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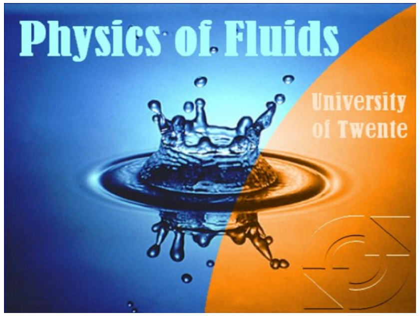
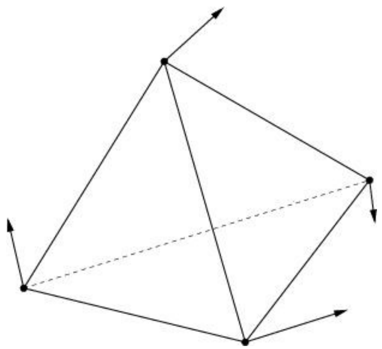


Geometry and statistics in fully developed turbulence



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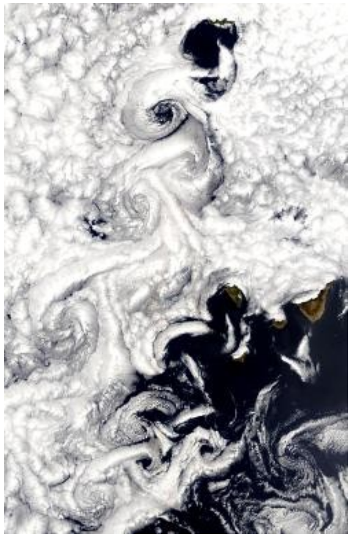
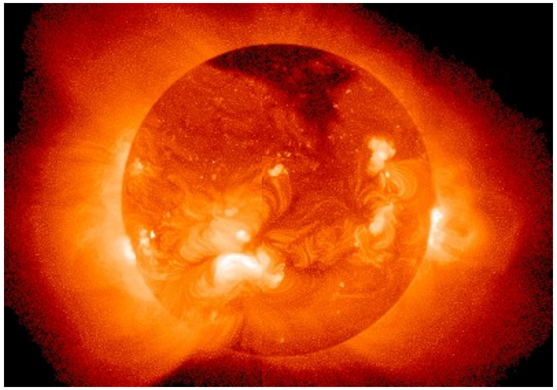
★ A fluid flow becomes turbulent when the source of energy which excites the motions in the fluid is sufficiently intense compared to the viscous resistance of the fluid.

★ Turbulence is characterized by very disordered motions leading to a very important mixing of the fluid.

★ Turbulence is ubiquitous:



(NASA)



(SOHO)

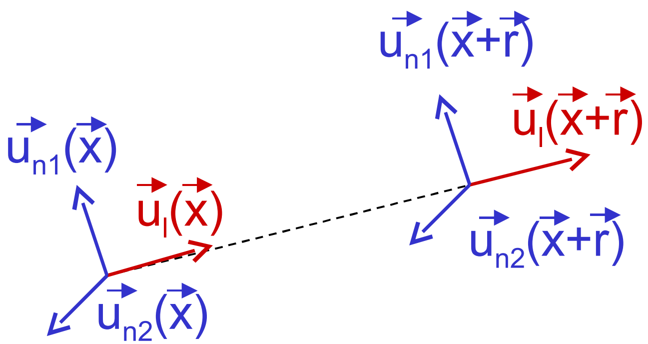
(NASA)

Roughly speaking, most investigations of turbulence consider separately either of the two aspects :

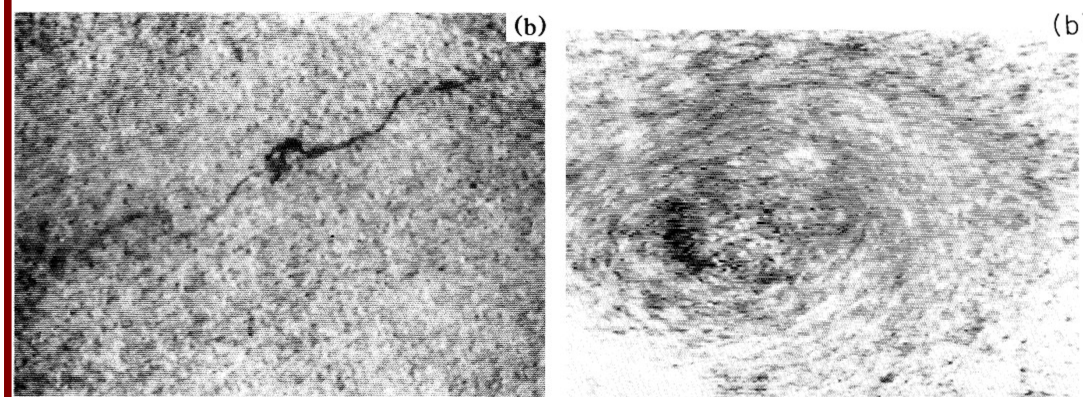
structure functions

$$S_n(r) = \left\langle \left(u_l(\vec{x}+\vec{r}) - u_l(\vec{x}) \right)^n \right\rangle$$

and their dependence with respect to the scale r



the local structure (or geometry) of intermittent regions of the flow (vorticity filaments, …)



Douady, Couder and Brachet, PRL 91

→ Aim here: capture both aspects, by investigating multipoint correlation functions.

Objective of the work:

Develop a theoretical understanding and a description of the fluctuating velocity field that captures both the scaling and structural aspects of the flow.

Idea: describe the velocity gradient tensor:

$$m_{ab} = \partial_a u_b$$

or its coarse-grained generalization:

$$M_{ab} = \frac{1}{V} \int d\vec{r} \partial_a u_b(\vec{r})$$ « coarse-grained »

Statistics as a function of r

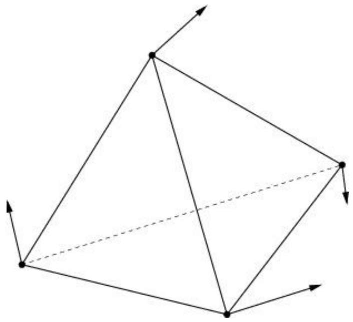
Topology:

$$\omega_i = \varepsilon_{ijk} m_{jk}$$

$$S = \frac{1}{2}\left(m + m^t\right)$$

- Lagrangian concepts have shown recently to be extremely useful, in particular to the passive scalar problem (see Shraiman and Siggia, 2000, and Falkovich, Gawedzki and Vergassola, 2001).

- Extension to the problem of the velocity field: follow 4-points - a tetrahedron- (or more) to construct the finite difference approximation of the velocity derivative tensor M.



→ « tetrad model »

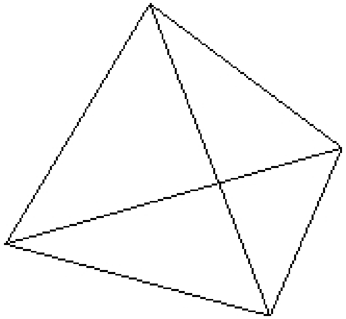
- This phenomenological model describes the dynamics of:
  - M : coarse-grained (filtered) velocity gradient tensor
  - g : moment of inertia tensor (geometrical deformation of the tetrad)

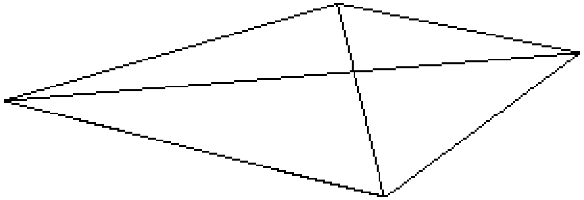
Ref.: M. Chertkov, A. Pumir and B.I. Shraiman, Phys. Fluids **11**, 2394 (1999).

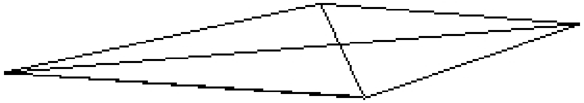
g: moment of inertia tensor

- The eigenvalues of g give information about the shape of the volume:

$g_1 \approx g_2 \approx g_3$: $g_1 \approx g_2 >> g_3$: $g_1 >> g_2 \approx g_3$:







- and about its size:

$$Tr(g) = g_1 + g_2 + g_3 = r^2$$

- **Purpose of the work here:** the evolution of the tetrahedron and of M can be modelled in terms of a system of stochastic differential equations (Chertkov et al, 1999).

Study this system of stochastic differential equations.

Potential pay-offs of this approach:

- get insight into the energy transfer processes between scale (Pumir, Shraiman and Chertkov, 2001)
- potentially, particle-based Large-Eddy Simulation schemes (Pumir and Shraiman, 2003)
- possible applications to different kinds of turbulent flows
- new way of thinking about the issue of turbulence (Chevillard and Meneveau, 2006)

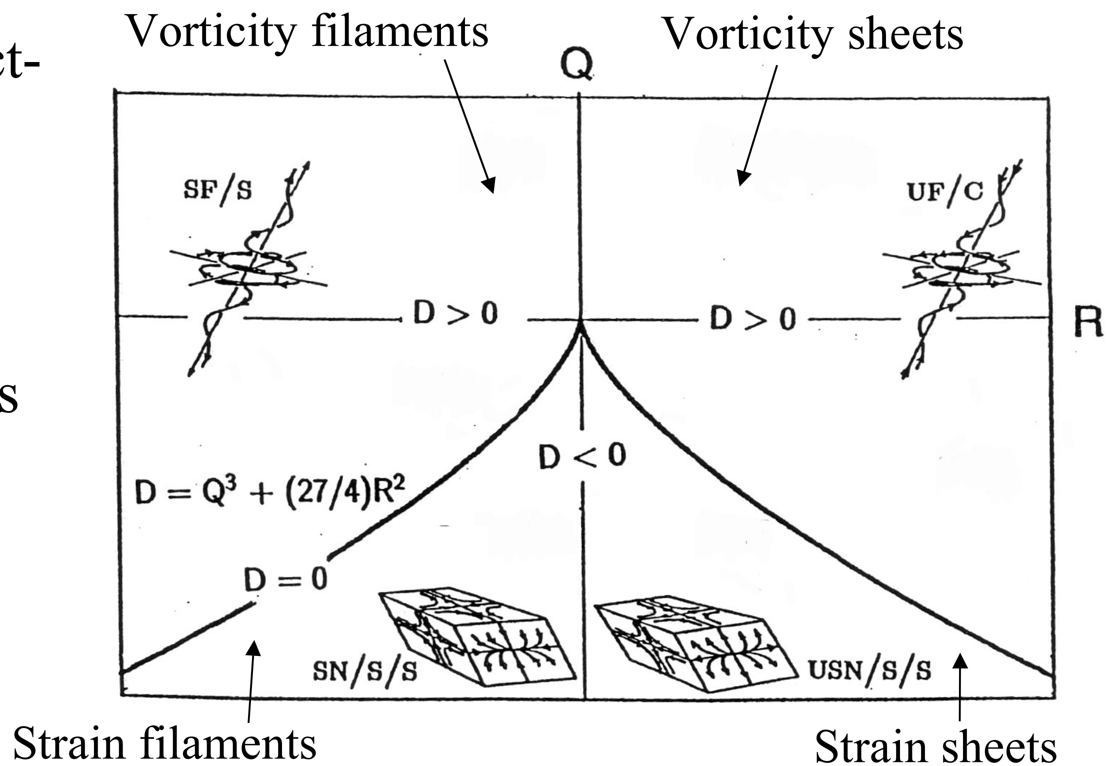
M as a diagnostics of flow topology

- The eigenvalues of M characterize the local topology of the flow (at the considered scale)

- They depend on 2 parameters only (Cayley-Hamilton):

$$Q = -\frac{1}{2} TrM^2$$

$$R = -\frac{1}{3} TrM^3$$



(from Cantwell)

Interesting flow diagnostics: calculate as a function of scale the joint probability distributions of Q and R, as well as densities of dynamical quantities in the (R,Q) plane.

Outline of the presentation

- Derivation and definition of the « tetrad model »
- Semiclassical method of resolution of the system
- Semiclassical solutions with isotropic forcing
- Semiclassical solutions with large scale shear
- Recent developments
- Conclusion and outlook

1. Derivation and definition

of the model

(3D, incompressible,
homogeneous, stationary turbulence)

★ Write the Navier-Stokes equation for the velocity gradient tensor $m = \partial v$:

$$\frac{dm_{ab}}{dt} + m_{ab}^2 = -\partial_{ab} p + \text{viscosity} + \text{forcing}$$

Closing issue: **pressure Hessian**

★ Isotropic approximation (Restricted Euler dynamics) (Vieillefosse, Cantwell):

$$\partial_{ab} p = -\frac{1}{3} Tr\left(m^2\right)\delta_{ab}$$

The resulting system can be completely solved, with the help of the invariants Q and R ($Q = -\frac{1}{2} Tr\left(m^2\right)$; $R = -\frac{1}{3} Tr\left(m^3\right)$)

$\longrightarrow$ finite time singularity !

To go beyond the Vieillefosse singularity, one needs to introduce the geometry of the Lagrangian set of points.

