Turbulence - Structures and Statistics

Michael Wilczek

Institute for Theoretical Physics, University of Münster

30.08.2007

Overview

- Turbulence Simulation
- Coherent Structures
- Statistical Description

Introduction

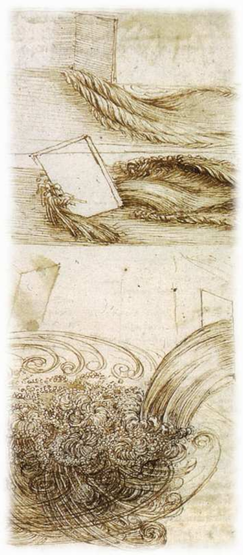
Turbulence ...

- is ubiquitous in nature
- remains one of the major challenges of classical physics
- is a paradigm for a complex system
- is governed by coherent structures
- requires a statistical description

Introduction

Problem: turbulence ...

- is described by nonlinear equations
- exhibits spatio-temporal chaos
- involves large space- and time-scales



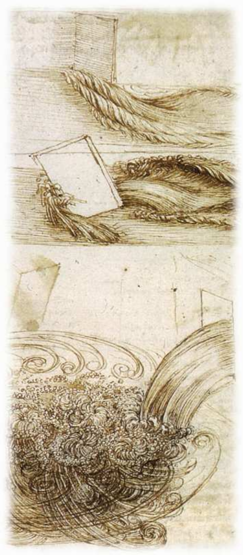
Introduction

Problem: turbulence . . .

- is described by nonlinear equations
- exhibits spatio-temporal chaos
- involves large space- and time-scales

Possible solutions:

- understanding of structures
- formulating a statistical theory



Introduction

Problem: turbulence . . .

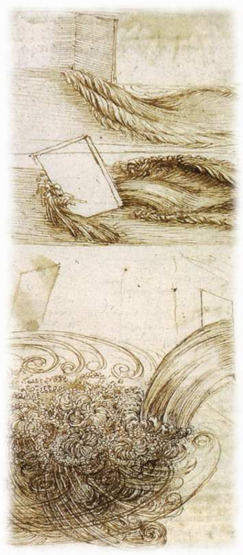
- is described by nonlinear equations
- exhibits spatio-temporal chaos
- involves large space- and time-scales

Possible solutions:

- understanding of structures
- formulating a statistical theory

Tools:

- any kind of mathematics, that will do
- computer simulations



DNS

DNS: equations

Navier-Stokes equations:

$$\frac{\partial \boldsymbol{u}}{\partial t}(\boldsymbol{x},t) + \boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla\boldsymbol{u}(\boldsymbol{x},t) = -\nabla p(\boldsymbol{x},t) + \nu\Delta\boldsymbol{u}(\boldsymbol{x},t) + \boldsymbol{\hat{f}(x,t)}$$

$$\nabla\cdot\boldsymbol{u}(\boldsymbol{x},t) = 0$$

DNS: equations

Navier-Stokes equations:

$$\frac{\partial \boldsymbol{u}}{\partial t}(\boldsymbol{x},t) + \boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla\boldsymbol{u}(\boldsymbol{x},t) = -\nabla p(\boldsymbol{x},t) + \nu\Delta\boldsymbol{u}(\boldsymbol{x},t) + \hat{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{t})$$

$$\nabla\cdot\boldsymbol{u}(\boldsymbol{x},t) = 0$$

Vorticity: $\boldsymbol{\omega}(\boldsymbol{x},t) = \nabla\times\boldsymbol{u}(\boldsymbol{x},t)$

Vorticity equation:

$$\frac{\partial \boldsymbol{\omega}}{\partial t}(\boldsymbol{x},t) = \nabla\times\big(\boldsymbol{u}(\boldsymbol{x},t)\times\boldsymbol{\omega}(\boldsymbol{x},t)\big) + \nu\Delta\boldsymbol{\omega}(\boldsymbol{x},t) + \boldsymbol{f}(\boldsymbol{x},t)$$

DNS: numerics I

- aim: forced (stationary) homogeneous, isotropic turbulence
- temporal discretization: RK3 TVD
- spatial discretization: box-length 2π, dim grid points, periodic boundary conditions
- pseudospectral code

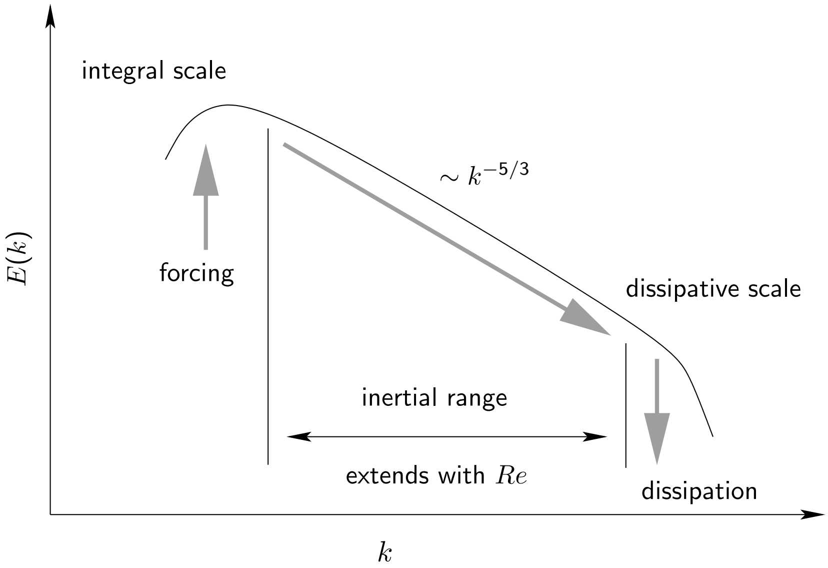
Thanx to H. Homann and R. Grauer for many hints and tips!

