Chapter 4
The Wave Equation

Another classical example of a hyperbolic PDE is a wave equation. The wave equation is a second-order linear hyperbolic PDE that describes the propagation of a variety of waves, such as sound or water waves. It arises in different fields such as acoustics, electromagnetics, or fluid dynamics. In its simplest form, the wave equation refers to a scalar function $u = u(\mathbf{r}, t)$, $\mathbf{r} \in \mathbb{R}^n$ that satisfies:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u\,. \tag{4.1}$$

Here ∇^2 denotes the Laplacian in $\mathbb{R}^n$ and c is a constant speed of the wave propagation. An even more compact form of Eq. (4.1) is given by

$$\Box^2 u = 0\,,$$

where $\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ is the d'Alembertian.

4.1 The Wave Equation in 1D

The wave equation for the scalar u in the one dimensional case reads

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{4.2}$$

The one-dimensional wave equation (4.2) can be solved exactly by d'Alembert's method, using a Fourier transform method, or via separation of variables. To illustrate the idea of the d'Alembert method, let us introduce new coordinates (ξ, η) by use of the transformation

$$\xi = x - ct\,, \qquad \eta = x + ct\,. \tag{4.3}$$

In the new coordinate system one can write

$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\,, \qquad \frac{1}{c^2}u_{tt} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}\,,$$

and Eq. (4.2) becomes

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = 0\,. \tag{4.4}$$

That is, the function u remains constant along the curves (4.3), i.e., Eq. (4.3) describes characteristic curves of the wave equation (4.2) (see App. B). Moreover, one can see that the derivative $\partial u/\partial\xi$ does not depends on η, i.e.,

$$\frac{\partial}{\partial\eta}\left(\frac{\partial u}{\partial\xi}\right) = 0 \Leftrightarrow \frac{\partial u}{\partial\xi} = f(\xi)\,.$$

After integration with respect to ξ one obtains

$$u(\xi,\eta) = F(\xi) + G(\eta)\,,$$

where F is the primitive function of f and G is the "constant" of integration, in general the function of η. Turning back to the coordinates (x,t) one obtains the general solution of Eq. (4.2)

$$\boxed{u(x,t) = F(x-ct) + G(x+ct)\,.} \tag{4.5}$$

4.1.1 Solution of the IVP

Now let us consider an initial value problem for Eq. (4.2):

$$\begin{aligned} u_{tt} &= c^2 u_{xx}\,, \quad t \geq 0\,, \\ u(x,0) &= f(x)\,, \\ u_t(x,0) &= g(x)\,. \end{aligned} \tag{4.6}$$

To write down the general solution of the IVP for Eq. (4.2), one needs to exspress the arbitrary function F and G in terms of initial data f and g. Using the relation

$$\frac{\partial}{\partial t}F(x-ct) = -cF'(x-ct)\,, \quad \text{where} \quad F'(x-ct) := \frac{\partial}{\partial\xi}F(\xi)$$

one becomes:

$$\begin{aligned} u(x,0) &= F(x) + G(x) = f(x)\,; \\ u_t(x,0) &= c(-F'(x) + G'(x)) = g(x)\,. \end{aligned}$$

After differentiation of the first equation with respect to x one can solve the system in terms of $F'(x)$ and $G'(x)$, i.e.,

$$F'(x) = \frac{1}{2}\left(f'(x) - \frac{1}{c} g(x) \right), \qquad G'(x) = \frac{1}{2}\left(f'(x) + \frac{1}{c} g(x) \right).$$

Hence

$$F(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(y)\, dy + C, \quad G(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(y)\, dy - C,$$

where the integration constant C is chosen in such a way that the initial condition $F(x) + G(x) = f(x)$ is fullfield. Alltogether one obtains:

$$\boxed{u(x,t) = \frac{1}{2}\Big(f(x-ct) + f(x+ct) \Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\, dy}. \tag{4.7}$$

4.1.2 *Numerical Treatment*

4.1.2.1 A Simple Explicit Method

The first idea is just to use central differences for both time and space derivatives, i.e.,

$$\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{\triangle t^2} = c^2 \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{\triangle x^2}, \tag{4.8}$$

or, with $\alpha = c\triangle t / \triangle x$

$$\boxed{u_i^{j+1} = -u_i^{j-1} + 2(1-\alpha^2)u_i^j + \alpha^2 (u_{i+1}^j + u_{i-1}^j)}. \tag{4.9}$$

Schematical representation of the scheme (4.9) is shown on Fig. 4.1.

