# Filaments

Michael Köpf

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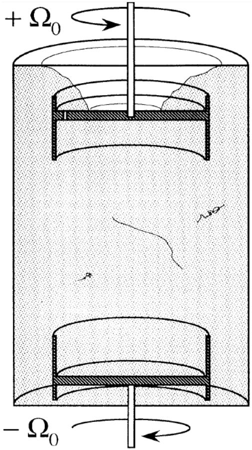
# Motivation: Experimental Evidence

Douady, Couder, Brachet (1991):

- Turbulence ($\mathrm{Re} \sim 80000$) is created within water seeded with a large number of very small bubbles
- Regions of high vorticity and low dissipation are low pressure regions

$$2\Delta p + \epsilon^2 - \omega^2 = 0$$

$\rightarrow$ Bubbles migrate to these regions



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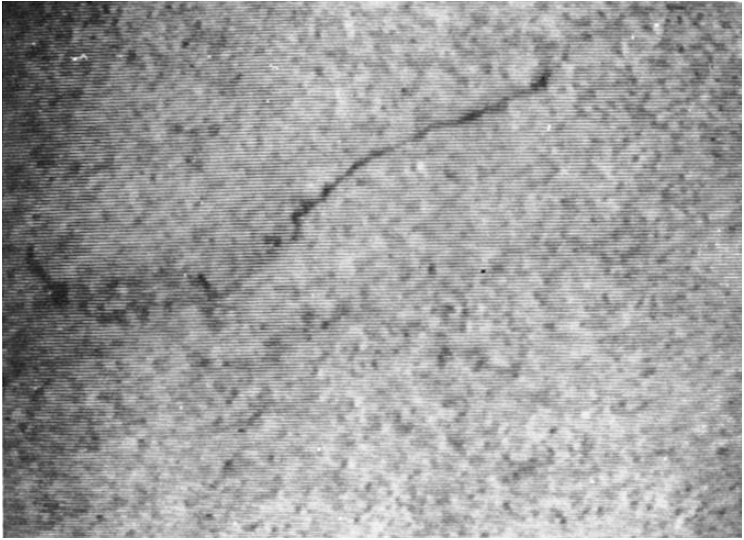

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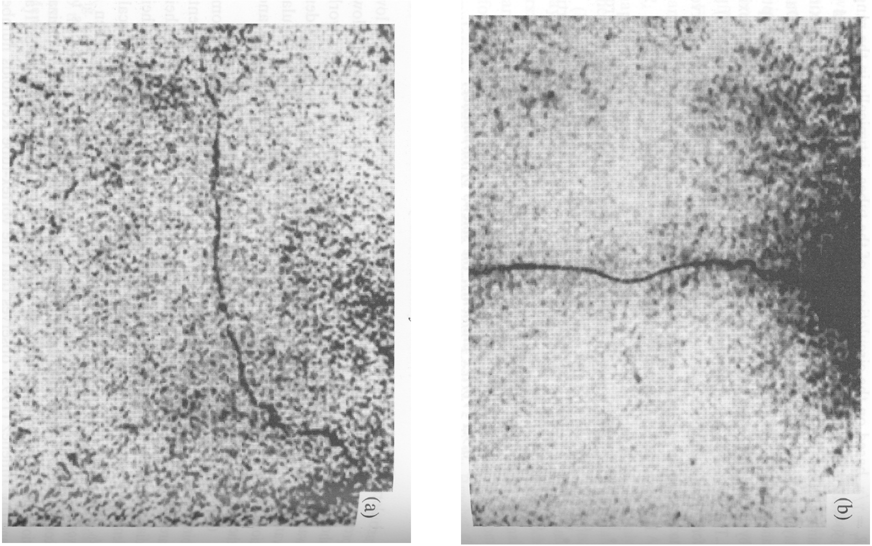

# Motivation: Experimental Evidence

- Now that we know that filaments are there, what role do they play in turbulence? („Filaments - Dog or Tail?“)
- A good starting point might be to learn about vortex filaments in general.