DNS: numerics II

$$\frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t}(\boldsymbol{k},t) + \nu\, k^2\, \tilde{\boldsymbol{\omega}}(\boldsymbol{k},t) = i\boldsymbol{k} \times \mathcal{F}\{\boldsymbol{u}(\boldsymbol{x},t) \times \boldsymbol{\omega}(\boldsymbol{x},t)\} + \tilde{f}(\boldsymbol{k},t)$$

- adaptive time-stepping (Courant-Friedrichs-Levy criterion)
- *pseudo*spectral: forward/backward FFT is computationally cheaper than convolution ($N \log N$ vs. N^2)
- aliasing: spherical mode truncation
- viscosity is treated exactly (integrating factor)
- forcing: freezing of low modes
- code is currently OpenMP parallelized

DNS: computational costs I



DNS: computational costs II

- forcing scale and dissipative scale should be well seperated
- inertial range extends with increasing Re
- size of smallest structures decreases with Re
- smallest structures should be well-resolved by the grid
- turbulent field should be accurately advanced in time

DNS: computational costs III

to be more precisely ...

$$\eta = \left(\frac{uL}{\nu}\right)^{-3/4} L = Re^{-3/4} L$$

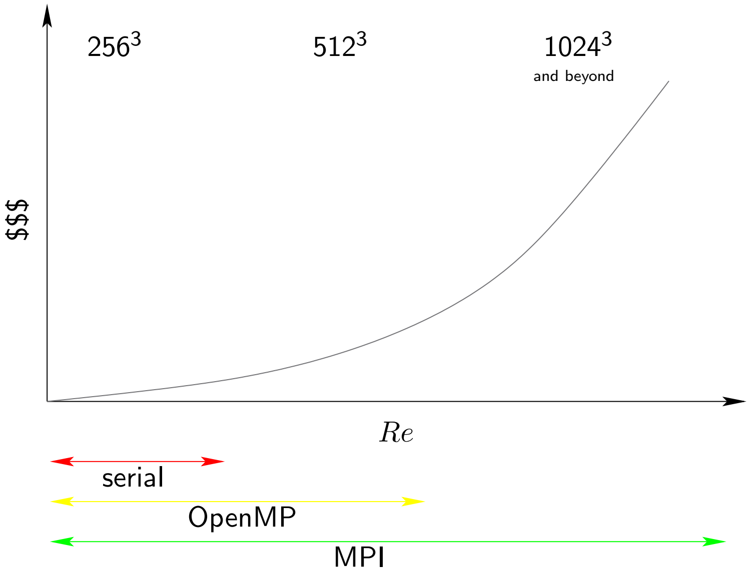
$$\Delta x \sim \eta$$

$$N_x \sim \left(\frac{2\pi}{\Delta x}\right)^3 \sim \left(\frac{2\pi}{L}\right) Re^{9/4} \quad \longrightarrow Re \sim \left(\frac{L}{2\pi}\right)^{4/3} N_x^{4/9}$$

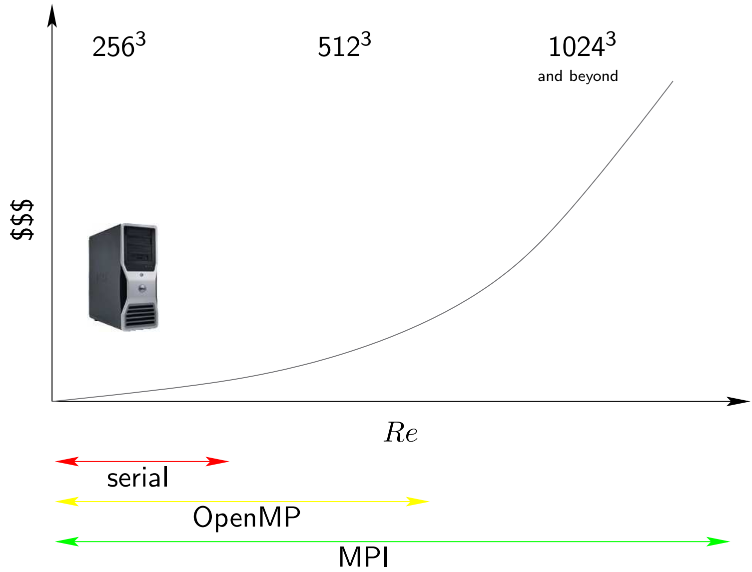
$$N_t \sim \frac{T}{\Delta t} \sim \frac{T}{\Delta x/u} \sim \frac{T}{l/u} Re^{3/4}$$

$$\$\$\$ \sim N_x N_t \sim \left(\frac{T}{l/u}\right)\left(\frac{2\pi}{l}\right)^3 Re^3$$