Equation for the geometry, derived from:

$$\frac{d\rho}{dt} = v = \rho M + \xi \quad \begin{cases} \rho M & \text{: coherent component of the velocity } (k \sim 1/R) \\ \xi & \text{: fluctuating component } (k >> 1/R) \end{cases}$$

or

$$\begin{aligned} \vec{\rho}_1 &= (\vec{r}_1 - \vec{r}_2)/\sqrt{2} \\ \vec{\rho}_2 &= (\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3)/\sqrt{6} \\ \vec{\rho}_3 &= (\vec{r}_1 + \vec{r}_2 + \vec{r}_3 - 3\vec{r}_4)/\sqrt{12} \end{aligned}$$

and $\rho_i^a = (\vec{\rho}_i)_a$

Introduce the moment of inertia tensor: $g = \rho^t \rho$

★ Equation for the coarse-grained velocity gradient tensor (obtained from an approximation of the pressure Hessian, based on analytical and numerical results):

$$\frac{dM}{dt} + \left(M^2 - \Pi \, Tr\, M^2\right) = \alpha \left(M^2 - \Pi \, Tr\, M^2\right) + \eta$$

local component of the pressure

$$\left(\Pi = \frac{g^{-1}}{Tr g^{-1}}\right)$$

nonlocal part of the pressure (analytical + numerical evidence) → reduction of the nonlinearity

fluctuating component

★ Reduction of the nonlinearity through the pressure Hessian: the importance of this effect is measured by α.

One finally obtains the following system of stochastic differential equations:

$$\begin{cases} \dfrac{dM}{dt} + (1-\alpha)\left(M^2 - \Pi \, TrM^2\right) = \eta \\ \dfrac{dg}{dt} - gM - M^t g - \beta\sqrt{Tr(MM^t)}\left(g - Tr(g)Id\right) = 0 \end{cases}$$

$$\Pi = \frac{g^{-1}}{Tr\left(g^{-1}\right)}$$

One assumes that the major effect of the noise acting on g is to (essentially) prevent the growth of anisotropy of the tetrad.

η is modelled by a Gaussian white noise term, obeying K41 scaling ($\rho^2 = Tr(g)$):

$$\left\langle \eta_{ab}(\rho;t).\eta_{cd}(0;0)\right\rangle = \gamma\left(\delta_{ac}\delta_{bd} - \frac{1}{3}\delta_{ab}\delta_{cd}\right)\frac{\varepsilon}{\rho^2}\delta(t)$$

Summary: the problem reduces to a set of stochastic differential equations, with 3 dimensionless parameters:

- 'reduction of nonlinearity' by pressure: α
- intensity of fluctuations in the g-equation that tend to isotropize g: β
- intensity of fluctuations in the M-equation: γ

Energy balance

- Define the energy at scale ρ by $E = Tr\left(VV^t\right)/2$ with $V_i^a = \rho_i^b M_{ba}$

Equation of evolution of the energy:

$$\partial_t E(\rho) = -\frac{\partial}{\partial \rho_i^a} < V_i^a Tr\left(VV^t\right) >_\rho + \alpha < Tr\left(VV^t M\right) >_\rho + \text{coupling with the small scales}$$

- Physical interpretation:
  - $-\frac{\partial}{\partial \rho_i^a} < V_i^a Tr\left(VV^t\right) >_\rho$: large scale energy flux
  - $\alpha < Tr\left(VV^t M\right) >_\rho$: eddy damping term

(see Borue and Orszag, 1998, Meneveau and Katz, 2000, ...)

The model should provide a way to compute the statistical properties of the M tensor as a function of scale !

What is the qualitative behavior of the solutions of this system of equations ?

n.b. : it depends on three dimensionless parameters, **α**, **β**, **γ**.

2. Resolution of the system

in the semiclassical approximation

The equation satisfied by the Eulerian PDF ...

A Fokker-Planck equation for the Eulerian PDF can be derived from this stochastic system:

$$\partial_t P(M,g,t) = L\,P(M,g,t)$$

The stationary solutions must satisfy the system:

* $L\,P = 0$
* $\int dM\,P(M,g) = 1$
* $P(M,g = L^2 Id) \sim \exp\left[-\frac{Tr(MM^t)}{(\varepsilon L^{-2})^{2/3}}\right]$ (Gaussian distribution at the integral scale)

... and its solution in terms of path integral:

This system can be solved by using Green's functions method:

$$P(M,g) = \int dM' \int dT\, G_{-T}(M;g \mid M';g')\, P(M',g')$$

(G : Green function) (boundary condition)

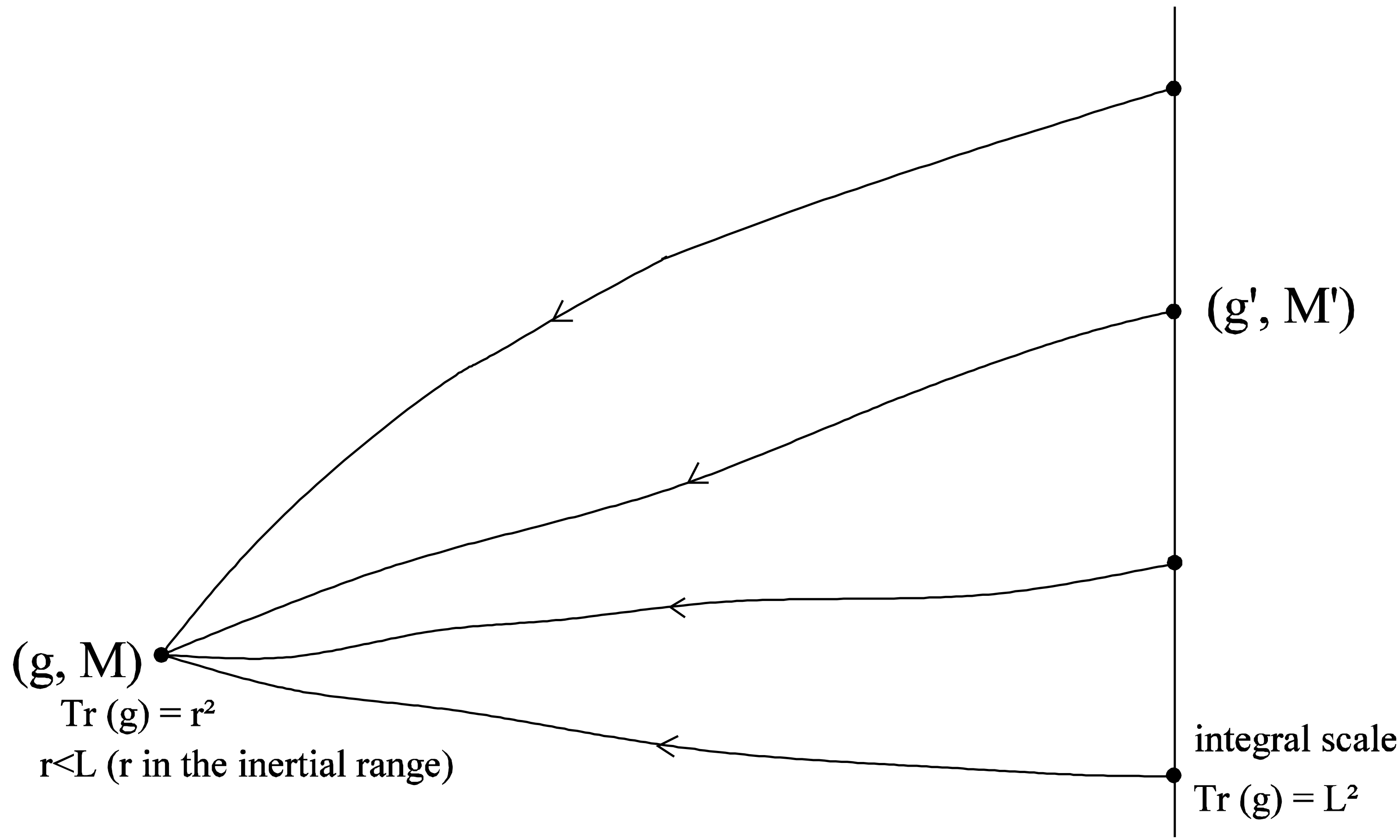
With:

$$G_{-T}(M;g \mid M';g') = \int [DM''] \int [Dg''] \exp[-S(M'';g'')]$$

Hence:

$$P(M,g) = \int dM' \int dT \int [DM''] \int [Dg''] \exp - \left[S(M'';g'') + Tr(M' M'^{t}) / (\varepsilon L^{-2})^{2/3} \right]$$

(Green function) (boundary condition)



Starting from an initial condition at the integral scale, one integrates the system until a fixed scale r (in the inertial range). It is in principle necessary to take into account all these trajectories in phase space.

(Approximate) methods of resolution:

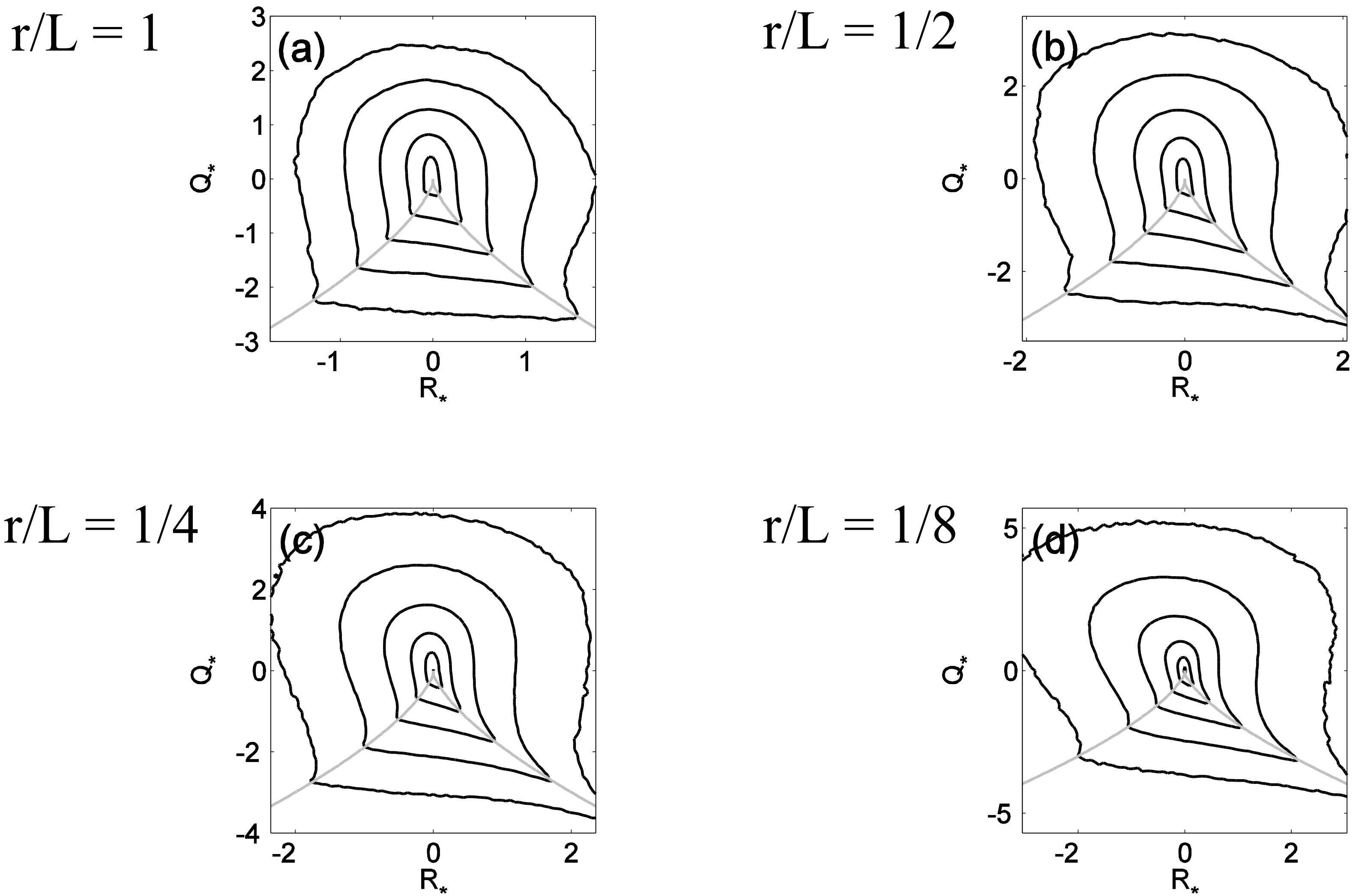
- We could use a straightforward Monte-Carlo method (in principle, exact)

Difficulty: * This method is relatively inefficient because configurations have statistical weights which vary by (many) orders of magnitude.
* It is therefore hard to obtain reliable results, particularly at small scales.

