Worldline Variational Approximation for the Bound-State Problem in Field Theory

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Literature:

K. Barro-Bergflödt, R. Rosenfelder and M. Stingl, Variational Worldline Approximation for the Relativistic Two-Body Bound State in a Scalar Model, Few-Body Systems **39**, 193 - 253 (2006) [arXiv:hep-ph/0601220]

Background:

R. P. Feynman, Slow electrons in a Polar Crystal, Phys. Rev. **97**, 660 - 665 (1955); reprinted in: L.M. Brown (Ed.), Selected Papers of Richard Feynman With Commentary, World Scientific, Singapore (2000), pp. 237- 242.

Yu. A. Simonov and J. A. Tjon, The Feynman-Schwinger Representation for the Relativistic Two-Particle Amplitude in Field Theory, Ann. Phys. (N.Y.) **228**, 1 - 18 (1993).

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1. The field-theory bound-state problem

Scalar fields $\hat{a}(x)$, $\hat{b}(y)$; particle masses m_a, m_b.
Two-body bound state $|\, B \,\rangle$:

$$\langle\, 0 \,|\, \hat{a}(x)\, \hat{b}(y) \,|\, B \,\rangle \;\neq\; 0.$$

signalled by pole in 4-point correlation fn.

$$\int d^4x_1 \ldots d^4x_4 e^{i\,(\,p_1\cdot x_1 + p_2\cdot x_2 + p_3\cdot x_3 + p_4\cdot x_4\,)} \langle\, 0 \,|\, \mathcal{T}\{\, \hat{a}(x_1)\, \hat{b}(x_2)\, \hat{b}(x_3)\, \hat{a}(x_4)\,\} \,|\, 0 \,\rangle$$

$$= (2\pi)^4\, \delta^4(p_1 + p_2 + p_3 + p_4)\; \tilde{G}_4\left(p',\, q,\, p\right)$$

$$\begin{aligned} q &= p_1 + p_2 \quad = -\,(p_3 + p_4)\,, \\ p' &= p_2 - p_1,\;\; p = p_4 - p_3 \end{aligned}$$

at timelike total momentum

$$0 \;<\; q^2 \;=\; M_B^2 \;<\; (m_a + m_b)^2\,.$$

"Binding energy"

$$\varepsilon \;=\; M_B \;-\; (m_a + m_b)\,.$$

Standard approach:

Bethe-Salpeter (B.-S.) equation:

Formally linear integral equation for either full 4-point function G_4 ("inhomogeneous" eq.) or B.-S. amplitude ("homogeneous" eq.), using as inputs noninteracting $G_2 \otimes G_2$ and interaction term K_4 ("1- and 2-particle-irreducible kernel") *with respect to one of the 3 channels*. But K_4 only definable as functional of G_2, G_3, and G_4 itself $\Rightarrow$ nonlinearity. Problems:

- Derivable only as member of a whole system of integral eqs., in which G_4 gets coupled to G_2, G_3, itself, and higher functions. All known approximations replace G_2, G_3 by zeroth-order perturbative forms or simplified *ansätze* to force closed form of G_4 – not justifiable.

- Kernel K_4 *formally* accounts for coupling to higher fns. exactly, but at price of becoming extremely complicated, infinite series ("dressed-skeletons expansion"). Approximations studied – usually "ladder approximation" ($K_4 = 1-$particle-exchange graph = lowest-order term) or at most "ladder-plus-singly-crossed-" approximation – usually justified as long-distance approxs., but unconvincing for e.g. s-wave bound states.

- Combined approximations suffer from all sorts of defects : violation of crossing principle, families of unphysical states with negative probabilities, wrong one-heavy-particle limit and/or nonrelativistic limit, ...

- $\Rightarrow$ Good reasons for exploring alternatives.

Testing ground for bound-state theories:

Wick-Cutkosky (W.-C.) model:

Φ_1, Φ_2 : complex scalar fields ("baryons")
χ : real scalar field ("meson')

$$S[\Phi_1, \Phi_2, \chi] = \int d^4x \left\{ \frac{1}{2} \left[(\partial_\mu \chi)^2 - m^2 \chi^2 \right] + \sum_{i=1}^{2} \left[|\partial_\mu \Phi_i|^2 - M_i^2 |\Phi_i|^2 + g_i |\Phi_i|^2 \chi \right] \right\}$$

M_1, M_2, m : bare masses (later: $M_1 = M_2 = M_0$)
g_1, g_2 : coupling constants

$[g_i] = +1 \implies$ Superrenormalizable theory
(only 1-loop 2-point function UV-divergent)

Coupling term indefinite $\implies$ Instability (G. Baym)

Nonrelativistic limit:
"Baryons" interacting through Yukawa potential

$$\propto \; g_i g_j \exp(-m r_{ij}) \,/\, r_{ij}$$

$\Rightarrow$ expect $\Phi_1 \Phi_2$ bound state(s) for g_i sufficiently large.

2. Baryon-current correlation function

Current-correlation function, or polarization propagator :

$$\Pi_{ij}(q^2) := -i \int d^4x \, e^{iq\cdot x} \, \langle 0|\mathcal{T}\{\, \widehat{C}_{ij}(x)\, \widehat{C}^{\dagger}_{ij}(0)\,\}|0\rangle_{conn}$$

"Current" (interpolating) operators :

$$\widehat{C}^{\dagger}_{ij}(x) = \widehat{\Phi}^{\dagger}_{i}(x)\widehat{\Phi}^{\dagger}_{j}(x)$$

(Note: Same starting point as in sum-rule & lattice approaches)

Insert complete set of 4-momentum eigenstates :

$$\mathbf{1} = \sum_n |P_n\rangle\langle P_n| \qquad (P_n \cdot P_n = M_n^2 \text{ for discrete n}).$$

⇒ Spectral representation :

$$\Pi_{ij}(q^2) = (2\pi)^3 \sum_n \frac{1}{q^2 + i0 - M_n^2}$$
$$\times\Big\{\, \delta^3(\mathbf{q} - \mathbf{P}_n)\,(q^0 + P_n^0)\,|\langle P_n|\widehat{C}^{\dagger}_{ij}(0)|0\rangle|^2$$
$$- \,\delta^3(\mathbf{q} + \mathbf{P}_n)\,(q^0 - P_n^0)\,|\langle P_n|\widehat{C}_{ij}(0)|0\rangle|^2\Big\}$$

Scalar $\Phi_i\Phi_j$ bound state ⇔ pole at timelike q^2.

Generating functional:

$$Z[J^*,J] = \frac{1}{\mathcal{N}} \int \prod_{i=1}^{2} \mathcal{D}\Phi_i \, \mathcal{D}\Phi_i^\dagger \, \mathcal{D}\chi$$
$$\times e^{i\{ S[\Phi,\Phi^\dagger,\chi] + \sum_j [(J_j^*,\Phi_j) + (\Phi_j^*, J_j)]\}}$$

Integrate out bilinearly occurring baryon fields:

$$Z = \frac{1}{\mathcal{N}} \int \mathcal{D}\chi \, e^{iS^{(0)}[\chi]} \prod_{i=1}^{2} \Big\{ \frac{1}{\det \mathcal{O}_i(\chi)}$$
$$\times e^{\left[-i \int d^4x d^4y J_i^*(x) \left(x|\mathcal{O}_i^{-1}(\chi)|y\right) J_i(y)\right]} \Big\}$$

Notation :

$$S^{(0)}[\chi] := \frac{1}{2} \int d^4x \, \left[(\partial\chi)^2 - m^2\chi^2\right],$$

$$\mathcal{O}_i(\chi) := -\partial^2 - M_i^2 + g_i \, \chi$$

Generate Π by functional differentiations :

$$\Pi_{ij} = -i \int d^4x \, e^{iq\cdot x} \frac{\delta^4 \ln Z}{\delta J_i(x) \delta J_i^*(x) \delta J_j(0) \delta J_j^*(0)} \Big|_{J=J^*=0}$$

