Zero Modes In Turbulence

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Intention

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$\longrightarrow$ Statistical conservation laws could help to understand the evolution of multiparticle configurations.

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That means:

$$\frac{d}{dt}\left\langle f(\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_N)\right\rangle = \frac{d}{dt}\left\langle f(\underline{\mathbf{R}})\right\rangle = 0$$

Example

Brownian Motion:

$$\langle R^2(t) \rangle = R^2(0) + \kappa t$$

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$$\implies \quad \langle R_{12}^2(t) - R_{34}^2(t) \rangle = const.$$

Conclusions of the existence of those functions

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- The existence of those multiparticle conservation laws indicates the presence of a long-time-memory.
- It is a reflection of the coupling among the particles due to the fact that they are all in the same velocity field.
- For the mean distance of the particles is growing in time the conserved quantities need to involve the geometry of the cloud.
  $\longrightarrow$ We might learn something about the shape evolution of the cloud.

Definition of the Kraichnan Model

$$\left\langle v^i(\mathbf{r},t)v^j(\mathbf{r}',t')\right\rangle = 2D^{ij}(\mathbf{r}-\mathbf{r}')\,\delta(t-t')$$

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$$D^{ij}(\mathbf{r}) = D_0\,\delta_{ij} - \frac{1}{2}d^{ij}(\mathbf{r})$$

Scaling of d^{ij}

d^{ij} scales as r^{ξ}, $0 \leq \xi \leq 2$, in the intertial interval $\eta << r << L$.
d^{ij} scales as r^2 in the viscous range $r << \eta$.
d^{ij} tends to $2D_0\,\delta^{ij}$ for $r \longrightarrow \infty$.

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d^{ij} scales as r^2 in the viscous range $r << \eta$.
d^{ij} tends to $2D_0\,\delta^{ij}$ for $r \longrightarrow \infty$.
In the limit $\eta \longrightarrow 0$, $L \longrightarrow \infty$:

$$\lim_{\eta \longrightarrow 0, L \longrightarrow \infty} d^{ij}(\mathbf{r}) = D_1 r^{\xi}\big((d - 1 + \xi - \wp\xi)\,\delta^{ij} + (\wp d - 1)\xi \frac{r^i r^j}{r^2}\big)$$

$\wp = \langle (\nabla_i v^i)^2 / (\nabla_i v^j)^2 \rangle$ is the degree of compressibility.

Calculating functions with constant expectation value

$$\langle f(\underline{\mathbf{R}}(t))\rangle = \int f(\underline{\mathbf{R}})\mathcal{P}(\underline{\mathbf{r}};\underline{\mathbf{R}};t)dR$$

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$\longrightarrow$ One needs to determine $\mathcal{P}(\underline{\mathbf{r}};\underline{\mathbf{R}};t)$.

Determination of $\mathcal{P}$

$$\mathcal{P}_N(\underline{\mathbf{r}}; \underline{\mathbf{R}}; t) = \langle p(\mathbf{r}_1, 0; \mathbf{R}_1, t|\mathbf{v}) \ldots p(\mathbf{r}_N, 0; \mathbf{R}_N, t|\mathbf{v})\rangle$$

Determination of $\mathcal{P}$

$$\mathcal{P}_N(\underline{\mathbf{r}}; \underline{\mathbf{R}}; t) = \langle p(\mathbf{r}_1, 0; \mathbf{R}_1, t|\mathbf{v}) \dots p(\mathbf{r}_N, 0; \mathbf{R}_N, t|\mathbf{v}) \rangle$$

From the advection-diffusion-equation

$$[\frac{\partial}{\partial t} - \nabla_{\mathbf{R}} \cdot \mathbf{v}(\mathbf{R}, t) - \kappa \Delta_{\mathbf{R}}] p(\mathbf{r}, s; \mathbf{R}, t|\mathbf{v}) = 0$$

one gets an equation for $\mathcal{P}_N$ via the phase space integral representation of the solution of this equation .

Determination of $\mathcal{P}$

$$\left(\frac{\partial}{\partial t} - \mathcal{M}_N\right)\mathcal{P}_N(\underline{\mathbf{r}}; \underline{\mathbf{R}}; t) = \delta(t)\,\delta(\underline{\mathbf{R}} - \underline{\mathbf{r}})$$

with

$$\mathcal{M}_N = \sum_{n,m=1}^{N} \nabla_{r_n^i} \nabla_{r_m^j} D^{ij}(\mathbf{r}_{nm}) + \kappa \sum_{n=1}^{N} \Delta_{\mathbf{r}_n}$$

Back to our intention

Our intention was to determine functions whose expectation value does not change in time. We already know that

$$\langle f(\underline{\mathbf{R}}(t))\rangle = \int f(\underline{\mathbf{R}})\tilde{\mathcal{P}}_N(\underline{\mathbf{r}};\underline{\mathbf{R}};t)dR$$

With our knowledge about the evolution equation of $\tilde{\mathcal{P}}_N$ we can find a partial differential equation for f.