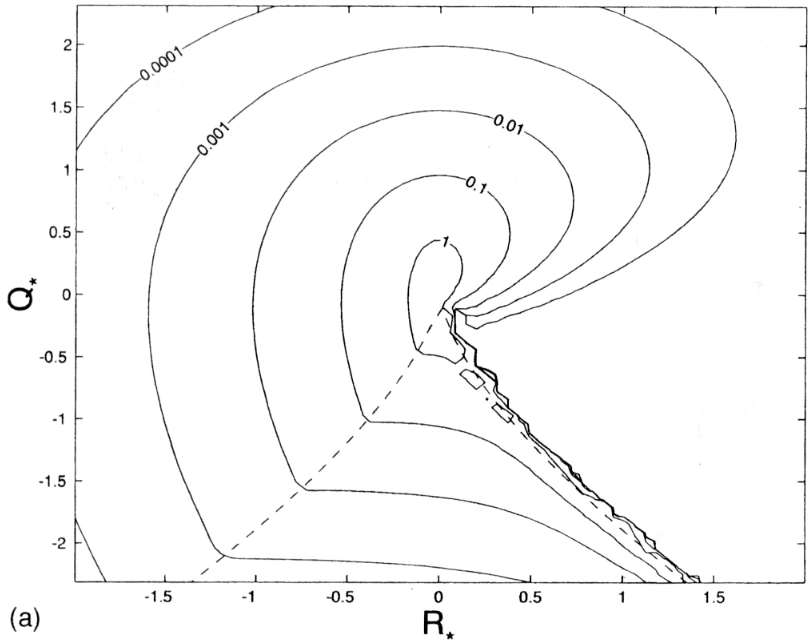
- Or look for the solutions in the deterministic approximation (γ=0)

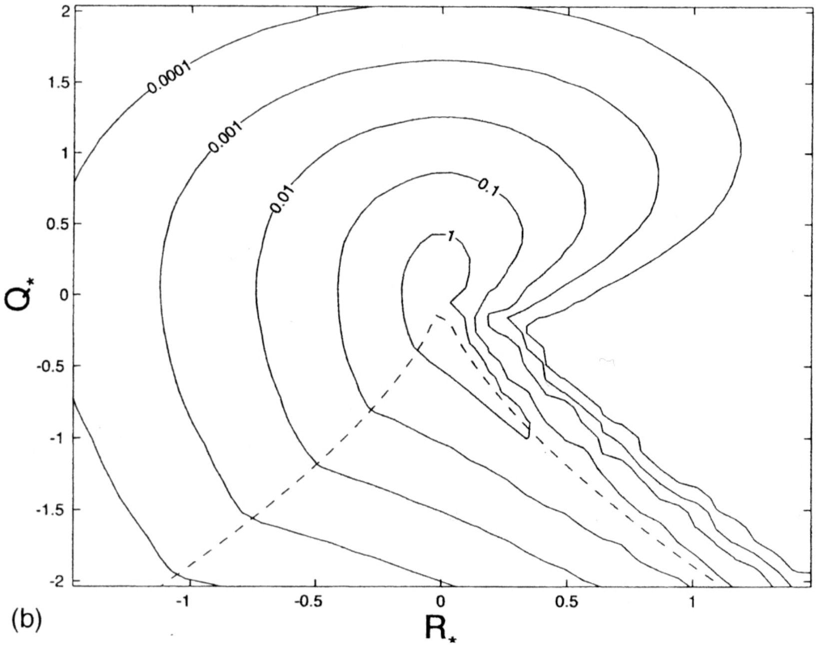
$\longrightarrow$ encouraging results when compared with DNS (see Chertkov et al, 1999)

Evolution of P(R,Q) as a function of scale: solutions calculated by **DNS** (R_λ=130; 256^3)



P(R,Q) solutions of the model in the **deterministic approximation**





(a) r = L/5

(b) r = L/2

(Chertkov *et al*, 1999)

- One rather uses the semiclassical approximation (saddle approximation on the path integral).

- **Method:** one considers only the trajectory for which the action is minimal (the one with the largest statistical weight).

- **Aim:** This method should provide important information, especially since many trajectories do not contribute much.

- The method is not rigorous; it is difficult to control the errors made

=> A better algorithm has to be implemented to understand the effect of fluctuations (~Monte-Carlo), and to really estimate the errors made by using the semi-classical approximation.

3. Numerical solutions of the system in the semiclassical approximation with isotropic forcing

-

Comparison with experimental and DNS data

A. Naso and A. Pumir, Phys. Rev. E **72**, 056318 (2005)

Model's prediction:

- The parameter that has the most crucial influence is α (reduction of nonlinearity).

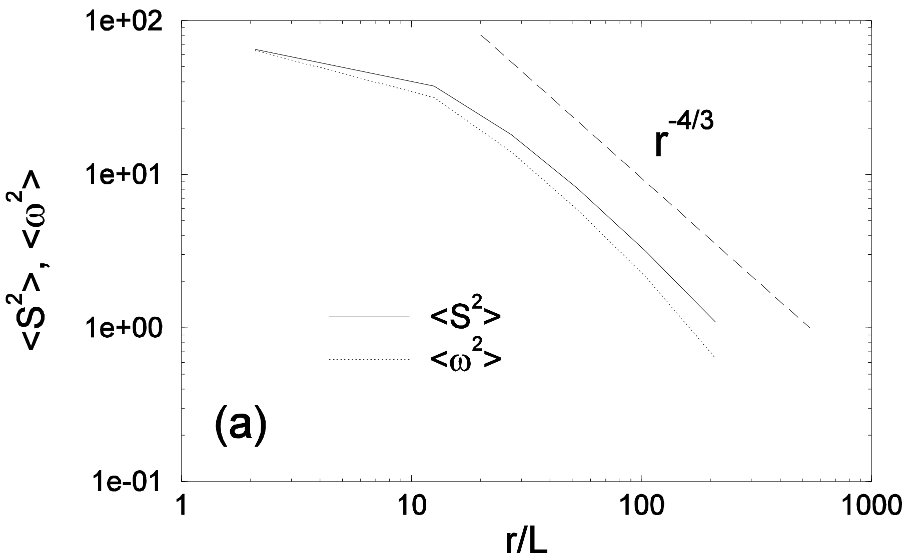
- The predictions of the model are in agreement with DNS results provided α is in a small interval, **α** ~ 0.45

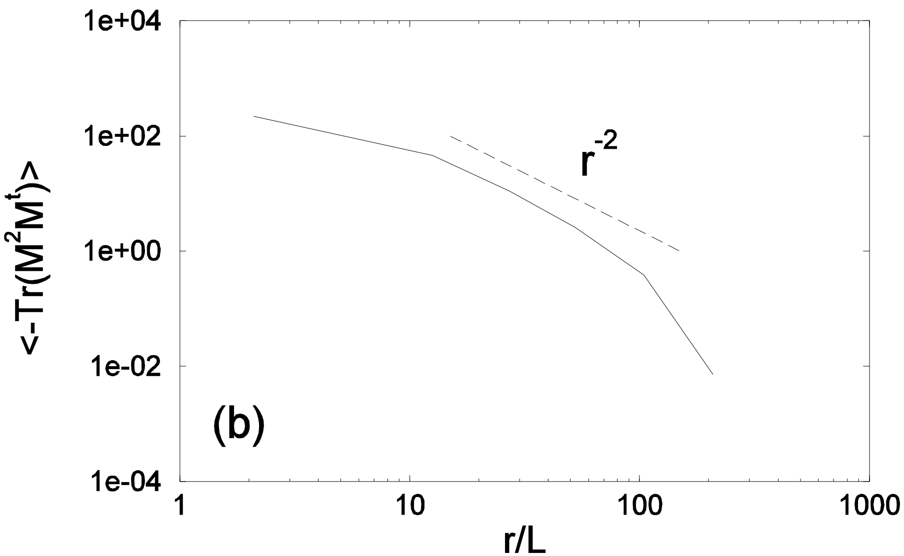
Scaling laws of 2nd and 3rd order moments of M: solutions calculated by DNS (R_λ=130; 256^3)

According to the K41 law, $\langle \Delta u(r) \rangle \propto r^{1/3}$ therefore $\langle M(r) \rangle \propto r^{-2/3}$

and $\langle \omega^2 \rangle$ et $\langle S^2 \rangle \propto r^{-4/3}$ $\langle - Tr(M^2 M^t) \rangle \propto r^{-2}$

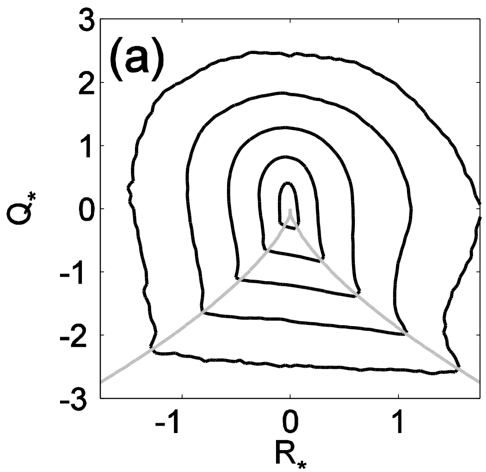
DNS results: these three quantities satisfy the Kolmogorov scaling:



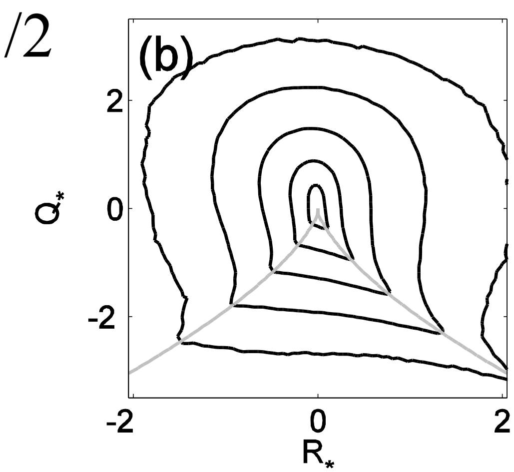


Evolution of P(R,Q) as a function of scale: solutions calculated by **DNS** ($R_\lambda = 130$; 256^3)

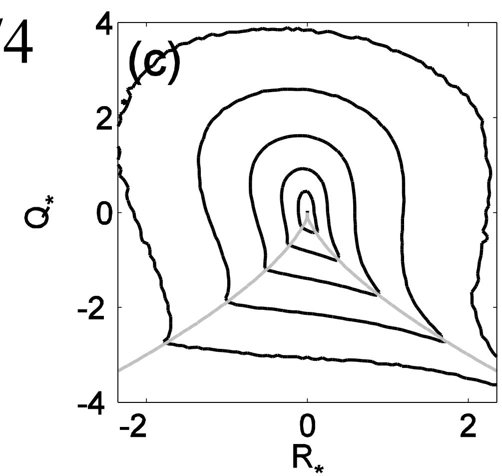
r/L = 1



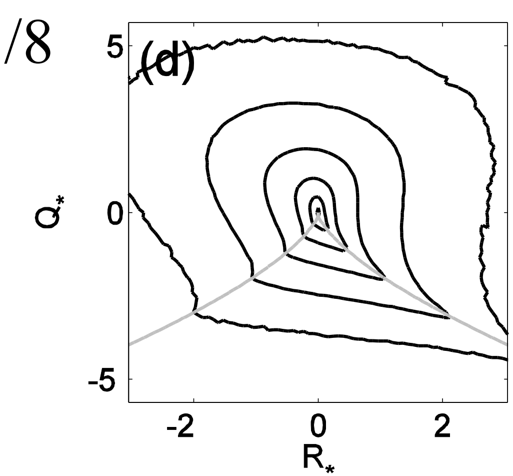
r/L = 1/2



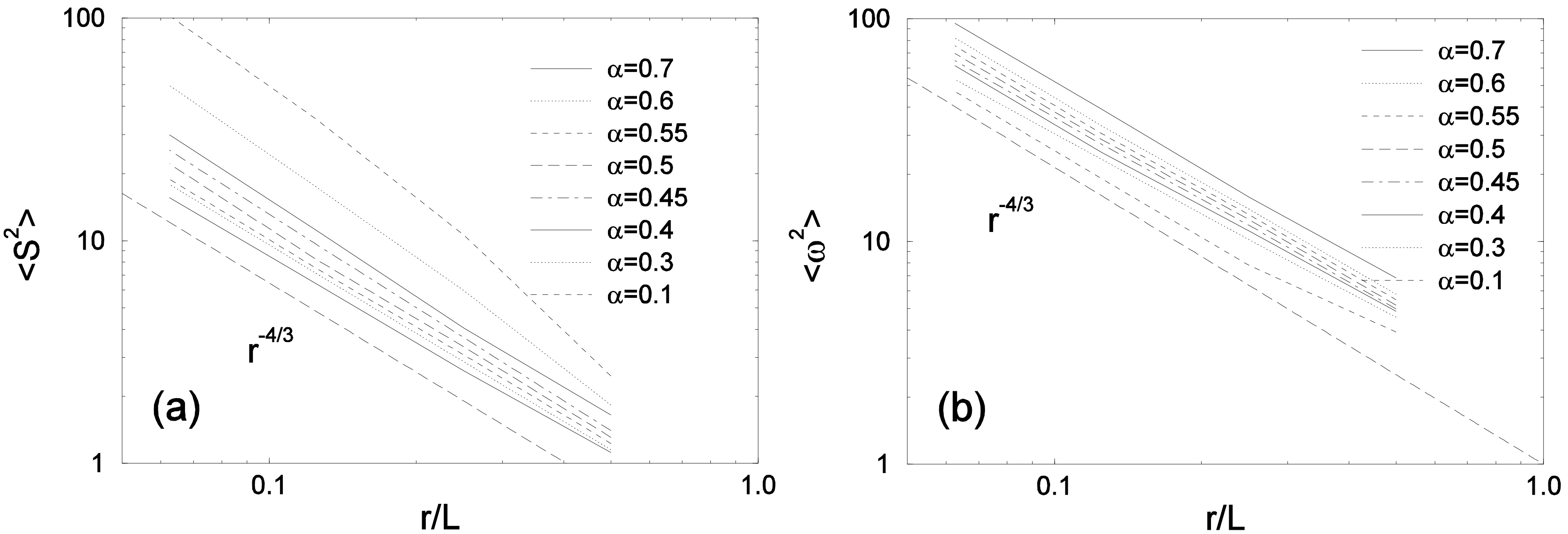
r/L = 1/4



r/L = 1/8

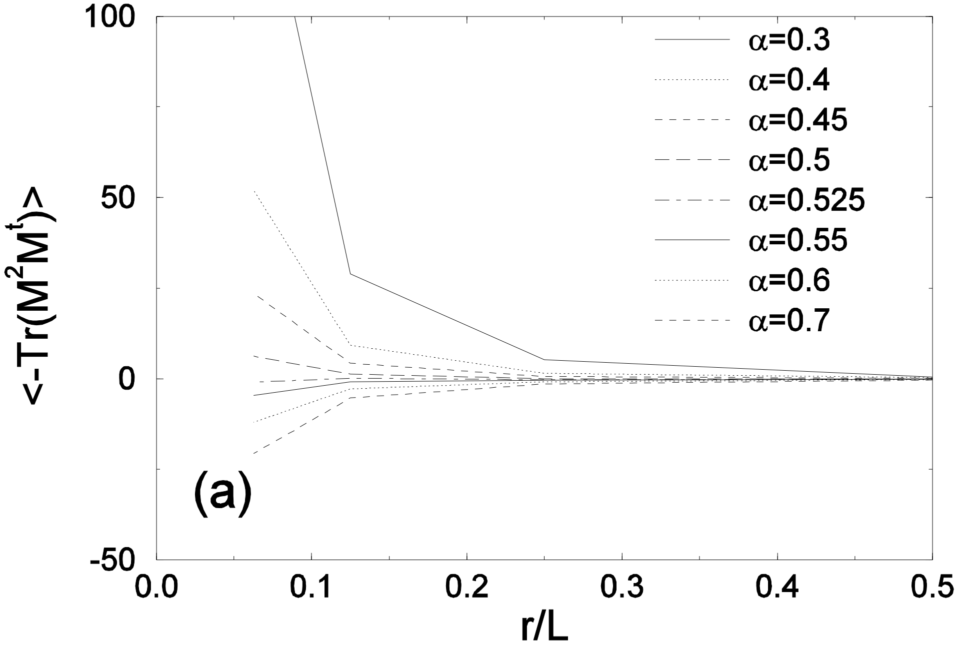


Scaling laws of 2nd order moments of M: **semiclassical** solutions of the model



The second moment of M has the right scaling provided the "nonlinearity reduction" α is not too small !

