Chapter 2
Advection Equation

Let us consider a continuity equation for the one-dimensional drift of incompressible fluid. In the case that a particle density $u(x,t)$ changes only due to convection processes one can write

$$u(x, t + \triangle t) = u(x - c\triangle t, t).$$

If $\triangle t$ is sufficient small, the Taylor-expansion of both sides gives

$$u(x,t) + \triangle t \frac{\partial u(x,t)}{\partial t} \simeq u(x,t) - c\triangle t \frac{\partial u(x,t)}{\partial x}$$

or, equivalently

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0\,. \tag{2.1}$$

Here $u = u(x,t)$, $x \in \mathbb{R}$, and c is a nonzero constant velocity. Equation (2.1) is called to be *an advection equation* and describes the motion of a scalar u as it is advected by a known velocity field. According to the classification given in Sec. 1.1, Eq. (2.1) is a hyperbolic PDE. The unique solution of (2.1) is determined by an initial condition $u_0 := u(x,0)$

$$u(x,t) = u_0(x - ct)\,, \tag{2.2}$$

where $u_0 = u_0(x)$ is an arbitrary function defined on $\mathbb{R}$.
One way to find this exact solution is the method of characteristics (see App. **??**). In the case of Eq. (2.1) the coefficients $A = c$, $B = 1$, $C = 0$ and Eqn. (**??**) read

$$\begin{aligned}
\frac{dt}{ds} &= 1 \Leftrightarrow |t(0) = 0| \Leftrightarrow t = s,\\
\frac{dx}{ds} &= c \Leftrightarrow |x(0) = x_0| \Leftrightarrow x = x_0 + ct.
\end{aligned}$$

That is, for the advection equation (2.1) characteristic curves are represented by straight lines (see Fig. 2.1). Hence, Eq. (**??**) becomes

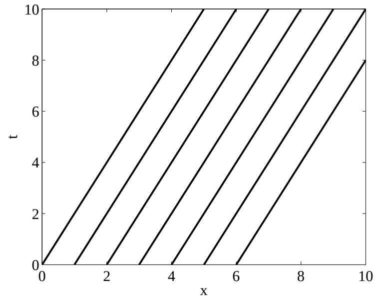


Fig. 2.1 Chareacteristic curves $x = x_0 + cs,\ s = t$ for advection equation (2.1) are shown for different values of c.

$$\frac{du}{ds} = 0 \quad \text{with} \quad u(0) = u_0(x_0)\,.$$

Alltogether the solution of (2.1) takes the form (2.2). The solution (2.2) is just an initial function u_0 shifted by ct to the right (for $c > 0$) or to the left ($c < 0$), which remains constant along the characteristic curves ($du/ds = 0$).

2.1 FTCS Method

Now we focus on different explicit methods to solve advection equation (2.1) numerically on the periodic domain $[0, L]$ with a given initial condition $u_0 = u(x, 0)$.

We start the discussion of Eq. (2.1) with a so-called FTCS (forward in time, centered in space) method. As discussed in Sec. 1.2 we introduce the discretization in time on the uniform grid

$$t_j = t_0 + j\triangle t\,, \qquad j = 0 \ldots T\,.$$

Furthermore, in the x-direction, we the uniform grid in the same manner

$$x_i = a + i\triangle x\,, \qquad i = 0 \ldots M\,, \quad \triangle x = \frac{L}{M}\,.$$

Adopting a forward temporal difference scheme (1.3), and a centered spatial difference scheme (1.7), Eq. (2.1) yields

$$\begin{aligned} \frac{u_i^{j+1} - u_i^j}{\triangle t} &= -c\frac{u_{i+1}^j - u_{i-1}^j}{2\triangle x} \Leftrightarrow \\ u_i^{j+1} &= u_i^j - \frac{c\triangle t}{2\triangle x}\left(u_{i+1}^j - u_{i-1}^j\right). \end{aligned} \tag{2.3}$$

Here we use a notation $u_i^j := u(x_i, t_j)$. Shematic representation of the FTCS approximation (2.3) is shown on Fig. 2.2.

