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Connecting Statistics and Dynamics of Turbulent Rayleigh–Bénard Convection

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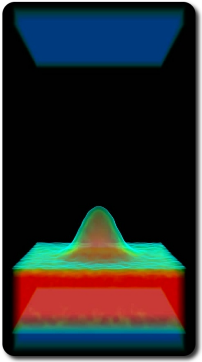

- Introduction
- Part I: Temperature Statistics and PDF Equations
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- Summary and Conclusion

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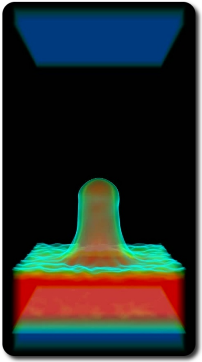
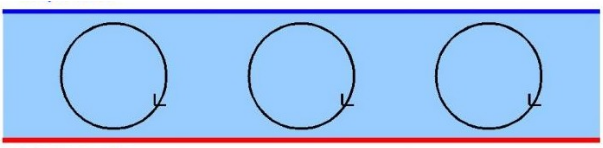


Phenomenon

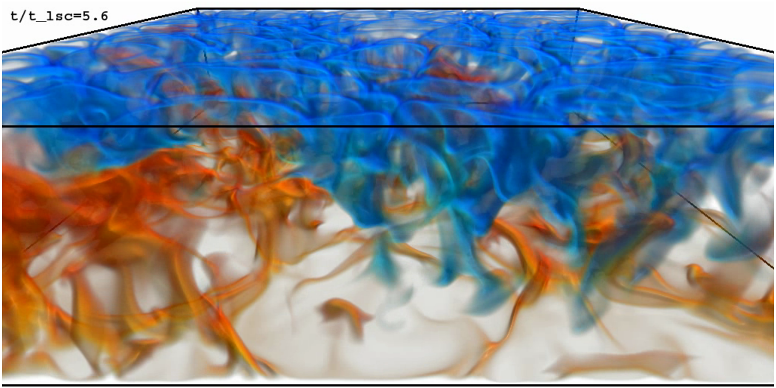
Rayleigh–Bénard Convection:

- Heated from below, cooled from above
- Ubiquitous in nature: Atmosphere, ozeans, plate tectonics, ...
- Different patterns, from stable laminar to highly turbulent flows
- Turbulent flows common in nature and applications, yet hard to handle analytically
- Our focus: Statistical Description that connects to the Dynamics



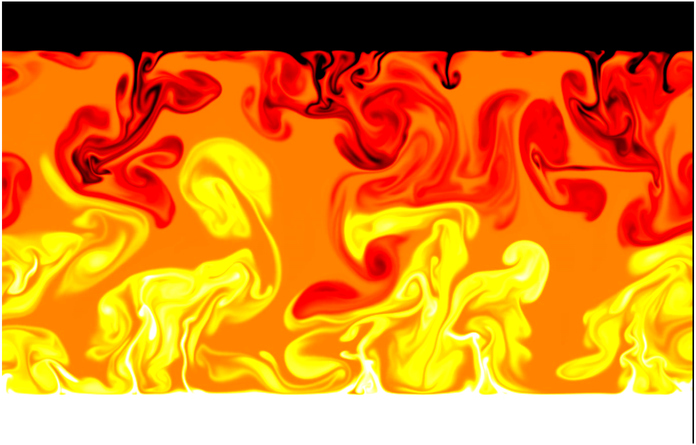
Phenomenon

Turbulent Rayleigh–Bénard Convection



Phenomenon

Turbulent Rayleigh–Bénard Convection



Governing Equations

Oberbeck-Boussinesq Equations in Non-dimensional Form

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \Delta T$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mathrm{Pr}\, \Delta \mathbf{u} + \mathrm{Pr}\, \mathrm{Ra}\, T \mathbf{e}_z \quad , \quad \nabla \cdot \mathbf{u} = 0$$

