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Description of Turbulent Rayleigh–Bénard Convection by PDF Methods Exhibits Limit Cycle Behavior

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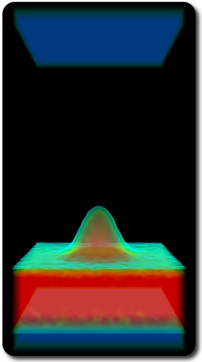

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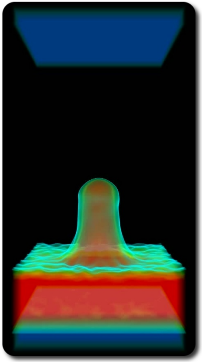
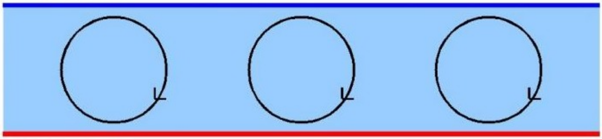


Phenomenon

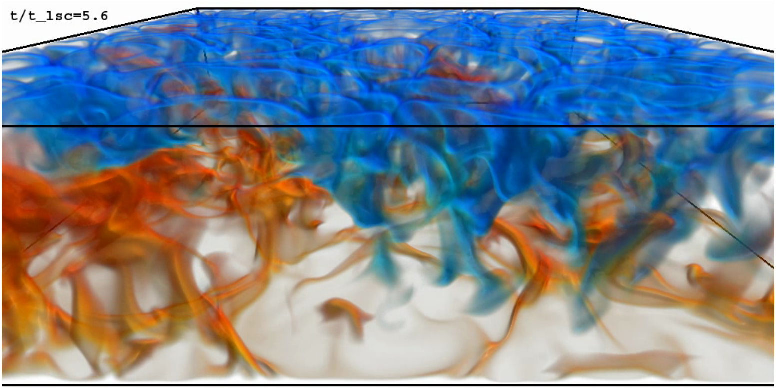
Rayleigh–Bénard Convection:

- Heated from below, cooled from above
- Ubiquitous in nature: Atmosphere, ozeans, plate tectonics, . . .
- Different patterns, from stable laminar to highly turbulent flows
- Turbulent flows common in nature and applications, yet hard to handle analytically
- Our focus: Statistical Description that connects to the Dynamics



Phenomenon

Turbulent Rayleigh–Bénard Convection



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Governing Equations

Oberbeck-Boussinesq Equations in Non-dimensional Form

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \Delta T$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mathrm{Pr}\, \Delta \mathbf{u} + \mathrm{Pr}\, \mathrm{Ra}\, T \mathbf{e}_z \quad , \quad \nabla \cdot \mathbf{u} = 0$$

Control Parameters

- Ra – Rayleigh number ($\propto$ temperature difference)
- Pr – Prandtl number ($= \frac{\text{kinematic viscosity}}{\text{thermal diffusivity}}$, material parameter)
- Γ – Aspect ratio (geometry parameter)

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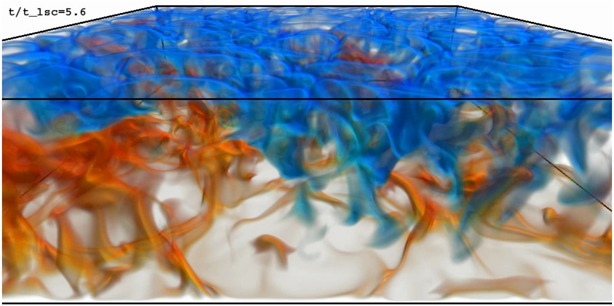
[a] JL, M. Wilczek, R. Friedrich, NJP 13(1):015002, 2011

Rayleigh–Bénard System

- Homogenous in horizontal direction
- No-slip bottom and top plates $\mathbf{u} = \mathbf{0}$
- Parameters: $Ra = 2 \cdot 10^7$, $Pr = 1$, $\Gamma = 4$
- Numerics: Pseudospectral and volume penalization methods



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Statistical Description in Terms of Temperature PDF

Deriving an Evolution Equation for Temperature PDF:

- Define temperature PDF as ensemble average $\langle\cdot\rangle$ of δ-distribution: $f(T,\mathbf{x},t) = \big\langle \delta\big(T - T(\mathbf{x},t)\big)\big\rangle$
- Calculate and put together derivatives of PDF
- Introduce conditional averages $\langle\cdot|T,\mathbf{x},t\rangle$
- Plug in Oberbeck-Boussinesq equations

Evolution Equation for Temperature PDF

$$\frac{\partial}{\partial t}f + \nabla\cdot\Big(\Big\langle \mathbf{u}\Big|T,\mathbf{x},t\Big\rangle f\Big) = -\frac{\partial}{\partial T}\Big(\Big\langle \frac{\partial}{\partial t}T + \mathbf{u}\cdot\nabla T\Big|T,\mathbf{x},t\Big\rangle f\Big)$$
$$= -\frac{\partial}{\partial T}\Big(\Big\langle \Delta T\Big|T,\mathbf{x},t\Big\rangle f\Big)$$

Deriving an Evolution Equation

Simplify equation by using statistical symmetries:

- Homogeniety: $f(T, \mathbf{x}, t) = f(T, z, t)$
- Stationarity: $f(T, z, t) = f(T, z)$

$\Rightarrow$ *x*-, *y*- and *t*-derivatives vanish!