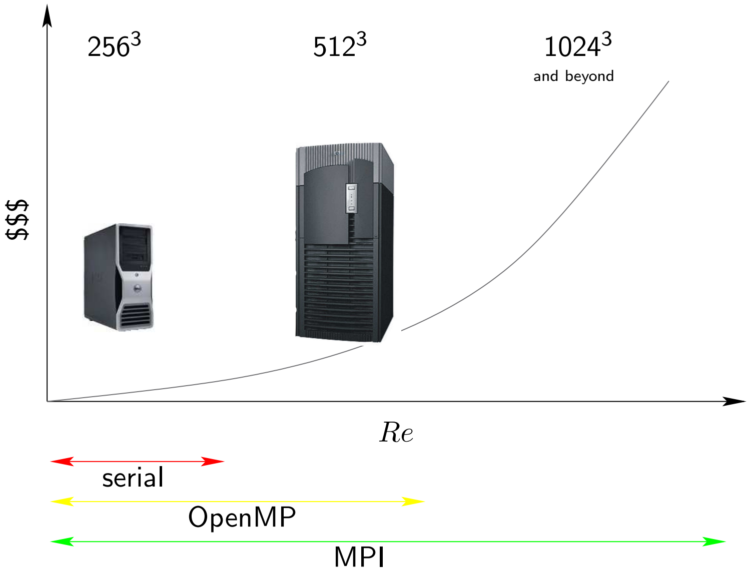
DNS: computational costs IV



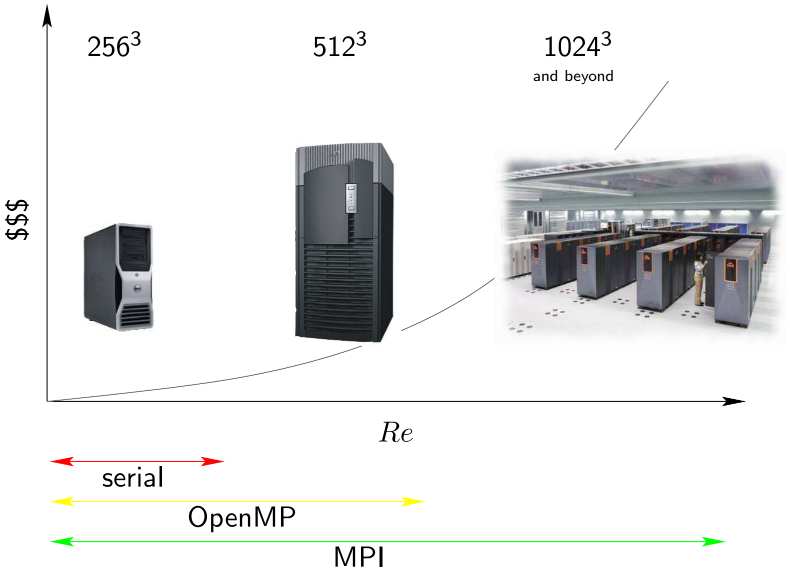
DNS: computational costs IV



DNS: computational costs IV



DNS: computational costs IV



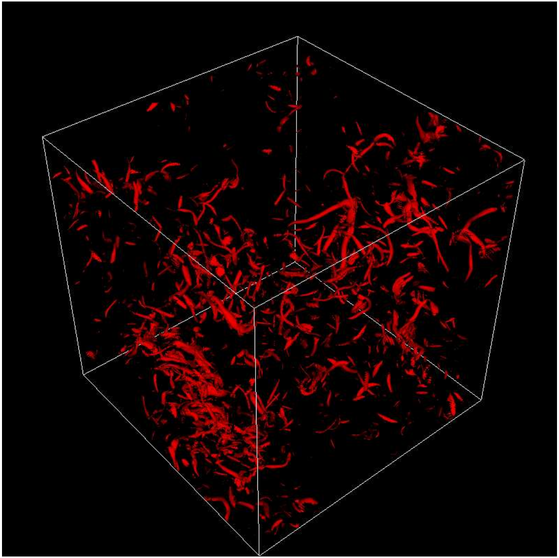
Coherent Structures

Coherent Structures: introduction

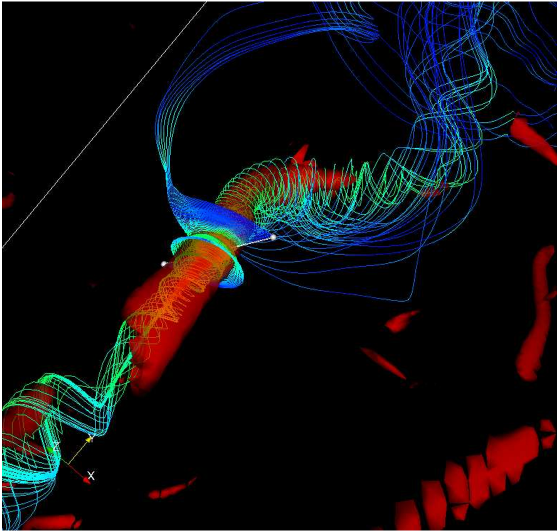
- turbulent fields are not random
- display a complex spatial structure
- structures severely influence statistical properties (intermittency)

Coherent Structures: phenomenology

- vortex tubes/ filaments
- intermittent spatial distribution



Coherent Structures: streamlines



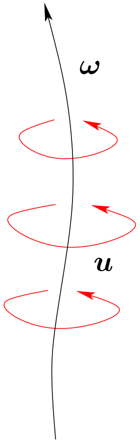
- spiraling streamlines
- vortex trapping

Coherent Structures: isolated tube

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} + \nu \Delta \boldsymbol{\omega}$$

$$\boldsymbol{\omega} = \omega_r(r, \varphi, z)\boldsymbol{e}_r + \omega_\varphi(r, \varphi, z)\boldsymbol{e}_\varphi + \omega_z(r, \varphi, z)\boldsymbol{e}_z$$

with $\omega_r, \omega_\varphi << \omega_z$



Coherent Structures: isolated tube

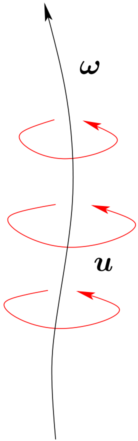
$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} + \nu \Delta \boldsymbol{\omega}$$

$$\boldsymbol{\omega} = \omega_r(r,\varphi,z)\boldsymbol{e}_r + \omega_\varphi(r,\varphi,z)\boldsymbol{e}_\varphi + \omega_z(r,\varphi,z)\boldsymbol{e}_z$$

with $\omega_r, \omega_\varphi << \omega_z$

$$\boldsymbol{u} = u_r(r,\varphi,z)\boldsymbol{e}_r + u_\varphi(r,\varphi,z)\boldsymbol{e}_\varphi + u_z(r,\varphi,z)\boldsymbol{e}_z$$

with $u_r, u_z << u_\varphi$



Coherent Structures: isolated tube

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} + \nu \Delta \boldsymbol{\omega}$$

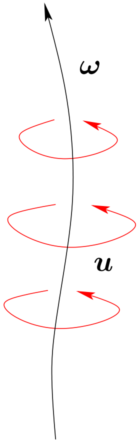
$$\boldsymbol{\omega} = \omega_r(r,\varphi,z)\boldsymbol{e}_r + \omega_\varphi(r,\varphi,z)\boldsymbol{e}_\varphi + \omega_z(r,\varphi,z)\boldsymbol{e}_z$$

with $\omega_r, \omega_\varphi << \omega_z$

$$\boldsymbol{u} = u_r(r,\varphi,z)\boldsymbol{e}_r + u_\varphi(r,\varphi,z)\boldsymbol{e}_\varphi + u_z(r,\varphi,z)\boldsymbol{e}_z$$

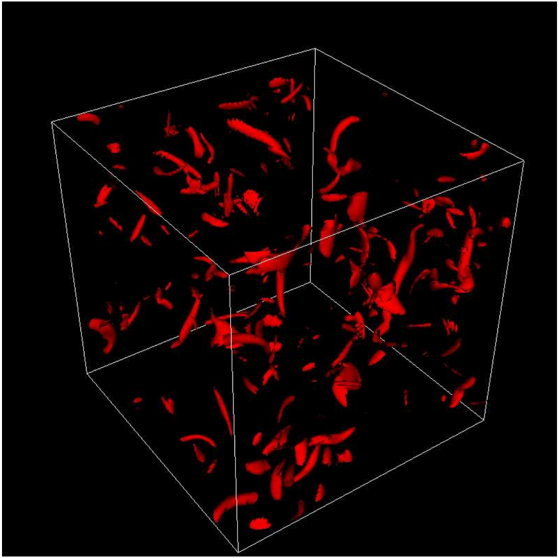
with $u_r, u_z << u_\varphi$

$$\Longrightarrow \frac{\partial \omega_z}{\partial t} + \underbrace{\frac{u_\varphi}{r}\frac{\partial \omega_z}{\partial \varphi}}_{\text{advection}} = \underbrace{\omega_z \frac{\partial u_\varphi}{\partial z}}_{\text{stretching}} + \underbrace{\nu \Delta \omega_z}_{\text{diffusion}}$$