# Overview

1. Dynamics of an Isolated Filament
   - Basic Definitions
   - A Special Case: Smoke Ring Dynamics
   - Localized Induction Approximation (LIA)
   - Beyond LIA: Global Induction
2. Application to Vortex Knots
   - Knots, Torus Knots
   - Numerical Results

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# Basic Equations

- Euler equation of an incompressible fluid

$$\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} = -\nabla p \qquad \nabla \cdot \vec{u} = 0$$

- Evolution of vorticity

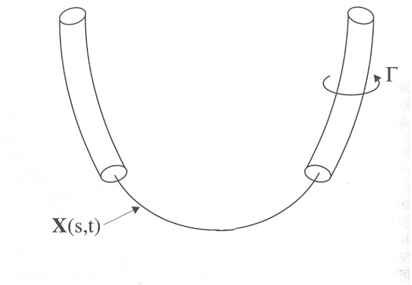
$$\partial_t \vec{\omega} + \vec{u} \cdot \nabla \vec{\omega} = \vec{\omega} \cdot \nabla \vec{u} \qquad \text{with vorticity } \vec{\omega} = \nabla \times \vec{u}$$

- Vortex-stretching due to the term $\vec{\omega} \cdot \nabla \vec{u}$ vanishes in two dimensions.

# Filaments

**Definition**

A *filament* is an isolated vortex tube of small diameter $dA$ and constant vorticity $\omega$.

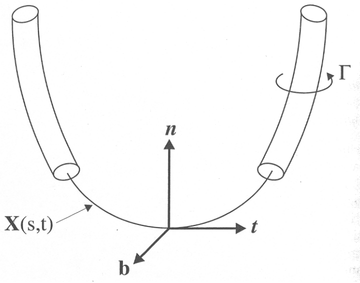



- Let $\vec{X}(s,t)$ be the centerline of a filament and $s$ a parameter of arc length.
- This is conveniently described by means the Serret-Frenet equations from differential geometry.

# Filaments

**Serret-Frenet Equations**

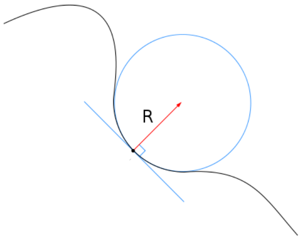
$$\begin{aligned}\vec{X}' &= \vec{t} \\ \vec{t}' &= \kappa \vec{n} \\ \vec{n}' &= \tau \vec{b} - \kappa \vec{t} \\ \vec{b}' &= -\tau \vec{n}\end{aligned}$$




$' \equiv \frac{\partial}{\partial s}$

Curvature $\kappa = \frac{\|\vec{t}'\|}{\|\vec{t}\|}$, $\kappa = \frac{1}{R}$

Torsion $\tau = \frac{\vec{b} \cdot \vec{n}'}{\|\vec{X}'\|} = \frac{\vec{X}' \times \vec{X}'' \cdot \vec{X}'''}{\|\vec{X}' \times \vec{X}''\|}$



# Dynamics of a Single Filament

- Formal analogue to magnetostatics:

$$\begin{aligned} \nabla \cdot \vec{u} &= 0 \\ \nabla \times \vec{u} &= \vec{\omega} \end{aligned} \qquad \qquad \begin{aligned} \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{B} &= \mu_0 \vec{j} \end{aligned}$$

- Biot-Savart Law:

$$\vec{u}(\vec{x}) = -\frac{1}{4\pi} \int d^3x' \frac{(\vec{x} - \vec{x}') \times \vec{\omega}(\vec{x}')}{\|\vec{x} - \vec{x}'\|^3}$$

- Consider ideal filaments

$$\vec{\omega}(\vec{x}) = \Gamma \int ds' \delta \left( \vec{x} - \vec{X}(s') \right) \vec{t}$$

$\Rightarrow$ This is analogous to the concept of filamentary wires

# Dynamics of a Single Filament

- Theorem of Biot-Savart for an ideal filament:

$$\vec{u}(\vec{x}) = -\frac{\Gamma}{4\pi}\int_{\vec{X}} ds' \frac{\left[\vec{x} - \vec{X}(s')\right] \times \vec{t}}{\|\vec{x} - \vec{X}(s')\|^3}$$

- Artifact of the idealization: $\vec{u}(\vec{x})$ diverges like $\frac{1}{\|\vec{x}-\vec{X}\|}$
- Self-induction:

$$\vec{u}(\vec{X}(s)) = -\frac{\Gamma}{4\pi}\int_{\vec{X}} ds' \frac{\left[\vec{X}(s) - \vec{X}(s')\right] \times \vec{t}}{\|\vec{X}(s) - \vec{X}(s')\|^3}$$