Notation for functional averages :

$$\langle\, A \,\rangle := \frac{\int\mathcal{D}\chi\, \left[\prod_{i=1}^{i=2} \frac{1}{\det \mathcal{O}_i(\chi)}\right] e^{iS^{(0)}[\chi]} A[\chi]}{\int\mathcal{D}\chi\, \left[\prod_{i=1}^{i=2} \frac{1}{\det \mathcal{O}_i(\chi)}\right] e^{iS^{(0)}[\chi]}}$$

Result current correlation :

$$\Pi_{ij} = i \int d^4x\, e^{iq\cdot x} \Big\{ \Big\langle\, (x|\mathcal{O}_i^{-1}(\chi)|0)(x|\mathcal{O}_j^{-1}(\chi)|0)\, \Big\rangle - \Big\langle\, (x|\mathcal{O}_i^{-1}(\chi)|0)\, \Big\rangle \Big\langle\, (x|\mathcal{O}_j^{-1}(\chi)|0)\, \Big\rangle \Big\}$$

For bound-state pole, 2nd line can be omitted
$\Rightarrow$ simplified Π_{ij}.

3. Worldline representation

"Quenched" approximation :

$$\det \mathcal{O}_i(\chi) = \underbrace{\det(-\partial^2 - M_i^2)}_{\text{drops out}} \underbrace{\det\left(1 + \frac{g_i}{-\partial^2 - M_i^2}\chi(x)\right)}_{\to 1 \text{ for large } M_i}$$

Physics : suppression of closed baryon loops;
good for heavy baryons.

$$\langle A \rangle \to \langle A \rangle_{quench} := \frac{\int \mathcal{D}\chi \, e^{iS^{(0)}[\chi]} A[\chi]}{\int \mathcal{D}\chi \, e^{iS^{(0)}[\chi]}}$$

$$\Pi_{ij} = i \int d^4x \, e^{iq\cdot x} \left\langle (x|\mathcal{O}_i^{-1}(\chi)|0)(x|\mathcal{O}_j^{-1}(\chi)|0) \right\rangle_{quench}$$

Schwinger-parameter ("proper-time") representation:

$$\mathcal{O}_i^{-1}(\chi) = \frac{1}{2i\kappa_0} \int_0^\infty dT\, e^{\left[\frac{iT}{2\kappa_0}\left(-\partial^2 - M_i^2 + i0 + g_i\chi\right)\right]}$$

Choice of κ_0 arbitrary ("reparametrization invariance").

Change to Euclidean (real) *formally* through $\kappa_0 \to i\kappa_E$ (and $-\partial^2 \to \partial_E^2$) . O. k. for stable theory.

Here: need complex form because $g_i\chi$ indefinite.

Dynamical eqs. will allow $\kappa_0 \to i\kappa_E$, become real.
But real *solutions* may exist only in limited domains.

$$(x|\mathcal{O}_i^{-1}(\chi)|0) = \frac{1}{2i\kappa_0} \int_0^\infty dT\, e^{-iT\frac{M_i^2 - i0}{2\kappa_0}} (x|e^{-iT\left(\frac{\partial^2 - g_i\chi}{2\kappa_0}\right)}|0)$$

In analogy to *quantum-mechanical* path integral

$$< x_b| \text{ exp } \left[-i\widehat{H}(t_b - t_a)/\hbar\right] \, |x_a >= \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x(t) \, e^{iS[x(t)]/\hbar}$$

over 3-*dimensional* paths $\mathbf{x}(t)$, use
integral over 4-dim. paths $x^\mu(t)$ (“worldlines”) :
(only 2 baryon worldlines because of quenched approx. !)

$$\left(x \left| e^{-iT\left[\frac{-\widehat{p}^\mu \widehat{p}_\mu}{2\kappa_0} - \frac{g_i}{2\kappa_0}\chi(\widehat{x})\right]} \right| y \right)$$

$$= \int_{x(0)=y}^{x(T)=x} \mathcal{D}x \text{ exp } \left\{ i \int_0^T dt \left[-\frac{\kappa_0}{2}\dot{x}^2 + \frac{g_i}{2\kappa_0}\chi(x(t)) \right] \right\}$$

Functional χ integration now Gaussian :

$$\int \mathcal{D}\chi \, e^{i\{S^{(0)}[\chi]+(b,\chi)\}} = \text{ const. exp } \left[-\frac{i}{2} \left(b, \, \frac{1}{-\partial^2 - m^2} b \right) \right]$$

$$b(z) \; := \sum_{i=1}^{2} \frac{g_i}{2\kappa_0} \int_0^{T_i} dt \, \delta\,(\, z \; - \; x_i(t))$$

Result for Π_{12} : "worldline path integral"

$$\Pi(q^2) = i \int_0^\infty \frac{dT_1 dT_2}{(2i\kappa_0)^2}\, e^{\left[-\frac{i}{2\kappa_0}(M_1^2 T_1 + M_2^2 T_2)\right]} \times$$

$$\underbrace{\left(\int d^4x \prod_{i=1}^{2} \int_{x_i(0)=0}^{x_i(T_i)=x} \mathcal{D}x_i\right)}_{\widetilde{\mathcal{D}}_{12}} e^{i\{q\cdot x + S_0[x_1] + S_0[x_2] + S_{int}[x_1,x_2]\}}$$

$$S_0[x_i] := \int_0^{T_i} dt \left[-\frac{\kappa_0}{2}\, \dot{x}_i^2(t)\right]$$

Functional variables reduced :$\{\Phi_i(x), \chi(x)\} \to \{x_i^\mu(t)\}$!

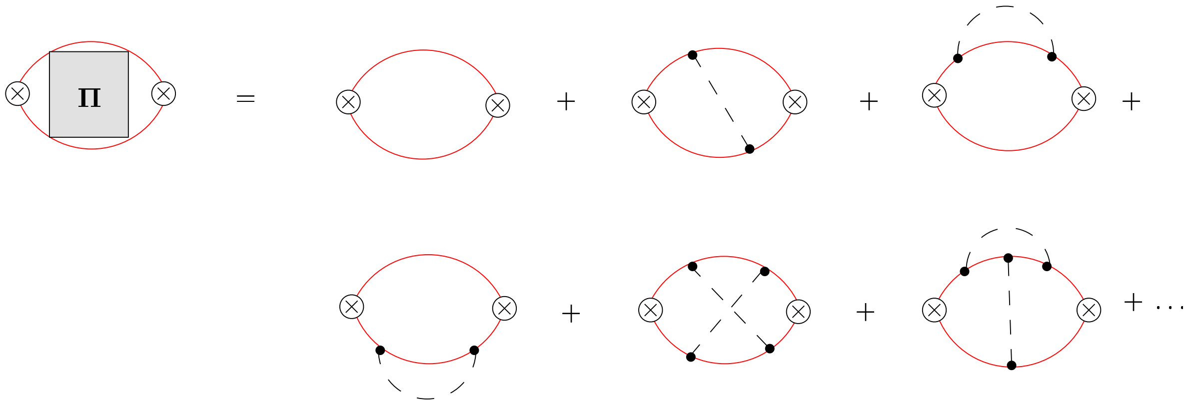
– Effective interaction (nonlocal and retarded) :

$$S_{int}[x_1, x_2] := -\sum_{i,j=1}^{2} \frac{g_i g_j}{8\kappa_0^2} \int_0^{T_i} dt \int_0^{T_j} dt' \left(x_i(t) \left| \frac{1}{-\partial^2 - m^2} \right| x_j(t')\right)$$