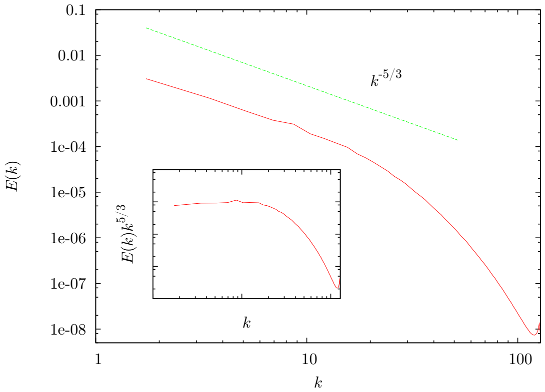


Coherent Structures: vortex dynamics

- complex vortex interaction



Coherent vs. Random Fields



dm^3	Re	R_λ	u_{rms}	ν	L	T	$k_{\max}\eta$
256	1674	158	0.084	0.0001	1.98	23.5	0.96

Coherent vs. Random Fields: randomization

$$\boldsymbol{\omega}(\boldsymbol{x},t) = \sum_{\boldsymbol{k}} \tilde{\boldsymbol{\omega}}_{\boldsymbol{k}}(t)\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$\Longrightarrow \quad \boldsymbol{u}(\boldsymbol{x},t) \quad \boldsymbol{a}(\boldsymbol{x},t)$$

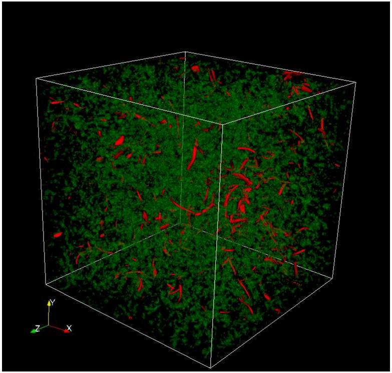
Coherent vs. Random Fields: randomization

$$\boldsymbol{\omega}(\boldsymbol{x},t) = \sum_{\boldsymbol{k}} \tilde{\boldsymbol{\omega}}_{\boldsymbol{k}}(t)\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

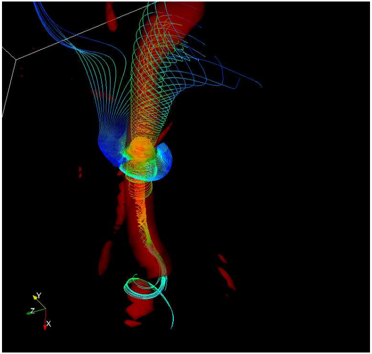
$$\Longrightarrow \quad \boldsymbol{u}(\boldsymbol{x},t) \quad \boldsymbol{a}(\boldsymbol{x},t)$$

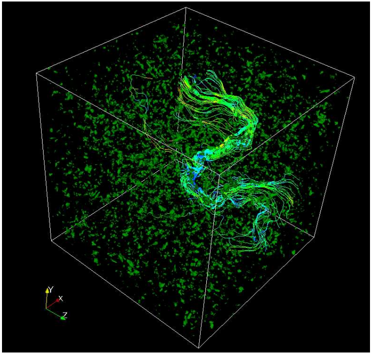
$$\boldsymbol{\omega}^r(\boldsymbol{x},t) = \sum_{\boldsymbol{k}} \tilde{\boldsymbol{\omega}}_{\boldsymbol{k}}(t)\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}+\varphi_{\boldsymbol{k}}}$$

$$\Longrightarrow \quad \boldsymbol{u}^r(\boldsymbol{x},t) \quad \boldsymbol{a}^r(\boldsymbol{x},t)$$



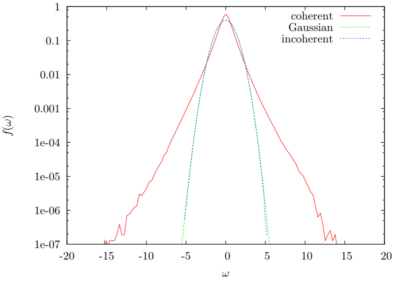
Coherent vs. Random Fields: streamlines





- no trapping events in incoherent vorticity
- different transport/diffusion properties (?)

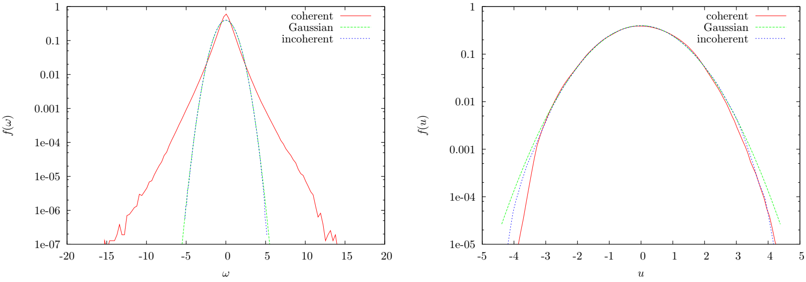
Coherent vs. Random Fields: pdf's



coherence as the origin of:

- non-Gaussianity

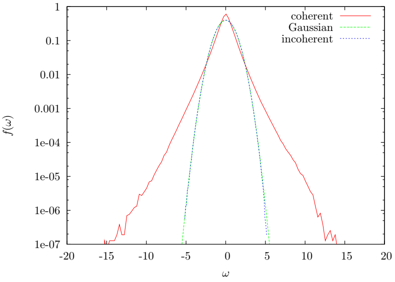
Coherent vs. Random Fields: pdf's

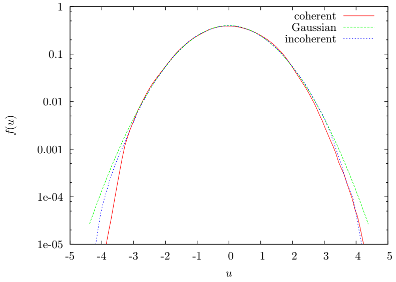


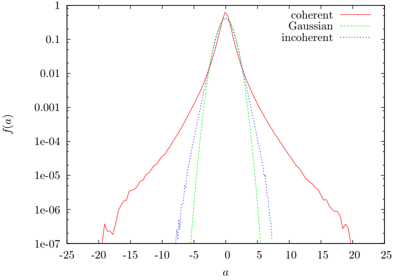
coherence as the origin of:

- non-Gaussianity

Coherent vs. Random Fields: pdf's



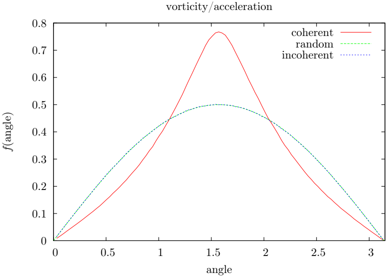




coherence as the origin of:

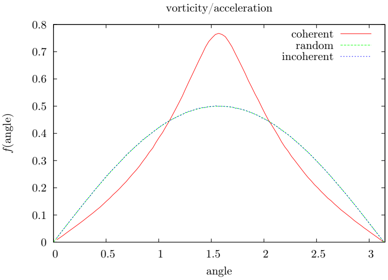
- non-Gaussianity
- intermittency

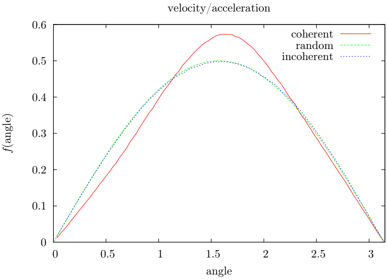
Coherent vs. Random Fields: alignment pdf's



- characteristic peaks
- similar to an ensemble of vortex tubes

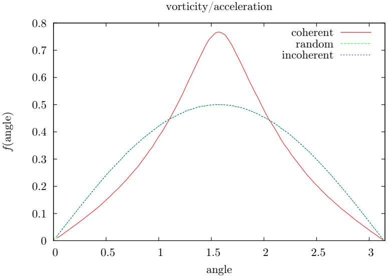
Coherent vs. Random Fields: alignment pdf's

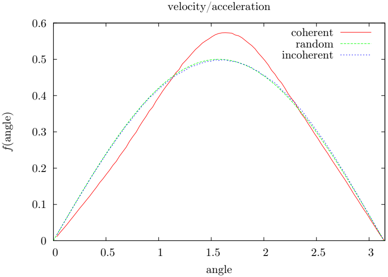


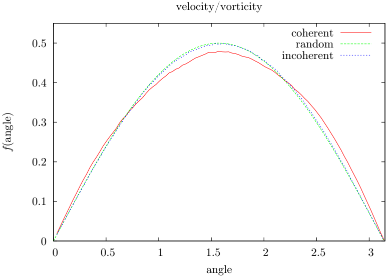


- characteristic peaks
- similar to an ensemble of vortex tubes

Coherent vs. Random Fields: alignment pdf's







- characteristic peaks
- similar to an ensemble of vortex tubes

Statistical Description

Lundgren's Hierarchy Revisited

Statistics: introduction

Consider an ideal gas:

- deterministic description impossible
- statistical description needed

Statistics: introduction

Consider an ideal gas:

- deterministic description impossible
- statistical description needed
- $\cdots$ and sufficient!

Statistics: introduction

Consider an ideal gas:

- deterministic description impossible
- statistical description needed
- $\cdots$ and sufficient!

- the same applies for turbulence
- Lundgren 1967: statistical description of turbulence similar to BBGKY hierarchy

Statistics: notation

fine-grained one-point distribution:

$$\hat{f}^1 = \hat{f}^1(\boldsymbol{v}^1; \boldsymbol{x}^1, t) := \delta(\boldsymbol{u}(\boldsymbol{x}^1, t) - \boldsymbol{v}^1) = \prod_{i=1}^{3} \delta(u_i(\boldsymbol{x}^1, t) - v_i^1)$$

Statistics: notation

fine-grained one-point distribution:

$$\hat{f}^1 = \hat{f}^1(\boldsymbol{v}^1; \boldsymbol{x}^1, t) := \delta(\boldsymbol{u}(\boldsymbol{x}^1, t) - \boldsymbol{v}^1) = \prod_{i=1}^{3} \delta(u_i(\boldsymbol{x}^1, t) - v_i^1)$$

two-point distribution:

$$\hat{f}^{12} = \hat{f}^{12}(\boldsymbol{v}^1, \boldsymbol{v}^2; \boldsymbol{x}^1, \boldsymbol{x}^2, t) := \delta(\boldsymbol{u}(\boldsymbol{x}^1, t) - \boldsymbol{v}^1)\delta(\boldsymbol{u}(\boldsymbol{x}^2, t) - \boldsymbol{v}^2)$$

Statistics: notation

fine-grained one-point distribution:

$$\hat{f}^1 = \hat{f}^1(\boldsymbol{v}^1; \boldsymbol{x}^1, t) := \delta(\boldsymbol{u}(\boldsymbol{x}^1, t) - \boldsymbol{v}^1) = \prod_{i=1}^{3} \delta(u_i(\boldsymbol{x}^1, t) - v_i^1)$$

two-point distribution:

$$\hat{f}^{12} = \hat{f}^{12}(\boldsymbol{v}^1, \boldsymbol{v}^2; \boldsymbol{x}^1, \boldsymbol{x}^2, t) := \delta(\boldsymbol{u}(\boldsymbol{x}^1, t) - \boldsymbol{v}^1)\delta(\boldsymbol{u}(\boldsymbol{x}^2, t) - \boldsymbol{v}^2)$$

pdf via ensemble average

$$f(\boldsymbol{v}; \boldsymbol{x}, t) = \langle \hat{f}(\boldsymbol{v}; \boldsymbol{x}, t) \rangle = \int \mathrm{d}\boldsymbol{v}' \, \delta(\boldsymbol{v}' - \boldsymbol{v}) \, f(\boldsymbol{v}'; \boldsymbol{x}, t)$$