Control Parameters

- Ra – Rayleigh number ($\propto$ temperature difference)
- Pr – Prandtl number ($= \frac{\text{kinematic viscosity}}{\text{thermal diffusivity}}$, material parameter)
- Γ – Aspect ratio (geometry parameter)

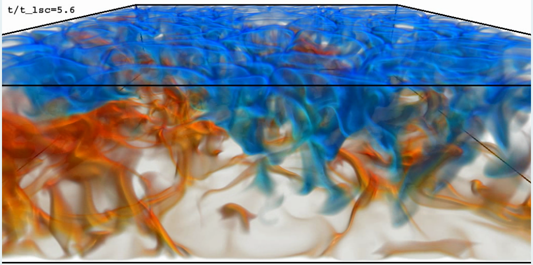
Contents

[a] JL, M. Wilczek, R. Friedrich

System and Numerics

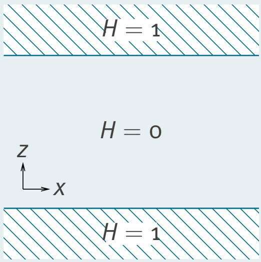
Rayleigh–Bénard System

- Homogenous in horizontal direction
- No-slip bottom and top plates $\mathbf{u} = 0$
- Parameters: $\mathrm{Ra} = 2 \cdot 10^7$, $\mathrm{Pr} = 1$, $\Gamma = 4$



System and Numerics

Numerics



- Tri-periodic pseudospectral code
- Velocity and temperature boundary conditions enforced via volume penalization[a]:
  - Add damping terms $-\frac{1}{\epsilon}H(\mathbf{x})\mathbf{u}$ and $-\frac{1}{\epsilon}H(\mathbf{x})(T - T_{\text{plate}})$ to Oberbeck-Boussinesq equations
  - $H(\mathbf{x})$ separates fluid and wall regions
  - For $\epsilon \to 0$, desired boundary conditions are obtained

[a] K. Schneider, *Comp. & Fluids*, 34(10):1223, 2005

Goal: A Statistical Description from First Principles

Laminar convection: Analytical solution of basic equations is possible

Turbulent convection: Analytical solution is (up to now) impossible

But:

- Analytical solution not necessarily needed
- Compare ideal gas: We can't predict 10^{23} particles, but we still can predict temperature, pressure, energy, ...

Goal:

- Achieve statistical description for turbulent Rayleigh–Bénard convection in terms of probability density function (PDF) of temperature!

Statistical Description in Terms of Temperature PDF

Deriving an Evolution Equation for Temperature PDF:

- Define temperature PDF as ensemble average $\langle\cdot\rangle$ of δ-distribution: $f(T,\mathbf{x},t) = \big\langle \delta\big(T - T(\mathbf{x},t)\big)\big\rangle$
- Calculate and put together derivatives of PDF
- Introduce conditional averages $\langle\cdot|T,\mathbf{x},t\rangle$
- Plug in Oberbeck-Boussinesq equations

Evolution Equation for Temperature PDF

$$\frac{\partial}{\partial t}f + \nabla\cdot\Big(\Big\langle \mathbf{u}\Big|T,\mathbf{x},t\Big\rangle f\Big) = -\frac{\partial}{\partial T}\Big(\Big\langle \frac{\partial}{\partial t}T + \mathbf{u}\cdot\nabla T\Big|T,\mathbf{x},t\Big\rangle f\Big)$$
$$= -\frac{\partial}{\partial T}\Big(\Big\langle \Delta T\Big|T,\mathbf{x},t\Big\rangle f\Big)$$

Deriving an Evolution Equation

Simplify equation by using statistical symmetries:

- Homogeneiety: $f(T, \mathbf{x}, t) = f(T, z, t)$
- Stationarity: $f(T, z, t) = f(T, z)$

$\Rightarrow$ x-, y- and t-derivatives vanish!