Note that one should also implement initial conditions (4.6). In order to implement the second initial condition one needs the virtual point u_i^{-1},

$$u_t(x_i, 0) = g(x_i) = \frac{u_i^1 - u_i^{-1}}{2\triangle t} + \mathcal{O}(\triangle t^2).$$

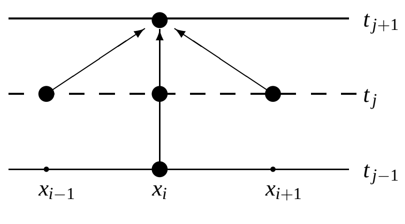

Fig. 4.1 Schematical visualization of the numerical scheme (4.9) for (4.2).

With $g_i := g(x_i)$ one can rewrite the last expression as

$$u_i^{-1} = u_i^1 - 2\triangle t g_i + \mathscr{O}(\triangle t^2),$$

and the second time row can be calculated as

$$\boxed{u_i^1 = \triangle t g_i + (1-\alpha^2) f_i + \frac{1}{2}\alpha^2 (f_{i-1} + f_{i+1}),} \tag{4.10}$$

where $u(x_i, 0) = u_i^0 = f(x_i) = f_i$.

von Neumann Stability Analysis

In order to investigate the stability of the explicit scheme (4.9) we start with the usual ansatz (1.21)

$$\varepsilon_i^{j+1} = g^j e^{ikx_i},$$

which leads to the following expression for the amplification factor $g(k)$

$$g^2 = 2(1-\alpha^2)g - 1 + 2\alpha^2 g \cos(k\triangle x).$$

After several transformations the last expression becomes just a quadratic equation for g, namely

$$g^2 - 2\beta g + 1 = 0, \tag{4.11}$$

where

$$\beta = 1 - 2\alpha^2 \sin^2\big(\frac{k\triangle x}{2}\big).$$

Solutions of the equation for $g(k)$ read

$$g_{1,2} = \beta \pm \sqrt{\beta^2 - 1}.$$

Notice that if $\beta > 1$ then at least one of absolute value of $g_{1,2}$ is bigger that one. Therefor one should desire for $\beta < 1$, i.e.,

$$g_{1,2} = \beta \pm i\sqrt{\beta^2 - 1}$$

and

$$|g|^2 = \beta^2 + 1 - \beta^2 = 1.$$

That is, the scheme (4.9) is conditional stable. The stability condition reads

$$-1 \le 1 - 2\alpha^2 \sin^2\left(\frac{k\triangle x}{2}\right) \le 1,$$

what is equivalent to the standart CFL condition (2.7)

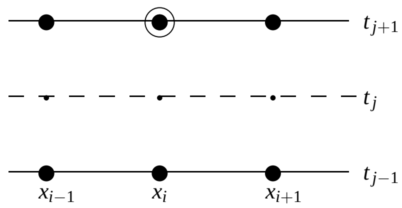

Fig. 4.2 Schematical visualization of the implicit numerical scheme (4.12) for (4.2).

$$\alpha = \frac{c\triangle t}{\triangle x} \leq 1\,.$$

4.1.2.2 An Implicit Method

One can try to overcome the problems with conditional stability by introducing *an implicit scheme*. The simplest way to do it is just to replace all terms on the right hand side of (4.8) by an average from the values to the time steps $j+1$ and $j-1$, i.e,

$$\frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{\triangle t^2} = \frac{c^2}{2\triangle x^2}\left(u_{i+1}^{j-1} - 2u_i^{j-1} + u_{i-1}^{j-1} + u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}\right). \quad (4.12)$$

Schematical diagramm of the numerical scheme (4.12) is shown on Fig. (4.2).

