Westfälische Wilhelms-Universität Münster
institute for theoretical physics

PDF Equations in Turbulent Rayleigh–Bénard Convection

Johannes Lülff[1,‡]
M. Wilczek[2,1]
† R. Friedrich[1]
R. Stevens[2,3]
D. Lohse[3]

[1]University of Münster
[2]University of Baltimore
[3]University of Twente

‡johannes.luelff@uni-muenster.de

Introduction

- Rayleigh–Bénard convection is the buoyancy-driven flow of a fluid enclosed between two horizontal plates.
- We investigate statistical properties of the fluctuating temperature field in the turbulent regime, i. e. at high Rayleigh number.
- For this, we derive evolution equations determining the probability density function (PDF) of temperature from first principles.
- Direct numerical simulation (DNS) of the basic equations governing the flow is used to obtain the statistical quantities.
- The statistical framework allows to identify how heat is transported in different regions of the convection cell and shows the appearance of a limit cycle.
- Analysis is performed for RB convection in periodic horizontal boundaries (2nd column) and in a cylindrical vessel (3rd column).

Governing Equations

- The nondimensionalized equations governing the Rayleigh–Bénard system in Oberbeck–Boussinesq approximation read

$$\frac{\partial}{\partial t}\boldsymbol{u}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}=-\nabla p+\mathrm{Pr}\Delta\boldsymbol{u}+\mathrm{Pr}\,\mathrm{Ra}\,T\boldsymbol{e}_z,\quad \nabla\cdot\boldsymbol{u}=0$$
$$\frac{\partial}{\partial t}T+\boldsymbol{u}\cdot\nabla T=\Delta T \qquad (1)$$

with the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$, the temperature field $T(\boldsymbol{x},t)$ and the pressure field $p(\boldsymbol{x},t)$.
- The nondimensional control parameters are the Rayleigh number Ra, the Prandtl number Pr and the aspect ratio Γ

$$\mathrm{Ra}=\frac{\alpha g\delta T h^3}{\nu\kappa},\quad \mathrm{Pr}=\nu/\kappa,\quad \Gamma=d/h$$

with thermal expansion coefficient α, gravitational acceleration g, outer temperature difference δT, vertical distance of top and bottom plate h, diameter of cylindrical cell d, kinematic viscosity ν and heat conductivity κ.

Deriving Evolution Equations for PDFs

- Within the framework of PDF equations, we derive evolution equations for PDFs from first principles, i. e. from the basic RB equations (1).
- Following the steps suggested in [1, 2, 3], we define the temperature PDF as an ensemble average over all possible realizations of the temperature field $T(\boldsymbol{x},t)$:

$$f(T,\boldsymbol{x},t)=\langle\delta(T-T(\boldsymbol{x},t))\rangle \qquad (2)$$

- Calculating spatial and temporal derivatives of the PDF and plugging in the basic RB equations (1) leads to an evolution equation for the temperature PDF $f=f(T,\boldsymbol{x},t)$:

$$\frac{\partial}{\partial t}f+\nabla\cdot\left(\left\langle\boldsymbol{u}\middle|T,\boldsymbol{x},t\right\rangle f\right)=-\frac{\partial}{\partial T}\left(\left\langle\frac{\partial}{\partial t}T+\boldsymbol{u}\cdot\nabla T\middle|T,\boldsymbol{x},t\right\rangle f\right)=-\frac{\partial}{\partial T}\left(\left\langle\Delta T\middle|T,\boldsymbol{x},t\right\rangle f\right) \qquad (3)$$