Back to our intention

$$\frac{d}{dt}\left\langle f(\underline{\mathbf{R}}(t))\right\rangle = \frac{d}{dt}\int f\left(\underline{\mathbf{R}}\right)\tilde{\mathcal{P}}_N(\underline{\mathbf{r}};\underline{\mathbf{R}};t)dR$$

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$$= \int f(\underline{\mathbf{R}})\frac{\partial}{\partial t}\tilde{\mathcal{P}}_N(\underline{\mathbf{r}};\underline{\mathbf{R}};t)dR = \int f(\underline{\mathbf{R}})\tilde{\mathcal{M}}_N\tilde{\mathcal{P}}_N(\underline{\mathbf{r}};\underline{\mathbf{R}};t)dR$$

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$$\frac{d}{dt}\langle f(\underline{\mathbf{R}}(t))\rangle = \frac{d}{dt}\int f(\underline{\mathbf{R}})\tilde{\mathcal{P}}_N(\underline{\mathbf{r}};\underline{\mathbf{R}};t)dR$$
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$$= \int \tilde{\mathcal{M}}_N^* f(\underline{\mathbf{R}})\tilde{\mathcal{P}}_N(\underline{\mathbf{r}};\underline{\mathbf{R}};t)dR$$
$$\stackrel{!}{=} 0$$

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$$= \int \tilde{\mathcal{M}}_N^* f(\underline{\mathbf{R}})\tilde{\mathcal{P}}_N(\underline{\mathbf{r}};\underline{\mathbf{R}};t)dR$$
$$\stackrel{!}{=} 0$$

Due to the fact, that $\tilde{\mathcal{P}}$ ist a density, we just need to solve the equation

$$\tilde{\mathcal{M}}_N^* f(\underline{\mathbf{R}}) = 0$$

Definition of the zero modes

Let $\tilde{\mathcal{M}}_N$ be defined as above.
We call a function f fulfilling the equation $\tilde{\mathcal{M}}_N f = 0$ a zero mode of $\tilde{\mathcal{M}}_N$.

Involving geometry

We consider especially scaling functions of dimension ζ, that means functions fulfilling

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Then one expects intuitively that $\left\langle f\left(\underline{\mathbf{R}}(t)\right)\right\rangle \sim t^{\frac{\zeta}{2-\xi}}$ because an N-particle generalization of the Richardson-type behaviour gives $\left\langle R(t)^{\zeta}\right\rangle \sim t^{\frac{\zeta}{2-\xi}}$.

Involving geometry

If one finds scaling functions not fulfilling this dimensional prediction the reason must be, that $f(\underline{\mathbf{R}}) = R^{\zeta} g(\hat{\underline{\mathbf{R}}})$ with g being a function of the shape of the cloud and balancing the growth of R.

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The functions not fulfilling the dimensional prediction turn out to be the zero modes and the so called slow modes. (Slow modes are zero modes of $\tilde{\mathcal{M}}_N^{k+1}$ for $k \in \mathbb{N}$.)

Example

For $\xi = 0$ we have $\mathcal{M}_N = \Delta$.

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$$\mathcal{M}_N = R^{-Nd+1}\frac{\partial}{\partial R}R^{Nd-1}\frac{\partial}{\partial R} + R^{-2}\hat{\Delta}$$

with $\hat{\Delta}$ being the angular Laplacian.

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with $\hat{\Delta}$ being the angular Laplacian.
The eigenvalues of $\hat{\Delta}$ are $0, 1, 2, \ldots$ and the eigenfunctions are the hermite polynomials H_j.
Thus the zero modes in this case are $f_{j,0}(\underline{\mathbf{R}}) = R^j H_j(\underline{\hat{\mathbf{R}}})$.

Asymptotic expansion of $\tilde{\mathcal{P}}_N$

The zero modes an the slow modes show up in the asymptotic behaviour of $\tilde{\mathcal{P}}_N$ when the initial points get close:

$$\tilde{\mathcal{P}}_N(\lambda\underline{\mathbf{r}};\underline{\mathbf{R}};t) = \sum_a \sum_{k=0}^{\infty} \lambda^{\zeta_a+(2-\xi)k} f_{a,k}(\underline{\mathbf{r}}) g_{a,k}(\underline{\mathbf{R}},t)$$

for small λ.