Let us check the stability of the implicit scheme (4.12). To this aim we use the standart ansatz

$$\varepsilon_i^{j+1} = g^j e^{ikx_i}$$

leading to the equation for $g(k)$

$$\beta g^2 - 2g + \beta = 0$$

with

$$\beta = 1 + \alpha^2 \sin^2\left(\frac{k\triangle x}{2}\right).$$

One can see that $\beta \geq 1$ for all k. Hence the solutions $g_{1,2}$ take the form

$$g_{1,2} = \frac{1 \pm i\sqrt{1-\beta^2}}{\beta}$$

and

$$|g|^2 = \frac{1-(1-\beta^2)}{\beta^2} = 1\,.$$

That is, the implicit scheme (4.12) is *absolute stable*.

Now, the question is, whether the implicit scheme (4.12) is better than the explicit scheme (4.9) form numerical point of view. To answer this question, let us analyse dispersion relation for the wave equation (4.2) as well as for both schemes (4.9) and

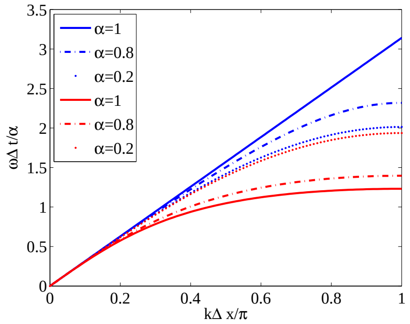

Fig. 4.3 Dispersion relation for the one-dimensional wave equation (4.2), calculated using the explicit (blue curves) and implicit (red curves) methods (4.9) and (4.12).

(4.12). The exact dispersion relation is

$$\omega = \pm ck\,,$$

i.e, all Fourier modes propagate without dispersion with the same phase velocity $\omega/k = \pm c$. Using the ansatz $u_i^j \sim e^{ikx_i - i\omega t_j}$ for the explicit method (4.9) one obtains:

$$\cos(\omega\triangle t) = 1 - \alpha^2(1-\cos(k\triangle x))\,, \tag{4.13}$$

while for the implicit method (4.12)

$$\cos(\omega\triangle t) = \frac{1}{1+\alpha^2(1-\cos(k\triangle x))}\,. \tag{4.14}$$

One can see that for $\alpha \to 0$ both methods provide the same result, otherwise the explicit scheme (4.9) always exceeds the implicit one (see Fig. (4.3)). For $\alpha = 1$ the scheme (4.9) becomes exact, while (4.12) deviates more and more from the exact value of ω for increasing α. Hence, for Eq. (4.2) there are no motivation to use implicit scheme instead of the explicit one.

4.1.3 Examples

Example 1.

Use the explicit method (4.9) to solve the one-dimansional wave equation (4.2):

$$u_{tt} = 4\,u_{xx} \quad \text{for} \quad x\in[0,L] \quad \text{and} \quad t\in[0,T] \tag{4.15}$$

with boundary conditions

$$u(0,t) = 0 \qquad u(L,t) = 0\,.$$

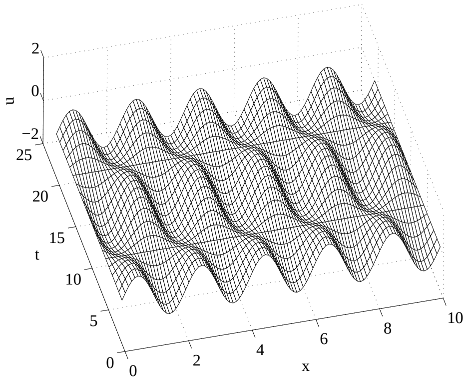

Fig. 4.4 Space-time evolution of Eq. (4.15) with the initial distribution $u(x,0) = \sin(\pi x)$, $u_t(x,0) = 0$.