Scaling laws of 3rd order moments of M: **semiclassical** solutions of the model



The sign of the energy transfer is positive, as it should, provided the "nonlinearity reduction" **α** is not too large !

Influence of the other parameters

✓ Influence of the parameter β:

Not much effect provided β is large enough.

✓ Influence of the parameter γ:

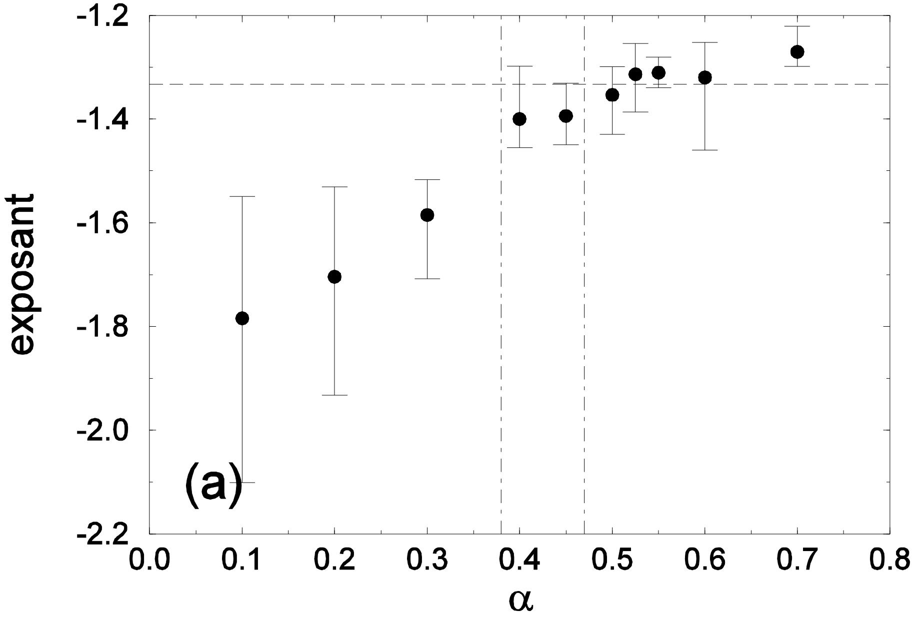
Main effect : change the numerical value of

$$\langle \omega^2 \rangle \times r^{4/3}$$

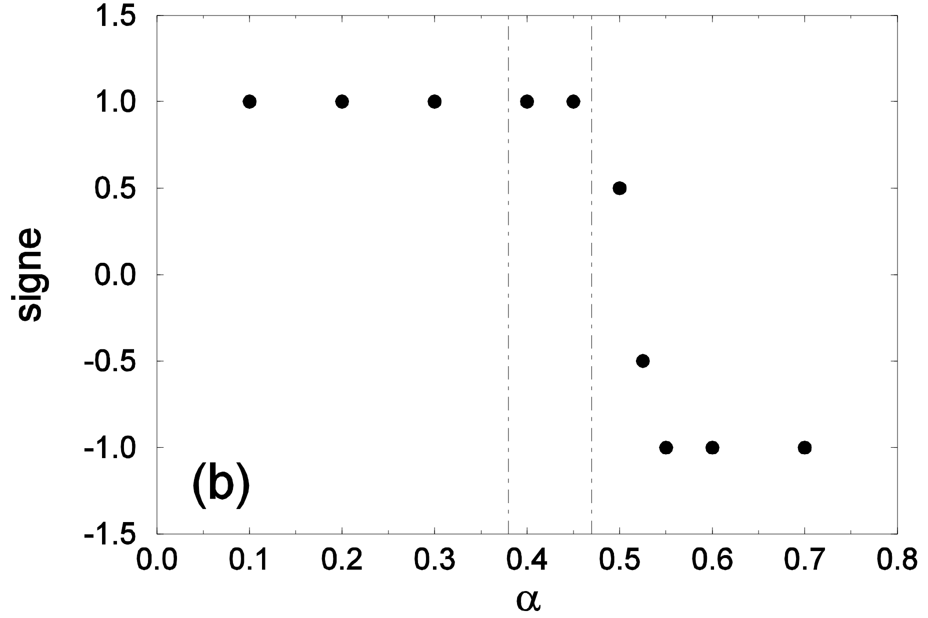
Scaling laws of 2nd and 3rd order moments of M: **semiclassical** solutions of the model

- $\langle \omega^2 \rangle$ scales according to K41 for any value of α.

- $\langle S^2 \rangle$ scales according to K41 for large values of α:



- The energy transfer, $\langle - r^2 Tr(M^2 M^t) \rangle$, has the right sign for small values of α:

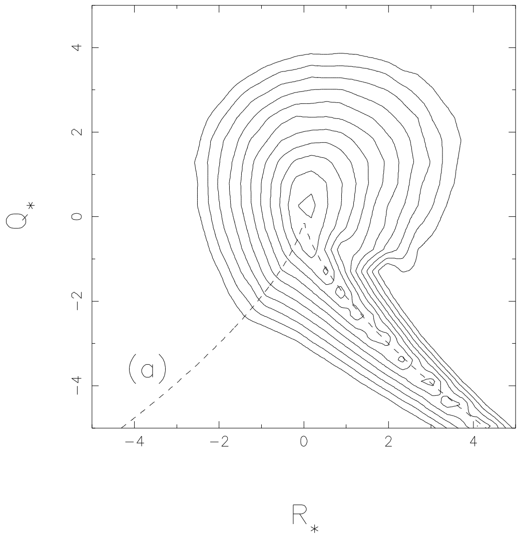


The solutions are quantitatively acceptable if α is in a narrow interval ~ 0.4-0.5 !

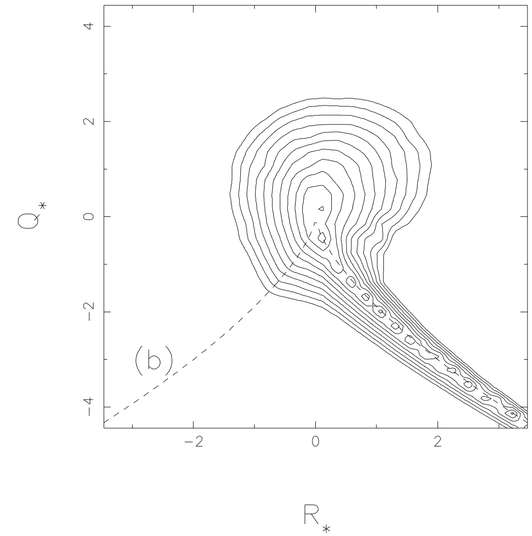
Evolution of P(R,Q) as a function of scale: **semiclassical** solutions of the model (1)

Parameters: **$\alpha = 0.2$**, **$\beta = 0.4$**, **$\gamma = 0.25$**

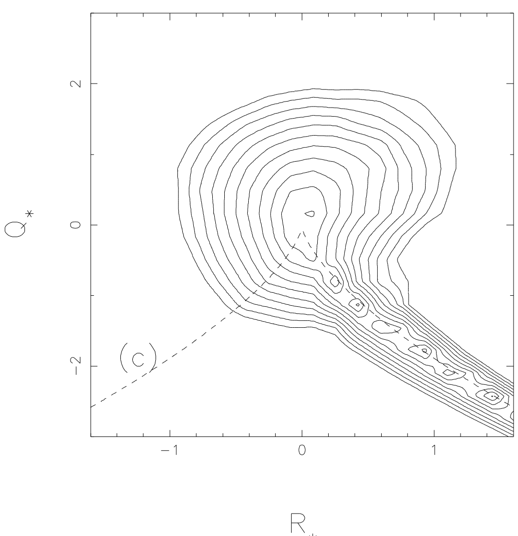
r/L = 1/2



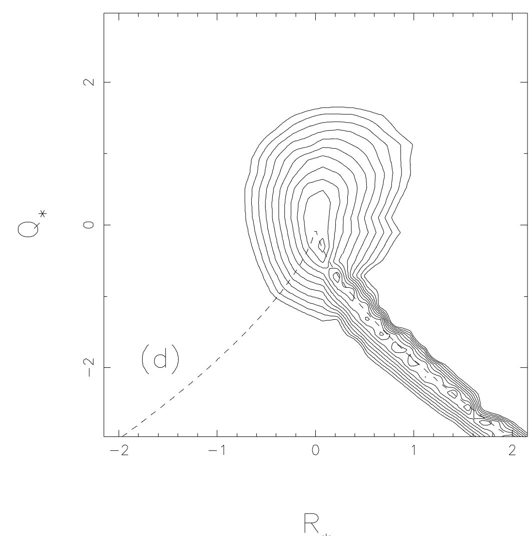
r/L = 1/4



r/L = 1/8



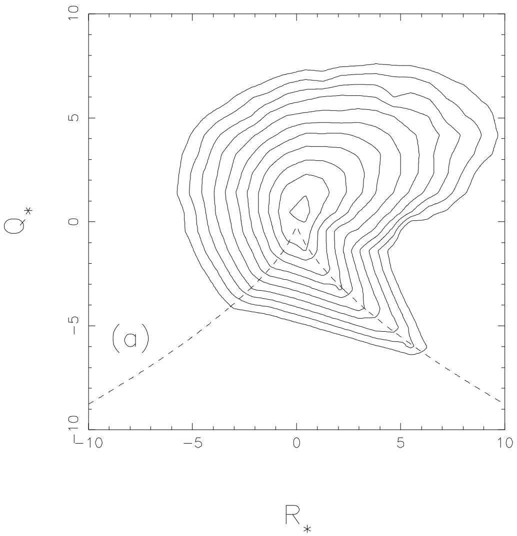
r/L = 1/16



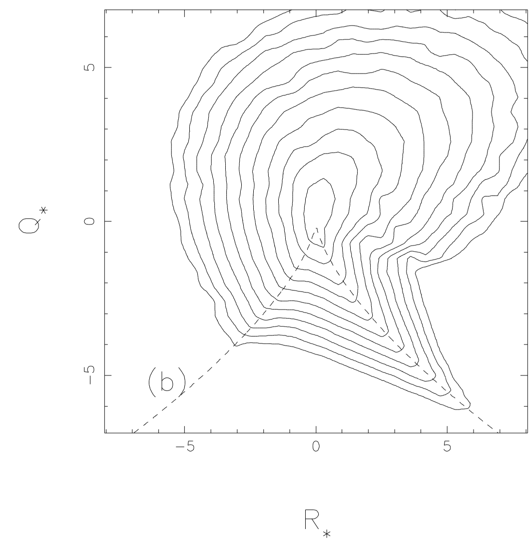
Evolution of P(R,Q) as a function of scale: **semiclassical** solutions of the model (2)

