Identifying Heat Transport Processes in Turbulent Rayleigh-Bénard Convection via a PDF Equation Approach

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Introduction

- Rayleigh-Bénard convection is the buoyancy-driven flow of a fluid enclosed between two horizontal plates.
- We investigate statistical properties of the fluctuating temperature field in a closed cylinder in the turbulent regime, i.e. at high Rayleigh number.
- For this, we derive evolution equations determining the probability density function (PDF) of temperature from first principles.
- Direct numerical simulation (DNS) of the basic equations governing the flow is used to obtain the statistical quantities.
- The statistical framework allows to identify how heat is transported in different regions of the convection cell.

Governing Equations

- The nondimensionalized equations governing the Rayleigh-Bénard system in Oberbeck-Boussinesq approximation read

$$\frac{\partial}{\partial t}\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}=-\nabla p+\mathrm{Pr}\Delta\boldsymbol{u}+\mathrm{Pr}\,\mathrm{Ra}\,T\boldsymbol{e}_z,\quad \nabla\cdot\boldsymbol{u}=0$$
$$\frac{\partial}{\partial t}T+\boldsymbol{u}\cdot\nabla T=\Delta T \qquad (1)$$

with the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$, the temperature field $T(\boldsymbol{x},t)$ and the pressure field $p(\boldsymbol{x},t)$.
- The nondimensional control parameters are the Rayleigh number Ra, the Prandtl number Pr and the aspect ratio Γ

$$\mathrm{Ra}=\frac{\alpha g\delta T h^3}{\nu\kappa}\quad,\quad \mathrm{Pr}=\nu/\kappa\quad,\quad \Gamma=d/h$$

with thermal expansion coefficient α, gravitational acceleration g, outer temperature difference δT, vertical distance of top and bottom plate h, diameter of cylindrical cell d, kinematic viscosity ν and heat conductivity κ.
- The spatial domain is described in cylindrical coordinates $\boldsymbol{x}=(r,\varphi,z)\in[0,1/2]\times[0,2\pi]\times[0,1]$. Boundary conditions are adiabatic sidewalls and horizontal plates of constant temperature, i.e.

$$T(z{=}0)=1\quad,\quad T(z{=}1)=0\quad,$$

and all surfaces are no slip.

Deriving Evolution Equations for PDFs

- Within the framework known as the LMN hierarchy [1,2,3], one can derive evolution equations for PDFs from first principles, i.e. from the basic RB equations (1).
- Following the steps suggested in [4,5,6], we define the temperature PDF as an ensemble average over all possible realizations of the temperature field $T(\boldsymbol{x},t)$:

$$f(\tau;\boldsymbol{x},t)=\langle\delta(\tau-T(\boldsymbol{x},t))\rangle \qquad (2)$$

- Calculating spatial and temporal derivatives of the PDF and plugging in the basic RB equations (1) leads to an evolution equation for the temperature PDF $f=f(\tau;\boldsymbol{x},t)$:

$$\frac{\partial}{\partial t}f+\nabla\cdot\left(\left\langle\boldsymbol{u}|\tau,\boldsymbol{x},t\right\rangle f\right)=-\frac{\partial}{\partial\tau}\left(\left\langle\frac{\partial}{\partial t}T+\boldsymbol{u}\cdot\nabla T|\tau,\boldsymbol{x},t\right\rangle f\right)=-\frac{\partial}{\partial\tau}\left(\left\langle\Delta T|\tau,\boldsymbol{x},t\right\rangle f\right)$$

Here, unclosed terms have been expressed as conditional averages, denoted as $\langle\cdot|\tau,\boldsymbol{x},t\rangle$.
- According to the symmetries of the system, the statistical quantities cannot depend on the azimuthal coordinate φ or the time t. Thus, when rewritten in cylindrical coordinates, the evolution equation of the PDF $f(\tau;r,z)$ becomes