Evolution Equation under Statistical Symmetries

$$\frac{\partial}{\partial z}\Big(\langle u_z|T,z\rangle\, f\Big) = -\frac{\partial}{\partial T}\Big(\langle \Delta T|T,z\rangle\, f\Big)$$

- The two conditional averages are estimated from numerics (more on that later on)

Method of Characteristics yields average behaviour

- Apply Method of Characteristics to evolution equation of temperature PDF
- One obtains characteristic curves, i. e. trajectories along which the evolution equation transforms from a PDE into an ODE
- The characteristics describe the average transport in phase space, spanned by T and z

Method of Characteristics

- Characteristics follow the vectorfield that is determined by the conditional averages:

Characteristics

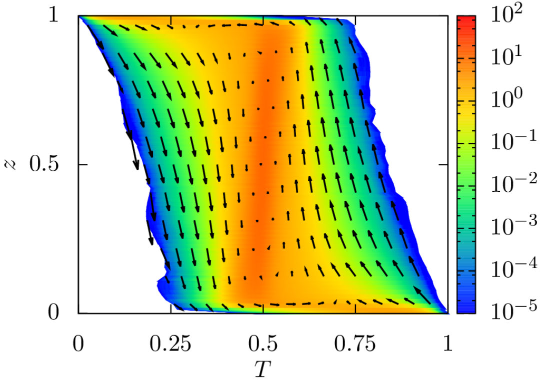
$$\begin{pmatrix} \dot{T} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \langle \Delta T | T, z \rangle \\ \langle u_z | T, z \rangle \end{pmatrix}$$

- $\left\langle \cdot \middle| T, z \right\rangle$ are "known" quantities
- Characteristics show average behavior of a fluid particle in phase space

$\Rightarrow$ Quasi-Lagrangian view obtained from snapshots of Eulerian fields!

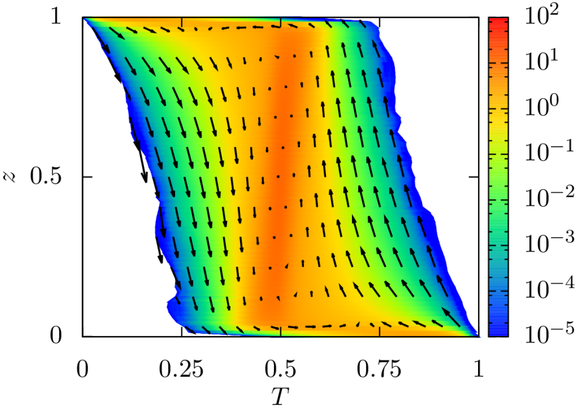
Results from the Numerics

- Color coded: Temperature PDF $f(T,z)$
- Black arrows: Vectorfield of Characteristics $\begin{pmatrix} \langle \Delta T | T,z \rangle \\ \langle u_z | T,z \rangle \end{pmatrix}$



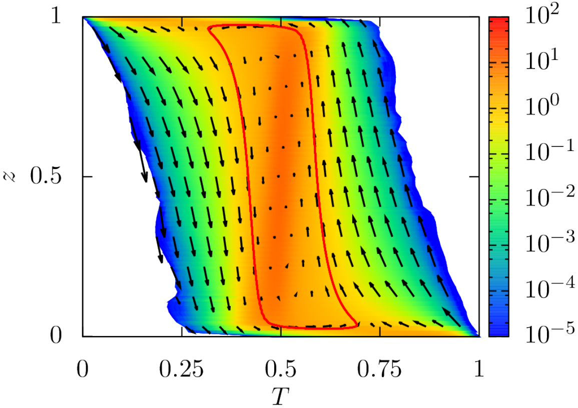
Results from the Numerics

- Integration of vectorfield $\begin{pmatrix} \langle \Delta T | T, z \rangle \\ \langle u_z | T, z \rangle \end{pmatrix}$ yields Characteristics