Parameters: **$\alpha = 0.6$**, **$\beta = 0.4$**, **$\gamma = 0.25$**

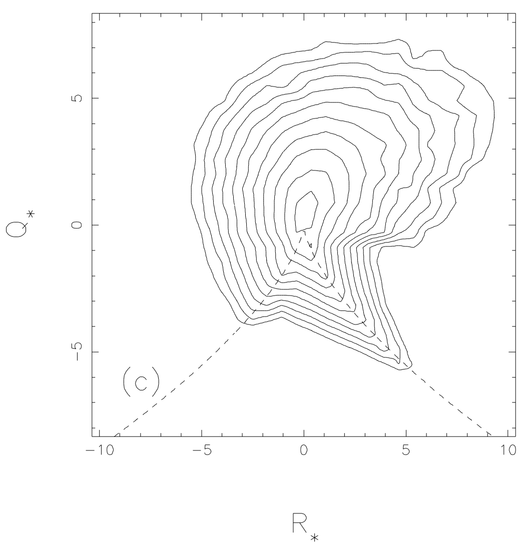
r/L = 1/2



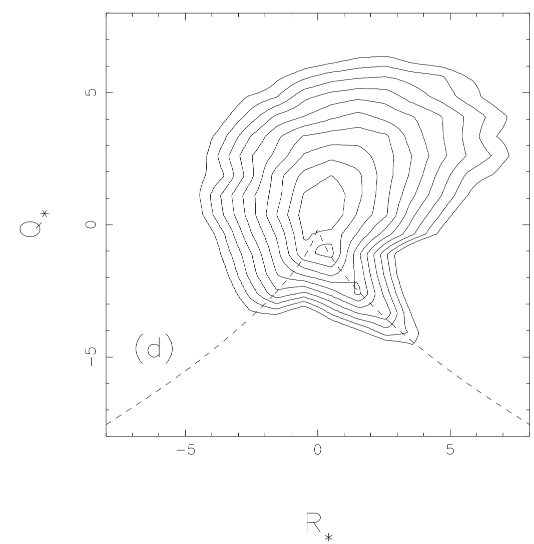
r/L = 1/4



r/L = 1/8



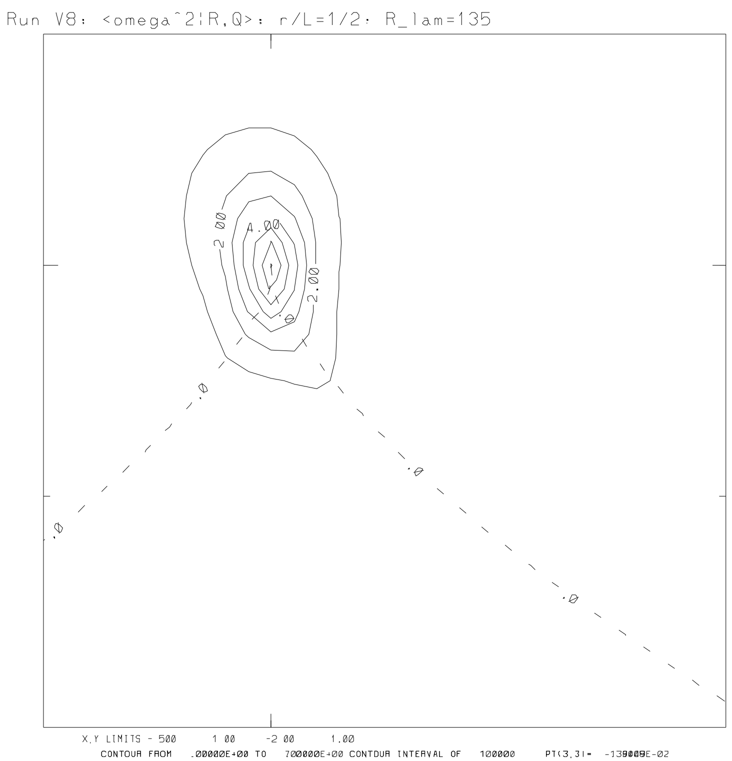
r/L = 1/16



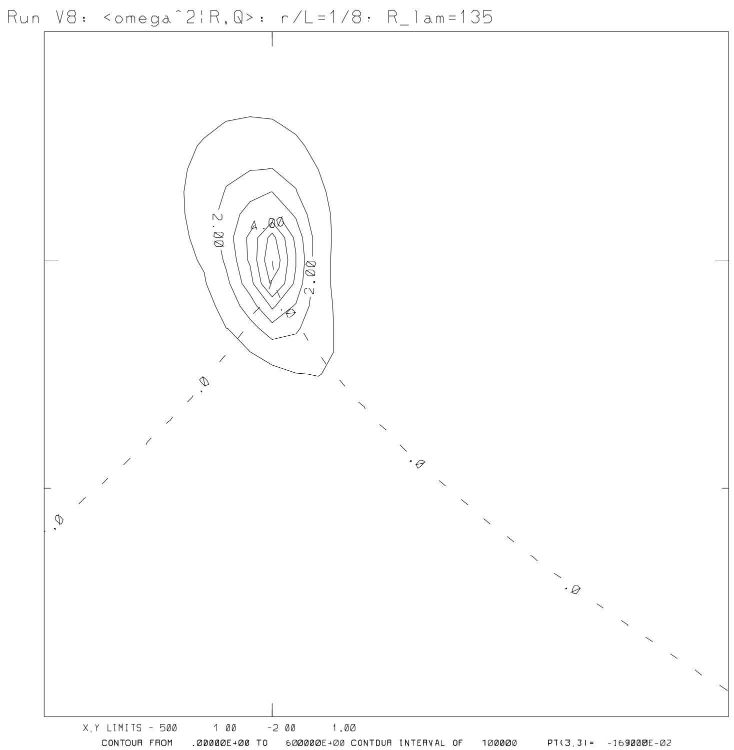
Scale dependence of the enstrophy density:

solutions calculated by ***DNS***

r/L = 1/2

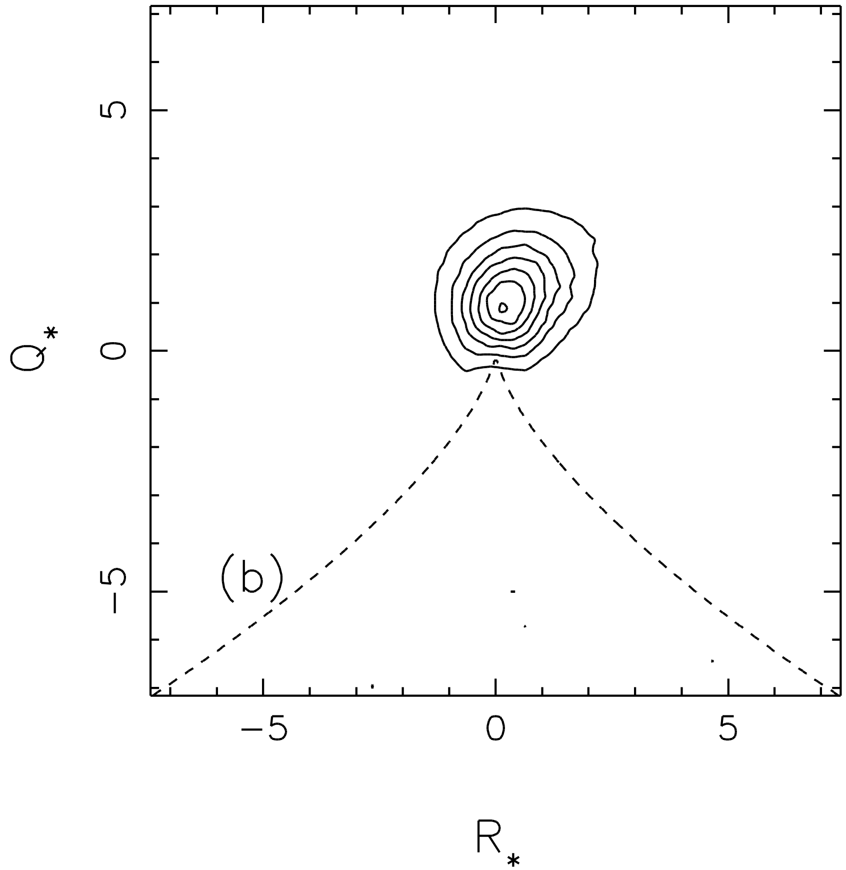


r/L = 1/8



Scale dependence of the enstrophy density: *semiclassical solution of the model*

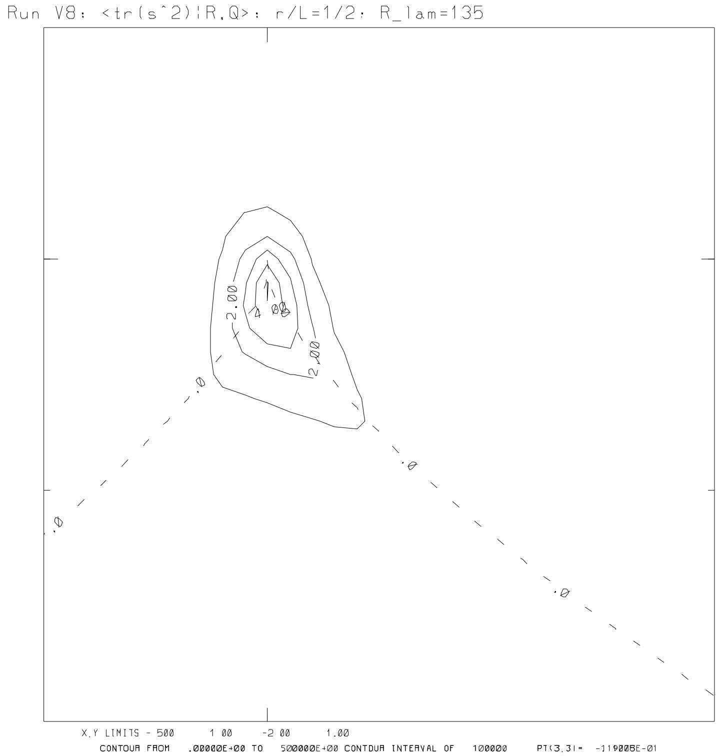
r/L = 1/8



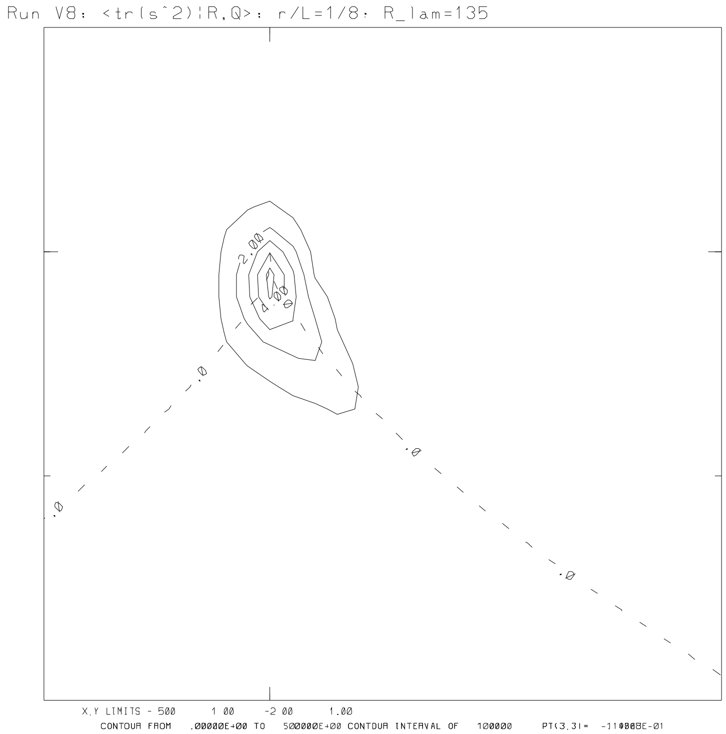
Scale dependence of the strain variance ($Tr(S^2)$) density:

solutions calculated by ***DNS***

r/L = 1/2

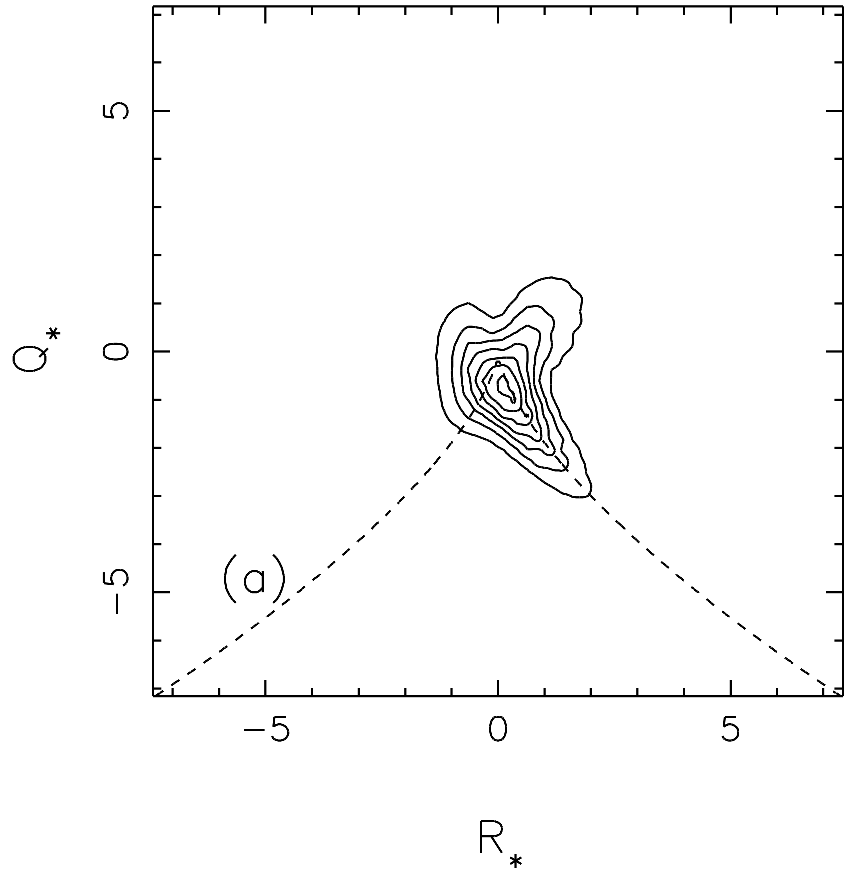


r/L = 1/8



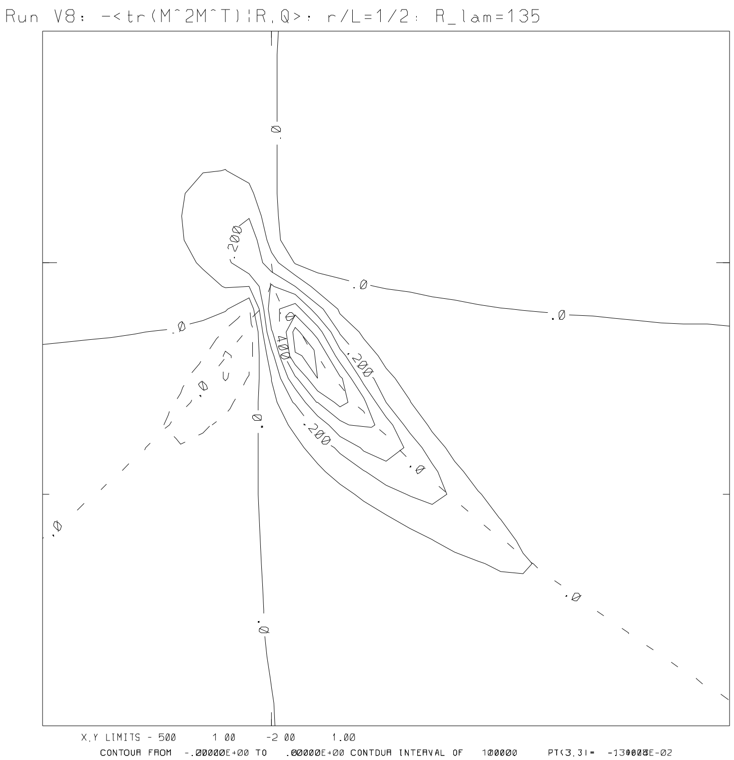
Strain variance ($Tr(S^2)$) density:
semiclassical *solution of the model*