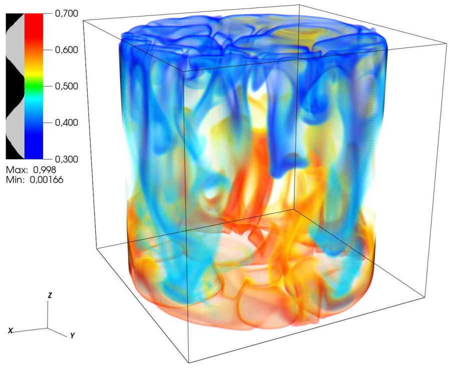
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\left\langle u_r|\tau,r,z\right\rangle f\right)+\frac{\partial}{\partial z}\left(\left\langle u_z|\tau,r,z\right\rangle f\right)=-\frac{\partial}{\partial\tau}\left(\left\langle\Delta T|\tau,r,z\right\rangle f\right) \qquad (3)$$

with the radial velocity u_r and the vertical velocity u_z.
- The three appearing conditional averages are unclosed and estimated from DNS.

3D Direct Numerical Simulation

- The governing equations (1) are discretized on a cylindrical grid via a second order central finite difference scheme.
- The grid spacing is non-uniform with clustering near the vertical sidewalls and the horizontal plates.
- Sufficient resolution of the boundary layers and the dissipative scales is ensured.
- Time advancement is achieved by a low-storage third-order Runge-Kutta scheme.
- The numerical scheme is the one implemented by R. Verzicco, described in [7].

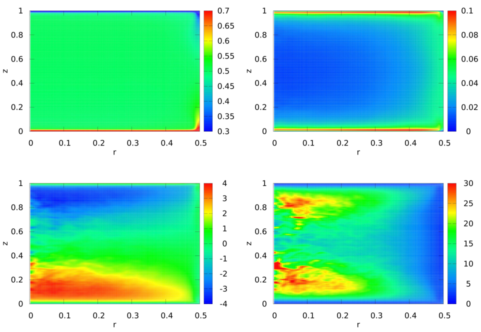
Simulation Details



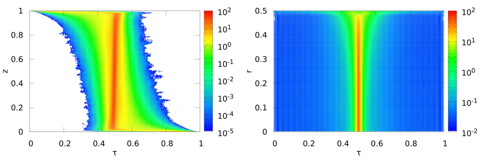
- Volume rendering of a snapshot of the temperature field used for the analysis presented here. Red corresponds to hot, blue to cold parts of the fluid. The volume rendering is done with the open source interactive parallel visualization tool VisIt [8].
- Physical parameters are $\mathrm{Ra}=2\cdot10^8$, $\mathrm{Pr}=1$ and $\Gamma=1$; the Nusselt number is estimated to $\mathrm{Nu}=40.4$.
- The grid size is $N_r\times N_\varphi\times N_z=192\times384\times384$; the horizontal boundary layers are resolved with 17 gridpoints.
- The ensemble to obtain the statistics consists of 870 snapshots, separated by 1 turnover time of the large scale circulation.

Basic Statistical Quantities

- Estimated first moments of the r- and z-resolved temperature statistics, i.e. averaged azimuthally and in time. Clockwise from top left: Mean, standard deviation, curtosis and skewness of temperature distribution:



The first two moments show a clear separation into bulk and horizontal and vertical boundary parts; however, this separation becomes less evident in the higher moments, esp. in the curtosis.
- Estimated temperature PDF $f(\tau;r,z)$, averaged over r and z, respectively, to get a two-dimensional projection, i.e. $f(\tau;z)=\langle f(\tau;r,z)\rangle_r$ (left) and $f(\tau;r)=\langle f(\tau;r,z)\rangle_z$ (right):



PDF Equations Revisited: The Method of Characteristics

- According to the partial differential equation (PDE) (3), the temperature PDF f is determined by the three conditional averages of heat diffusion ΔT, radial velocity u_r and vertical velocity u_z, and solutions of the PDE live in the phase space spanned by τ, r and z.
- Utilizing the method of characteristics, one finds trajectories $(\tau(s),r(s),z(s))$ in τ,r,z-phase space (the so-called *characteristics*) along which the PDE (3) transforms into an ordinary differential equation (ODE).
- These characteristics are determined by the conditional averages:

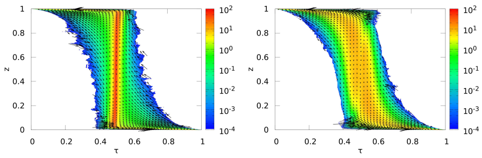
$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix}\tau(s)\\ r(s)\\ z(s)\end{pmatrix}=\begin{pmatrix}\langle\Delta T|\tau,r,z\rangle\\ \langle u_r|\tau,r,z\rangle\\ \langle u_z|\tau,r,z\rangle\end{pmatrix}\Bigg|_{\substack{\tau=\tau(s)\\ r=r(s)\\ z=z(s)}} \qquad (4)$$

Here, the characteristics are parametrized by s, which can be identified as the time.
- Along these characteristics, the resulting ODE describes how the PDF $f(s)=f(\tau(s);r(s),z(s))$ changes according to the divergence of the vector field on the right hand side:

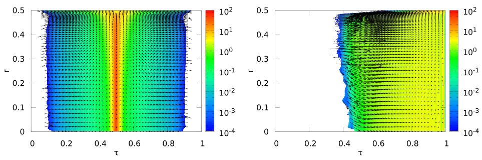
$$\frac{\mathrm{d}}{\mathrm{d}s}f(s)=-\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\langle u_r|\tau,r,z\rangle\right)+\frac{\partial}{\partial z}\langle u_z|\tau,r,z\rangle+\frac{\partial}{\partial\tau}\langle\Delta T|\tau,r,z\rangle\right)f(s) \qquad (5)$$

Method Of Characteristics: Results from DNS

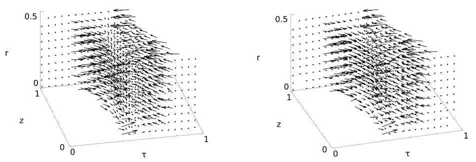
- With the three conditional averages estimated from the numerics, one can visualize the vectorfield (4) that defines the characteristics; the characteristics lie tangent to the vectorfield. These characteristics represent the average path a fluid parcel travels through the phase space.
- As a plot of the whole vectorfield in 3D becomes quite cumbersome, we show projections onto τ,z- and the τ,r-plane by averaging over different regions of the r- and z-direction, respectively. The underlying color plots show the temperature PDFs in the considered regions:
- Trajectories and temperature PDF in the τ,z-plane; averaging over inner ($r\in[0,0.25]$, left) and outer boundary ($r\in[0.485,0.5]$, right) regions of the cylinder:



Tracing the vector field in this graph, one can qualitatively reconstruct the characteristics and thus the typical *Rayleigh-Bénard cycle* of fluid heating up at the bottom, rising up, cooling down at the top plate, falling down and heating up again. One can also see that the inner regions of the cylinder are responsible for most of the vertical transport.
- Trajectories and temperature PDF in the τ,r-plane; averaging over bulk ($z\in[\delta_T,1-\delta_T]$, left) and boundary layer ($z\in[0,\delta_T]$, right) regions of the cylinder, with $\delta_T=\frac{1}{2\mathrm{Nu}}$ being the thermal boundary layer thickness:



In the bulk, hot and cold fluid is transported to the mean temperature and outwards, while in the bottom boundary layer, cold downfalling fluid is transported inward and heated up.
- To combine the discussion of the simultaneously occurring transport in r- and z-direction and to give an impression of the overall structure of the vectorfield, we plot the full data as a stereoscopic image; cross your eyes to view the full vectorfield below in 3D!



Conclusions and Future Work

- The LMN hierarchy and symmetry considerations allow us to derive equations determining PDFs from first principles.
- The evolution equation for the temperature PDF is derived, which contains unclosed terms in the form of conditional averages.
- These conditional averages are estimated from the numerics, which performs DNS of the RB equations in a closed cylinder.
- Utilizing the method of characteristics, the DNS results are used to construct trajectories which show the average path a fluid parcel travels through phase space.
- Tracing these paths through phase space, one can reproduce the typical *Rayleigh-Benard cycle* in a qualitative manner.
- To finalize the cumbersome discussion of the 3D vector field, the next step is to integrate these fields to obtain the actual characteristics. Also, the divergence of the vectorfield which describes the deformation of the temperature PDF needs to be evaluated and discussed.

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