  Taylor-Expansion yields

$$\vec{u}(\vec{X}(s)) = \frac{\Gamma}{4\pi}\left[\vec{X}' \times \vec{X}'' \int \frac{ds'}{|s - s'|} + \mathcal{O}(1)\right]$$

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Taylor-Expansion yields

$$\vec{u}(\vec{X}(s)) = \frac{\Gamma}{4\pi} \left[ \underbrace{\vec{X}' \times \vec{X}''}_{\sim \vec{b}} \int \frac{ds'}{|s - s'|} + \mathcal{O}(1) \right]$$

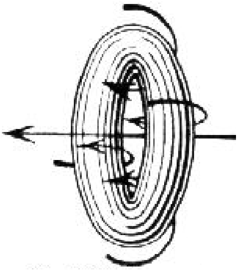
# A Special Case: Smoke Ring Dynamics

- Assume constant vorticity inside the ring:

$$\omega_0(\vec{x}) = \begin{cases} \frac{\Gamma}{\pi\sigma^2}, & |\vec{x}| < \sigma \\ 0, & |x| > \sigma. \end{cases}$$

- Lamb (1932): Ring moves perpendicular to its plane at velocity

$$u_\perp = \frac{\Gamma}{4\pi R}\left[\ln\left(\frac{8R}{\sigma}\right) - \frac{1}{4} + \mathcal{O}\left(\frac{\sigma}{R}\right)\right]$$



With $R/\sigma \gg 1$:

$$\ln\left(\frac{8R}{\sigma}\right) \approx const.$$

$$\Rightarrow u_\perp = \frac{C_1}{R} \sim \kappa.$$

# Local Induction Approximation (LIA)

- To render the self-induction integral finite, introduce a cut-off so that $|s' - s| > \epsilon$
- Further, self-induction by distant parts of the filament are neglected by limiting the Taylor-expansion to the leading order.

$$\frac{\partial \vec{X}(s,t)}{\partial t} = \ln\left(\frac{L}{\epsilon}\right)\frac{\Gamma}{4\pi}\left(\vec{X}'(s,t) \times \vec{X}''(s,t)\right)$$

- Rescaling $t$ yields the local induction evolution (LIE)

$$\Rightarrow \frac{\partial \vec{X}}{\partial t} = \kappa(s,t)\vec{b}(s,t).$$

  $\rightarrow$ Around $s$ the filament looks like a ring-vortex of radius $1/\kappa(s,t)$, the local curvature radius of the filament.
- This approximation is known as *Local Induction Approximation* (LIA), sometimes also called „Smoke Ring Approximation“

# Hasimoto-Transformation

- Starting from the LIA evolution equation $\dot{\vec{X}}(s,t) = \kappa(s,t)\vec{b}$, introduce the following formal substitutions

$$\begin{aligned}
\vec{N}(s,t) &= \left(\vec{n} + \mathrm{i}\vec{b}\right)\mathrm{e}^{\mathrm{i}\Phi} \text{ mit } \Phi = \int_0^s ds\,\tau(s,t) \\
\psi(s,t) &= \kappa(s,t)\mathrm{e}^{\mathrm{i}\Phi} \\
\kappa = |\psi| & \qquad \tau = \Phi'
\end{aligned}$$

$$\Rightarrow -\mathrm{i}\dot{\psi} = \psi'' + \frac{1}{2}|\psi|^2\psi.$$

- Time evolution in LIA can be related to a cubic Schrödinger equation.

# Soliton Solutions to the LIA Equation

- Like the non-linear SE the LIA has soliton solutions
- Consider the simplest case: a wave travels along the filament at constant speed $c$

$$\psi(s-ct) = \kappa(s-ct)\mathrm{e}^{\mathrm{i}\int_0^s ds\tau(s-ct)}.$$

Into the NLSE

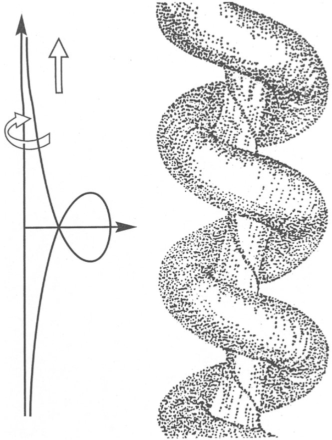
$$\Rightarrow (c-2\tau)\kappa^2 = 0 \qquad \tau = \tau_0 = \frac{c}{2} = const.$$

- Integration yields

$$\kappa = \pm 2\tau_0 \mathrm{sech}(\pm\tau_0(s-ct)).$$

# Soliton Solutions to the LIA Equation

- $\kappa$ and $\tau$ together with the Serret-Frenet equations yield parametric expressions of $(\vec{X}, \vec{t}, \vec{n}, \vec{b})$.