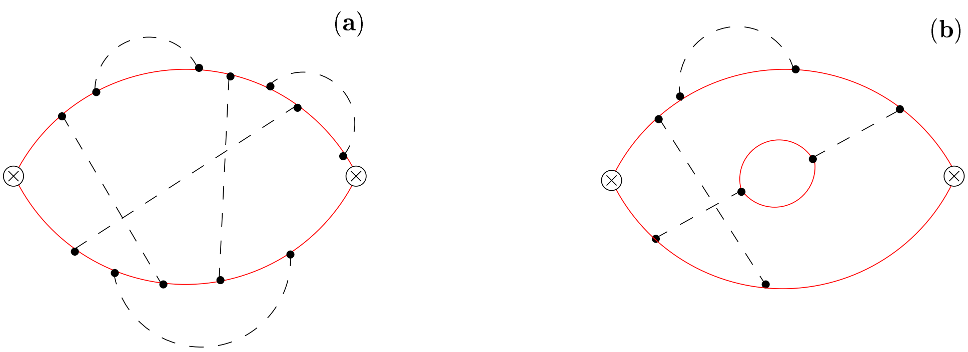
Decomposition

$$S_{int}[x_1, x_2] = \underbrace{S_{int}^{(1,1)}[x_1] + S_{int}^{(2,2)}[x_2]}_{\text{self-energy terms}} + \underbrace{2S_{int}^{(1,2)}[x_1, x_2]}_{\text{2-baryon interaction}}$$

Perturbative expansion of correlator:



Sums all-orders self-energy, vertex-corrections, ladder and crossed-ladder diagrams (a) but no vacuum polarization (b) :



quenched unquenched

Normalization : multiply & divide $T_1 T_2$ integrand with exact path integral

$$i \int \tilde{\mathcal{D}}_{12} e^{i\left(q\cdot x + \sum_{n=1}^{2} S_0[x_n]\right)} = \left[\frac{\kappa_0}{2\pi(T_1+T_2)}\right]^2 \exp\left(i\frac{q^2}{2\kappa_0}\frac{T_1 T_2}{T_1+T_2}\right)$$

Result for correlator

$$\Pi(q^2) = -\int_0^\infty \frac{dT_1 dT_2}{(4\pi)^2 (T_1+T_2)^2} \exp\left\{\frac{i}{2\kappa_0}\Big[-(M_1^2 T_1 + M_2^2 T_2) + q^2 \left(\frac{T_1 T_2}{T_1+T_2}\right)\Big]\right\} \times \frac{\int \tilde{\mathcal{D}}_{12} \exp\{ i\tilde{S}\ [x_1, x_2]\}}{\int \tilde{\mathcal{D}}_{12} \exp\{ i\tilde{S}_0[x_1, x_2]\}}$$

$$\begin{aligned} \tilde{S}_0 : &= q\cdot x + \sum_{i=1}^{2} S_0[x_i] \\ \tilde{S}\ : &= \tilde{S}_0 + S_{\text{int}}[x_1, x_2] \end{aligned}$$

"Free" correlator ($S_{\text{int}} = 0$) has branch cut along real axis of complex q^2 plane at

$$q^2 \geq q^2_{\text{thr}} := (M_1 + M_2)^2$$

corresp. to continuum 2-baryon states. For $S_{\text{int}} \neq 0$ expect same, plus bound-state pole(s) at $0 < q^2 < q^2_{\text{thr}}$.

Digression: **Feynman's Polaron**

Polaron: = "dressed" electron slowly moving through polar crystal (e.g.NaCl), dragging polarization cloud

Model Hamiltonian (H. Fröhlich, 1954) :

$$\hat{H} = \frac{1}{2m}\hat{\mathbf{p}}^2 + \sum_{\mathbf{k}} \omega\, \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \sqrt{\alpha} \sum_{\mathbf{k}} \frac{1}{|\mathbf{k}|} \left[\hat{a}^\dagger_{\mathbf{k}} e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}} + h.c. \right]$$

α : dimless electron-phonon coupling ($\approx 1 \ldots 10$) .
m :electron mass, ω :phonon frequency (indep. of $\mathbf{k}$).

Feynman (1955) : phonons can be integrated out exactly in path integral for partition fn.:

$$\begin{aligned} S_{\text{eff}} &= \int_0^\beta dt\, \frac{1}{2} m\dot{x}^2 + \alpha \int_0^\beta dt dt'\, e^{-\omega|t-t'|} \\ &\quad \times \underbrace{\int d^3k\, \frac{1}{k^2} \exp\left[i\mathbf{k}\cdot\left(\mathbf{x}(t) - \mathbf{x}(t')\right)\right]}_{=\text{const}\cdot|\mathbf{x}(t)-\mathbf{x}(t')|^{-1}} \end{aligned}$$

One-particle problem, but two-time (retarded) action!

Variational principle (from Jensen inequality) for

$$\begin{aligned} Z(\beta) &= \int \mathcal{D}x\, e^{-S_{\text{eff}}(\beta)} \quad \overset{\beta\to\infty}{\longrightarrow} \quad e^{-\beta E_0} \\ &= \int \mathcal{D}x\, e^{-S_t} \cdot \underbrace{\frac{\int \mathcal{D}x\, \exp\left(-S_t - (S_{\text{eff}} - S_t)\right)}{\int \mathcal{D}x\, \exp(-S_t)}}_{=:\langle e^{-\Delta S}\rangle_t} \end{aligned}$$

$\Longrightarrow$ ground-state energy of polaron at rest :

$$E_0 \le E_t + \lim_{\beta\to\infty} \frac{1}{\beta} \left\langle S_{\text{eff}} - S_t \right\rangle$$

Feynman's trial action :

$$\begin{aligned} S_t &= \int_0^\beta dt\, \frac{\dot{x}^2}{2} \\ &+ \int_0^\beta dt\, dt'\, f\left(|t-t'|\right) \left[\mathbf{x}(t) - \mathbf{x}(t')\right]^2 \end{aligned}$$

$\Rightarrow$ nonlinear variational eq. for retardation fn. $f(\sigma)$.

Best analytical method, works for **all** α :

$\alpha \to 0$:

$$\begin{aligned} E_0 &= -\alpha - 0.0159\,\alpha^2 - 0.000806\,\alpha^3 + \ldots \\ E_F &= -\alpha - 0.0123\,\alpha^2 - 0.000634\,\alpha^3 + \ldots \end{aligned}$$

$\alpha \to \infty$:

$$\begin{aligned} E_0 &= -0.1085\,\alpha^2 - 2.84 + \mathcal{O}\left(\alpha^{-2}\right) \\ E_F &= -0.1061\,\alpha^2 - 2.83 + \mathcal{O}\left(\alpha^{-2}\right) \end{aligned}$$

– confirmed by Monte-Carlo calculations [Alexandrou & Rosenfelder, Phys. Rep. **215** (1992)].

