Scale Dependent Renormalization and the Schrödinger Functional

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Outline

Some Conventions

We Are Lattice People

We do Monte Carlo (MC) simulations with

- Lattice size L
- Lattice spacing a
- $(L/a)^4$ lattice points (in 4 dimensions)
- The lattice introduces a momentum cutoff a^{-1}

First we will consider pure Yang-Mills-Theory, later switch to QCD.

The Running Coupling

QCD

Bare coupling constant

$$g_0$$

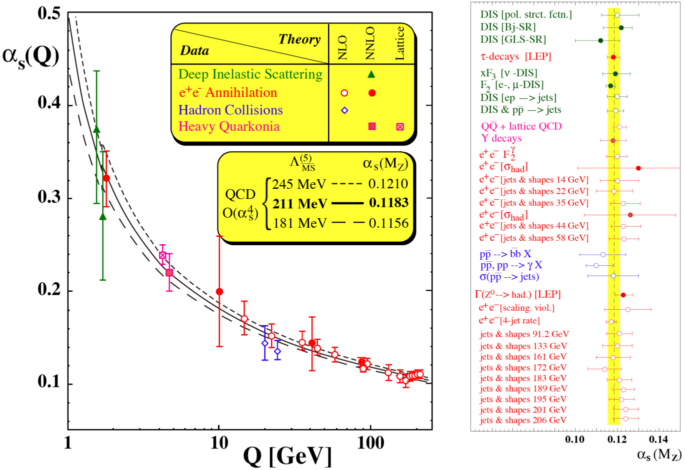
has to be renormalized.

The Real World

Physical, scale dependant coupling $\alpha(\mu)$, e.g.

$$\alpha(\mu) \propto \frac{\sigma(e^+e^- \to \bar{q}qg)}{\sigma(e^+e^- \to \bar{q}q)}$$

A Picture



Stolen from S. Bethke: α_s 2002

The Perturbation Theory Side

For high energies μ, one can use perturbation theory (PT) to make predictions. The renormalization group (RG) tells us that

$$\mu \frac{\partial \overline{g}}{\partial \mu} = \beta(\overline{g}),$$

PT then yields

$$\beta(\overline{g}) \overset{\overline{g}\to 0}{\sim} -\overline{g}^3(b_0 + \overline{g}^2 b_1 + \ldots)$$

But QCD should also describe low energy phenomena ...

What Do We Want To Do?

A Test For QCD

- Determine $\alpha(\mu)$ non-perturbatively on the lattice
- Make connection to PT in the high energy sector
- i.e. connect **low**- and **high**-energy regimes of QCD, predict e.g. Λ/F_π or simply Λ trough hadronic input
- Use PT (or some other effective theory) for 'real world predictions'
- Compare with experiments

How To Do This?

Define a physical coupling, e.g.

$$\alpha_{\overline{q}q}(\mu) := \frac{1}{C_F} r^2 F(r)\Big|_{\mu=1/r}$$

and measure it on the lattice!

Simple?

How To Do This?

Define a physical coupling, e.g.

$$\alpha_{\overline{q}q}(\mu) := \frac{1}{C_F} r^2 F(r) \Big|_{\mu=1/r}$$

and measure it on the lattice!

This doesn't work!

Why Doesn't It Work?

We have to satisfy constraints:

- $\mu \geq 10\,\mathrm{GeV}$ for PT matching
- $\mu \ll a^{-1}$ to control discretization errors
- $L \gg \frac{1}{m_\pi}, r_0$ to control finite size effects

This leads to

$$L \gg r_0, \frac{1}{m_\pi} \sim \frac{1}{0.14GeV} \gg \frac{1}{\mu} \sim \frac{1}{10GeV} \gg a$$

$\Rightarrow$ Simulate $L/a \gg 70$ lattice points in MC simulation
$\rightarrow$ (today) not possible

The Way Out (Lüscher, Weisz, Wolf, 1991)

Besides a^{-1}, another energy scale is accessible in MC simulations, namely L

The L Trick

- Identify $\mu = \frac{1}{L}$, i.e. choose finite size effects as observable
- Find a clever definition for $\alpha(L)$
- Split up the
  - Renormalization of $\alpha(L)$ for fixed L and
  - Computation of the scale dependence of α