von Neumann Stability Analysis

To investigate stability of the scheme (2.3) we follow the concept of von Neumann, introduced in Sec. 1.3. The usual ansatz

$$\varepsilon_i^j \sim e^{ikx_i}$$

leads to the following relation

$$\varepsilon_i^{j+1} = e^{ikx_i} - \frac{c\triangle t}{2\triangle x}\left(e^{ik(x_i+\triangle x)} - e^{ik(x_i-\triangle x)}\right) = \underbrace{\left(1 - \frac{c\triangle t}{2\triangle x}\left(e^{ik\triangle x} - e^{-ik\triangle x}\right)\right)}_{g(k)}\varepsilon_i^j\,,$$

where ε_i^{j+1} stands for the cumulative rounding error at time t_j. The von Neumann's stability condition (1.22) for the amplification factor $g(k)$ reads:

$$|g(k)| \leq 1 \qquad \forall k.$$

In our case one obtains:

$$|g(k)|^2 = 1 + \frac{c^2\triangle t^2}{\triangle x^2}\sin^2(k\triangle x),$$

One can see that the magnitude of the amplification factor $g(k)$ is greater than unity for all k. This implies that the instability occurs for all given c, $\triangle t$ and $\triangle x$, i.e., the FTCS scheme (2.3) is *unconditionally unstable*.

2.2 Upwind Methods

The next simple scheme we are intersted in belongs to the class of so-called *upwind methods* – numerical discretization schemes for solving hyperbolic PDEs. The idea of this method is that the spatial differences are skewed in the "upwind" direction, i.e., the direction from which the advecting flow originates. The origin of upwind methods can be traced back to the work of R. Courant et al. [2].
The simplest upwind schemes possible are given by

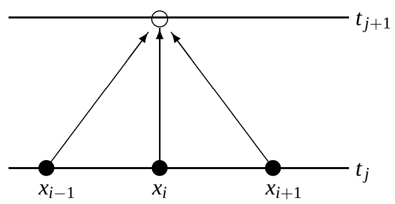


Fig. 2.2 Schematic visualization of the FTCS-method (2.3).

$$
\begin{aligned}
\frac{u_i^{j+1}-u_i^j}{\triangle t} &= -c\frac{u_i^j-u_{i-1}^j}{\triangle x} \Leftrightarrow \\
u_i^{j+1} &= u_i^j - \frac{c\triangle t}{\triangle x}\left(u_i^j-u_{i-1}^j\right), \qquad (c>0)\,.
\end{aligned} \tag{2.4}
$$

and

$$
\begin{aligned}
\frac{u_i^{j+1}-u_i^j}{\triangle t} &= -c\frac{u_{i+1}^j-u_i^j}{\triangle x} \Leftrightarrow \\
u_i^{j+1} &= u_i^j - \frac{c\triangle t}{\triangle x}\left(u_{i+1}^j-u_i^j\right) \qquad (c<0)\,.
\end{aligned} \tag{2.5}
$$

Note that the upwind scheme (2.4) corresponds to the case of positive velocities c, whereas Eq. (2.5) stands for the case $c < 0$. The next point to emphasize is that both schemes (2.4)–(2.5) are only first-order in space and time. Shematic representations of both upwind methods is presented on Fig. 2.3

In the matrix form the upwind scheme (2.4) takes the form

$$
\mathbf{u}^{j+1} = A\mathbf{u}^j, \tag{2.6}
$$

where $\mathbf{u}^j$ is a vector on the time step j and A is a $n \times n$ matrix ($h := \triangle t/\triangle x$),

$$
A = \begin{pmatrix}
1-ch & 0 & 0 & \ldots \boxed{ch} \\
ch & 1-ch & 0 & \ldots 0 \\
0 & ch & 1-ch & \ldots 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & \ldots & ch & 1-ch
\end{pmatrix}
$$

The boxed element A_{1n} indicates the influence of the periodic boundary conditions. Similary, one can also represent the scheme (2.5) in the form (2.6) with matrix

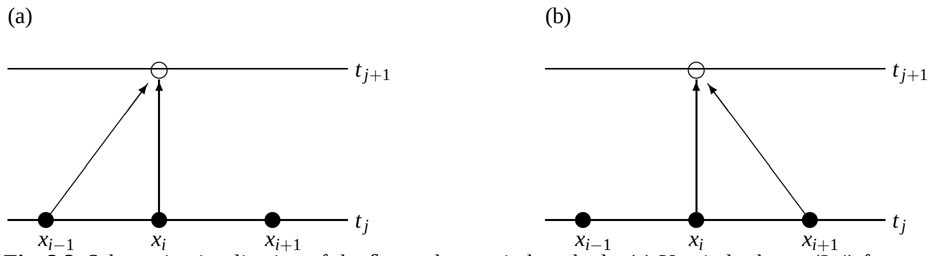


Fig. 2.3 Schematic visualization of the first-order upwind methods. (a) Upwind scheme (2.4) for $c > 0$. (b) Upwind scheme (2.5) for $c < 0$.