$r/L = 1/8$

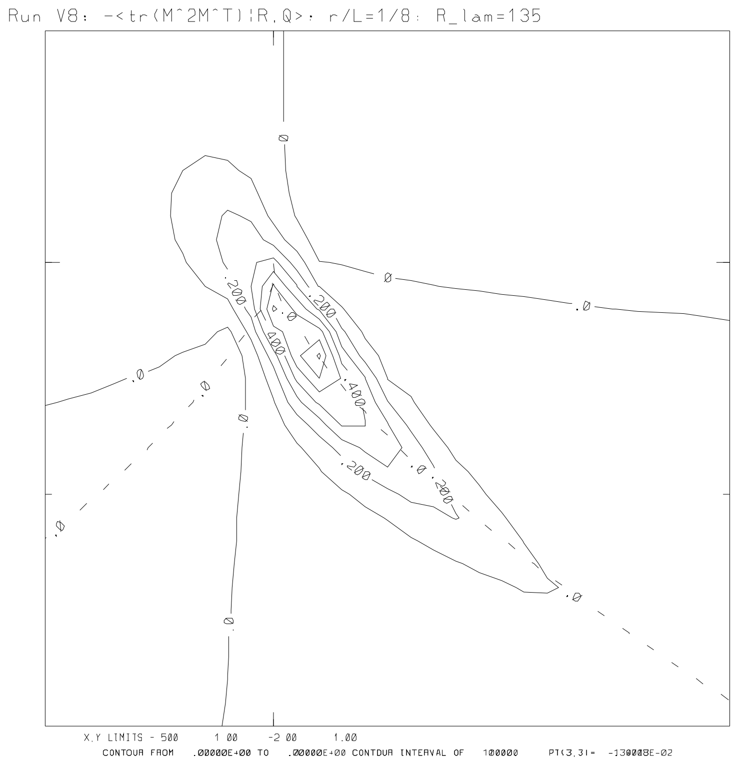


Scale dependence of the energy transfer density:
*solutions calculated by **DNS***

r/L = 1/2

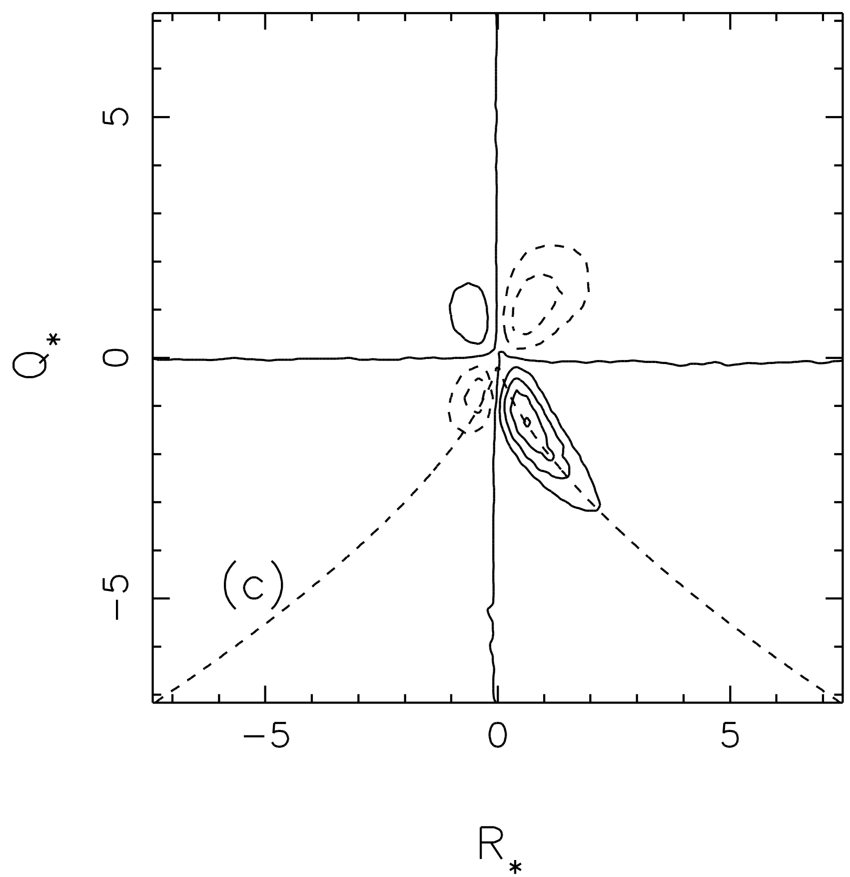


r/L = 1/8



Scale dependence of the energy transfer density:
semiclassical solution of the model

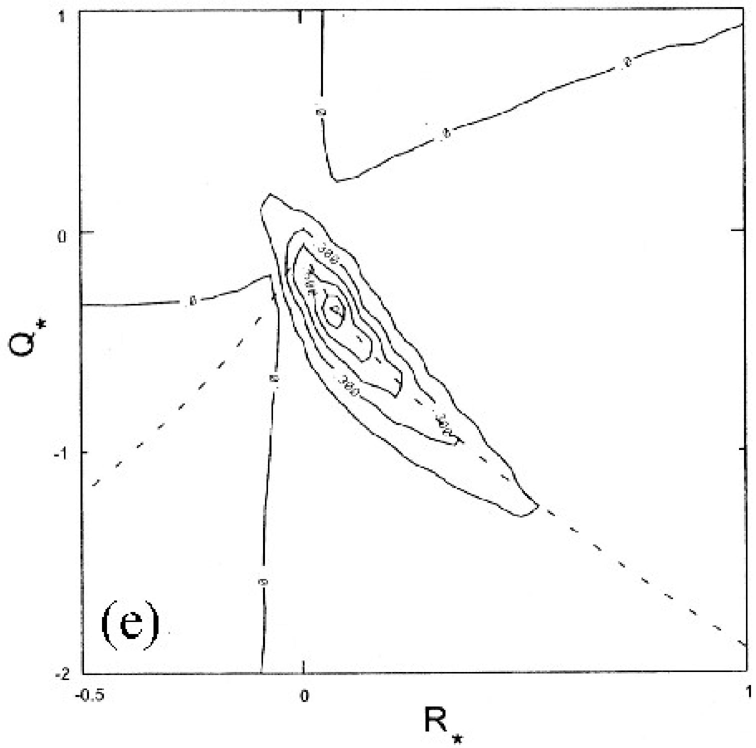
r/L = 1/8



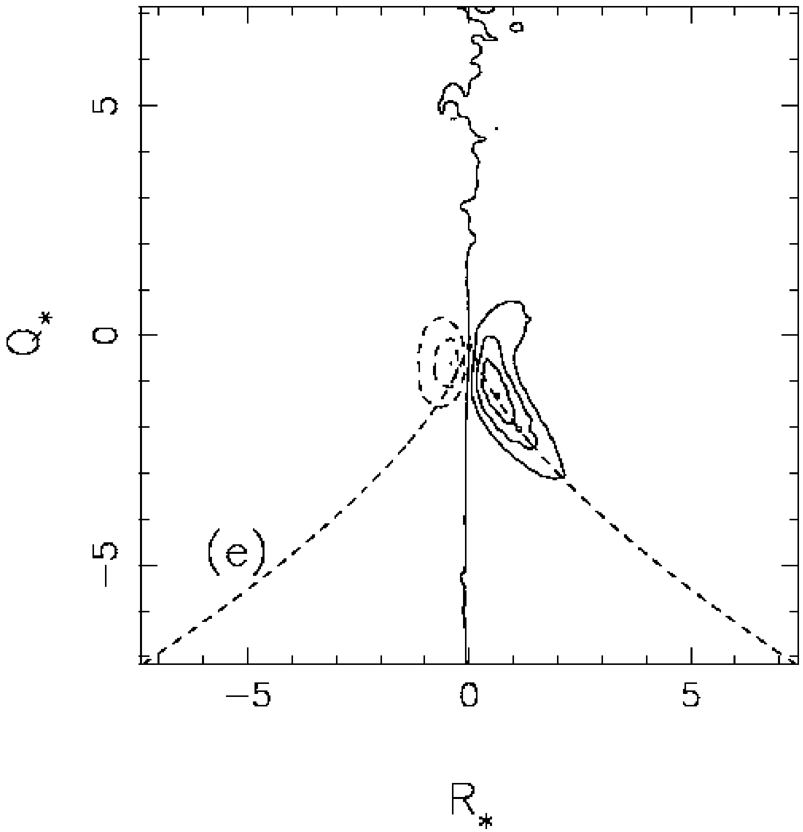
Strain skewness $-\mathrm{Tr}(S^3)$ density:
DNS *and* ***semiclassical*** *solution of the model*

r/L = 1/8

DNS



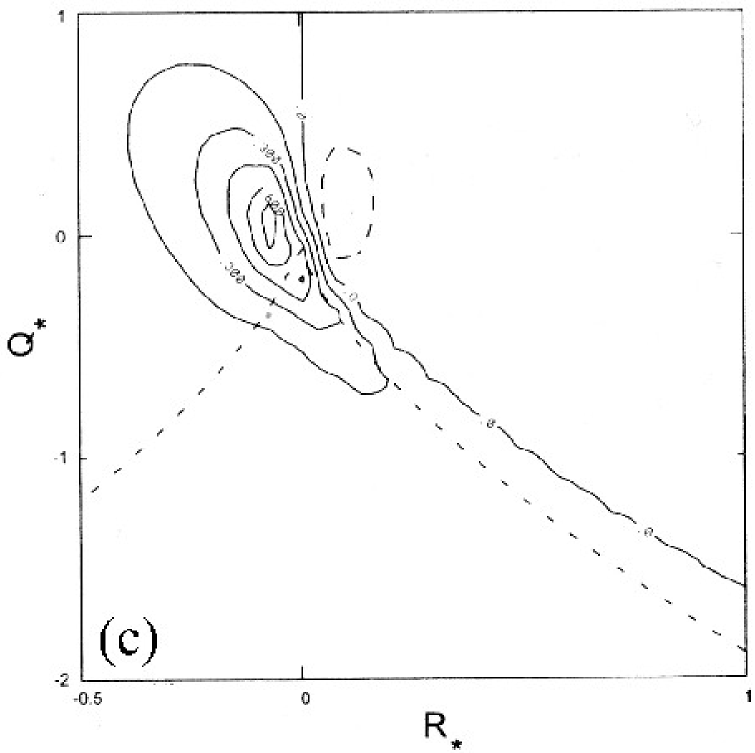
SC



Enstrophy production density:
DNS *and* ***semiclassical*** *solution of the model*

r/L = 1/8

DNS



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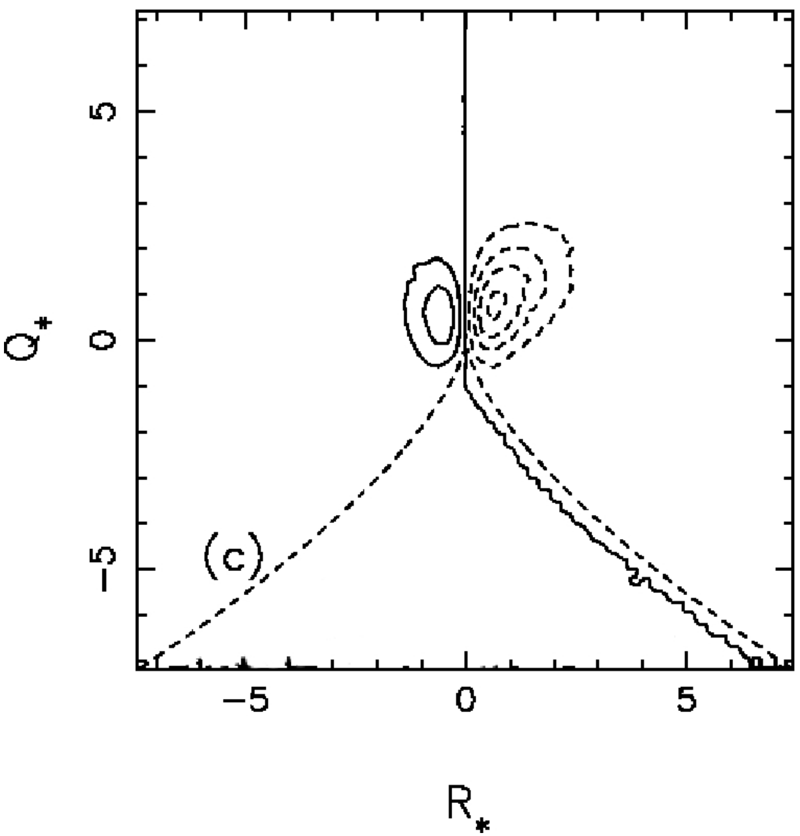


4. Numerical solutions of the system in the semiclassical approximation with large scale shear

A. Naso, M. Chertkov and A. Pumir, J. Turbul. **7**, N41 (2006)

- One of the postulates of turbulence theory is the universality of small scale velocity fluctations. In particular, these fluctuations should be isotropic for any large scale forcing (at least for large enough Re). Equivalently, the effect of anisotropy should diminish for decreasing scales.