Step By Step: The Step Scaling Function (SSF)

To investigate the scale evolution of α, define the step scaling function σ

The Step Scaling Function

- Choose starting point $u_0 = \overline{g}^2(L)$
- Choose a scaling factor s
- Define $\sigma(s, u_0) = \overline{g}^2(sL)$

This is a discrete integrated β-function

The SSF on a Lattice

REMEMBER: We Are Lattice People

- Obtained on a lattice, σ will carry a dependence on a/L
- So define

$$\Sigma(s, u, a/L) = \overline{g}^2(sL)\big|_{\overline{g}^2(L)=u, g_0 \text{ fixed}, \text{a/L fixed}}$$

- Calculate $\Sigma(s, u, a/L)$ for several lattice resolutions and take the limit

$$\sigma(s, u) = \lim_{a/L \to 0} \Sigma(s, u, a/L)$$

σ in Three ^{n} Steps

How To Obtain σ?

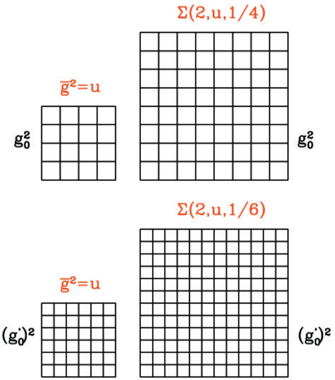
1. Choose initial $(L/a)^4$ lattice
2. Tune β such that $\overline{g}(L) = u$ is where you want to start
3. Compute $\overline{g}(2L)$ with the same bare parameters and get $\Sigma(2, u, a/L)$

Repeat for several resolutions a/L and extrapolate $a/L \to 0$

Note:

- Step 2) takes care of renormalization
- Step 3) computes the scale-evolution of of the renormalized coupling

σ: A Comic Approach



Stolen from ALPHA Collaboration

Does It Work?

One finds that

$$\frac{\Sigma(2, u, a/L) - \sigma(2, u)}{\sigma(2, u)} = \delta_1(a/L)u + \delta_2(a/L)u^2 + \dots$$

where

$$\delta_n = O(a/L).$$

This looks good, the continuum limit is reached with errors of $O(a/L)$.

What About Universality?

Question

Does σ depend on the choice of the action?

Answer

It seems not ...

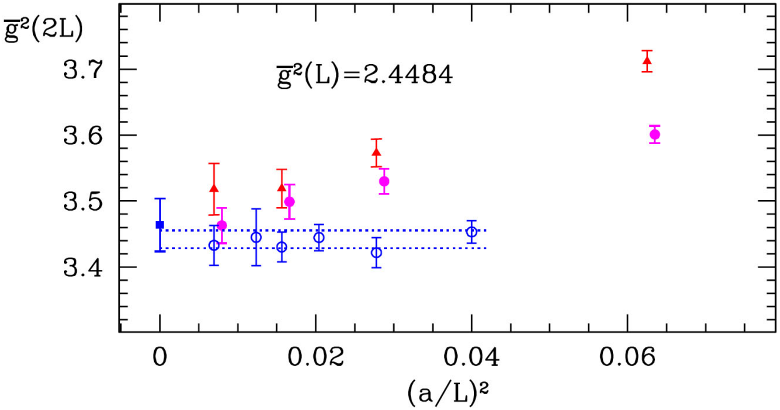
Strategy

Improve Σ

$$\Sigma^{(k)}(2, u, a/L) = \frac{\Sigma(2, u, a/L)}{1 + \sum_{i=1}^{k} \delta_i(a/L) u}$$

and calculate σ for different actions.