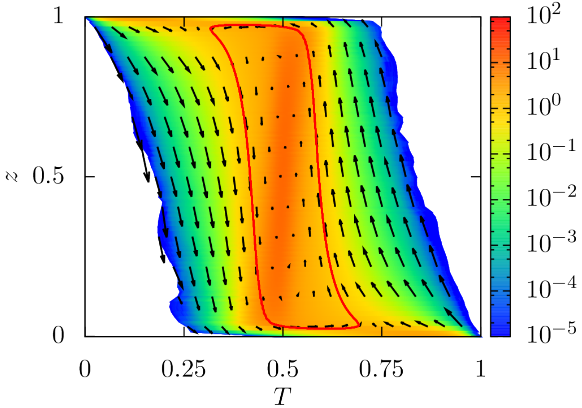
Results from the Numerics show Limit Cycle

- Characteristic curves starting from any point (T_o, z_o) converge towards limit cycle!



Results from the Numerics show Limit Cycle

- Limit cycle and vectorfield show typical Rayleigh–Bénard cycle!



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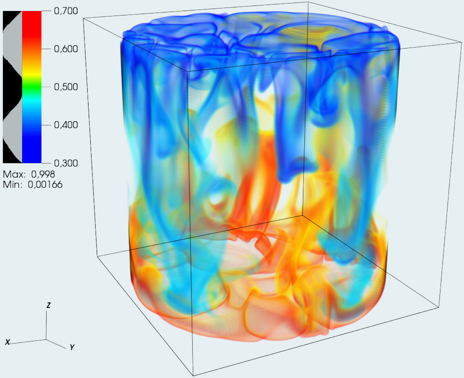
[a] JL, M. Wilczek, R. Friedrich, R. Stevens, D. Lohse

System and Numerics

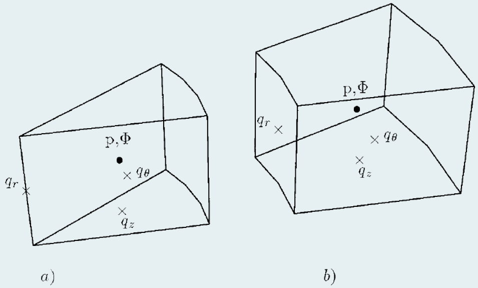
Rayleigh–Bénard System



- Cylindrical Vessel with insulating sidewalls
- Cylindrical coordinates: $\mathbf{x} = (r, \varphi, z)$
- All surfaces are no-slip $\mathbf{u} = \mathbf{0}$
- Parameters: $\mathrm{Ra} = 2 \cdot 10^8$, $\mathrm{Pr} = 1$, $\Gamma = 1$

System and Numerics

Numerics[1]



- 2nd order finite difference scheme
- non-uniform cylindrical grid
- gridpoint clustering near horizontal plates and sidewalls

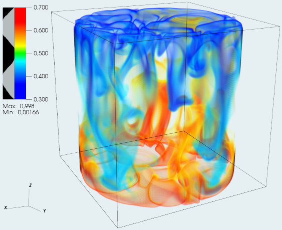
[1]R. Verzicco and R. Camussi, *J. Fluid Mech.*, 477:19–49, 2003

Evolution Equation for Temperature PDF

Evolution Equation

$$\frac{\partial}{\partial t} f + \nabla \cdot \left(\left\langle \mathbf{u} \middle| T, \mathbf{x}, t \right\rangle f \right) = -\frac{\partial}{\partial T} \left(\left\langle \Delta T \middle| T, \mathbf{x}, t \right\rangle f \right)$$

Symmetries



- Homogeneous in azimuthal (φ-) direction: $f(T, r, \varphi, z, t) = f(T, r, z, t)$
- Stationary in time: $f(T, r, z, t) = f(T, r, z)$