$$A = \begin{pmatrix} 1+ch & -ch & 0 & \dots 0 \\ 0 & 1+ch & -ch & \dots 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 1+ch & -ch \\ \boxed{-ch} & \dots & 0 & 1+ch \end{pmatrix}$$

Again, the boxed element A_{n1} displays the influence of periodic boundary conditions.

von Neumann Stability Analysis

In order to investigate the stability of the upwind scheme (2.4) (or (2.5)) we start with the usual ansatz

$$\varepsilon_i^j \sim e^{ikx_i},$$

leading to the equation for the cumulative rounding error at time t_{j+1}

$$\varepsilon_i^{j+1} = g(k)\varepsilon_i^j,$$

where the amplification factor $g(k)$ for, e.g., the upwind scheme (2.4) is given by

$$g(k) = 1 - \frac{c\triangle t}{\triangle x}\left(1 - e^{-ik\triangle x}\right) = \left|\alpha = \frac{c\triangle t}{\triangle x}, \varphi = -k\triangle x\right| = 1 - \alpha + \alpha e^{i\varphi}.$$

The stability condition (1.22) is fulfilled for all k as long as

$$|g(k)| \leq 1 \Leftrightarrow 1 - \alpha \leq 0 \Leftrightarrow \boxed{\frac{c\triangle t}{\triangle x} \leq 1 \Leftrightarrow c \leq \frac{\triangle x}{\triangle t}.} \tag{2.7}$$

That is, the method (2.4) is *conditionally stable*, i.e., is stable if and only if the "physical" velocity c is not bigger than the spreading velocity $\triangle x/\triangle t$ of the numerical method. This is equivalent to the condition that the time step, $\triangle t$, must be smaller than the time taken for the wave to travel the distance of the spatial step, $\triangle x$. Schematic illustration of stability condition (2.7) is shown on Fig. . Condition (2.7) is called *a Courant-Friedrichs-Lewy (CFL)* stability criterion whereas α is. The condition (2.7) is named after R. Courant, K. Friedrichs, and H. Lewy, who described it in their paper in 1928 [8].

Numerical results

Figure 2.5 shows an example of the calculation in which the upwind scheme (2.4) is used to advect a Gaussian pulse. Parameters of the calculation are choosen as

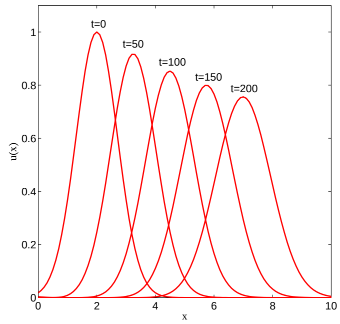


Fig. 2.4 Advection of a one-dimensional Gaussian shaped pulse $u_0 = \exp(-(x-0.2)^2)$ with the scheme (2.4). Numerical calculation performed on the interval $x \in [0, 10]$ using $c = 0.5$, $\triangle t = 0.05$, $\triangle x = 0.1$. Numerical solutions at different times $t = 0$, $t = 50$, $t = 100$, $t = 150$, $t = 200$ are shown.

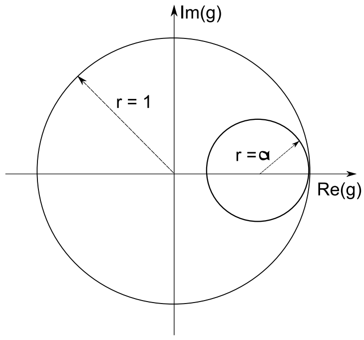


Fig. 2.5 Scematic illustration of the stability condition (2.7) for the upwind-method (2.4).

`Space interval`	L=10
`Initial condition`	$u_0(x) = \exp(-(x-2)^2)$
`Space discretization step`	$\triangle x = 0.1$
`Time discretization step`	$\triangle t = 0.05$
`Velocity`	$c = 0.5$
`Amount of time steps`	$T = 200$

For parameter values given above the stability condition (2.7) is fulfilled, so the scheme (2.4) is stable. On the other hand, one can see, that the wave-form shows evidence of dispersion. We discuss this problem in details in the next section.