# LIA: Properties & Shortcomings

Assumptions:

- Filaments have to be sufficiently thin ($\sigma / R \ll 1$)
- Induction by distant parts of the filament have to be neglectable

Consequences:

- Calculations are drastically simplified (especially useful for numerical simulations)
- There are several conserved quantities:
  - Maximum projected area of a closed vortex filament $\mathcal{C}$

$$\vec{P}(t) = \frac{1}{2} \int_{\mathcal{C}} ds (\vec{X} \times \vec{X}')$$

    (Momentum conservation) $\Rightarrow$ A non-circular vortex ring can never become perfectly circular
  - Total torsion
  - Arc length $\Rightarrow$ No vortex-stretching!
  - ...

# Beyond LIA: Global Induction Effects

General case: „Full“ Biot-Savart law

- Remove the divergent part of the Biot-Savart integral

$$\vec{u}(X(\vec{s},t)) = -\frac{\Gamma}{4\pi}\int_{\vec{X}} ds' \frac{\left[\vec{X}(s,t) - \vec{X}(s',t)\right] \times \vec{t}}{\|\vec{X}(s,t) - \vec{X}(s',t)\|^3}$$

  by cutting off the line integral (just as in LIA).
- Take higher orders of the Taylor expansion into account
- Influences on the filament evolution can thus be divided in two portions:
  - local influences: LIA
  - global influences: Higher order terms

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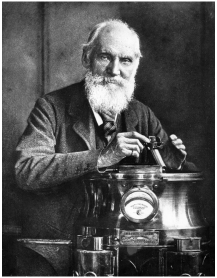
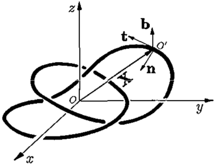
# Vortex Knots

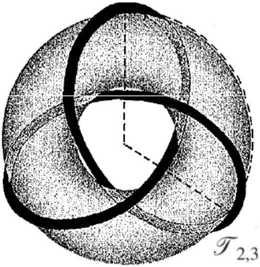
- A *knot* is a circle embedded in $\mathbb{R}^3$
- Lord Kelvin (1875) was the first to investigate the existence and stability of vortex knots in Euler flow.





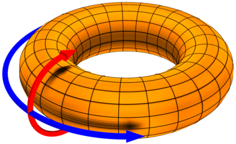

- Knots are useful for modelling of (topologically) complex structures.

# Torus Knots

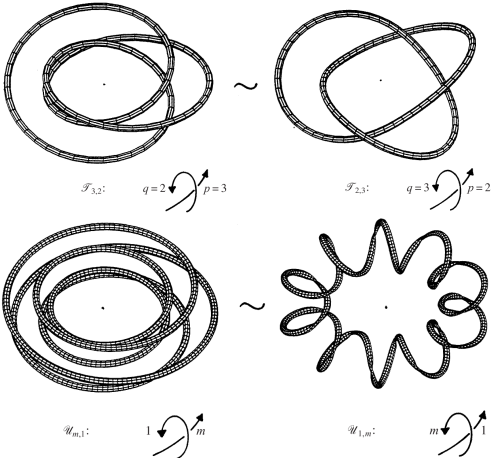



- Closed curves which can be drawn on a torus' surface

- Classification by number of toroidal ($p$) and poloidal ($q$) windings: $\mathcal{T}_{p,q}$



# Torus Knots



$\mathcal{T}_{3,2}$:
$q = 2$
$p = 3$
$\mathcal{T}_{2,3}$:
$q = 3$
$p = 2$
$\mathcal{U}_{m,1}$:
1
$m$
$\mathcal{U}_{1,m}$:
$m$
1