Temperature PDF

Evolution Equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\left\langle u_r|T,r,z\right\rangle f\right)+\frac{\partial}{\partial z}\left(\left\langle u_z|T,r,z\right\rangle f\right)=-\frac{\partial}{\partial T}\left(\left\langle \Delta T|T,r,z\right\rangle f\right)$$

Vectorfield of Characteristics

$$\begin{pmatrix}\dot{T}\\ \dot{r}\\ \dot{z}\end{pmatrix}=\begin{pmatrix}\left\langle \Delta T|T,r,z\right\rangle\\ \left\langle u_r|T,r,z\right\rangle\\ \left\langle u_z|T,r,z\right\rangle\end{pmatrix}$$

Characteristics show Limit Cycle (LC)!

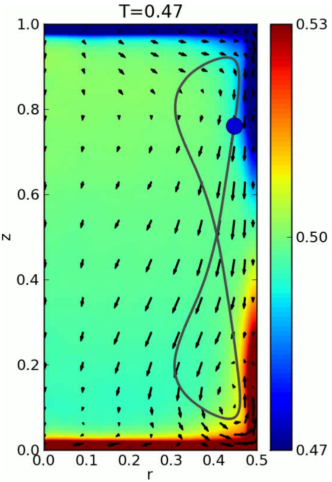
- Integration of vectorfield

$$\left\langle \begin{pmatrix} \Delta T \\ u_r \\ u_z \end{pmatrix} \middle| T, r, z \right\rangle$$

yields characteristics

- Characteristics for different starting positions converge to unique limit cycle in *T*-*r*-*z* phase space

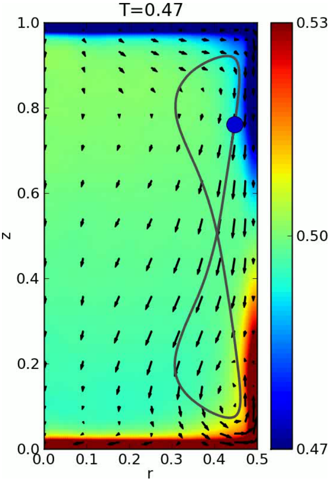
Shown on right:
- (Mean) temperature color coded
- Projection of limit cycle into *r*-*z* plane
- Fluid parcel traveling along the LC
- Vectorfield slice in *r*-*z* plane at *T* coordinate of fluid parcel



Limit Cycle

Features of Limit Cycle:

- RB Cycle:
  - Heating up / moving outwards at the bottom
  - Moving up / inwards in the bulk
  - Cooling down / moving outwards at the top, ...
- Cornerflows w/o need for a LSC
- Limit Cycle lies in outer regions, $0.3 < r < 0.5$, and around the mean temperature, $0.47 < T < 0.53$



Summary

Part I

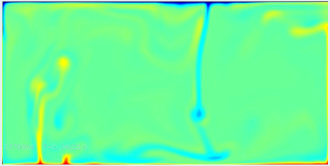
- We derived an evolution equation for the PDF of temperature from first principles
- Unclosed terms are expressed via conditional averages, which are estimated from DNS
- The Method of Characteristics is used to link statistics and dynamics of the system
- The framework allows to identify a limit cycle, which shows the average transport processes in Rayleigh-Bénard convection in both cases (2D/3D phase space)
- Outlook: Further investigation of limit cycle

Contents

[a] K. Petschel, M. Wilczek, U. Hansen, R. Friedrich

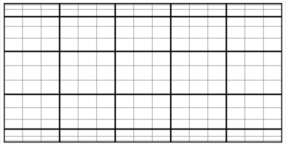
System and Numerics

Rayleigh–Bénard System



- 2D convection, insulating sidewalls
- $\mathrm{Ra} = 10^8$, $\Gamma = 2$, $\mathrm{Pr} = \infty$ ($\rightarrow$ Earth's mantle: $\mathrm{Pr} \approx 10^{25}$)
- All surfaces are stress-free