4. Variational Approximation

Choose class of trial actions $\tilde{S}_t$, exactly calculable

$$\frac{\int \tilde{\mathcal{D}}_{12} \exp\{ i\tilde{S}\ [x_1,x_2]\}}{\int \tilde{\mathcal{D}}_{12} \exp\{ i\tilde{S}_0[x_1,x_2]\}} = \frac{\int \tilde{\mathcal{D}}_{12} \exp\{ i\tilde{S}_t\}}{\int \tilde{\mathcal{D}}_{12} \exp\{ i\tilde{S}_0\}}$$

$$\times \underbrace{\frac{\int \tilde{\mathcal{D}}_{12} \exp\{ i\tilde{S}_t\} \exp\{ i(\tilde{S}-\tilde{S}_t)\}}{\int \tilde{\mathcal{D}}_{12} \exp\{ i\tilde{S}_t\}}}_{=: \left\langle \exp\, i(\tilde{S}-\tilde{S}_t)\right\rangle_t}$$

Feynman-Jensen variational principle
(complex version: heuristic only !) :

$$\left\langle \exp\, i(\tilde{S}-\tilde{S}_t)\right\rangle_t \overset{\text{optimal}}{\approx} \exp\left\{ i\left\langle \tilde{S}-\tilde{S}_t \right\rangle_t \right\}_{\text{stat}}$$

where $\tilde{S}_t$ fulfills stationarity condition,

$$\frac{\delta}{\delta \tilde{S}_t} \langle \tilde{S}-\tilde{S}_t \rangle_t \Big|_{\text{stat}} = 0.$$

Real case : Jensen's inequality

Convexity of $\exp(-x)$ on real axis $\Rightarrow$

$$e^{-\frac{1}{2}(x_1+x_2)} \leq \frac{1}{2}\left(e^{-x_1}+e^{-x_2}\right)$$

More generally : for any average $\langle \ldots \rangle$ w. r. t. positive, normalized measure :

$$e^{-\langle A \rangle} \leq \left\langle e^{-A} \right\rangle .$$

Generalization to functional measure :

$$\exp\left\{-\int \mathcal{D}[x]\, A[x]\right\} \leq \int \mathcal{D}[x]\, \exp\{-A[x]\}\ , \qquad \text{if } \int \mathcal{D}[x] = 1.$$

$\Rightarrow$ Maximum of $\exp\left\{-\left\langle \tilde{S}-\tilde{S}_t \right\rangle_t\right\}$ w. r. t. $\tilde{S}_t$ is best approximation for $\left\langle \exp -(\tilde{S}-\tilde{S}_t)\right\rangle_t$ in given class of trial actions $\tilde{S}_t$.

(Carried over to complex $A[x]$ or $(\tilde{S}-\tilde{S}_t)$: widespread **heuristic** procedure, but not rigorous !)

Choice of trial action: $\langle \ldots \rangle_t$ doable analytically $\Leftrightarrow$ $\tilde{S}_t$ bilinear in x_1, x_2.

Parametrization of paths with $x_i(0) = 0$, $x_i(T_i) = x$:

$$x_i^\mu(\tau) = x^\mu \cdot \tau + \sum_{k=1}^{\infty} \left(\frac{\sqrt{2T_i}}{k\pi} \right) a_k^{(i)\,\mu} \sin\left(k\pi\tau\right)$$

$$\mu = 0 \ldots 3; \qquad \tau := \frac{t}{T_i} \in [0, 1]$$

Functional measure now :

$$\int \tilde{\mathcal{D}}_{12} = \text{const.} \int d^4x \int \mathcal{D}^d a_1 \mathcal{D}^d a_2$$

Free action

$$S_0 = -\frac{\kappa_0}{2} \sum_{i=1}^{2} \left[\frac{1}{T_i} x^2 + \sum_{k=1}^{\infty} \left(a_k^{(i)} \right)^2 \right]$$

taken as guideline for choice of S_t :

$$\tilde{S}_t = \tilde{\lambda} q \cdot x - \frac{\kappa_0}{2} \sum_{i=1}^{2} \left[A_0 \frac{x^2}{T_i} + \sum_{k=1}^{\infty} A_k^{(i)} \left(a_k^{(i)} \right)^2 \right]$$

$$+ \kappa_0 \sum_{k=1}^{\infty} B_k \left(a_k^{(1)} \cdot a_k^{(2)} \right)$$

Coefficients to be determined variationally:

$$\tilde{\lambda}, \quad A_0, \quad A_k^{(i)}\,(\,i = 1,2\,), \quad B_k.$$

More generally possible, but not used here :

- Nondiagonal terms $a_k^{(i)} \cdot a_l^{(j)}$, $l \neq k$
  (found negligible in limit $T_1 + T_2 \longrightarrow \infty$)
- Tensorial coefficients
  $$A_k \longrightarrow A_k^{\mu\nu} = \underbrace{A_k^T \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right)}_{\text{transverse}} + \underbrace{A_k^L \left(\frac{q^\mu q^\nu}{q^2} \right)}_{\text{longitudinal}}$$
  (improve non-relativistic limit quantitatively)

With $\tilde{S}_t$, all path integrals in

$$\Pi^{\text{var}} = -\int_0^\infty \frac{dT_1 dT_2}{(4\pi)^2 (T_1+T_2)^2} \exp\left\{ \frac{i}{2\kappa_0} \Big[-(M_1^2 T_1 + M_2^2 T_2) + q^2 \left(\frac{T_1 T_2}{T_1+T_2} \right) \Big] \right\} \times \frac{\int \tilde{\mathcal{D}}_{12} \exp\{ i \tilde{S}_t \}}{\int \tilde{\mathcal{D}}_{12} \exp\{ i \tilde{S}_0 \}} \exp\left\{ i \langle \, \tilde{S} - \tilde{S}_t \, \rangle_t \right\}$$

Gaussian, exactly calculable.

Klein-Gordon kernel handled by

$$\left(x_i(t) \left| \frac{1}{-\partial^2 - m^2} \right| x_j(t') \right) = \int \frac{d^4p}{(2\pi)^4} \frac{\exp\left[-ip\cdot(x_i(t) - x_j(t'))\right]}{p^2 - m^2 + i0}.$$

Correlator in variational approximation:

$$\Pi^{\rm var}(q^2) = -\int_0^\infty \frac{dT_1 dT_2}{(4\pi)^2 (T_1+T_2)^2} \exp\left\{ \frac{i}{2\kappa_0} \left[-(M_1^2 T_1 + M_2^2 T_2) \right.\right.$$

$$\left.\left. + q^2 \left(\frac{T_1 T_2}{T_1+T_2}\right)(2\lambda - \lambda^2) - (T_1+T_2)\left(\Omega_{12}(q^2; T_{1,2}) + V(q^2; T_{1,2})\right) \right] \right\}$$

Ω_{12} (from $\det^{-1/2}$ and S_0 terms),
V (from S_{int} term) :
functions of q^2, T_1, T_2, $\lambda := \frac{\tilde{\lambda}}{A_0}$
and complicated functionals of $A_k^{(1,2)}$, B_k.

5. Bound-state pole : Mano's equation

For function defined by proper-time integral

$$\Pi^{\text{var}}(q^2) = \frac{1}{i}\int_0^{\infty} dT e^{iT\left[N(q^2)\right]} g(q^2, T),$$

whose convergence at $T \to \infty$ is due to the oscillating exponential, and where $g(q^2, T)$ is smooth in q^2, a pole can develop only when

$$N = 0 \quad \text{at} \quad q^2 = M_B^2.$$

What is T ? Restriction to equal-mass case

$$M_1 = M_2 =: M_0.$$

Here, answer is to transform T_1, T_2 integration to

$$\begin{aligned} T \, : &= \frac{T_1 + T_2}{2} = 0 \ \ldots \ \infty \\ s \, : &= T_1 - T_2 = -2T \ \ldots \ +2T. \end{aligned}$$

Possibility of b.-s. pole develops because

as $T \to \infty$, $\Omega_{12} + V \to$ constant $\Omega_{12}^{\infty} + V^{\infty}$,
independent of T and s.

Integrand for $T \to \infty$:

$$\exp\left\{ \frac{i}{\kappa_0} \underbrace{\left[\frac{q^2}{4}(2\lambda - \lambda^2) - M_0^2 - (\Omega_{12} + V)^{\infty} \right]}_{\text{N}} \cdot T - \ln(2T) + [\,\text{nonincreasing}\,](T, s) \right\}$$

M_B^2 determined by **Mano's equation** :

$$\frac{q^2}{4}(2\lambda - \lambda^2) - M_0^2 - (\Omega_{12} + V)^{\infty}\Big|_{q^2 = M_B^2} = 0.$$

(K. Mano, 1955, for one-body problem).