ζ_a are the scaling dimensions of the zero modes.

Shape evolution

To examine the shape evolution one first trades time in the relative N-particle evolution $\underline{\tilde{\mathbf{R}}}(t)$ for the size variable R.
That means one starts with a configuration of size r and shape $\underline{\tilde{\mathbf{r}}}$ and denotes by $\underline{\tilde{\mathbf{R}}}(R)$ the shape the first time the cloud reaches the size R.

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That means one starts with a configuration of size r and shape $\underline{\tilde{\mathbf{r}}}$ and denotes by $\underline{\tilde{\mathbf{R}}}(R)$ the shape the first time the cloud reaches the size R.
Then the density of the shapes $\underline{\tilde{\mathbf{R}}}(R)$ is:

$$\mathcal{P}_N(\underline{\hat{\mathbf{r}}}; \underline{\hat{\mathbf{R}}}; \frac{r}{R}) = \sum_a \left(\frac{r}{R}\right)^{\zeta_a} f_a(\underline{\hat{\mathbf{r}}}) h_a(\underline{\hat{\mathbf{R}}})$$

h_a are the eigenfunctions of$\mathcal{P}_N(\frac{r}{R})$.

Shape evolution

One then gets

$$\left\langle f(\hat{\underline{\mathbf{R}}}(R)) \right\rangle = \sum_a \left(\frac{r}{R}\right)^{\zeta_a} f_a(\hat{\underline{\mathbf{r}}}) \int f(\hat{\underline{\mathbf{R}}}) h_a(\hat{\underline{\mathbf{R}}}) d\hat{\underline{\mathbf{R}}}$$

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That means that

- The zero modes are the relaxation modes of the shape relaxation.

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That means that

- The zero modes are the relaxation modes of the shape relaxation.
- For the zero modes themselves we obtain

$$\left\langle f_a(\underline{\hat{\mathbf{R}}}(R))\right\rangle = \left(\frac{r}{R}\right)^{\zeta_a} f_a(\underline{\hat{\mathbf{r}}})$$

Structure functions

Consider a passive scalar in a Kraichnan velocity field. Structure functions are differences of correlation functions:

$$S_N(r) = \left\langle \left(\theta(\mathbf{r}) - \theta(\mathbf{0})\right)^N \right\rangle$$

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$$S_N(r) = \left\langle \left(\theta(\mathbf{r}) - \theta(\mathbf{0})\right)^N \right\rangle$$

They decrease when r grows and this decrease is given by the laws governing the decay of shape fluctuations. Thus the scaling exponents of the zero modes coincide with the scaling exponents of the structure functions:

$$S_N(r) \sim r^{\zeta_N} L^{\frac{N}{2}(2-\xi)-\zeta_N}$$

L is the forcing scale of the scalar and ζ_N ist the lowest scaling exponent of an irreducible N-particle-zero mode.

The model

Let us take a look at a one-dimensional model with three particles.

The model

Let us take a look at a one-dimensional model with three particles. In this case the equation $\frac{\partial}{\partial t}\mathcal{P}_N = \mathcal{M}_N\mathcal{P}_N$ simplifies to

$$\frac{\partial}{\partial t} f(x_1, x_2, x_3, t) = \{\frac{\partial^2}{\partial x_1^2}C(0) + \frac{\partial^2}{\partial x_2^2}C(0) + \frac{\partial^2}{\partial x_3^2}C(0)$$

$$+ 2\frac{\partial^2}{\partial x_1 \partial x_2}C(x_1 - x_2) + 2\frac{\partial^2}{\partial x_1 \partial x_3}C(x_1 - x_3)$$

$$+ 2\frac{\partial^2}{\partial x_2 \partial x_3}C(x_2 - x_3)\} f(x_1, x_2, x_3, t)$$