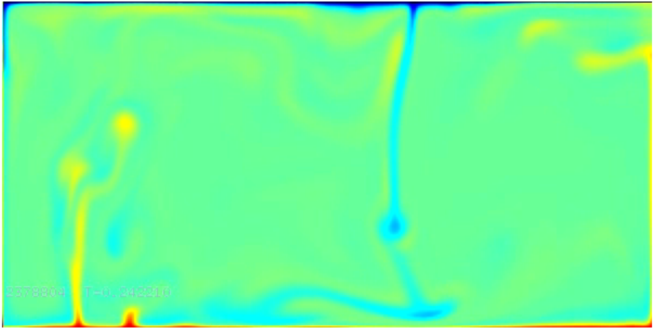
Numerics



- Finite volume multigrid method[a]
- Grid refinement near plates

[a] R. Trompert, U. Hansen, *Geophys. Astrophys. Fluid Dyn.*, 83:261, 1996

Reversal in Large-Scale Circulation (LSC)



Reversal mechanism:

- Stable one-cell structure
- Plumes create two-cell structure
- Two-cell structure breaks down and forms reversed one-cell structure

Describing the Reversal Mechanism

Idea

- Expand velocity field $\mathbf{u}(\mathbf{x}, t)$ in orthogonal basis $\{\hat{\mathbf{u}}_{lm}\}$:

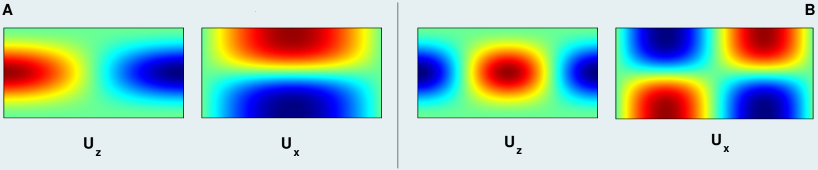
$$\mathbf{u}(\mathbf{x}, t) = \sum_{l,m} \xi_{lm}(t)\hat{\mathbf{u}}_{lm}(\mathbf{x})$$

  with time-dependent amplitudes $\xi_{lm}(t)$
- Identify modes involved in reversal
- Analyze and describe dynamics of these modes

Describing the Reversal Mechanism

Orthogonal Basis $\{\hat{\mathbf{u}}_{lm}\}$

- $\hat{\mathbf{u}}_{lm}$ are Eigenfunctions of Laplacian $\Delta\mathbf{u}$, satisfy b.c.
- Important modes for reversal mechanism:



One-Cell structure $\hat{\mathbf{u}}_{11}$ Two-Cell structure $\hat{\mathbf{u}}_{21}$

$\Rightarrow$ Amplitudes $\xi_{11}(t)$ and $\xi_{21}(t)$ are used to describe reversal!

Statistics and Dynamics of Main Modes

Analyzation of 2D subspace spanned by modes ξ_{11} and ξ_{21}

- Joint PDF

$$f(\xi_{11}, \xi_{21})$$

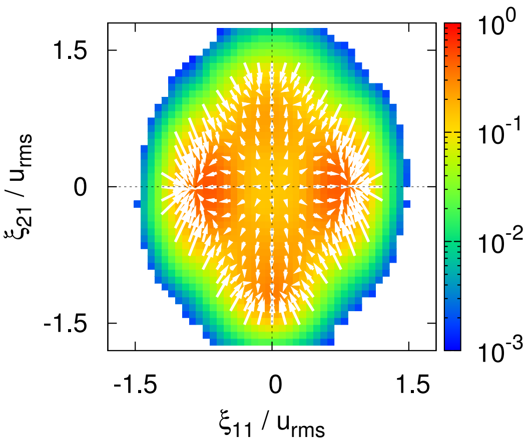
describes probability of (simultaneous) appearance of modes