"Kinetic" (Ω) and "interaction" (V) terms for $T \to \infty$:
Fourier sums $\sum_{k=1}^{\infty} \to$ integrals over $E = \frac{k\pi}{T}$:

$$\sum_{k=1}^{\infty} f(A_k) \longrightarrow \frac{T}{\pi} \int_0^{\infty} dE\, f\big(A(E)\big), \quad \text{etc.}$$

For simplicity consider case

$$g_1 = g_2 = g \;\rightarrow\; A^{(1)}(E) = A^{(2)}(E) = A(E).$$

Use $A_{\pm}(E) = A(E) \pm B(E)$; choose $\kappa_0 = i\kappa_E$:

$$\begin{aligned} \Omega_{12}^{\infty} &= \Omega[A_+] + \Omega[A_-] \\ \Omega[A] &= \frac{\kappa_E\, d}{2\pi} \int_0^{\infty} dE \left[\ln A(E) + \frac{1}{A(E)} - 1 \right] \end{aligned}$$

$$V_{ii}^{\infty} = \frac{-\,g^2}{32\pi^2} \int_0^{\infty} d\sigma \frac{1}{\mu_{11}^2(\sigma)} \int_0^1 du$$

$$\times \exp\left\{ -\frac{1}{2\kappa_E} \left[m^2 \mu_{11}^2(\sigma) \frac{1-u}{u} + \frac{\lambda^2 \sigma^2 q^2}{4\mu_{11}^2(\sigma)} u \right] \right\}$$

$$V_{12}^{\infty} = \frac{-Z g^2}{32\pi^2} \int_0^{\infty} d\sigma \frac{1}{\mu_{12}^2(\sigma)} \int_0^1 du$$

$$\times \exp\left\{ -\frac{1}{2\kappa_E} \left[m^2 \mu_{12}^2(\sigma) \frac{1-u}{u} + \frac{\lambda^2 \sigma^2 q^2}{4\mu_{12}^2(\sigma)} u \right] \right\}$$

"Pseudotimes" μ_{ij}^2 are functionals of
"profile functions" $A(E)$, $B(E)$ or $A_+(E)$, $A_-(E)$:

$$\mu_{11}^2[A_\pm;\sigma) = \frac{2}{\pi}\int_0^\infty dE\,\frac{1}{E^2}\left[\frac{1}{A_-(E)}+\frac{1}{A_+(E)}\right]\sin^2(\frac{E\sigma}{2}) = \mu_{22}^2$$

$$\mu_{12}^2[A_\pm;\sigma) = \frac{2}{\pi}\int_0^\infty dE\,\frac{1}{E^2}\left[\frac{\sin^2(\frac{E\sigma}{2})}{A_-(E)}+\frac{\cos^2(\frac{E\sigma}{2})}{A_+(E)}\right]$$

To be noted :

$\mu_{12}^2 \quad \rightarrow \quad$ finite as $\sigma \rightarrow 0$ $\quad \left(\frac{1}{A_+} \propto E^2 \text{ at } E = 0 \right)$

$\mu_{11}^2 \quad \rightarrow \quad 0$ ($\propto \sigma$) as $\sigma \rightarrow 0$

Leads to divergence in $V_{11} = V_{22}$ at $\sigma = 0$.

This is the expected UV divergence calling for

6. Mass Renormalization

Regularize divergence of V_{ii}^{∞} integrand at $\sigma = 0$,
e.g. by dimensional regularization in $d = 4 - 2\epsilon$:

$$V_{ii}^{\infty} = [V_{ii}^{\infty}]_{sing}(\epsilon) + [V_{ii}^{\infty}]_{reg}(q^2) \; ;$$

$$[V_{ii}^{\infty}]_{sing}(\epsilon) = \frac{(g_i \mu^{\epsilon})^2}{4\kappa_0} \int_0^{\infty} d\sigma \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 - m^2 + i0} \exp\left(\frac{i}{2\kappa_0} p^2 \sigma\right)$$

$$= \frac{-(g_i)^2}{2(4\pi)^2} \left\{ \frac{1}{\epsilon} + \underbrace{\left[1 - \gamma + \ln\left(4\pi \frac{\mu^2}{m^2}\right)\right]}_{C(\mu)} + \mathcal{O}(\epsilon) \right\} ,$$

independent of κ_0 and q^2. $\left[V_{ii}^{\infty}\right]_{reg}$ is regular at $d = 4$.

Mano's equation only contains sum $M_0^2 + 2V_{ii}^{\infty}$
$\Rightarrow$ absorb $1/\epsilon$ divergence in modification of bare mass :

$$M_0^2 + 2V_{ii}^{\infty} = \underbrace{\left\{ M_0^2 - \frac{(g_i)^2}{(4\pi)^2} \left[\frac{1}{\epsilon} + C(\mu)\right] \right\}}_{\text{renormalized mass } \bar{M}(\mu)^2} + 2\,[V_{ii}^{\infty}]_{reg}(q^2) \; .$$

After solution of variational problem for trial quantities λ and $A_\pm(E)$, intermediate, μ-dependent mass $\bar{M}^2$ can be eliminated in favor of physical baryon mass M^2 by *renormalization condition* $M_B \to 2M$ as $V_{12} \to 0$, i.e.

$$N(q^2 = 4M^2) = 0 \quad \text{at} \quad Z = 0 \ (V_{12} = 0).$$

Note: $\left[V_{ii}^{\infty}\right]_{sing}$ independent of trial quantities $\Rightarrow$ Choice of $C(\mu)$ does not affect variational equations.

7. Stationarity Equations

For $T \to \infty$, stationarity condition $\frac{\delta}{\delta S_t} \langle \ldots \rangle_t = 0$ simplifies to $\frac{\delta}{\delta S_t} N[A_\pm, \lambda; q^2) = 0$, i.e.,

$$\frac{\partial}{\partial \lambda} N = 0 \, , \qquad \frac{\delta}{\delta A_\pm(E)} N = 0 \, .$$

Give system of 3 one-dimensional, but strongly nonlinear integral equations :

$$\frac{1}{\lambda} = 1 + \left(\frac{\alpha}{2\pi\kappa_E}\right) M^2 \int_0^\infty d\sigma \, \sigma^2 \int_0^1 du \, u \sum_{n=1}^{2} \frac{Z^{n-1}}{\mu_{1n}^4(\sigma)}$$
$$\times \exp\left\{ -\frac{1}{2\kappa_E} \left[m^2 \mu_{1n}^2(\sigma) \frac{1-u}{u} + \frac{\lambda^2 \sigma^2 q^2}{4\mu_{1n}^2(\sigma)} u \right] \right\},$$

$$A_+(E) = 1 + \frac{2}{\kappa_E E^2} \int_0^\infty d\sigma \left\{ \sin^2\!\left(\frac{E\sigma}{2}\right) \cdot \frac{\delta V_{11}^\infty}{\delta \mu_{11}^2}[A_\pm; \lambda] \right.$$
$$\left. + \cos^2\!\left(\frac{E\sigma}{2}\right) \cdot \frac{\delta V_{12}^\infty}{\delta \mu_{12}^2}[A_\pm; \lambda] \right\},$$

$$A_-(E) = 1 + \frac{2}{\kappa_E E^2} \int_0^\infty d\sigma \left\{ \sin^2\!\left(\frac{E\sigma}{2}\right) \cdot \sum_{n=1}^{2} \frac{\delta V_{1n}^\infty}{\delta \mu_{1n}^2}[A_\pm; \lambda] \right\}.$$