- One of the simplest flow configurations to study this effect is the homogeneous shear turbulence.

- Experimental (Shen and Warhaft, 2000) and numerical (Pumir and Shraiman, 1995; Pumir, 1996) studies of homogeneous shear turbulent flows actually show that the decrease of the shear effects is much slower than naively expected...

- **Idea**: the tetrad model can be applied to all kinds of forcing, simply by changing the large scale condition
→ impose a large scale shear, and calculate the scale dependence of P(R,Q) and of the dynamical quantities, for different values of the shear intensity.

- The equations are the same than those considered in the isotropic case. We simply change the large scale condition for the velocity field:

$$P(M, g = L^2 Id) \sim \exp\left[\frac{-Tr\left[(M-\Sigma)(M-\Sigma)^t\right]}{(\varepsilon L^{-2})^{2/3}}\right]$$

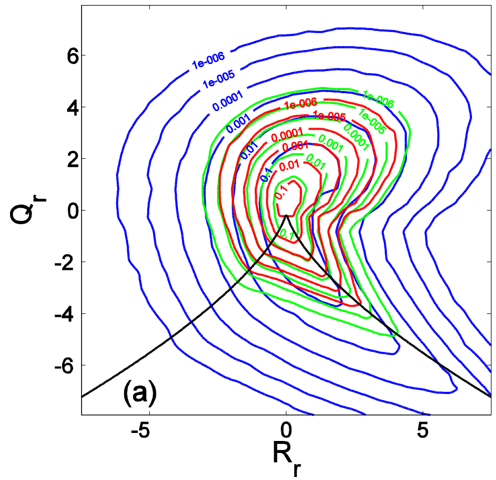
where:

$$\Sigma \equiv \begin{pmatrix} 0 & s & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$; s measures the shear intensity

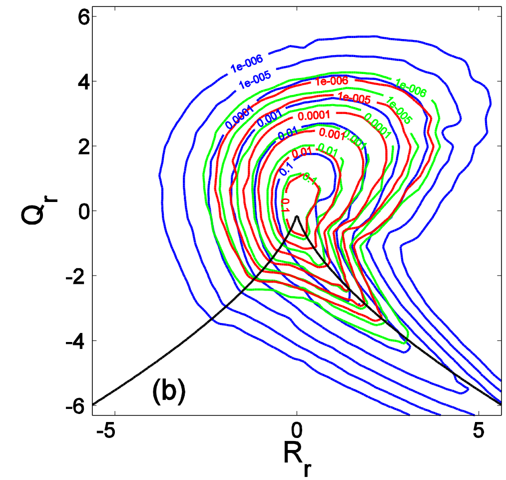
Scale dependence of P(R,Q): **semiclassical** solutions of the model with s=0, 1, 6

Parameters: $\alpha = 0.6$, $\beta = 0.4$, $\gamma = 0.25$

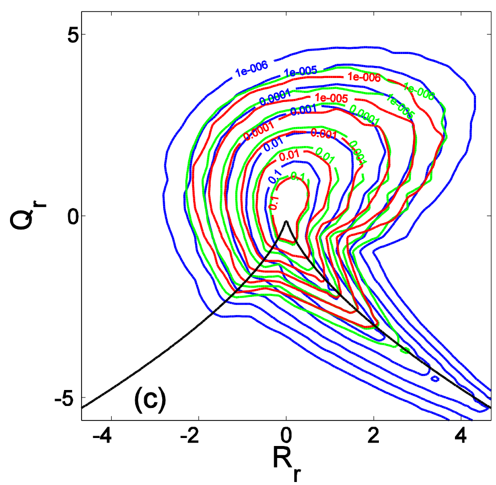
r/L = 1/2



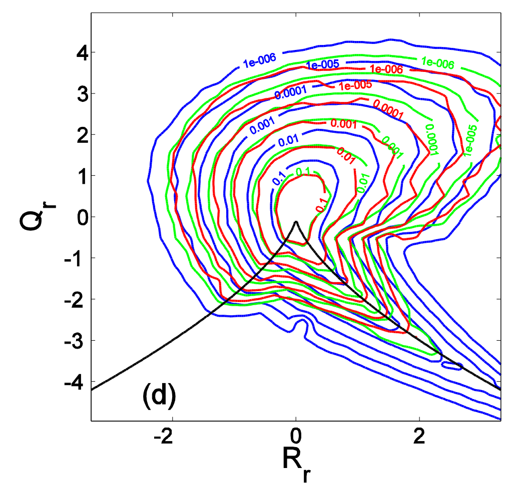
r/L = 1/4



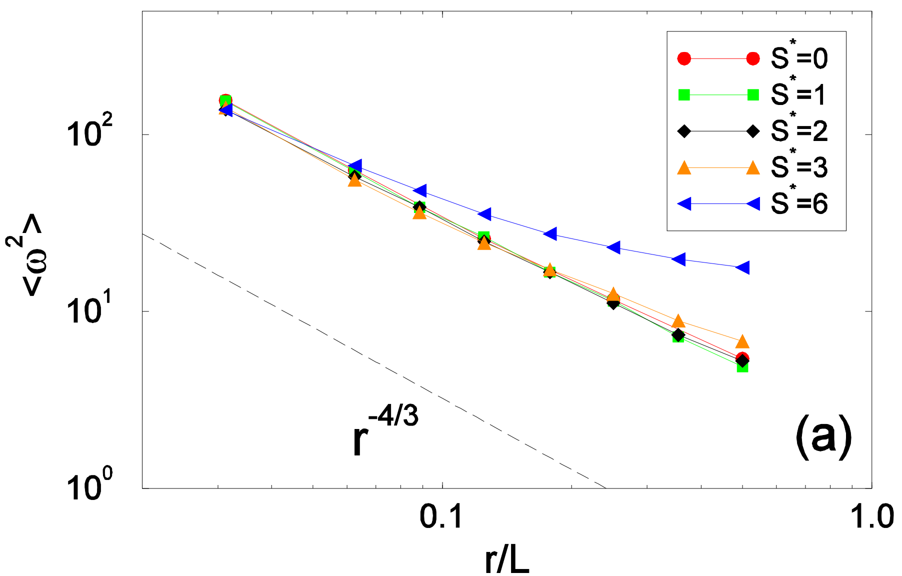
r/L = 1/8

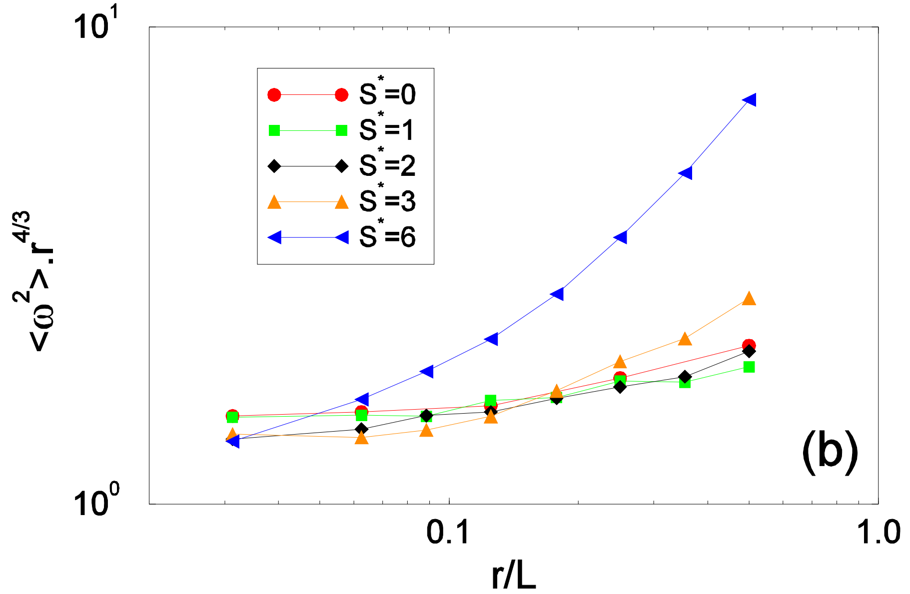


r/L = 1/16

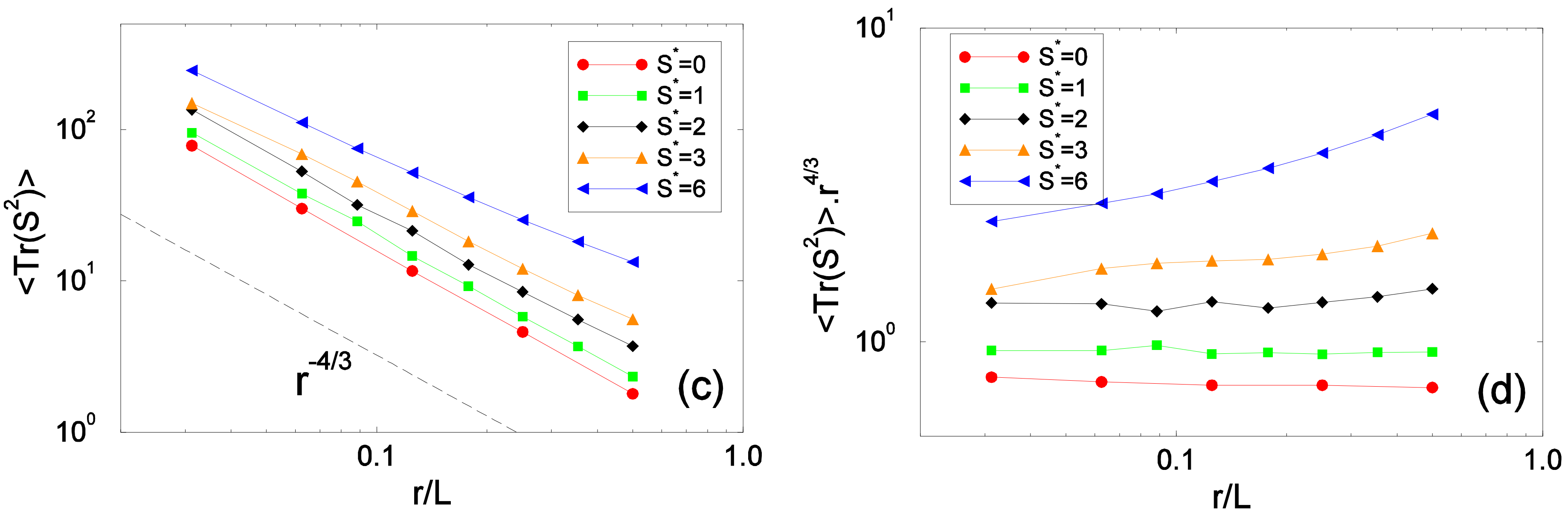


Scale dependence of <ω²> for different values of s

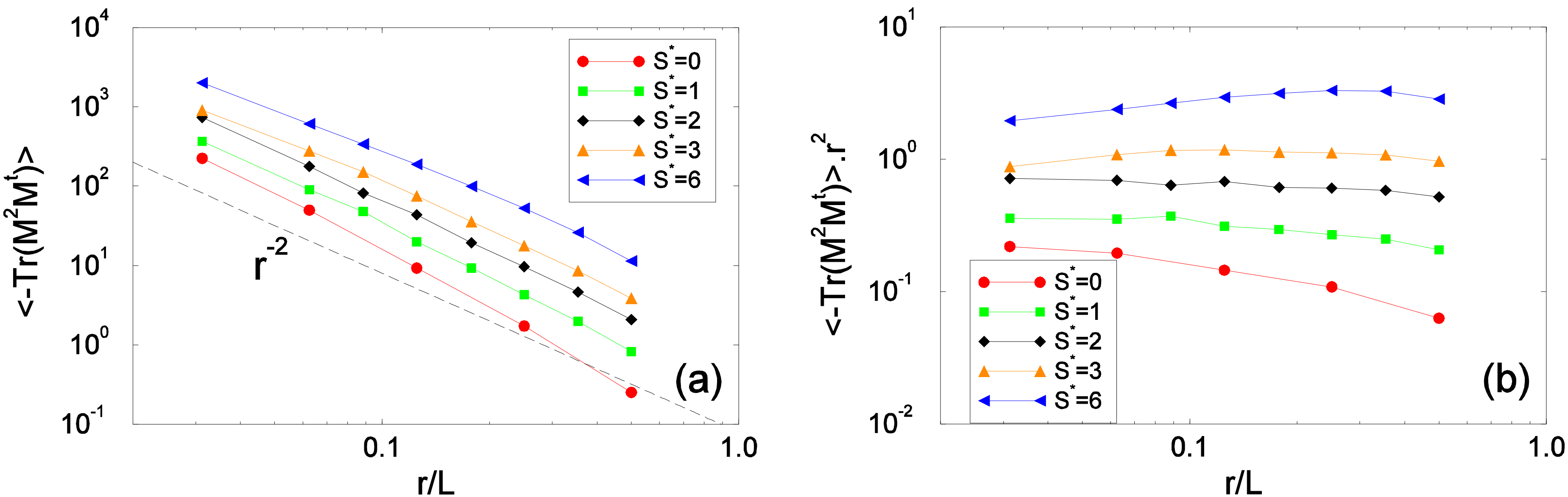




Scale dependence of <Tr(S²)> for different values of s



Scale dependence of the **energy transfer** for different values of s



- Our results are consistent with the accepted view that the effects of large scale anisotropy decrease when the scale decreases.