Statistics: properties

reduction property:

$$f^1 = \int \mathrm{d}\boldsymbol{v}^2 f^{12}$$

Statistics: properties

reduction property:

$$f^1 = \int \mathrm{d}\boldsymbol{v}^2 f^{12}$$

separation property:

$$\lim_{|\boldsymbol{x}^2-\boldsymbol{x}^1|\to\infty} f^{12} = f^1 f^2$$

Statistics: properties

reduction property:

$$f^1 = \int \mathrm{d}\boldsymbol{v}^2 f^{12}$$

separation property:

$$\lim_{|\boldsymbol{x}^2 - \boldsymbol{x}^1| \to \infty} f^{12} = f^1 f^2$$

coincidence property:

$$\lim_{\boldsymbol{x}^1 \to \boldsymbol{x}^2} f^{12} = f^1 \, \delta(\boldsymbol{v}^2 - \boldsymbol{v}^1)$$

Statistics: properties

reduction property:

$$f^1 = \int \mathrm{d}\boldsymbol{v}^2 f^{12}$$

separation property:

$$\lim_{|\boldsymbol{x}^2-\boldsymbol{x}^1|\to\infty} f^{12} = f^1 f^2$$

coincidence property:

$$\lim_{\boldsymbol{x}^1\to\boldsymbol{x}^2} f^{12} = f^1\,\delta(\boldsymbol{v}^2-\boldsymbol{v}^1)$$

joint/conditional averages:

$$\langle\phi(\boldsymbol{x},t)\hat{f}(\boldsymbol{v};\boldsymbol{x},t)\rangle = f(\boldsymbol{v};\boldsymbol{x},t)\,\langle\phi(\boldsymbol{x},t)|\boldsymbol{v}\rangle$$

Statistics: evolution equation for $\hat{f}(\boldsymbol{v};\boldsymbol{x},t)$

substantial derivative of the fine-grained pdf:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{f}(\boldsymbol{v};\boldsymbol{x},t) = \frac{\partial}{\partial t}\hat{f}(\boldsymbol{v};\boldsymbol{x},t) + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\hat{f}(\boldsymbol{v};\boldsymbol{x},t)$$

Statistics: evolution equation for $\hat{f}(\boldsymbol{v};\boldsymbol{x},t)$

substantial derivative of the fine-grained pdf:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{f}(\boldsymbol{v};\boldsymbol{x},t) = \frac{\partial}{\partial t}\hat{f}(\boldsymbol{v};\boldsymbol{x},t) + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\hat{f}(\boldsymbol{v};\boldsymbol{x},t)$$

together with

$$\frac{\partial}{\partial t}\hat{f}(\boldsymbol{v};\boldsymbol{x},t) = -\frac{\partial \boldsymbol{u}(\boldsymbol{x},t)}{\partial t}\cdot\nabla_{\boldsymbol{v}}\hat{f}(\boldsymbol{v};\boldsymbol{x},t)$$

(and a little rearrangement) one ends up with

Statistics: evolution equation for $\hat{f}(\boldsymbol{v};\boldsymbol{x},t)$

substantial derivative of the fine-grained pdf:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{f}(\boldsymbol{v};\boldsymbol{x},t) = \frac{\partial}{\partial t}\hat{f}(\boldsymbol{v};\boldsymbol{x},t) + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\hat{f}(\boldsymbol{v};\boldsymbol{x},t)$$

together with

$$\frac{\partial}{\partial t}\hat{f}(\boldsymbol{v};\boldsymbol{x},t) = -\frac{\partial \boldsymbol{u}(\boldsymbol{x},t)}{\partial t}\cdot\nabla_{\boldsymbol{v}}\hat{f}(\boldsymbol{v};\boldsymbol{x},t)$$

(and a little rearrangement) one ends up with

$$\frac{\partial}{\partial t}\hat{f}(\boldsymbol{v};\boldsymbol{x},t) + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\hat{f}(\boldsymbol{v};\boldsymbol{x},t) = -\nabla_{\boldsymbol{v}}\cdot\left[\hat{f}(\boldsymbol{v};\boldsymbol{x},t)\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}(\boldsymbol{x},t)\right].$$

Statistics: evolution equation for $f(\boldsymbol{v};\boldsymbol{x},t)$

additional ensemble averaging leads to:

$$\frac{\partial}{\partial t} f(\boldsymbol{v};\boldsymbol{x},t) + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f(\boldsymbol{v};\boldsymbol{x},t) = -\nabla_{\boldsymbol{v}}\cdot\left\langle \hat{f}(\boldsymbol{v};\boldsymbol{x},t)\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}(\boldsymbol{x},t)\right\rangle$$

Statistics: evolution equation for $f(\boldsymbol{v};\boldsymbol{x},t)$

additional ensemble averaging leads to:

$$\frac{\partial}{\partial t}f(\boldsymbol{v};\boldsymbol{x},t)+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}f(\boldsymbol{v};\boldsymbol{x},t)=-\nabla_{\boldsymbol{v}}\cdot\left\langle \hat{f}(\boldsymbol{v};\boldsymbol{x},t)\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}(\boldsymbol{x},t)\right\rangle$$

in terms of a conditional average:

$$\frac{\partial}{\partial t}f(\boldsymbol{v};\boldsymbol{x},t)+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}f(\boldsymbol{v};\boldsymbol{x},t)=-\nabla_{\boldsymbol{v}}\cdot\underbrace{\left\langle\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}(\boldsymbol{x},t)\middle|\boldsymbol{v}\right\rangle}_{\text{Here's the physics!}}f(\boldsymbol{v};\boldsymbol{x},t).$$

Statistics: coupling to f^{12} I

alternatively:

$$\left\langle \hat{f}(\boldsymbol{v};\boldsymbol{x},t)\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}(\boldsymbol{x},t)\right\rangle = \left\langle \hat{f}(\boldsymbol{v};\boldsymbol{x},t)\left(-\nabla_{\boldsymbol{x}}p(\boldsymbol{x},t)+\nu\Delta_{\boldsymbol{x}}\boldsymbol{u}(\boldsymbol{x},t)\right)\right\rangle .$$

with the identity $1=\int \mathrm{d}\boldsymbol{v}^2\,\delta(\boldsymbol{u}^2-\boldsymbol{v}^2)$ (and the Poisson equation for the pressure)

$$-\left\langle \hat{f}\,\nabla p\right\rangle =$$

$$\left\langle -\frac{\nabla_{\boldsymbol{x}^1}}{4\pi}\iint \mathrm{d}\boldsymbol{x}^2\mathrm{d}\boldsymbol{v}^2\frac{\nabla_{\boldsymbol{x}^2}(\boldsymbol{v}^2\cdot\nabla_{\boldsymbol{x}^2}\boldsymbol{v}^2)}{|\boldsymbol{x}^1-\boldsymbol{x}^2|}\,\delta(\boldsymbol{u}^1-\boldsymbol{v}^1)\,\delta(\boldsymbol{u}^2-\boldsymbol{v}^2)\right\rangle =$$