2.3 The Lax Method

Let us consider a minor modification of the FTCS-method (2.3), in which the term u_i^j has been replaced by an average over its two neighbours (see Fig. 2.6):

$$u_i^{j+1} = \frac{1}{2}\left(u_{i+1}^j + u_{i-1}^j\right) - \frac{c\triangle t}{2\triangle x}\left(u_{i+1}^j - u_{i-1}^j\right). \tag{2.8}$$

In this case the matrix A of the linear system (2.6) is given by a sparse matrix with zero main diagonal

$$A = \begin{pmatrix} 0 & a & 0 & 0 & \dots & 0 & 0 & \boxed{\mathrm{b}} \\ b & 0 & a & 0 & \dots & 0 & 0 & 0 \\ 0 & b & 0 & a & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & b & 0 & a \\ \boxed{\mathrm{a}} & 0 & 0 & 0 & \dots & 0 & b & 0 \end{pmatrix},$$

where

$$a = \frac{1}{2} - \frac{c\triangle t}{2\triangle x},$$
$$b = \frac{1}{2} + \frac{c\triangle t}{2\triangle x}.$$

and the boxed elements represent the influence of periodic boundary conditions.

von Neumann stability analysis

In the case of the scheme (2.8) the amplification factor $g(k)$ becomes

$$g(k) = \cos k\triangle x - i\frac{c\triangle t}{\triangle x}\sin k\triangle x.$$

With $\alpha = \frac{c\triangle t}{\triangle x}$ and $\varphi(k) = k\triangle x$ one obtains

$$|g(k)|^2 = \cos^2\varphi(k) + \alpha^2\sin^2\varphi(k) = 1 - (1-\alpha^2)\sin^2\varphi(k).$$

The stability condition (1.22) is fulfilled for all k as long as

$$1 - \alpha^2 \geq 0 \Leftrightarrow \frac{c\triangle t}{\triangle x} \leq 1,$$

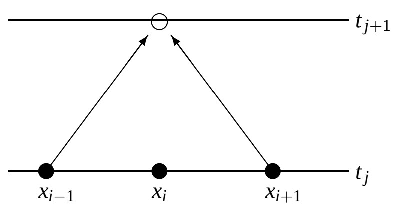


Fig. 2.6 Schematic visualization of the Lax method (2.8).

which is again the Courant-Friedrichs-Lewy condition (2.7). In fact, all stable *explicit* differencing schemes for solving the advection equation (2.1) are subject to the CFL constraint, which determines the maximum allowable time-step $\triangle t$.

Numerical results

Consider a realization of the Lax method (2.8) on the concrete numerical example:

`Space interval`	L=10
`Initial condition`	$u_0(x) = \exp(-10(x-2)^2)$
`Space discretization step`	$\triangle x = 0.05$
`Time discretization step`	$\triangle t = 0.05$
`Velocity`	$c = 0.5$
`Amount of time steps`	$T = 200$

As can be seen from Fig. 2.7 (a) like the upwind method (2.4), the Lax scheme introduces a spurious *dispersion* effect into the advection problem (2.1). Although the pulse is advected at the correct speed (i.e., it appears approximately stationary in the co-moving frame $x-ct$ (see Fig. 2.7 (b))), it does not remain the same shape as it should.

Fourier Analysis

One can try to understand the origin of the dispersion effect with the help of the dispersion relation. The ansatz of the Fourier mode of the form