Parametrization now through

$$M^2 : \quad = \quad \text{physical baryon mass (pole position);}$$

$$\alpha : \quad = \quad \frac{g^2}{16\pi M^2} \quad - \quad \text{dimless coupling constant.}$$

Functional dependence on profile functions through

$$\mu_{1n}^2(\sigma) = \mu_{1n}^2[A_+, A_-, \lambda; \sigma) \quad - \quad \text{functionals as before;}$$

$$\frac{\delta V_{1n}^{\infty}}{\delta \mu_{1n}^2}[A_\pm, \lambda; \sigma) = \frac{Z^{n-1}\alpha M^2}{\pi \mu_{1n}^4(\sigma)} \int_0^1 du\, u \left[1 - \frac{\lambda^2 q^2 \sigma^2}{16\kappa_E \mu_{1n}^2} u\right]$$
$$\times \exp\left\{-\frac{1}{2\kappa_E}\left[m^2 \mu_{1n}^2(\sigma)\frac{1-u}{u} + \frac{\lambda^2\sigma^2 q^2}{4\mu_{1n}^2(\sigma)} u\right]\right\}.$$

To be noted :

- Stationarity eqs. reparametrization invariant in κ_0, therefore solved “Euclidean” (at $\kappa_0 = i\kappa_E$) where they are real;
- Can be used, without solving them, to deduce properties of profile fns. and pseudotimes, e.g. for limit values, scaling properties, ...

In particular, for large & small arguments,

$$A_\pm(E) \underset{E\to\infty}{\longrightarrow} 1 + \left(\frac{\alpha M^2}{4\kappa_E}\right)\frac{1}{E} + \dots,$$

$$A_-(E) \underset{E\to 0}{\longrightarrow} A_-(0) \quad \text{regular at } E = 0$$

$$A_+(E) \underset{E\to 0}{\longrightarrow} \frac{\omega^2}{E^2} + \text{const.} + \dots \ , \qquad \omega^2 = \frac{2}{\kappa_E}\int_0^\infty d\sigma \frac{\delta V_{12}}{\delta\mu_{12}^2(\sigma)}$$

$$\mu_{1n}^2(\sigma) \underset{\sigma\to\infty}{\longrightarrow} \frac{\sigma}{2A_-(0)} + \frac{1}{\pi}\int_0^\infty dE\,\frac{1}{E^2}\Big[\frac{1}{A_-(E)} - \frac{1}{A_-(0)} + \frac{1}{A_+(E)}\Big] + \dots$$

$$\mu_{11}^2(\sigma) = \mu_{22}^2(\sigma) \underset{\sigma\to 0}{\longrightarrow} \sigma + \left(\frac{\alpha M^2}{4\pi\kappa_E}\right)\sigma^2 \ln\frac{\sigma}{\sigma_1} + \dots$$

$$\mu_{12}^2(\sigma) \underset{\sigma\to 0}{\longrightarrow} \mu_{12}^2(0) \quad \text{regular at } \sigma = 0 \ .$$

Scaling with respect to κ_E parameter :

$$A_\pm(\kappa_E, E) = A_\pm(\kappa_E \cdot E) \quad \Longrightarrow \quad \mu_{1n}^2(\kappa_E, \sigma) = \kappa_E\, \mu_{1n}^2(\frac{\sigma}{\kappa_E}).$$

The latter give reparametrization invariance of Ω_{12}, V_{1n} .

8. Solution & Results

Procedure for solution :

1. Input parameter : $\frac{m}{M} = 0.15 \quad (\approx \frac{m_{pion}}{m_{nucleon}})$.
   Physical baryon mass M serves as mass scale to express all other masses & energies.

2. Choose value of coupling constant α.

3. Solve variational eqs. for $A_+(E)$, $A_-(E)$, λ numerically by iteration (on grid of Gaussian integration points), first for $q^2/M^2 = 4$ and $Z = 0$ $(V_{12} = 0)$ to determine $\bar{M}^2$.

4. Choose values of q^2/M^2 below 4 in small steps.

5. For each value, solve again variational eqs. numerically and ...

6. ...calculate $N[A_+, A_-, \lambda; q^2/M^2]$ (in units of M^2), using previously determined $\bar{M}^2$.

7. Plot N/M^2 versus q^2/M^2 and look for zero.

8. If no zero yet, increase α and go back to step 3.

9. If zero at $q^2 = M_B^2$ found (regula falsi), plot binding energy $\epsilon/M = (M_B - 2M)/M$ as function of α.

Results:

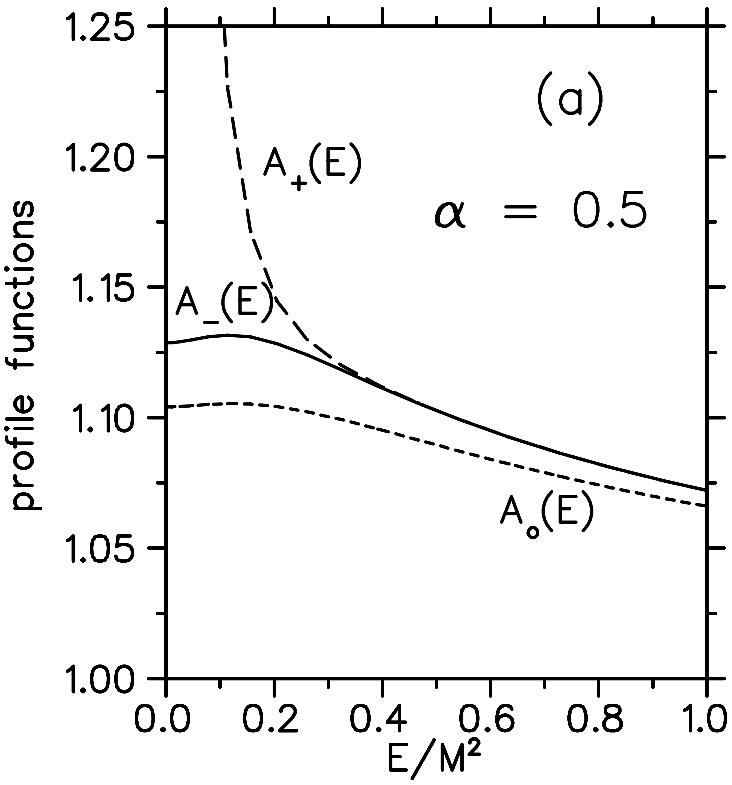
- Method does produce bound state for couplings $\alpha \gtrapprox 0.43$, larger than the non-relativistic (Yukawa) threshold value $\alpha \gtrapprox 0.40$, and still larger than threshold couplings observed in B.-S.-based approximations ($\alpha \gtrapprox 0.30$).
Reason is poor approximation of relativistic-Yukawa interaction by quadratic, oscillator-like one.