Statistics: coupling to f^{12} I

alternatively:

$$\left\langle \hat{f}(\boldsymbol{v};\boldsymbol{x},t)\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}(\boldsymbol{x},t)\right\rangle = \left\langle \hat{f}(\boldsymbol{v};\boldsymbol{x},t)\left(-\nabla_{\boldsymbol{x}}p(\boldsymbol{x},t)+\nu\Delta_{\boldsymbol{x}}\boldsymbol{u}(\boldsymbol{x},t)\right)\right\rangle .$$

with the identity $1=\int \mathrm{d}\boldsymbol{v}^2\,\delta(\boldsymbol{u}^2-\boldsymbol{v}^2)$ (and the Poisson equation for the pressure)

$$-\left\langle \hat{f}\,\nabla p\right\rangle =$$

$$\left\langle -\frac{\nabla_{\boldsymbol{x}^1}}{4\pi}\iint \mathrm{d}\boldsymbol{x}^2\mathrm{d}\boldsymbol{v}^2\frac{\nabla_{\boldsymbol{x}^2}(\boldsymbol{v}^2\cdot\nabla_{\boldsymbol{x}^2}\boldsymbol{v}^2)}{|\boldsymbol{x}^1-\boldsymbol{x}^2|}\,\delta(\boldsymbol{u}^1-\boldsymbol{v}^1)\,\delta(\boldsymbol{u}^2-\boldsymbol{v}^2)\right\rangle =$$

$$-\frac{\nabla_{\boldsymbol{x}^1}}{4\pi}\int \mathrm{d}\boldsymbol{x}^2\mathrm{d}\boldsymbol{v}^2\frac{(\boldsymbol{v}^2\cdot\nabla_{\boldsymbol{x}^2})^2}{|\boldsymbol{x}^1-\boldsymbol{x}^2|}\,f^{12}$$

Statistics: coupling to f^{12} II

similar treatment of diffusion term:

$$\left\langle \nu(\Delta_{\boldsymbol{x}^1}\boldsymbol{u}^1)\delta(\boldsymbol{u}^1-\boldsymbol{v}^1)\right\rangle = \lim_{\boldsymbol{x}^2\to\boldsymbol{x}^1}\left\langle \nu\Delta_{\boldsymbol{x}^2}\boldsymbol{u}^2\delta(\boldsymbol{u}^1-\boldsymbol{v}^1)\right\rangle$$

Statistics: coupling to f^{12} II

similar treatment of diffusion term:

$$\left\langle \nu(\Delta_{\boldsymbol{x}^1}\boldsymbol{u}^1)\delta(\boldsymbol{u}^1-\boldsymbol{v}^1)\right\rangle = \lim_{\boldsymbol{x}^2\to\boldsymbol{x}^1}\left\langle \nu\Delta_{\boldsymbol{x}^2}\boldsymbol{u}^2\delta(\boldsymbol{u}^1-\boldsymbol{v}^1)\right\rangle$$

$$= \lim_{\boldsymbol{x}^2\to\boldsymbol{x}^1}\left\langle \nu\Delta_{\boldsymbol{x}^2}\int \mathrm{d}\boldsymbol{v}^2\,\boldsymbol{v}^2\delta(\boldsymbol{u}^1-\boldsymbol{v}^1)\,\delta(\boldsymbol{u}^2-\boldsymbol{v}^2)\right\rangle$$

Statistics: coupling to f^{12} II

similar treatment of diffusion term:

$$\left\langle \nu(\Delta_{\boldsymbol{x}^1}\boldsymbol{u}^1)\delta(\boldsymbol{u}^1-\boldsymbol{v}^1)\right\rangle = \lim_{\boldsymbol{x}^2\to\boldsymbol{x}^1}\left\langle \nu\Delta_{\boldsymbol{x}^2}\boldsymbol{u}^2\delta(\boldsymbol{u}^1-\boldsymbol{v}^1)\right\rangle$$

$$= \lim_{\boldsymbol{x}^2\to\boldsymbol{x}^1}\left\langle \nu\Delta_{\boldsymbol{x}^2}\int \mathrm{d}\boldsymbol{v}^2\,\boldsymbol{v}^2\delta(\boldsymbol{u}^1-\boldsymbol{v}^1)\,\delta(\boldsymbol{u}^2-\boldsymbol{v}^2)\right\rangle$$

$$= \lim_{\boldsymbol{x}^2\to\boldsymbol{x}^1}\nu\Delta_{\boldsymbol{x}^2}\int \mathrm{d}\boldsymbol{v}^2\,\boldsymbol{v}^2 f^{12}$$

Statistics: coupling to f^{12} II

first equation of Lundgren's hierarchy:

$$\frac{\partial}{\partial t} f + \boldsymbol{v}^1 \cdot \nabla_{\boldsymbol{x}^1} f =$$
$$- \nabla_{\boldsymbol{v}^1} \left[- \frac{\nabla_{\boldsymbol{x}^1}}{4\pi} \int \mathrm{d}\boldsymbol{x}^2 \mathrm{d}\boldsymbol{v}^2 \frac{(\boldsymbol{v}^2 \cdot \nabla_{\boldsymbol{x}^2})^2}{|\boldsymbol{x}^1 - \boldsymbol{x}^2|} f^{12} + \lim_{\boldsymbol{x}^2 \to \boldsymbol{x}^1} \nu \Delta_{\boldsymbol{x}^2} \int \mathrm{d}\boldsymbol{v}^2 \, \boldsymbol{v}^2 f^{12} \right]$$

derivation of higher orders analogously

Statistics: increment pdf

velocity increments:

$$\delta \boldsymbol{u}(\boldsymbol{x}^1, \boldsymbol{x}^2, t) = \boldsymbol{u}(\boldsymbol{x}^2, t) - \boldsymbol{u}(\boldsymbol{x}^1, t)$$

fine-grained pdf:

$$\hat{h}(\delta \boldsymbol{v}; \boldsymbol{x}^1, \boldsymbol{x}^2, t) := \delta(\boldsymbol{u}(\boldsymbol{x}^2, t) - \boldsymbol{u}(\boldsymbol{x}^1, t) - \delta \boldsymbol{v})$$

substantial derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{h}(\delta \boldsymbol{v}; \boldsymbol{x}^1, \boldsymbol{x}^2, t) = \frac{\partial}{\partial t}\hat{h} + \boldsymbol{v}^1 \cdot \nabla_{\boldsymbol{x}^1}\hat{h} + \boldsymbol{v}^2 \cdot \nabla_{\boldsymbol{x}^2}\hat{h}$$