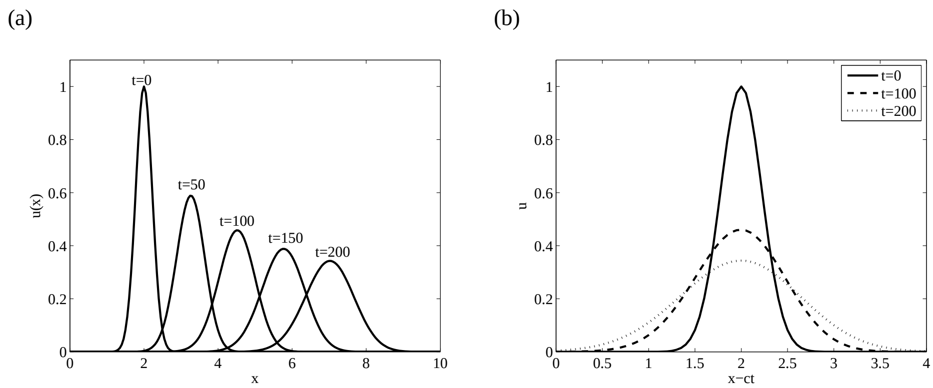


Fig. 2.7 Numerical implementation of the Lax method (2.8). Parameters: Advection velocity is $c = 0.5$, length of the space interval is $L = 10$, space and time discretization steps are $\triangle x = 0.05$ and $\triangle t = 0.05$, amount of time steps is $T = 200$, and initial condition is $u_0(x) = \exp(-10(x-2)^2)$. (a) Time evolution of $u(x,t)$ for different time moments. Solutions at $t = 0$, 100, 150, 200 are shown. (b) Time evolution in the co-moving frame $x-ct$ at $t = 0$, 100, 200.

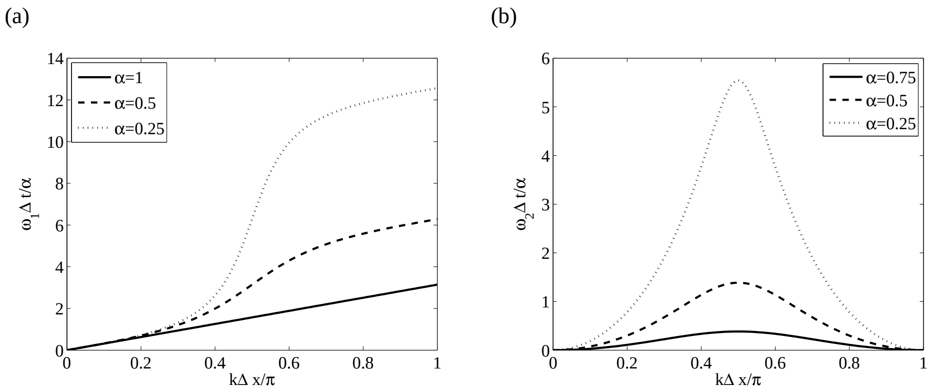


Fig. 2.8 Illustration of the dispersion relation for the Lax method calculated for different values of the Courant number α. (a) Real part of ω. (b) Imaginary part of ω.

$$u_i^j \sim e^{ikx_i - i\omega t_j}$$

for Eq. (2.8) results in the following relation

$$e^{-i\omega\triangle t} = \cos k\triangle x - i\alpha \sin k\triangle x\,,$$

where again $\alpha = c\triangle t/\triangle x$. For $\alpha = 1$ the right hand side of this relation is equal to $\exp(-ik\triangle x)$ and one otbaines

$$\omega = k\frac{\triangle x}{\triangle t} = k\cdot c\,.$$

That is, in this case the Lax method (2.8) is exact (the phase velocity ω/k is equal c). However, in general case one should suppose $\omega = \omega_1 - i\omega_2$, i.e., the Fourier modes are of the form

$$u(x,t) \sim e^{ikx - i(\omega_1 - i\omega_2)t} = e^{i(kx - \omega_1 t)}e^{-\omega_2 t}$$

and the corresponding dispersion relation reads

$$\omega\triangle t = (\omega_1 - i\omega_2)\triangle t = i\,\ln\big(\cos k\triangle x - i\alpha\sin k\triangle x\big)\,. \tag{2.9}$$

Hence, if $\omega_2 \geq 0$ one has deal with damped waves, that decay exponentially with the time contstant $1/\omega_2$. Furthermore, from Eq. (2.9) can be seen, that for $\alpha < 1$ Fouirer modes with wavelength about some grid constants ($\lambda = 2\pi/k \approx 4\triangle x$) are not only damped (see Fig 2.8 (b)) but, on the other hand, propagate with the essential greater phase velocity ω_1/k as long-wave components (see Fig. 2.8 (a)). Now the question we are interested in is what is the reason for this unphysical behavior? To answer this question let us rewrite the differencial scheme (2.8):