Assume that the initial position and velocity are

$$u(x,0) = f(x) = \sin(\pi x)\,, \quad \text{and} \quad u_t(x,0) = g(x) = 0\,.$$

Other parameters are:

`Space interval`	L=10
`Space discretization step`	$\triangle x = 0.1$
`Time discretization step`	$\triangle t = 0.05$
`Amount of time steps`	$T = 20$

First one can find the d'Alambert solution. In the case of zero initial velocity Eq. (4.7) becomes

$$u(x,t) = \frac{f(x-2t)+f(x+2t)}{2} = \frac{\sin\pi(x-2t)+\sin\pi(x+2t)}{2} = \sin(\pi x)\,\cos(2\pi t)\,,$$

i.e., the solution is just a sum of a travelling waves with initial form, given by $\frac{f(x)}{2}$. Numerical solution of (4.15) is shown on Fig. (4.4).

Example 2.

Solve Eq. (4.15) with the same boundary conditions. Assume now, that initial distributions of position and velocity are

$$u(x,0) = f(x) = 0 \quad \text{and} \quad u_t(x,0) = g(x) = \begin{cases} 0, & x \in [0, x_1]; \\ g_0, & x \in [x_1, x_2]; \\ 0, & x \in [x_2, L]\,. \end{cases}$$

Other parameters are:

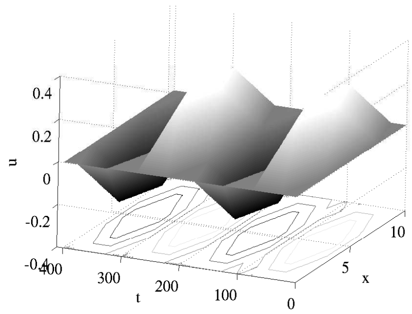

Fig. 4.5 Space-time evolution of Eq. (4.15) with the initial distribution $u(x,0)=0$, $u_t(x,0)=g(x)$.

`Initial nonzero velocity`	g_0=0.5
`Initial space intervals`	$x_1 = L/4,\ x_2 = 3L/4$
`Space interval`	L=10
`Space discretization step`	$\triangle x = 0.1$
`Time discretization step`	$\triangle t = 0.05$
`Amount of time steps`	$T = 400$

Numerical solution of the problem is shown on Fig. (4.5).

Example 3. Vibrating String

Use the explicit method (4.9) to solve the wave equation for a vibrating string:

$$u_{tt} = c^2 u_{xx} \quad \text{for} \quad x \in [0, L] \quad \text{and} \quad t \in [0, T], \tag{4.16}$$

where $c = 1$ with the boundary conditions

$$u(0,t) = 0 \qquad u(L,t) = 0 .$$

Assume that the initial position and velocity are

$$u(x,0) = f(x) = \sin(n\pi x/L), \quad \text{and} \quad u_t(x,0) = g(x) = 0, \quad n = 1, 2, 3, \ldots .$$

Other parameters are:

`Space interval`	L=1
`Space discretization step`	$\triangle x = 0.01$
`Time discretization step`	$\triangle t = 0.0025$
`Amount of time steps`	$T = 2000$

Usually a vibrating string produces a sound whose frequency is constant. Therefore, since frequency characterizes the pitch, the sound produced is a constant note. Vibrating strings are the basis of any string instrument like guitar or cello. If the speed of propagation c is known, one can calculate the frequency of the sound produced by the string. The speed of propagation of a wave c is equal to the wavelength multiplied by the frequency f:

$$c = \lambda f$$

If the length of the string is L, the fundamental harmonic is the one produced by the vibration whose nodes are the two ends of the string, so L is half of the wavelength of the fundamental harmonic, so

$$f = \frac{c}{2L}$$

Solutions of the equation in question are given in form of standing waves. The standing wave is a wave that remains in a constant position. This phenomenon can occur because the medium is moving in the opposite direction to the wave, or it can arise in a stationary medium as a result of interference between two waves traveling in opposite directions (see Fig. (4.6))