Transformation to relative coordinates

Transformation to relative coordinates gives:

$$\frac{\partial}{\partial t} f(r_1, r_2, t) = \{ \frac{\partial^2}{\partial r_1^2} Q(r_1) + \frac{\partial^2}{\partial r_2^2} Q(r_2)$$

$$+ (\frac{\partial^2}{\partial r_1 \partial r_2} (Q(r_1) + Q(r_2) - Q(r_1 - r_2))\big) f(r_1, r_2, t) =: M\, f(r_1, r_2, t)$$

with $Q(r) = 2\big(C(0) - C(r)\big)$

Calculating the adjoint operator

Calculating M^* with partial integration gives:

$$\int g(r_1, r_2) M f(r_1, r_2, t) dr_1 dr_2 = \int f(r_1, r_2, t) \Big(Q(r_1) \frac{\partial^2}{\partial r_1^2}$$

$$+ \; Q(r_2) \frac{\partial^2}{\partial r_2^2} \; + \; \big(Q(r_1) + Q(r_2)$$

$$- \; Q(r_1 - r_2) \big) \frac{\partial^2}{\partial r_1 \partial r_2} \Big) g(r_1, r_2) dr_2 dr_2$$

$$\implies M^* \; = \; Q(r_1) \frac{\partial^2}{\partial r_1^2} \; + \; Q(r_2) \frac{\partial^2}{\partial r_2^2}$$

$$+ \; \big(Q(r_1) + Q(r_2) - Q(r_1 - r_2) \big) \frac{\partial^2}{\partial r_1 \partial r_2}$$

Transformation to size and shape coordinates

We assume $Q(r) = |r|^\alpha$ and transform the equation to size and shape coordinates $r = \sqrt{r_1^2 + r_2^2}$ and $\rho = \frac{r_1}{r_2}$.

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We assume $Q(r) = |r|^\alpha$ and transform the equation to size and shape coordinates $r = \sqrt{r_1^2 + r_2^2}$ and $\rho = \frac{r_1}{r_2}$.
Presuming $f(r,\rho) = r^\gamma g(\rho)$ we obtain an ordinary differential equation for $g(\rho)$ (WLOG $r_1 > r_2 > 0$):

$$
\begin{aligned}
&\big(\gamma\rho^2 + \gamma(\gamma-2)\rho + \gamma(\gamma-1) + \gamma(\gamma-1)\rho^{\alpha+2} + \gamma(\gamma-2)\rho^{\alpha+1} \\
&+ \gamma\rho^\alpha + \gamma(2-\gamma)\rho(\rho-1)^\alpha\big)\, g(\rho) \\
&+ (1+\rho^2)\big(2\rho^3 - (1+\gamma)\rho^2 + 2(1-\gamma)\rho + \gamma - 1 - \\
&(1+\gamma)\rho^{\alpha+2} + 2\gamma\rho^{\alpha+1} + (\gamma-1)\rho^\alpha + \\
&(1-\gamma)(\rho-1)^\alpha + (1+\gamma)\rho^2(\rho-1)^\alpha\big)\, g'(\rho) \\
&+ \big(\rho^2 - \rho - \rho^{\alpha+1} + \rho^\alpha + \rho(\rho-1)^\alpha\big)\big(1+\rho^2\big)^2 g''(\rho) \;=\; 0
\end{aligned}
$$

The case $\alpha = 0$

- For $\alpha = 0$ we obtain a Brownian motion and the results are a special case of the example above.
- The scaling exponents of the zero modes are the natural numbers.
- For we are just interested in translation invariant zero modes we get two independent zero modes for every scaling exponent due to the second order differential equation above.
- The lowest scaling dimension of irreducible zero modes is 2.

The case $\alpha = 0$

The first zero modes:

$$\gamma = 2: \\ 2r_1^2 - 2r_1 r_2 - r_2^2\,,\; 2r_1 r_2 - r_2^2$$

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$$2r_1^2 - 2r_1r_2 - r_2^2\,,\; 2r_1r_2 - r_2^2$$
$$\gamma = 3:$$
$$2r_1^3 - 3r_1^2r_2 - 3r_1r_2^2 + 2r_2^3,$$
$$r_1r_2^2 - r_2r_1^2$$

The case $\alpha = 0$

The first zero modes:

$$\begin{aligned}
\gamma &= 2 : \\
& 2r_1^2 - 2r_1 r_2 - r_2^2 \,,\; 2r_1 r_2 - r_2^2 \\
\gamma &= 3 : \\
& 2r_1^3 - 3r_1^2 r_2 - 3r_1 r_2^2 + 2r_2^3, \\
& r_1 r_2^2 - r_2 r_1^2 \\
\gamma &= 4 : \\
& 2r_1^4 - 4r_1^3 r_2 - 6r_1^2 r_2^2 + 8r_1 r_2^3 - r_2^4 \\
& 4r_1^3 r_2 - 6r_1^2 r_2^2 + r_2^4 \\
& etc.
\end{aligned}$$

The case $\alpha = 2$

If we set $\alpha = 2$ the equation we get is rather simple.
We obtain

$$r^2 \frac{\partial^2 f}{\partial r^2} \stackrel{!}{=} 0$$

Thus we get

$$\gamma(\gamma - 1) r^{\gamma - 2} g(\rho) \stackrel{!}{=} 0$$

Hence the scaling exponents of the zero modes are just 0 and 1.
But for g an arbitrary function can be choosen.