- Drift vectorfield

$$\mathbf{D}(\xi_{11}, \xi_{21}) = \frac{1}{\tau} \left\langle \begin{pmatrix} \xi_{11} \\ \xi_{21} \end{pmatrix} (t+\tau) - \begin{pmatrix} \xi_{11} \\ \xi_{21} \end{pmatrix} (t) \middle| \xi_{11}, \xi_{21} \right\rangle$$

describes transition between modes (with τ much smaller than time between reversals)

Statistics and Dynamics of Main Modes

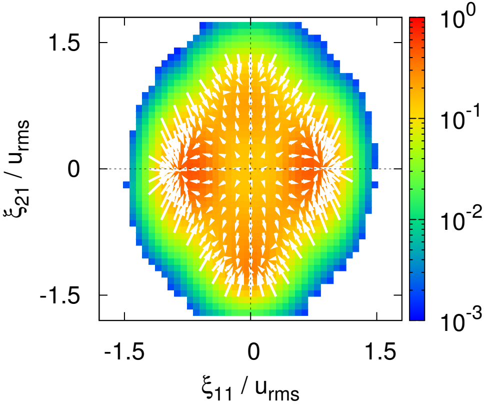


- Color: PDF $f(\xi_{11}, \xi_{21})$
- White arrows: Drift vectorfield $\mathbf{D}(\xi_{11}, \xi_{21})$

⇒ 4 attractive fixed points in the most probable configurations

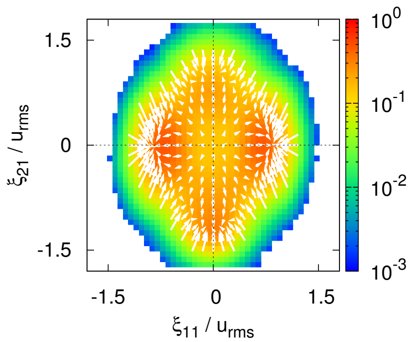
⇒ Unstable fixed point in the center

Statistics and Dynamics of Main Modes



1. Noise (plumes) drives system out of one-cell fixed point (FP)
2. Large-Scale Circulation breaks down, system settles in two-cell FP
3. Noise drives system out of two-cell FP, system settles in one-cell FP with reversed circulation (same direction equally possible)

Statistics and Dynamics of Main Modes



- Reversal involves transition over higher modes, induced by plumes acting as noise
- Direct reversal between one-cell FPs not possible (→ unstable FP in center)
- Transition over higher modes possible (not shown here)
- Reversal mechanism different from no-slip boundaries (→ corner rolls)

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Summary

Part II

- Flow reversal is understood as a transition over higher modes induced by plumes
- Primary modes ξ_{11} and ξ_{21} are used to describe reversal
- PDF and drift vectorfield of modes are used to identify stable configurations (fixed points) and transitions between them
- Plumes act as noise that drives the system out of the fixed points

Contents

Conclusion

Part I[1]

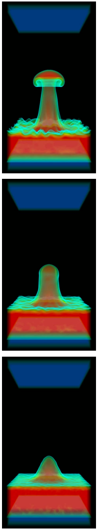
Starting from an evolution equation for the temperature PDF, we obtained the most probable behavior of a fluid parcel in phase space and found a limit cycle

Part II[2]

Analyzing the most important modes, we identified flow reversals as a transition between stable fixed points over higher modes that is triggered by plumes acting as noise

$\Rightarrow$ Connection between statistics and dynamics of RB convection!

[1] JL, M. Wilczek, R. Friedrich, *New J. Phys.*, 13(1):015002, 2011
[2] K. Petschel et. al., *Phys. Rev. E*, 84:026309, 2011



Thank you for your attention!

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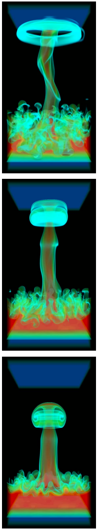
Rudolf Friedrich
† August 2012



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