Some Numerical Results



Stolen from CP-CACS Collaboration

Putting It Together

What We've Got So Far

Assume, one has

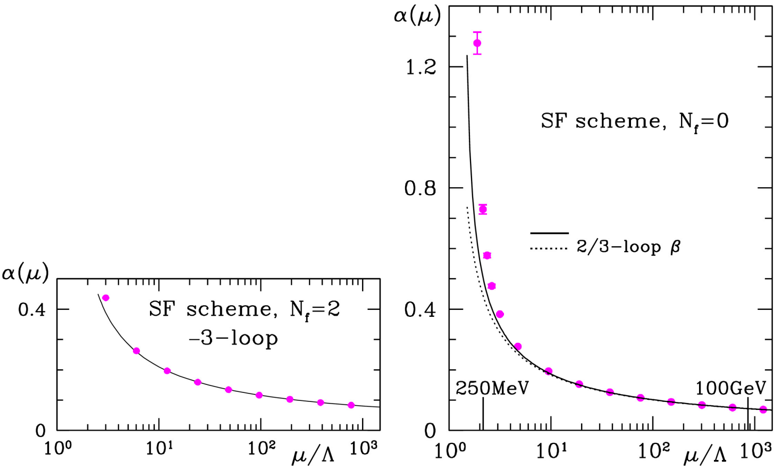
- Calculated $\sigma(u_i)$ for several u_i
- Interpolated a polynomial $\sigma(u)$

The Final Step

Then one can construct the running coupling $\overline{g}^2(2^{-i}L_0) = u_i$ via the recursion

$$u_0 = \overline{g}^2(L_0), \sigma(u_{i+1}) = u_i$$

Some Results



Done by ALPHA

What Still Has To Be Done

The Definition of $\alpha(L)$

We have to define $\alpha(L)$ such that it has

- An easy expansion in PT
- A small Monte Carlo variance
- Small discretization errors

Which leads us to ...

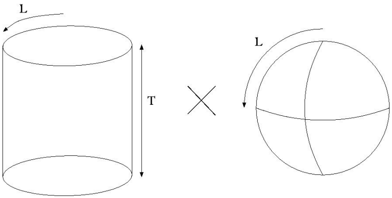
Introducing: The Schrödinger Functional

The SF ...

- Was first used by Symanzik for renormalization of the Schrödinger Picture in QFT
- Then by Lüscher and Narayanan, Weisz, Wolff for finite size scaling technique
- Is the propagation kernel of some field configuration C to another in euclidean time T

The Space-time

Our theory lives on a L^3-space-box with periodic boundary and finite time T, like this



The Players I: Gauge Fields

- On our space-time live $SU(N)$ gauge fields $A_k(\vec{x})$ on LS^3.
- We want for a $SU(N)$ gauge transformation Λ

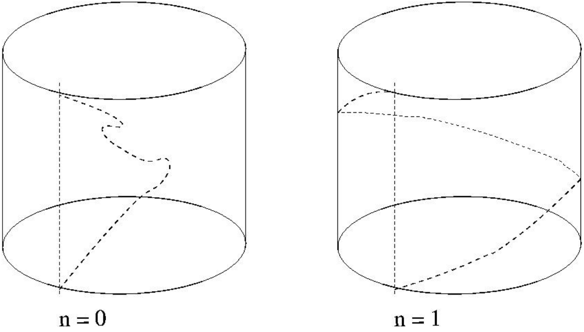
$$A_k^{\Lambda}(\vec{x}) = \Lambda(\vec{x})A_k(\vec{x})\Lambda(\vec{x})^{-1} + \Lambda(\vec{x})\partial_k\Lambda(\vec{x})^{-1}$$

  to be another gauge field.
- We only admit periodic gauge transformations Λ

The Winding Number Thing

The Operators $A : S^3 \rightarrow SU(N)$ fall in disconnected topological classes, labelled by their winding number n. A simple example:

$f : S^2 \rightarrow U(1) \simeq S^2$



The Players II: The States

A state is a wave functional $\psi[A]$. On the set of all states, a scalar product is given by

$$\langle\psi|\chi\rangle = \int \mathcal{D}[A]\psi[A]^*\chi[A]$$

with

$$\mathcal{D}[A] = \prod_{\vec{x},k,a} A_k^a(\vec{x})$$

Physical states satisfy $\psi[A^\Lambda] = \psi[A]$. We introduce the projector on the set of physical states through

$$\mathbb{P}\psi[A] = \int \mathcal{D}[\Lambda]\psi[A^\Lambda]$$

The Players III: The Boundary

How To Make Up a State ...