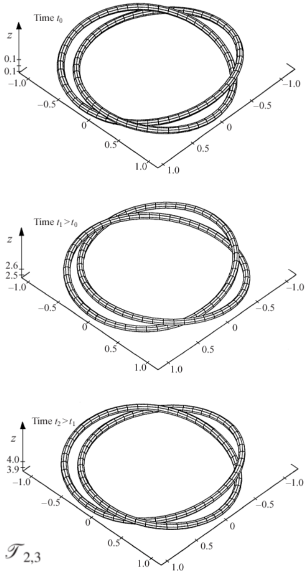
# Dynamics of Vortex Knots

Evolution in LIA

**Theorem (Ricca 1995)**

A torus knots $\mathcal{T}_{p,q}$ is linearly stable in LIA, if and only if $p < q$.

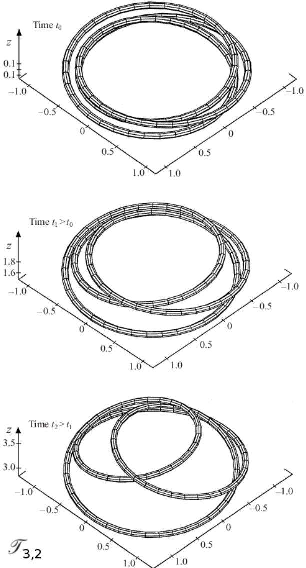
- Numerical simulations (Ricca, Samuels, Barenghi 1997) indicate, that vortex knots which satisfy this condition can travel distances several times their own size.



# Dynamics of Vortex Knots

Evolution in LIA

- In the reverse case, $q > p$, the vortex knot structure decays much faster.

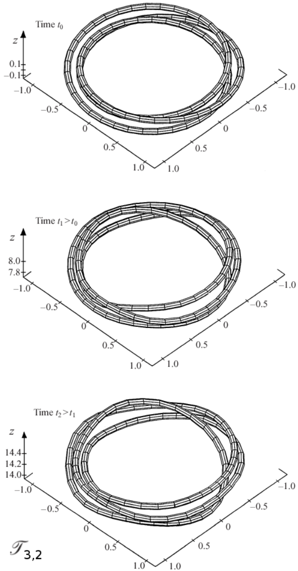



$\mathscr{T}_{3,2}$

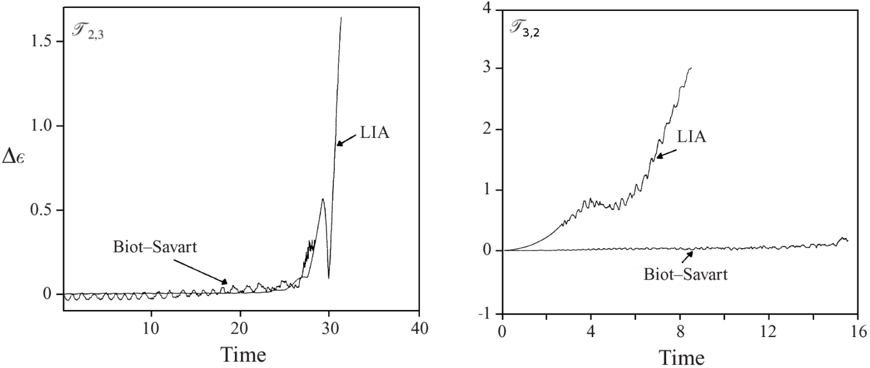
# Dynamics of Vortex Knots

Development due to the „full“ BSL

- Adding global influences stabilizes the knot evolution in both cases.
- This is due to a rotation of the „arms“ of the knot around each other (see Jiminez (1975) on stability of co-rotating vortices),



# LIA vs. Full Biot-Savart Law




$$\epsilon = \frac{R_{\max} - R_{\min}}{2}$$

# Summary

- In order to understand the role they play in turbulence filaments are investigated
- The dynamics of an isolated vortex filament in Euler flow is described by the Biot-Savart law $\rightarrow$ Divergence has to be handled
- From the LIA viewpoint, the filament is locally seen as part of a vortex ring of a radius corresponding to local curvature
- By Hasimoto transformation the LIA evolution equation can be related to a NLSE $\rightarrow$ Solitons
- The LIA is the local portion of the desingularized BS integral
- Knot vortex dynamics gain stability by taking into account the global portions of the BSI.