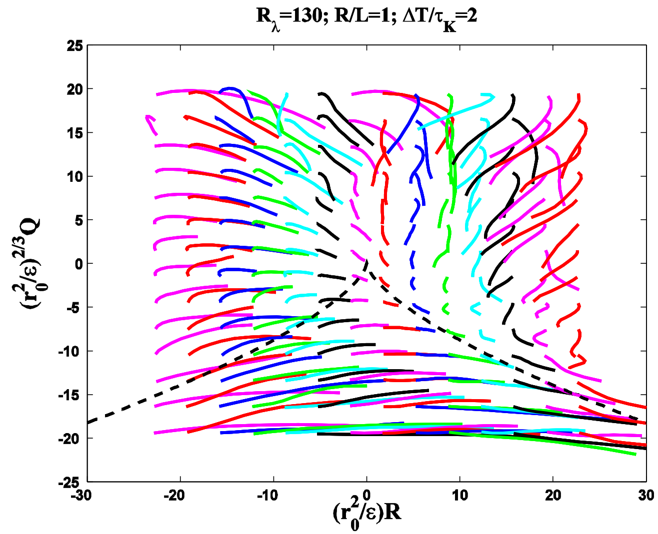
- **New result**: difference of behavior between vorticity dominated and strain dominated structures. The anisotropy effects diminish significantly faster for vorticity dominated structures (enstrophy) than for strain dominated ones (strain variance, energy transfer).

- The faster relaxation of vorticity dominated structures towards isotropy is consistent with the observation that vortical structures in turbulent flows are generally more intense than strain structures. They are therefore less sensitive to a moderate, large scale effect (imposed shear).

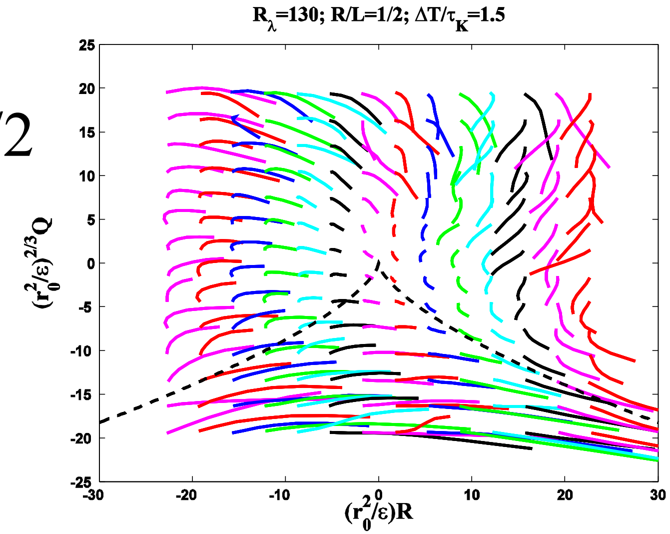
5. Lagrangian trajectories

Lagrangian trajectories in the (R,Q) plane: **DNS** results (1)

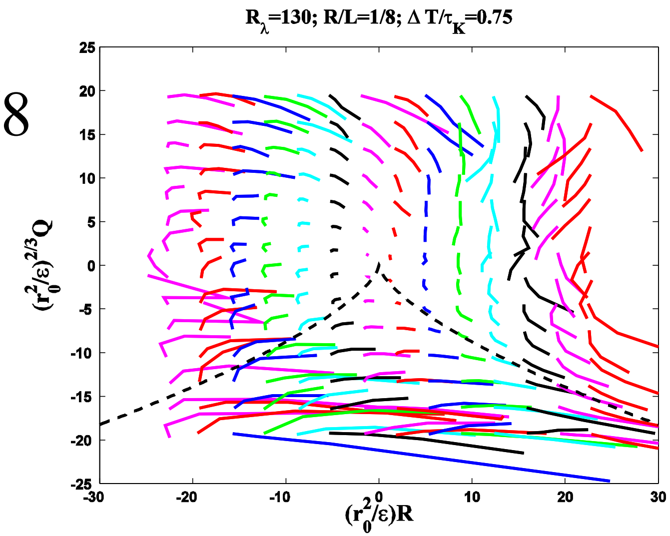
r/L = 1



r/L = 1/2



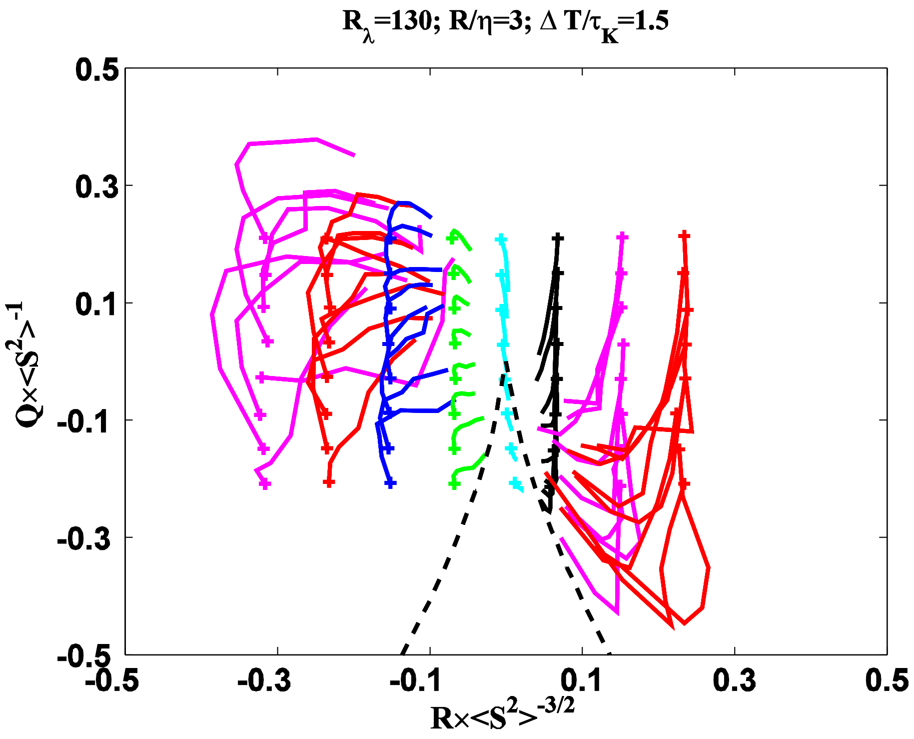
r/L = 1/8



Trajectories similar while r is in the inertial range

Lagrangian trajectories in the (R,Q) plane: **DNS** results (2)

r/L = 1/32 (r/η=3)

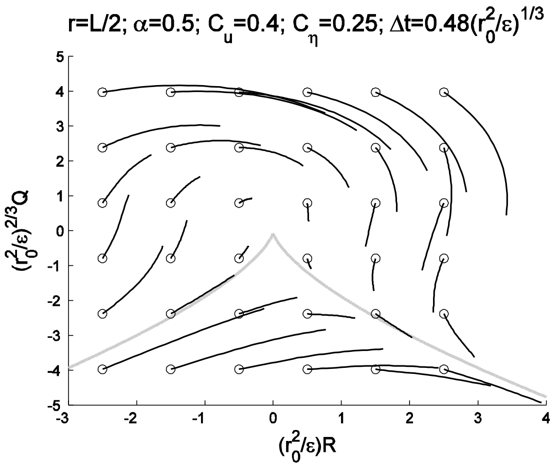


The character of the trajectories is not the same in the dissipative range !

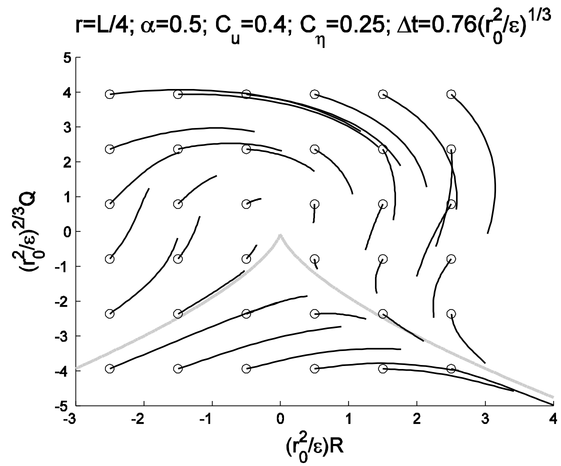
Difference with Chevillard and Meneveau, 2006

Lagrangian trajectories in the (R,Q) plane: **model solutions**

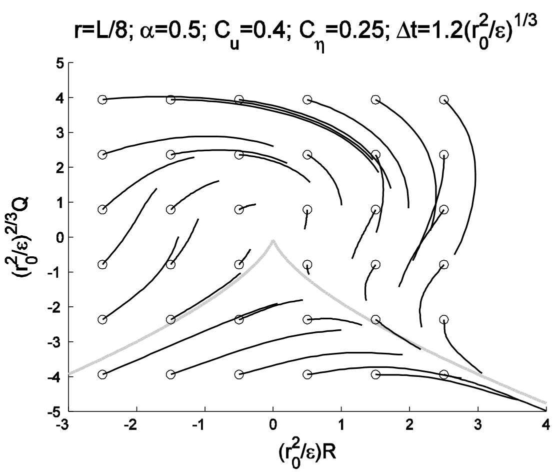
r/L = 1/2



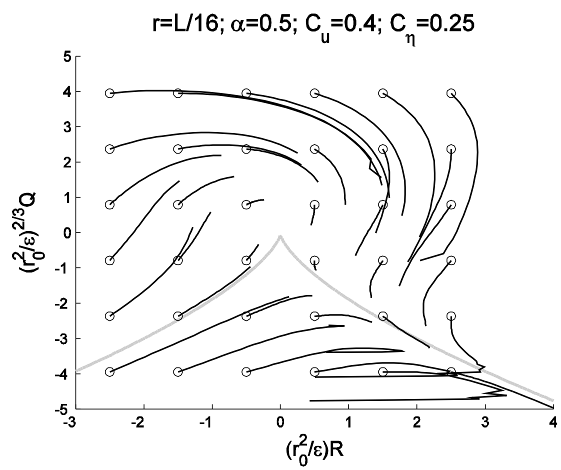
r/L = 1/4



r/L = 1/8



r/L = 1/16



6. Conclusion and Outlook

- We have formulated a dynamical model of turbulent velocity fluctuations, involving several key ingredients of fluid dynamics turbulence.

- Three dimensionless parameters are involved in this model, formulated as a set of stochastic ODE's.

- We have studied approximate solutions of this model:
  * with isotropic forcing: the numerical results show several possible behaviors. The nonlinearity reduction α plays the key role

  * with large scale shear: anisotropy effects decrease significantly faster for vorticity dominated structures than for strain dominated ones

- A major advantage of this model is the fact that it can be applied to all kinds of turbulent flows, simply by changing the boundary condition (at large scale)

→ easy to study the effect of large scale anisotropy on small scales (rotation, stretchings/contractions, Rayleigh-Taylor turbulence, …).

- Thanks to the semiclassical results, we have also designed a hybrid method of resolution (Monte-Carlo/saddle node) that incorporates more precisely the fluctuations in the dynamics (beyond the semiclassical approximation).

Expected output: find out about the importance of the fluctuations as a function of the flow structures.

see A. Naso, A. Pumir and M. Chertkov, to appear in J. Turbul. (2007)

- **Very recent development :** new experimental results from the Göttingen, Zürich, Risø and Lyon groups

=> exciting new developments expected !!!

Danke !!!

References:

- M. Chertkov, A. Pumir and B.I. Shraiman, Phys. Fluids **11**, 2394 (1999)
  → *derivation of the model*
- A. Naso and A. Pumir, Phys. Rev. E **72**, 056318 (2005)
  → *semiclassical solution in homogeneous and isotropic turbulence*
- A. Naso, M. Chertkov and A. Pumir, J. Turbul. **7**, N41 (2006)
  → *semiclassical solution in shear turbulence*
- A. Naso, A. Pumir and M. Chertkov, to appear in J. Turbul. (2007)
  → *hybrid method of resolution*

Method of resolution in the semiclassical approximation

In the solution, expressed as a path-integral, look for the trajectory for which the integrand of the exponential:

$$\exp - \left[S\left(m; g; \frac{dm}{dt}; \frac{dg}{dt} \right) + Tr\left(mm^t \right) \Big|_{r=L} \right]$$

is minimal.

☆ Solve the Euler-Lagrange equations for the Lagrangian associated to S, with arbitrary initial conditions, compatible with the constraints (R,Q,r).

• Find the minimum value over all possible initial conditions (*amebsa* ≡ simplex method + simulated annealing).