- Take a smooth classical gauge field $C_k(\vec{x})$
- Introduce a state $|C\rangle$ via

$$\langle C|\psi\rangle = \psi[C] \quad \forall \text{ states } \psi$$

- C can be made gauge invariant by applying $\mathbb{P}$

Putting It Together

Defining the Schrödinger Functional

- Let

$$Z[C', C] = \langle C' | e^{-HT} \mathbb{P} | C \rangle$$

- Invariant under gauge transformations due to $\mathbb{P}$

Putting It Together

Defining the Schrödinger Functional

- Let

$$
\begin{aligned}
Z[C', C] &= \langle C' | e^{-HT} \mathbb{P} | C \rangle \\
&= \sum_{n=0}^{\infty} e^{E_n T} \psi_n[C'] \psi_n[C]^*
\end{aligned}
$$

- Where ψ_n is the n-th (physical) energy eigenstate
- Invariant under gauge transformations due to $\mathbb{P}$

Going Functional

We Are Lattice People

We want a functional integral:

$$Z[C', C] = \int \mathcal{D}[\Lambda]\mathcal{D}[A] e^{S[A]}$$

(modulo renormalization factor) where

$$A_k(x) = \begin{cases} C_k^{\Lambda}(\vec{x}) & \text{at } \; x^0 = 0 \\ C_k'(\vec{x}) & \text{at } \; x^0 = T \end{cases}$$

and

$$S[A] = -\frac{1}{2g_0^2}\int \mathrm{d}^4x \, \mathrm{tr}(F_{\mu\nu}F_{\mu\nu})$$

The Topology Trick

After the $\mathcal{D}[A]$ integration, Z reads

$$Z[C', C] = \int \mathcal{D}[\Lambda] F[\Lambda]$$

and actually, F only depends on the winding number n. So we find that

$$Z[C', C] = \sum_{n=-\infty}^{\infty} \int \mathcal{D}[A] e^{S[A]}$$

where

$$A_k(x) = \begin{cases} C_k^{\Lambda_n}(\vec{x}) & \text{at } x^0 = 0 \\ C'_k(\vec{x}) & \text{at } x^0 = T \end{cases}$$

The Action And the Winding Number

A Boundary for the Action

- One finds that $S[A]$ is bounded by

$$S[A] \geq \frac{1}{2g_0^2} |S_{CS}[C] - S_{CS}[C'] + n|$$

- Where S_{CS} is the Chern-Simons action
- And n the winding number of A
- Only have to check a few topological sectors for minimal action gauge fields, which dominate the integral

The Action And the Winding Number

A Boundary for the Action

- One finds that $S[A]$ is bounded by

$$S[A] \geq \frac{1}{2g_0^2}|S_{CS}[C] - S_{CS}[C'] + n| = \frac{1}{2g_0^2}|\text{some number} + n|$$

- Where S_{CS} is the Chern-Simons action
- And n the winding number of A
- Only have to check a few topological sectors for minimal action gauge fields, which dominate the integral

Finding The Minimum

How To Obtain a Minimal Action Configuration B ?

- Generally difficult
- Easy if we
  - Take a known solution B of the field eqns. and
  - Define C, C' as

$$C_k(\vec{x}) = B_k(x)\big|_{x^0=0} \qquad C'_k(\vec{x}) = B_k(x)\big|_{x^0=T}$$

- If
  - $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$ is self dual and
  - $S_{SC}[C'] - S_{CS}[C] < 1/2$ and
  - $n(B) = 0$
- Then B is the unique (up to gauge transformations) minimal action configuration

A Simple Example

A One-Parameter Family of Background Fields

Consider the BG-field

$$B_0(x) = 0 \quad B_k(x) = b(x^0) I_k \quad [I_k, I_l] = \epsilon_{klj} I_j.$$

Self-duality condition reduces to

$$\partial_0 b = b^2 \quad \Rightarrow \quad b(x^0) = (\tau - x^0)^{-1}.$$

We just found a family of globally stable background fields!

A Simple Example

A One-Parameter Family of Background Fields

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Self-duality condition reduces to

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We will need this for α!

What About Renormalization?

Question: Is the SF Renormalizable?