Here, unclosed terms have been expressed as conditional averages, denoted as $\langle\cdot|T,\boldsymbol{x},t\rangle$, which are later estimated from DNS.
- To solve the above first order PDE (3), the Method of Characteristics is used to obtain most probable evolution of a fluid parcel in phase space, described by trajectories through phase space (the so-called characteristics); the characteristics follow the vectorfield of the conditional averages, i. e. are solutions $\begin{pmatrix}T(t)\\ \boldsymbol{x}(t)\end{pmatrix}$ to

$$\begin{pmatrix}\dot{T}\\ \dot{\boldsymbol{x}}\end{pmatrix}=\begin{pmatrix}\langle\Delta T|T,\boldsymbol{x},t\rangle\\ \langle\boldsymbol{u}|T,\boldsymbol{x},t\rangle\end{pmatrix} \qquad (4)$$

- This framework is applied to different RB systems with different geometries and different symmetries that allow for simplifications:
  - Stationary convection with periodic horizontal boundaries: No dependence of statistics on horizontal coordinates or time (2nd column)
  - Stationary convection in cylindrical vessel: No dependence of statistics on azimuthal coordinate or time (3nd column)

In the following, these two systems will be handled individually.

References

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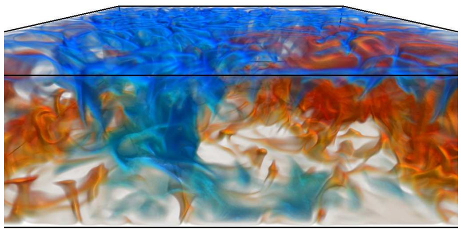
Periodic Horizontal Boundaries (2D Phase Space)



- Volume rendering of a snapshot of the temperature field, done with *Vapor* [4]
- Homogenous in horizontal direction
- No-slip bottom and top plates $\boldsymbol{u}=\boldsymbol{0}$
- Parameters: $\mathrm{Ra}=2\cdot10^7$, $\mathrm{Pr}=1$, $\Gamma=4$
- Numerics: Pseudospectral and volume penalization methods [2]
- For a video of the flow, scan this QR code:

Symmetries and Evolution Equation (2D)

- Statistical quantities can not depend on horizontal coordinates or time but only on T and z ⇒ phase space becomes 2D
- Evolution equation (3) simplifies to

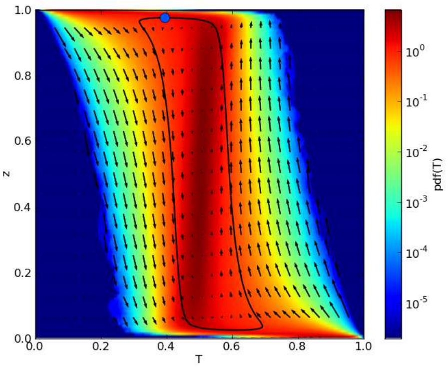
$$\frac{\partial}{\partial z}\left(\langle u_z|T,z\rangle f\right)=-\frac{\partial}{\partial T}\left(\langle\Delta T|T,z\rangle f\right) \qquad (5)$$

- Characteristics, i. e. solutions $\begin{pmatrix}T(t)\\ z(t)\end{pmatrix}$, follow the vector field defined by the conditional averages:

$$\begin{pmatrix}\dot{T}\\ \dot{z}\end{pmatrix}=\begin{pmatrix}\langle\Delta T|T,z\rangle\\ \langle u_z|T,z\rangle\end{pmatrix} \qquad (6)$$

Characteristic Curves and Limit Cycle (2D)

- Conditional averages are estimates from snapshots of numerical simulation
- Integrating (6) for arbitrary initial conditions: All characteristics converge towards limit cycle!



- Horizontal axis: temperature coordinate T, vertical axis: vertical coordinate z
- Background color: PDF of temperature
- Black arrows: vector field
- Black curve: limit cycle
- For a video of a fluid parcel traveling along the limit cycle, scan this QR code:

Features of Limit Cycle (2D)

- Main movement through phase space in horizontal direction near the boundary layers, i. e. cold fluid heating near the bottom plate and hot fluid cooling near the top plate
- Main movement through phase space in vertical direction in the bulk, i. e. hot fluid moving up and cold fluid moving down
- Additionally, slight horizontal movement in the bulk, i. e. hot fluid beginning to cool while moving down and vice versa