$$\underbrace{\frac{1}{2}u_i^{j+1}+\frac{1}{2}u_i^{j+1}}_{u_i^{j+1}}\underbrace{-\frac{1}{2}u_i^{j-1}+\frac{1}{2}u_i^{j-1}}_{0}=\frac{1}{2}\left(u_{i+1}^j+u_{i-1}^j\right)+\underbrace{u_i^j-u_i^j}_{0}-\frac{c\triangle t}{2\triangle x}\left(u_{i+1}^j-u_{i-1}^j\right)\Leftrightarrow$$

$$\frac{1}{2}\left(u_i^{j+1}-u_i^{j-1}\right)=\frac{1}{2}\left(u_{i+1}^j-2u_i^j+u_{i-1}^j\right)-\frac{c\triangle t}{2\triangle x}\left(u_{i+1}^j-u_{i-1}^j\right)-\frac{1}{2}\left(u_i^{j+1}-2u_i^j+u_i^{j-1}\right),$$

or, in the continuous limit,

$$\frac{\partial u}{\partial t}=\frac{\triangle x^2}{2\triangle t}\frac{\partial^2 u}{\partial x^2}-c\frac{\partial u}{\partial x}-\frac{\triangle t^2}{2}\frac{\partial^2 u}{\partial t^2} \tag{2.10}$$

Althougth the last term in (2.10) tends to zero as $\triangle t\to 0$, the behavior of the first term depends on the behavior of $\triangle t$ and $\triangle x$. That is, the Lax method is not a consistent way to solve Eq. (2.1). This message becomes clear if one calculates the partial derivative

$$\frac{\partial^2 u}{\partial t^2}\overset{(2.1)}{=}c^2\frac{\partial^2 u}{\partial x^2}.$$

Substitution of the last expresssion into Eq. (2.10) relults in the equation, which in addition to the advection term includes diffusion term as well,

$$\frac{\partial u}{\partial t}=-c\frac{\partial u}{\partial x}+D\frac{\partial^2 u}{\partial x^2},$$

where

$$D=\frac{\triangle x^2}{2\triangle t}-c^2\frac{\triangle t}{2}$$

is a positive diffusion constant. Now the unphysical behavior of the Fourier modes becomes clear–we have integrated *the wrong equation!* That is, other numerical approximations should be used to solve Eq. (2.1) in a more correct way.

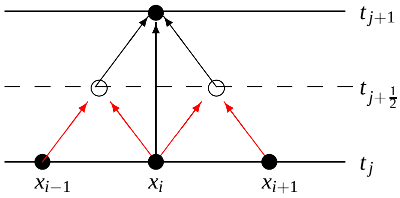
2.4 The Lax-Wendroff method

The Lax-Wendroff method, named after P. Lax and B. Wendroff [5], can be derived in a variety of ways. Let us consider two of them. The first way is based on the idea of so-called *multistep* methods. First of all let us calculate u_i^{j+1} using the information on the half time step:

$$u_i^{j+\frac{1}{2}}=u_i^j+\frac{\triangle t}{2}\left(-c\frac{\partial u}{\partial x}\bigg|_{(i,j)}\right),$$
$$u_i^{j+1}=u_i^j+\triangle t\left(-c\frac{\partial u}{\partial x}\bigg|_{(i,j+\frac{1}{2})}\right).$$

Now we use the central difference to approximate the derivative $u_x|_{i,j+\frac{1}{2}}$, i.e.,

Fig. 2.9 Schematical visualization of the Lax-Wendroff method (2.11).



$$u_i^{j+1} = u_i^j - \frac{c\triangle t}{\triangle x}\left(u_{i+\frac{1}{2}}^{j+\frac{1}{2}} - u_{i-\frac{1}{2}}^{j+\frac{1}{2}}\right).$$

On the second step, both quantities $u_{i\pm\frac{1}{2}}^{j+\frac{1}{2}}$ can be calculated using the Lax method (2.8). As the result, following two-steps scheme is obtained (see Fig. 2.9):

$$\begin{aligned} u_{i-\frac{1}{2}}^{j+\frac{1}{2}} &= \frac{1}{2}\left(u_i^j + u_{i-1}^j\right) - \frac{c\triangle t}{2\triangle x}\left(u_i^j - u_{i-1}^j\right), \\ u_{i+\frac{1}{2}}^{j+\frac{1}{2}} &= \frac{1}{2}\left(u_i^j + u_{i+1}^j\right) - \frac{c\triangle t}{2\triangle x}\left(u_{i+1}^j - u_i^j\right), \\ u_i^{j+1} &= u_i^j - \frac{c\triangle t}{\triangle x}\left(u_{i+\frac{1}{2}}^{j+\frac{1}{2}} - u_{i-\frac{1}{2}}^{j+\frac{1}{2}}\right). \end{aligned} \tag{2.11}$$