Evolution Equation under Statistical Symmetries

$$\frac{\partial}{\partial z}\Big(\langle u_z | T, z\rangle f\Big) = -\frac{\partial}{\partial T}\Big(\langle \Delta T | T, z\rangle f\Big)$$

- The two conditional averages are estimated from numerics (more on that later on)

Method of Characteristics yields average behaviour

- Apply Method of Characteristics to evolution equation of temperature PDF
- One obtains characteristic curves, i. e. trajectories along which the evolution equation transforms from a PDE into an ODE
- The characteristics describe the average transport in phase space, spanned by T and z

Method of Characteristics

- Characteristics follow the vector field that is determined by the conditional averages:

Vector Field of Characteristics

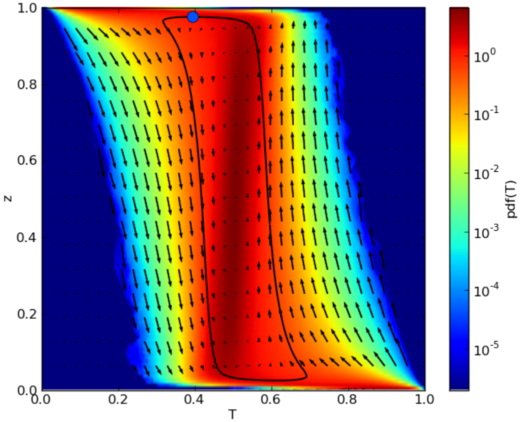
$$\begin{pmatrix} \dot{T} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \langle \Delta T | T, z \rangle \\ \langle u_z | T, z \rangle \end{pmatrix}$$

- Solutions / characteristics $\begin{pmatrix} T(t) \\ z(t) \end{pmatrix}$ show average behavior of a fluid particle in phase space

Results from the Numerics show Limit Cycle

Characteristic curves starting from any point converge towards limit cycle (black curve)!

- Color: pdf of T
- Arrows: Vector field
- Colored Circle: Solution $\begin{pmatrix} T(t) \\ z(t) \end{pmatrix}$ following limit cycle; Temperature color coded



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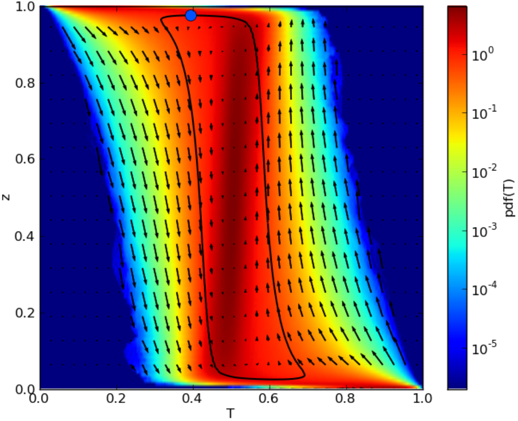

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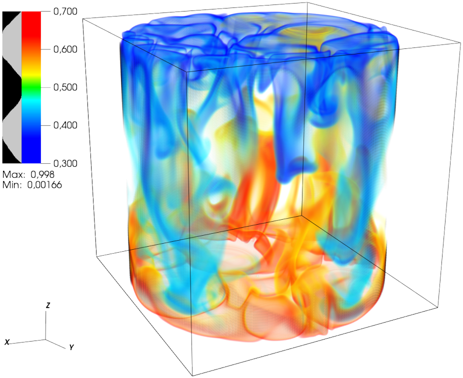
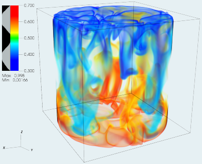
Rayleigh–Bénard System



- Cylindrical Vessel with insulating sidewalls
- Cylindrical coordinates: $\mathbf{x} = (r, \varphi, z)$
- All surfaces are no-slip $\mathbf{u} = \mathbf{0}$
- Parameters: $\mathrm{Ra} = 2 \cdot 10^8$, $\mathrm{Pr} = 1$, $\Gamma = 1$
- Numerics: Finite differences w/ gridpoint clustering

System and Characteristics

Symmetries



- Homogeneous in azimuthal (φ-) direction: $f(T, r, \varphi, z, t) = f(T, r, z, t)$
- Stationary in time: $f(T, r, z, t) = f(T, r, z)$

Vector field of Characteristics

$$\begin{pmatrix} \dot{T} \\ \dot{r} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \langle \Delta T | T, r, z \rangle \\ \langle u_r | T, r, z \rangle \\ \langle u_z | T, r, z \rangle \end{pmatrix}$$

- Additional phase space dimension (radial movement)

Characteristics show Limit Cycle (LC)!