The case $\alpha = 2$

It is interesting that from $\gamma = 0$ follows that

$$\langle g(\rho)\rangle = 0$$

for arbitrary g.
Consequently the shape is identically distributed.

The case $\alpha = 1$

This case is the most complicated of the three.
Assuming $f(r,\rho) = r^{\gamma}\, g(\rho)$ we obtain

$$g(\rho) \;=\; \left(1+\rho^2\right)^{-\frac{\gamma}{2}} (\rho - 1)\, h(\rho)$$

where h is a solution of the hypergeometric differential equation with certain parameters:

$$\rho(\rho-1)h''(\rho) + (2 - 2\gamma - \big((2-\gamma) + (1-\gamma) + 1\big)\rho)h'(\rho)$$
$$-\,(2-\gamma)(1-\gamma)h(\rho) \;=\; 0$$

The case $\alpha = 1$

Solving the resulting hypergeometric equation we get two linearly independent solutions:

$$h_1(\rho) = \rho^{\gamma-2} \sum_{n=0}^{\infty} a_n \frac{1}{\rho}$$

$$h_2(\rho) = \rho^{\gamma-2} [\rho + \gamma(1-\gamma) \sum_{n=0}^{\infty} K_n a_n \frac{1}{\rho} - \gamma(1-\gamma) log(\rho) \sum_{n=0}^{\infty} a_n \frac{1}{\rho}]$$

with

$$a_n = \prod_{i=1}^{n} \frac{(2+i-\gamma)(1+i+\gamma)}{i(i+1)}$$

$$K_n = \sum_{i=1}^{n} (\frac{1}{2+i-\gamma} + \frac{1}{1+i+\gamma} - \frac{1}{1-\gamma} - \frac{1}{\gamma})$$

The case $\alpha = 1$

To find the scaling exponents we have to continue the above solution for the range $r_1 > r_2 > 0$ to the other ranges.
Using the property of the velocity field that two particles which start together will stay together during the whole evolution, mathematically formulated $f(r_1, r_2) = f(r_1)\,\delta(r_1 - r_2)$, we get that the zero modes have to be zero whenever two particles have the same coordinate, that means $r_1 = r_2$ or $r_1 = 0$ or $r_2 = 0$.
This leads us to the conclusion that the scaling exponents are the natural numbers and the zero modes are:

$$f(r,\rho) \;=\; r^{\gamma}\left(1+\rho^2\right)^{-\frac{\gamma}{2}}(\rho-1)\rho^{\gamma-2}\sum_{n=0}^{\infty} a_n \frac{1}{\rho}$$

The case $\alpha = 1$

The first zero modes $(WLOG\ r_1 > r_2 > 0)$:

$$\gamma = 2 :$$
$$r_1 r_2 - r_2^2$$

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The first zero modes $(WLOG\ r_1 > r_2 > 0)$:

$$\gamma = 2:$$
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$$\gamma = 3:$$
$$r_1^2 r_2 - 3 r_1 r_2^2 - 2 r_2^3$$

The case $\alpha = 1$

The first zero modes $(WLOG\ r_1 > r_2 > 0)$:

$$\begin{aligned} \gamma &= 2: \\ r_1 r_2 - r_2^2 \\ \gamma &= 3: \\ r_1^2 r_2 - 3r_1 r_2^2 - 2r_2^3 \\ \gamma &= 4: \\ r_1^3 r_2 - 6r_1^2 r_2^2 + 10r_1 r_2^3 - 5r_2^4 \\ etc. \end{aligned}$$

Outlook

- Finding the scaling exponents for $1 + \epsilon$ using perturbative methods

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- Finding the scaling exponents for $1 + \epsilon$ using perturbative methods
- Using the Stratonovich convention of stochastic differential equations instead of the Îto convention to see the differences

Summary

To sum up

- Functions of the coordinates of a cloud of particles whose expectation value does not change in time are predestined to examine the shape evolution of the cloud.

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- In the Kraichnan model they are given as zero modes of explicit evolution operators.

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- Functions of the coordinates of a cloud of particles whose expectation value does not change in time are predestined to examine the shape evolution of the cloud.
- In the Kraichnan model they are given as zero modes of explicit evolution operators.
- In very simple cases it is possible to calculate the zero modes and their scaling exponentes analytically, im others perturbation theory is needed.

The end

THANK YOU FOR LISTENING!