4/4

Which $\theta(0)$ shall we choose?

- The symmetric choice $\theta(0) = \frac{1}{2}$ (Stratanovich convention) is easiest, and leads to commutation of time derivatives and averaging.
- The pure backward definition is sometimes said to be more physically justified, but leads to complicated results for means.

$\rightarrow$ no general answer

Determinant - Version 2/2

We could also write Z as

$$Z \propto \int DvDuD\psi D\psi^* e^{-S[u,v,\psi,\psi^*]}, \tag{22}$$

with the action

$$\begin{aligned} S &= -i\int u((\partial_t - D_0)v + F[v]) + \frac{1}{2}\int uA^{-1}u \\ &\quad + \int \psi\left(\partial_t - D_0 + \frac{\delta F}{\delta v}\right)\psi^* \end{aligned} \tag{23}$$

by insertion of anticommuting ghost fields ψ, ψ^*.

The Ghosts

- Ghost fields can be integrated graph by graph, showing (of course) the above result on a perturbative level.
- The action is BRS-invariant under the changes:

$$\begin{aligned} \delta u &= 0, && (24) \\ \delta\psi^* &= 0, && (25) \\ \delta v &= \epsilon\psi^*, && (26) \\ \delta\psi &= i\epsilon u. && (27) \end{aligned}$$

This gives the desired result on a non-perturbative way!

$$\left\langle \psi \frac{\delta F}{\delta v} \psi^* \right\rangle = i \langle uF \rangle \qquad (28)$$

Back to Burgers Equation

But first - the operator ordering - where can it come from?
Only possibility in a classical theory:

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$\longrightarrow$ we can safely forget about DET :-)
$\longrightarrow$ bad news for NSE... :-(

Generating Functional for Burgers Equation

Two ways (at least) of writing it:

$$Z[J] = \int DuDve^{-S[u,v]+\int Jv} \tag{29}$$

with

$$S[u,v] = -i\int u(\partial_t v + v\partial_x v - \nu\nabla^2 v) + \frac{1}{2}\int uA^{-1}u, \tag{30}$$

or

$$Z[J] = \int Dve^{-S[v]+\int Jv} \tag{31}$$

with

$$S[v] = \frac{1}{2}\int(\partial_t v+v\partial_x v-\nu\nabla^2 v)A(\partial_t v+v\partial_x v-\nu\nabla^2 v). \tag{32}$$

Other Applications

- Search for Instanton-Solutions (Falkovich et al.)
- Operator Product Expansion and Fusion Rules for Structure Functions (Polyakov)

perhaps also

- Renormalization Group Analysis

MC-Considerations 1/2

For the identification of a physical theory, the lattice quantities are related to the continuum ones:

- Viscosity:

$$\nu = 2 * \frac{NT}{NX^2}$$

- Reynolds-Number:

$$\chi_0 = \mathbf{Re}^3 \nu^3 L^{-4}$$

These relations tell us how to perform the continuum limit.

MC-Considerations 2/2

Physics implies a smallest length scale λ, given by

$$\lambda = \left(\frac{\nu^3}{\chi_0}\right)^{\frac{1}{4}},$$

that has to be resolved (or else the simulations become unstable) $\rightarrow$ viscosity as regularization of singular structures!

MC-Simulations (so far)

Codes:

- (Metropolis single node and multiple node)
- Heat Bath single node and multiple nodes

Runs:

- Varying lattice sizes at constant viscosity / Reynolds-number
- Varying Reynolds-numbers at constant lattice size
- Varying viscosity at constant Reynolds-number

MC-Simulations (so far)

Architecture:

- in 1+1 dimension: single CPU sufficient for lattices up to 256×256
- tested also on the Cluster, and on
- nVidia graphics card using CUDA (not suitable for Heat Bath Algorithms)
- hopefully soon: JuMP

MC-Simulations (first results)

Configurations:

- Burgers solutions can be reproduced
- Kinks can be observed, and their movements observed
- Changing viscosity has dramatic impact on stability of calculations, as said above $\rightarrow \lambda$ can be "measured"
- Reynolds-number determines length / number of ramps

(see plots)

MC-Simulations first results)

Statistics:
first hints on linear scaling of the structure functions!
(see plots)

Summary

- MC-Calculations can give very direct insights into turbulence and the origin of intermittency
- Burgers equation can be studied in detail, shocks and intermittent exponents reproduced
- Results seem to be universal (not dependend on ν or $\mathbf{Re}$)
- Problems with finding a suitable discretization for derivatives close at the shocks

Outlook (1/2)

Concerning Burgulence:

- Measurement of more observables (energy spectrum, energy decay, prob. distr. for small velocity increments,...)
- $D = 2, 3$ (with constraint)
- (Localization of structures)

Outlook (2/2)

Concerning NSE:

- Non-local interactions
- Structure formation?
- Fundamentals of intermittency?