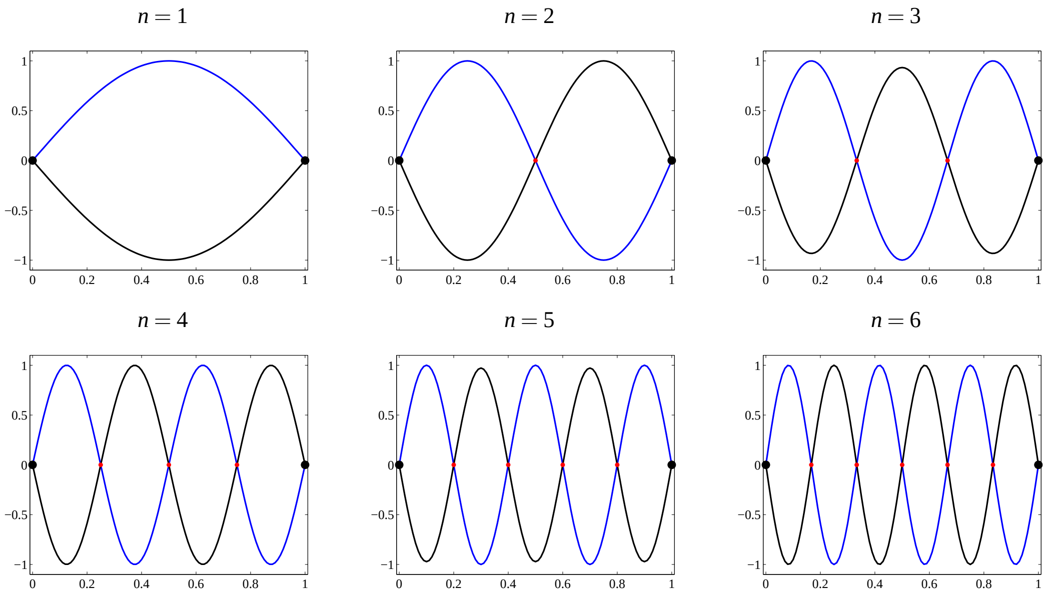

Fig. 4.6 Standing waves in a string. The fundamental mode and the first five overtones are shown. The red dots represent the wave nodes.

4.2 The Wave Equation in 2D

4.2.1 *Examples*

4.2.1.1 Example 1.

Use the standart five-point explicit method (4.9) to solve a two-dimansional wave equation

$$u_{tt} = c^2(u_{xx} + u_{yy}), \quad u = u(x,y,t)$$

on the rectangular domain $[0,L] \times [0,L]$ with Dirichlet boundary conditions. Other parameters are:

`Space interval`	L=1
`Space discretization step`	$\triangle x = \triangle y = 0.01$
`Time discretization step`	$\triangle t = 0.0025$
`Amount of time steps`	$T = 2000$
`Initial condition`	$u(x,y,0) = 4x^2 y(1-x)(1-y)$

Numerical solution of the problem for two different time moments $t = 0$ and $t = 500$ can be seen on Fig. (4.7)

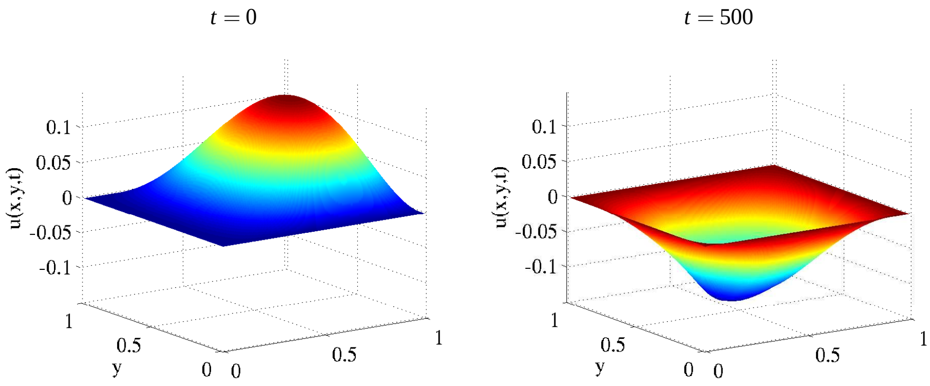

Fig. 4.7 Numerical solution of the two-dimensional wave equation, shown for $t = 0$ and $t = 500$.