The approximation scheme (2.11) can also be rewritten as

$$u_i^{j+1} = b_{-1}u_{i-1}^j + b_0 u_i^j + b_1 u_{i+1}^j\,, \tag{2.12}$$

where constants b_{-1}, b_0 and b_1 are given by

$$\begin{aligned} b_{-1} &= \frac{\alpha}{2}(\alpha+1)\,, \\ b_0 &= 1-\alpha^2\,, \\ b_1 &= \frac{\alpha}{2}(\alpha-1) \end{aligned}$$

and α is the Courant number. The matrix A of the linear system (2.6) is a sparse matrix of the form

$$A = \begin{pmatrix} b_0 & b_1 & 0 & 0 & \dots & 0 & 0 & \boxed{b_{-1}} \\ b_{-1} & b_0 & b_1 & 0 & \dots & 0 & 0 & 0 \\ 0 & b_{-1} & b_0 & b_1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & b_{-1} & b_0 & b_1 \\ \boxed{b_1} & 0 & 0 & 0 & \dots & 0 & b_{-1} & b_0 \end{pmatrix},$$

where boxed elements stays for influence of the periodic boundary conditions.

Notice that the three-point scheme (2.12) is second-order accurate in space and time. The distinguishing feature of the Lax–Wendroff method is, that for the linear advection equation (2.1) it is the only *explicit* scheme of second-order accuracy in space and time.
The second way to derive the Lax-Wendroff diffrential scheme is based on the idea that we would like to get a scheme with second-order accurate in space and time. First of all, we use Taylor series expansion in time, namely

$$u(x_i,t_{j+1}) = u(x_i,t_j) + \triangle t \partial_t u(x_i,t_j) + \frac{\triangle t^2}{2}\partial_t^2 u(x_i,t_j) + \mathscr{O}(\triangle t^2)\,.$$

In the next place one replaces time derivatives in the last expression by space derivatives according to the relation

$$\partial_t^{(n)} u = (-c)^n \partial_x^{(n)} u.$$

Hence

$$u(x_i,t_{j+1}) = u(x_i,t_j) - c\triangle t \partial_x u(x_i,t_j) + \frac{c^2\triangle t^2}{2}\partial_x^2 u(x_i,t_j) + \mathscr{O}(\triangle t^2)\,.$$

Finally, the space derivatives are approximated by central differences (1.7), (1.12), resulting in the Lax-Wendroff scheme (2.12).

von Neumann stability analysis

In the case of the method (2.12) the amplification factor $g(k)$ becomes

$$g(k) = \big(1 + \alpha^2(\cos(k\triangle x) - 1)\big) - i\alpha\sin(k\triangle x)$$

and

$$|g(k)|^2 = 1 - \alpha^2(1-\alpha^2)\big(1-\cos(k\triangle x)\big)^2\,.$$

Hence, the stability condition (1.22) reads

$$1 - \alpha^2 \geq 0 \Leftrightarrow \alpha = \frac{c\triangle x}{\triangle t} \leq 1\,,$$

and one becomes (as expected) the CFL-condition (2.7) again.

Fourier analysis

In order to check availability of dispersion, let us calculate the dispersion relation for the scheme (2.12). The ansatz of the form $\exp\big(i(kx_i - \omega t_j)\big)$ results in

$$e^{-i\omega\triangle t} = \big(1+\alpha^2(\cos(k\triangle x)-1)\big) - i\alpha\sin(k\triangle x)\,,$$

and with $\omega = \omega_1 + i\omega_2$ one obtaines

$$\omega\triangle t = \omega_1\triangle t - i\omega_2\triangle t = i\ln\bigg(\big(1+\alpha^2(\cos(k\triangle x)-1)\big) - i\alpha\sin(k\triangle x)\bigg)\,.$$

One can easily see, that in the case of (2.12) dispersion (see Fig. 2.10 (a)) as well as damping (diffusion) (see Fig. 2.10 (b)) of Fourier modes take place. However, as can be seen on Fig. 2.10 and Fig. 2.11, dispersion and diffusion are weaker as for the Lax method (2.8) and appear by much smaller wave lengths. Because of these properties and taking into account the fact that the method (2.12) is of the second order, it becomes a standard scheme to approximate Eq. (2.1). Moreover, the scheme (2.12) can be generalized to the case of conservation equation in general form.