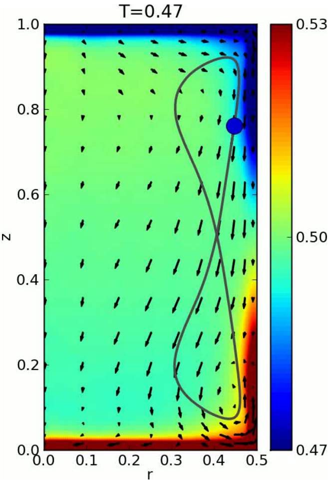
- Integration of vector field

$$\left\langle \begin{pmatrix} \Delta T \\ u_r \\ u_z \end{pmatrix} \middle| T, r, z \right\rangle$$

yields characteristics

- Characteristics for different starting positions converge to unique limit cycle in T-r-z phase space

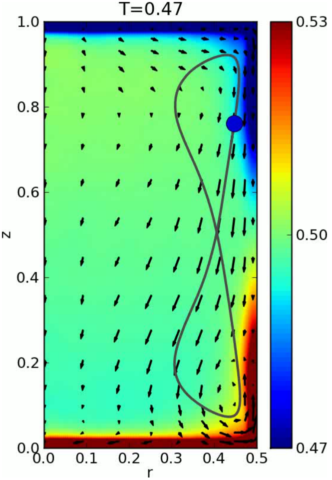
Shown on right:
- (Mean) temperature color coded
- Projection of limit cycle into r-z plane
- Fluid parcel traveling along the LC
- Vector field slice in r-z plane at T coordinate of fluid parcel



Limit Cycle

Features of Limit Cycle:

- RB Cycle:
  - Heating up / moving outwards at the bottom
  - Moving up / inwards in the bulk
  - Cooling down / moving outwards at the top, ...
- Cornerflows w/o need for a LSC
- Limit Cycle lies in outer regions, $0.3 < r < 0.5$, and around the mean temperature, $0.47 < T < 0.53$



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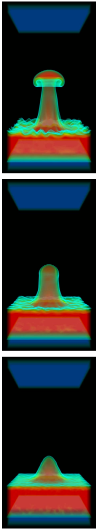
Summary

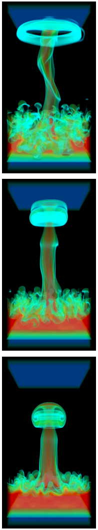
- We derived an evolution equation for the PDF of temperature from first principles
- Unclosed terms are expressed via conditional averages, which are estimated from DNS
- The Method of Characteristics is used to link statistics and dynamics of the system
- The framework allows to identify a limit cycle, which shows the average transport processes in Rayleigh-Bénard convection in both cases (2D/3D phase space)
- Outlook: Further investigation of limit cycle



Thank you for your attention!

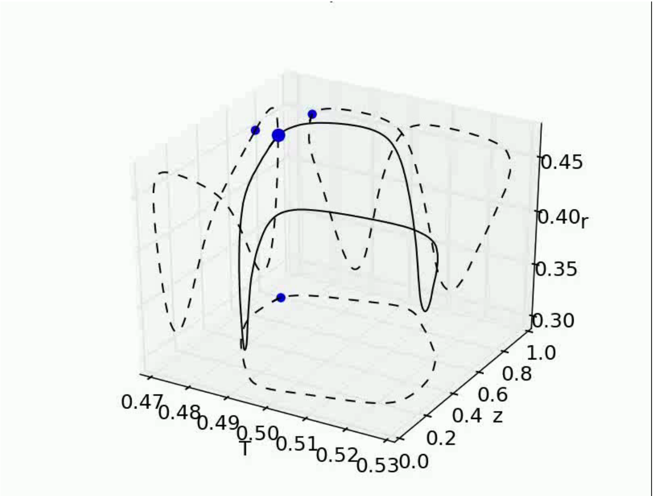
institut für
theoretische physik

Rudolf Friedrich
† August 2012



Supplementaries

Limit Cycle



Method of Characteristics

- Along the characteristics, the PDE transforms into an ODE:

ODE along the characteristic curves

$$\frac{\mathrm{d}}{\mathrm{d}s} f(s) = -\underbrace{\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\langle u_r|T,r,z\rangle\right) + \frac{\partial}{\partial z}\langle u_z|T,r,z\rangle + \frac{\partial}{\partial T}\langle \Delta T|T,r,z\rangle\right)}_{d(T,r,z)} f(s)$$

- The divergence $d(T,r,z)$ of the vector field describes the change of the PDF along a characteristic

Method of Characteristics

The ODE can be integrated along the characteristic curves, giving the following solution:

Integrated ODE along the characteristics

$$f(s) = f(s_0)\exp\left[-\int_{s_0}^{s} \mathrm{d}s \left(\frac{\partial}{\partial T}\langle \Delta T | T, z\rangle + \frac{\partial}{\partial z}\langle u_z | T, z\rangle \right)\Big|_{\substack{T=T(s)\\ z=z(s)}} \right]$$

- This relation describes the change of the PDF along a certain trajectory $\begin{pmatrix} T(s) \\ z(s) \end{pmatrix}$ in T, z-phase space, starting at $\begin{pmatrix} T(s_0) \\ z(s_0) \end{pmatrix}$