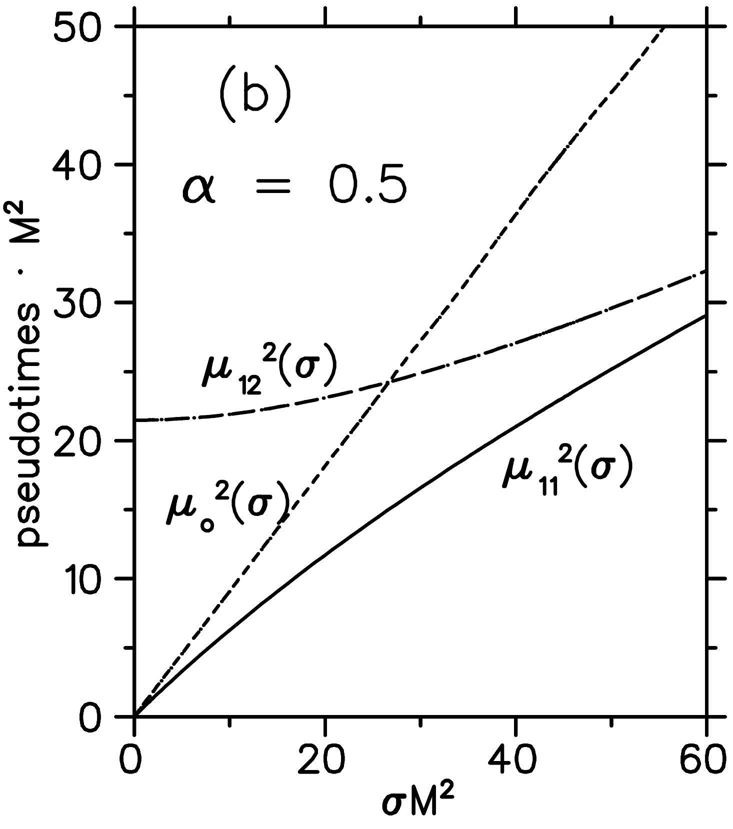
- Above threshold coupling, (absolute values of) binding energies are substantially larger than in B.-S.-based calculations (ladder and "generalized-ladder" approximations).
Detailed comparison shows this is mainly due to inclusion of self-energy and vertex-correction effects (quantum corrections to $G_{2-\text{baryon}}$ and $G_{2-\text{baryon}-1-\text{meson}}$ functions). Check by calculations with $\alpha = 0$ but $Z\alpha \neq 0$.

- Unlike perturbation theory or B.-S. ladder approximation, the method does account for the instability of the W.-C.-system, a genuinely nonperturbative property:

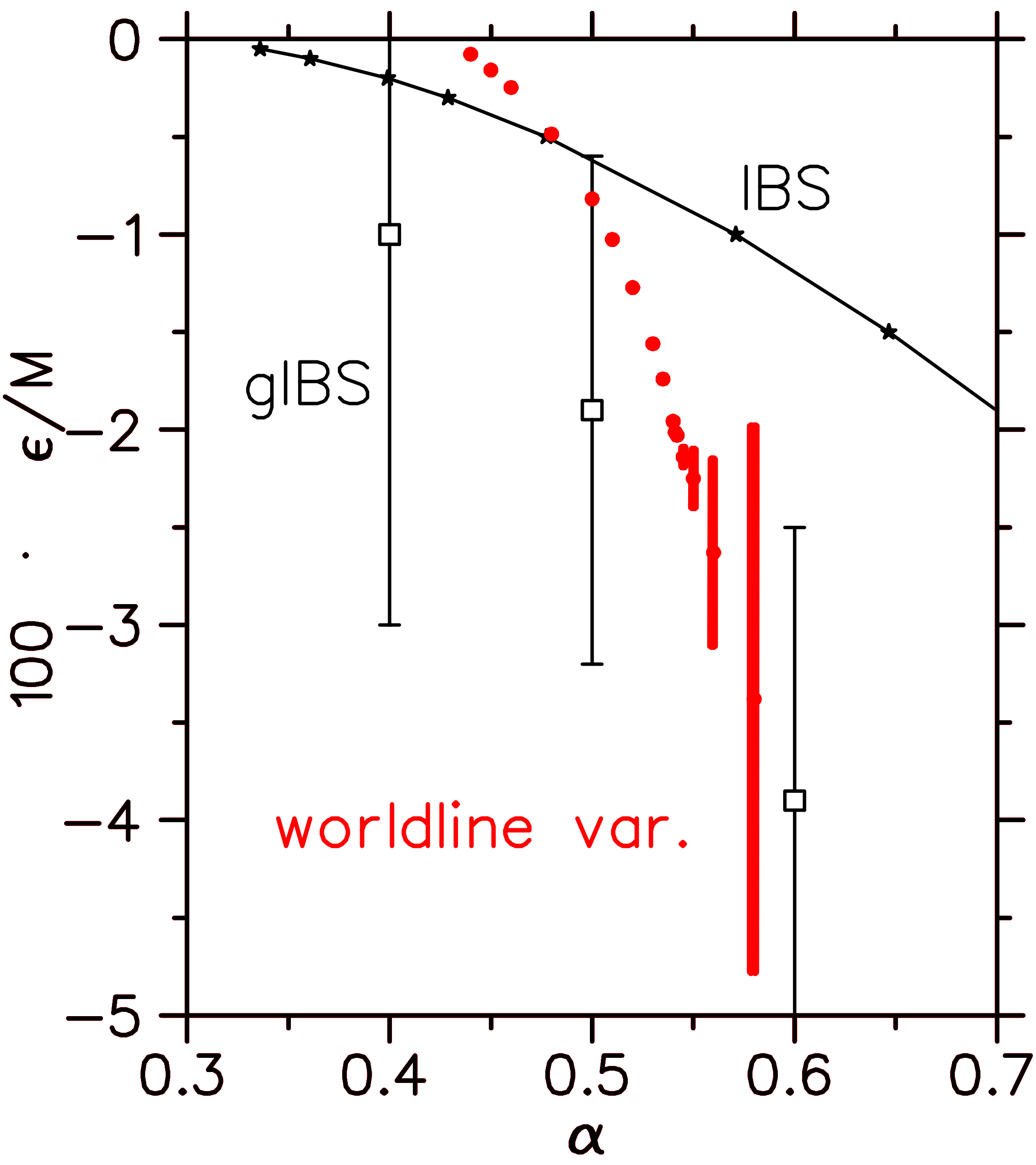
  at couplings $\alpha_{\text{crit}} \gtrapprox 0.542$, there are no real solutions any more to Mano's equation. (In fact, solutions become complex, with imaginary parts giving width of metastable state).
The α_{crit} value is markedly smaller than the $\alpha_{\text{crit}} = 0.817$ observed in *one*-baryon problem: states of unstable field system decay the faster the more particles they contain.

Profile functions and pseudotimes for bound-state solution at $\alpha = 0.5$ (parameter $\kappa_E/M = 1$) :





Binding energy $\frac{\epsilon}{M}$ vs. coupling constant α ($\frac{m}{M} = 0.15$) :



The charm of doing things analytically

Example: Massless mesons ($m = 0$) at weak inter-particle coupling $Z\alpha$.

Use worldline version of "Feynman-Hellmann theorem",

$$\frac{\partial M_B^2}{\partial Z} = \frac{8}{\lambda Z} V_{12}(q^2 = M_B^2)$$

to derive expansion,

$$\frac{E_0}{M/2} = -b_2 (Z\alpha)^2 \left[1 + r_1 \frac{\alpha}{\pi} + \ldots\right] - b_4 (Z\alpha)^4 \left[1 + \ldots\right] - \ldots$$

	variat.	exact	
b_2 :	$\frac{1}{\pi} = 0.318$	$\frac{1}{2}$	(Coulomb)
r_1 :	$\frac{7}{2} = 3.5$	4	(eff. field theory)
b_4 :	$\frac{1}{\pi^2} = 0.101$	$\frac{5}{32} = 0.156$	(Todorov's eq.)

– Numerical coefficients smaller, as expected from a variational calculation

9. Summary & Conclusions

1. Basing two-particle interaction problem on path-integral representation for suitable *current*-correlation function, rather than on Bethe-Salpeter type equations for full four-point function, leads to a formulation closer to sum-rule and lattice approaches.

2. *Worldline* formulation for this problem leads (in quenched approximation) to huge reduction in functional variables and highly visual picture in terms of (heavy-) particle trajectories.

3. Nonlocal, retarded effective interaction sums up in closed form noncrossed and crossed multi-meson exchanges of arbitrarily high order, along with self-energy and vertex-correction effects.