Lax-Wendroff method for 1D conservation equations

A typical one-dimensional evolution equation takes the form

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0\,, \tag{2.13}$$

where $u = u(x,t)$ and the form of a function $F(u)$ depends on the problem we are interested in. One can try to apply the Lax-Wendroff method (2.12) to Eq. (2.13). With $F_i^j := F(u_i^j)$ one obtains the following differential scheme

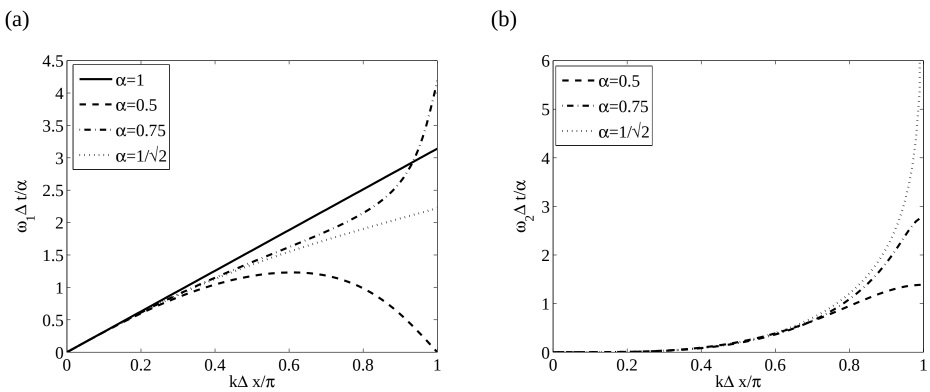


Fig. 2.10 Illustration of the dispersion relation for the Lax-Wendroff method calculated for different values of α. (a) Real part of ω (dispersion). (b) Imaginary part of ω (diffusion).

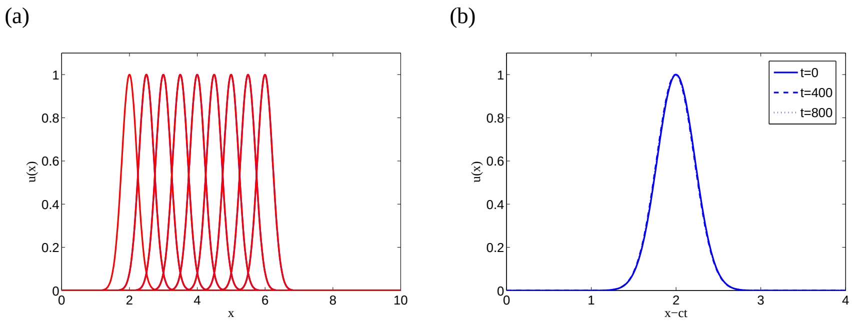


Fig. 2.11 Numerical implementation of the Lax-Wendroff method (2.12). Parameters are: Advection velocity is $c = 0.5$, length of the space interval is $L = 10$, space and time discretization steps are $\triangle x = 0.05$ and $\triangle t = 0.05$, amount of time steps is $T = 800$, and initial condition is $u_0(x) = \exp(-(x-2)^2)$. (a) Time evolution of $u(x,t)$ for different time moments. (b) Time evolution in the co-moving frame $x - ct$ at $t = 0, 400, 800$.

$$
\begin{aligned}
u_{i-\frac{1}{2}}^{j+\frac{1}{2}} &= \frac{1}{2}\left(u_i^j + u_{i-1}^j\right) - \frac{\triangle t}{2\triangle x}\left(F_i^j - F_{i-1}^j\right), \\
u_{i+\frac{1}{2}}^{j+\frac{1}{2}} &= \frac{1}{2}\left(u_i^j + u_{i+1}^j\right) - \frac{\triangle t}{2\triangle x}\left(F_{i+1}^j - F_i^j\right), \\
u_i^{j+1} &= u_i^j - \frac{\triangle t}{\triangle x}\left(F_{i+\frac{1}{2}}^{j+\frac{1}{2}} - F_{i-\frac{1}{2}}^{j+\frac{1}{2}}\right).
\end{aligned}
\tag{2.14}
$$

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