In the weak coupling domain, expand the SF around the induced background field and obtain for the effective action:

$$\Gamma[B] = -\ln Z[C', C]$$
$$= g_0^{-2}\Gamma_0[B] + \Gamma_1[B] + g_0^2\Gamma_2[B] + \dots$$

with $\Gamma_0[B] = g_0^2 S[B]$, divergent in each power of g_0

Answer: Most Probably ... Yes

- Of course, one has to renormalize g_0, (m)
- In general, one has to add boundary counter-terms
- This should be sufficient (checked up to 2-loop order in QCD)
- In Yang-Mills theory, no such counter-terms are needed

The Running Coupling (Finally)

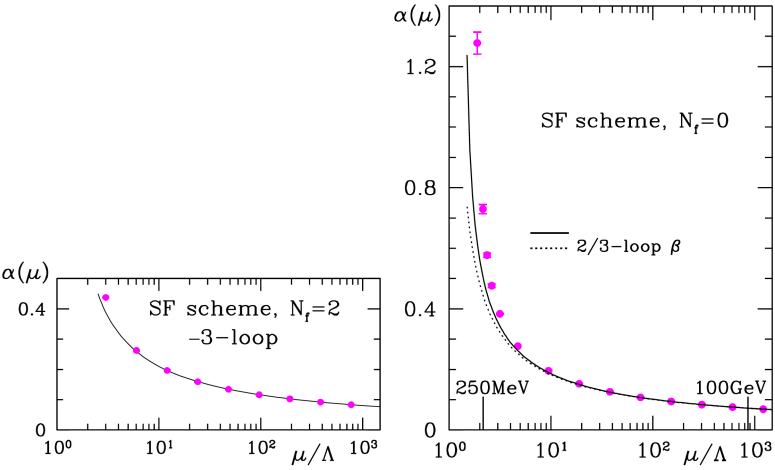
A Running Coupling Recipe

- Choose a background field B depending on a dimensionless parameter η
- Then $\Gamma'[B] = -\frac{\partial}{\partial\eta}\Gamma[B]$ is a renormalization group invariant.
- Set $T = L$ and define a physical coupling via

$$\overline{g}^2(L) := \Gamma'_0[B]/\Gamma'[B]$$

- This is a Casimir force between the boundary fields
- If the chosen field depends on parameters with dimension $\neq 1$, scale them proportional to L, e.g. in our example set $\tau = -L/\eta$

The Result



From ALPHA again

Let's Measure a Mass

Fermions

The next interesting quantities which needs scale dependent Renormalization are the quark Masses.

- Define N_f fermion fields ψ_s on our periodic space time
- Define boundary fields ζ, ζ' for quark fields
- Add counter terms for ψ at the boundary for renormalization

Definition for $\overline{m}$

Defining a Running Quark Mass

- Use the PCAC relation to define $\overline{m}$

$$\partial_\mu A_\mu^R(x) = (\overline{m}_s + \overline{m}_{s'})P^R(x)$$

with

$$A_\mu^R(x) = Z_A A_\mu(x) = Z_A \overline{\psi}_s(x)\gamma_\mu\gamma_5\psi_{s'}(x)$$
$$P^R(x) = Z_p P(x) = Z_p \overline{\psi}_s(x)\gamma_5\psi_{s'}(x)$$

- $A_\mu(x)$ is renormalized through current algebra relations
- Scale- & scheme-dependence arises through renormalization of $P(x)$, $Z_P = Z_P(\mu)$
- the corresponding RG function reads $\tau(\overline{g})\overline{m}_s = \mu\frac{\partial \overline{m}_s}{\partial \mu}$

Doing It All Over

A Definition for $Z_p(L)$

We drop s and define

$$Z_P(L) = \frac{\sqrt{3f_1}}{f_P(L/2)}$$

where $\sqrt{3f_1}$ is only a normalization factor, defined as

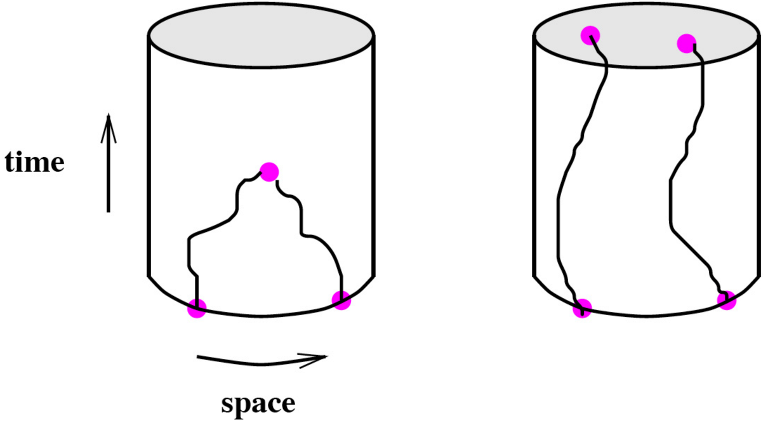
$$f_P(x) = -\frac{1}{3}\int \mathrm{d}^3y\ \mathrm{d}^3z\ \langle\overline{\psi}(x)\gamma_5\frac{1}{2}\tau^a\psi(x)\overline{\zeta}(y)\gamma_5\frac{1}{2}\tau^a\zeta(z)\rangle$$