4. Combination with *variational* method, based on trial actions bilinear in the Fourier amplitudes of trajectories, leads to path integral performable analytically. Contrary to all known approximations to Bethe-Salpeter equation, its evaluation implies neither a neglect of dressing of $(n < 4)-$point functions (self-energy and vertex- correction effects), nor a violation of the non-relativistic limit, although the quenched approx. still does imply a certain violation of crossing symmetry.

5. One drawback of the method is that *two* approximations, quenched and variational, are necessary. Both can be in principle be improved through series expansions in the neglected terms, but the resulting double series is clumsy.

6. Experience shows that the quadratic trial-action ansatz ("Yukawa approximated by oscillator") is not very accurate, particularly for the low-energy ("threshold") regime at weak coupling.

7. On the other hand the zeroth approximation furnished by this ansatz is structured and transparent; it allows many particular aspects and limiting situations to be studied analytically, and thus leads to better insight and understanding.

8. The "realistic" application best suited for this method would seem to be the physics of heavy quarkonia in QCD. There, the renormalization problem is more involved than in a superrenormalizable theory, but the existing application to QED makes it likely that (in the framework of the bilinear ansatz) it can still be handled.

A. Bosonic Gaussian integral

$$\int \mathcal{D}\phi \, \exp i\Big[\frac{1}{2}(\phi, A\phi) + (j,\phi)\Big]$$

$$= \frac{\text{const.}}{(\det A)^{1/2}} \exp\Big[-\frac{i}{2}(j, A^{-1}j)\Big]$$

Application to trial action : if

$$\tilde{S}_M := p\cdot x - \frac{\kappa_0}{2}A_x\, x^2 - \frac{\kappa_0}{2}\sum_{i=1}^{2}\sum_{k=1}^{\infty} A_k^{(i)}\big(a_k^{(i)}\big)^2$$

$$+ \kappa_0\sum_{k=1}^{\infty} B_k\big(a_k^{(1)}\cdot a_k^{(2)}\big) + \sum_{i=1}^{2}\sum_{k=1}^{\infty} \underbrace{f_k^{(i)}}_{sources} \cdot a_k^{(i)},$$

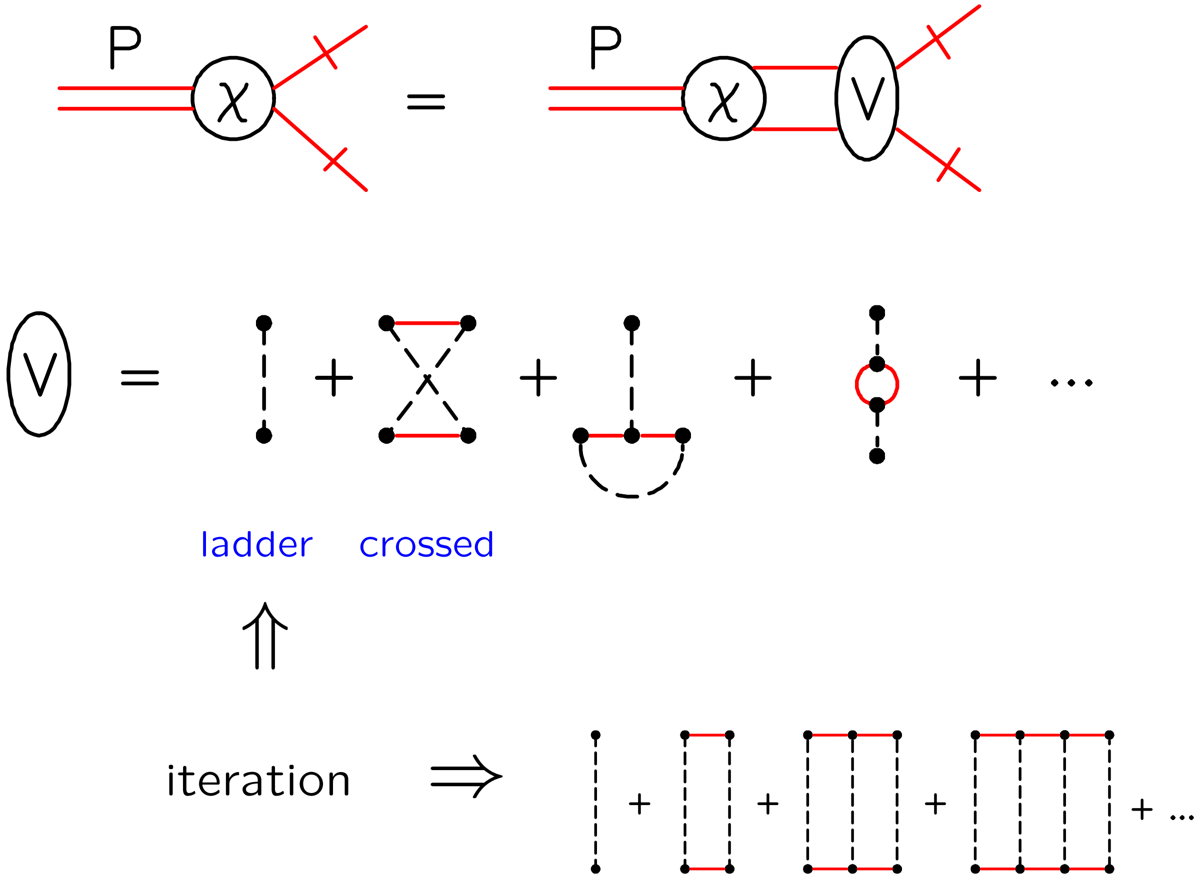
then

$$\int \mathcal{D}^d a^{(1)}\, \mathcal{D}^d a^{(2)}\, d^d x \, \exp\{i\tilde{S}_M\}$$

$$= \text{const.}\Big\{A_x \prod_{k=1}^{\infty}(A_k^{(1)}A_k^{(2)} - B_k^2)\Big\}^{-d/2} \times$$

$$\exp\Big\{\frac{i}{2\kappa_0}\Big[\frac{p^2}{A_x} + \sum_{k=1}^{\infty}\frac{A_k^{(1)}\big(f_k^{(2)}\big)^2 + A_k^{(2)}\big(f_k^{(1)}\big)^2 + 2B_k\big(f_k^{(1)}\cdot f_k^{(2)}\big)}{A_k^{(1)}A_k^{(2)} - B_k^2}\Big]\Big\}$$

B. Homogeneous Bethe-Salpeter equation:



C. A brief history of worldlines

Quantum Mechanics:

$$< x_b \,|\, \exp\left[-i\hat{H}(t_b - t_a)/\hbar\right] \,|\, x_a > \quad = \quad \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x(t)\, e^{iS[x(t)]/\hbar}$$

Heisenberg, Schrödinger, Dirac (1925 - 1927)		Dirac (1933), Feynman (1942)
operators, wavefunctions	$\longleftrightarrow$	path integrals, trajectories
“WAVES”	$\longleftrightarrow$	“PARTICLES”

Field Theory:

field operators $\widehat{\phi}(x)$, states	$\longleftrightarrow$	worldlines $x^{\mu}(t)$
Jordan, Heisenberg, Pauli ($\sim$ 1930)		Feynman ($\sim$ 1950)
"FIELDS"	$\longleftrightarrow$	"PARTICLES"
"second quantization"	$\longleftrightarrow$	"first quantization"

Dyson (1949)

$\Downarrow$

wins !

(see textbooks)

renaissance ... from string theory (!)

Bern & Kosower (1991)

Strassler (1992) showed how to derive the Bern-Kosower rules from the particle (worldline) representation of Quantum Field Theory

Advantages:

a) efficient way to calculate diagrams with many legs

b) new approximation methods for large couplings
(cf. Feynman's treatment of the polaron)