Statistics: increment pdf

basically repeating the same steps leads to

$$\frac{\partial}{\partial t}\hat{h} + \boldsymbol{v}^1 \cdot \nabla_{\boldsymbol{x}^1}\hat{h} + \boldsymbol{v}^2 \cdot \nabla_{\boldsymbol{x}^2}\hat{h} = -\nabla_{\delta \boldsymbol{v}} \cdot \left[\hat{h}\frac{\mathrm{d}\delta \boldsymbol{u}}{\mathrm{d}t}\right]$$

Statistics: increment pdf

basically repeating the same steps leads to

$$\frac{\partial}{\partial t}\hat{h} + \boldsymbol{v}^1 \cdot \nabla_{\boldsymbol{x}^1}\hat{h} + \boldsymbol{v}^2 \cdot \nabla_{\boldsymbol{x}^2}\hat{h} = -\nabla_{\delta\boldsymbol{v}} \cdot \left[\hat{h}\frac{\mathrm{d}\delta\boldsymbol{u}}{\mathrm{d}t}\right]$$

averaging:

$$\frac{\partial}{\partial t}h + \boldsymbol{v}^1 \cdot \nabla_{\boldsymbol{x}^1}h + \boldsymbol{v}^2 \cdot \nabla_{\boldsymbol{x}^2}h = -\nabla_{\delta\boldsymbol{v}} \cdot \left\langle \hat{h}\frac{\mathrm{d}\delta\boldsymbol{u}}{\mathrm{d}t} \right\rangle$$

Statistics: increment pdf

basically repeating the same steps leads to

$$\frac{\partial}{\partial t}\hat{h} + \boldsymbol{v}^1 \cdot \nabla_{\boldsymbol{x}^1}\hat{h} + \boldsymbol{v}^2 \cdot \nabla_{\boldsymbol{x}^2}\hat{h} = -\nabla_{\delta \boldsymbol{v}} \cdot \left[\hat{h}\frac{\mathrm{d}\delta \boldsymbol{u}}{\mathrm{d}t}\right]$$

averaging:

$$\frac{\partial}{\partial t}h + \boldsymbol{v}^1 \cdot \nabla_{\boldsymbol{x}^1}h + \boldsymbol{v}^2 \cdot \nabla_{\boldsymbol{x}^2}h = -\nabla_{\delta \boldsymbol{v}} \cdot \left\langle \hat{h}\frac{\mathrm{d}\delta \boldsymbol{u}}{\mathrm{d}t}\right\rangle$$

$$\frac{\partial}{\partial t}h + \boldsymbol{v}^1 \cdot \nabla_{\boldsymbol{x}^1}h + \boldsymbol{v}^2 \cdot \nabla_{\boldsymbol{x}^2}h = -\nabla_{\delta \boldsymbol{v}} \cdot \left\langle \frac{\mathrm{d}\delta \boldsymbol{u}}{\mathrm{d}t}\middle|\delta \boldsymbol{v}\right\rangle h.$$

Statistics: stationarity and homogeneity

stationarity:

$$\frac{\partial}{\partial t} h = 0$$

Statistics: stationarity and homogeneity

stationarity:

$$\frac{\partial}{\partial t} h = 0$$

homogeneity: pdf only depends on $\boldsymbol{r} := \boldsymbol{x}^2 - \boldsymbol{x}^1$

$$h(\delta \boldsymbol{v}; \boldsymbol{x}^1, \boldsymbol{x}^2, t) = h(\delta \boldsymbol{v}; \boldsymbol{r}, t),$$

which implies

$$\nabla_{\boldsymbol{x}^2} h = -\nabla_{\boldsymbol{x}^1} h = \nabla_{\boldsymbol{r}} h,$$

Statistics: stationarity and homogeneity

stationarity:

$$\frac{\partial}{\partial t} h = 0$$

homogeneity: pdf only depends on $\boldsymbol{r} := \boldsymbol{x}^2 - \boldsymbol{x}^1$

$$h(\delta \boldsymbol{v}; \boldsymbol{x}^1, \boldsymbol{x}^2, t) = h(\delta \boldsymbol{v}; \boldsymbol{r}, t),$$

which implies

$$\nabla_{\boldsymbol{x}^2} h = -\nabla_{\boldsymbol{x}^1} h = \nabla_{\boldsymbol{r}} h,$$

leading to

$$\delta \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} h = -\nabla_{\delta \boldsymbol{v}} \cdot \left\langle \frac{\mathrm{d}\delta \boldsymbol{u}}{\mathrm{d}t} \middle| \delta \boldsymbol{v} \right\rangle h.$$

Statistics: What's next?

- $\left\langle \frac{\mathrm{d}\delta \boldsymbol{u}}{\mathrm{d}t} \middle| \delta \boldsymbol{v} \right\rangle$ seems to be important quantity
- contains transition from large-scale Gaussianity to small-scale intermittency

Statistics: What's next?

- $\left\langle \frac{\mathrm{d}\delta \boldsymbol{u}}{\mathrm{d}t} \middle| \delta \boldsymbol{v} \right\rangle$ seems to be important quantity
- contains transition from large-scale Gaussianity to small-scale intermittency
- determine $\left\langle \frac{\mathrm{d}\delta \boldsymbol{u}}{\mathrm{d}t} \middle| \delta \boldsymbol{v} \right\rangle$ numerically
- ... and try to understand

Statistics: What's next?

- $\left\langle \frac{\mathrm{d}\delta \boldsymbol{u}}{\mathrm{d}t} \middle| \delta \boldsymbol{v} \right\rangle$ seems to be important quantity
- contains transition from large-scale Gaussianity to small-scale intermittency
- determine $\left\langle \frac{\mathrm{d}\delta \boldsymbol{u}}{\mathrm{d}t} \middle| \delta \boldsymbol{v} \right\rangle$ numerically
- ... and try to understand
- solve pdf equation analytically (method of characteristics)
- ... or numerically

Summary

- slender vortices are crucial to turbulent dynamics
- framework for a statistical description of turbulence