$$f_1 = -\frac{1}{3L^6}\int \mathrm{d}^3u\ \mathrm{d}^3v\ \mathrm{d}^3y\ \mathrm{d}^3z\ \langle\overline{\zeta}'(u)\gamma_5\frac{1}{2}\tau^a\zeta'(v)\overline{\zeta}(y)\gamma_5\frac{1}{2}\tau^a\zeta(z)\rangle$$

which look complicated, but...

f_p and f_1, an Illustration

... can be illustrated like this:



Calculating $M(m, \mu)$, Pt. 1

Yet Another Step Scaling Function

So far, we have

$$\overline{m}(\mu)_s = \frac{Z_A}{Z_P(L)} m_s$$

Define the step scaling function σ_P as

$$Z_P(2L) = \sigma_P(u) Z_P(L)$$

and compute $\sigma(L_0), \ldots, \sigma(2^k L_0)$. Use these for

$$\frac{M}{\overline{m}(2^k L_0)} = \underbrace{\frac{M}{\overline{m}(L_0)}}_{\text{accessible in PT}} \underbrace{\frac{\overline{m}(L_0)}{\overline{m}(2L_0)} \frac{\overline{m}(2L_0)}{\overline{m}(2^2 L_0)} \cdots \frac{\overline{m}(2^{k-1} L_0)}{\overline{m}(2^k L_0)}}_{\sim \text{SSF}^{-1}}$$

Calculating $M(m, \mu)$, Pt. 2

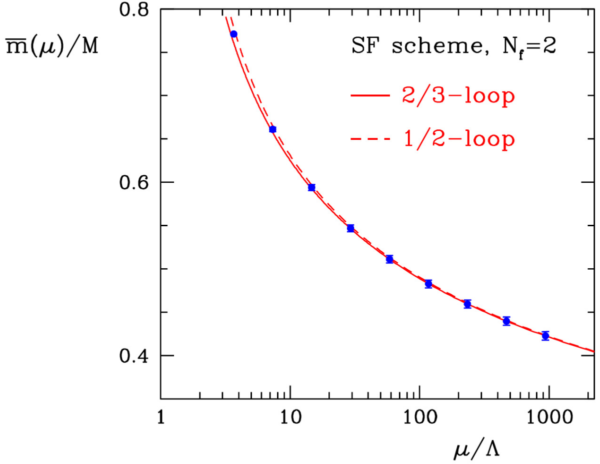
The Final Step

Finally, we can compute

$$\begin{aligned} M &= \frac{M}{\overline{m}(2^k L_0)} \overline{m}(2^k L_0) \\ &= \underbrace{\frac{M}{\overline{m}(2^k L_0)}}_{\text{(known from Pt. 1)}} \underbrace{\frac{Z_A}{Z_P\left(\mu = (2^k L_0)^{-1}\right)}}_{\text{(from simulations)}} m \\ &= Z(\mu) m \end{aligned}$$

We found the overall renormalization factor!

This Talk's Last Picture



ALPHA once more

Conclusions

- Important physical quantities like α and m require scale dependent renormalization
- Scale dependent renormalization is a difficult task, because a large variety of energy scales has to be covered
- This problem can be fixed by using a finite scaling technique
- The Schrödinger Functional provides a good framework for the definition of scale dependent quantities

Thank you!

Some literature:

- R. Sommer: Non-perturbative QCD [...], hep-lat/0611020
- Capitani, Lüscher, Sommer, Wittig: Non-perturbative quark mass renormalization in quenched lattice QCD, hep-lat/9810063