⇒ Typical Rayleigh–Bénard cycle of hot fluid moving up, cooling down at the top plate, falling down and heating up again at the bottom plate

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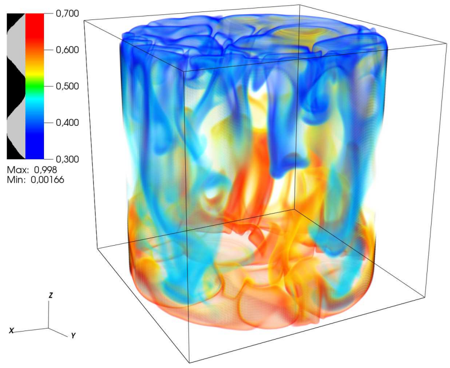
Center for Nonlinear Science
living.knowledge WWU Münster

Cylindrical Vessel (3D Phase Space)



- Volume rendering of a snapshot of the temperature field, done with *VisIt* [5]
- Insulating sidewalls, all surfaces no-slip $\boldsymbol{u}=\boldsymbol{0}$
- System described in cylindrical coordinates, $\boldsymbol{x}=(r,\varphi,z)$
- Parameters: $\mathrm{Ra}=2\cdot10^8$, $\mathrm{Pr}=1$, $\Gamma=1$
- Numerics: Finite differences on a staggered cylindrical grid w/ grid-point clustering [6]

Symmetries and Evolution Equation (3D)

- Statistical quantities can not depend on azimuthal coordinate φ or time but only on T, r and z ⇒ phase space becomes 3D
- Evolution equation (3) simplifies to

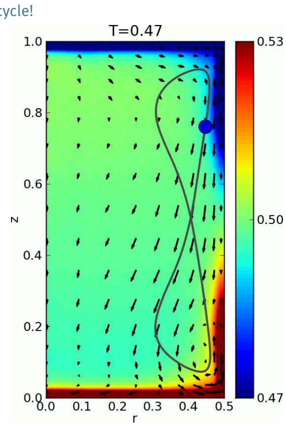
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\langle u_r|T,r,z\rangle f\right)+\frac{\partial}{\partial z}\left(\langle u_z|T,r,z\rangle f\right)=-\frac{\partial}{\partial T}\left(\langle\Delta T|T,r,z\rangle f\right) \qquad (7)$$

- Characteristics follow vector field defined by conditional averages:

$$\begin{pmatrix}\dot{T}\\ \dot{r}\\ \dot{z}\end{pmatrix}=\begin{pmatrix}\langle\Delta T|T,r,z\rangle\\ \langle u_r|T,r,z\rangle\\ \langle u_z|T,r,z\rangle\end{pmatrix} \qquad (8)$$

Characteristic Curves and Limit Cycle (3D)

- Integrating (8) for arbitrary initial conditions: All characteristics converge towards limit cycle!



- Horizontal axis: radial coordinate r, vertical axis: vertical coordinate z, color: temperature T
- Background color: mean temperature $\langle T(r,z)\rangle$
- Black arrows: vector field
- Gray curve: limit cycle
- For a video of a fluid parcel traveling along the limit cycle, scan this QR code:

Features of Limit Cycle (3D)

- Heating and cooling near the bottom and top plate
- Vertical and slight inward movement in the bulk
- Outward movement near the plates
- Cornerflows

⇒ Typical Rayleigh–Bénard cycle of hot fluid moving up, cooling down at the top plate, falling down and heating up again at the bottom plate; additional inward and outward movement

Conclusions and Future Work

- We derived an evolution equation for the PDF of temperature from first principles
- Unclosed terms are expressed via conditional averages, which are estimated from DNS
- The Method of Characteristics is used to link statistics and dynamics of the system
- The framework allows to identify a limit cycle, which shows the average transport processes in Rayleigh–Bénard convection in both cases (2D/3D phase space)
- Outlook: Further investigation of limit cycle