The Euler-Lagrange Problem in Turbulence

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Overview

- Introduction
- Relating Eulerian and Lagrangian two point pdfs
- First results (2D)

Introduction

The governing equation

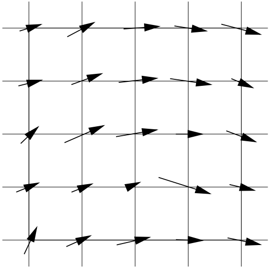
- Navier-Stokes-equation

$$\dot{\mathbf{u}} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{u}$$
$$\nabla \cdot \mathbf{u} = 0$$

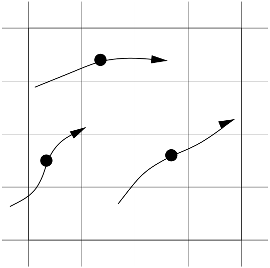
- Vortex-transport-equation in 2D $\omega = \textbf{curl } \mathbf{u}$

$$\dot{\omega} + \mathbf{u} \cdot \nabla \omega = \nu \Delta \omega$$

Euler vs. Lagrange



Euler



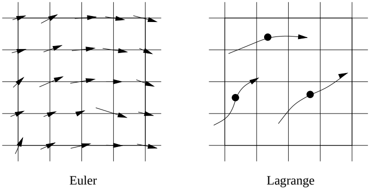
Lagrange

$$\frac{d\mathbf{X}(t)}{dt} = \mathbf{u}(\mathbf{X}(t), t)$$

$$\frac{d\mathbf{U}(t)}{dt} = [-\nabla p(\mathbf{x}, t) + \nu\Delta\mathbf{u}(\mathbf{x}, t) + f(\mathbf{x}, t)]_{\mathbf{x}=\mathbf{X}(t)}$$

Velocity increments

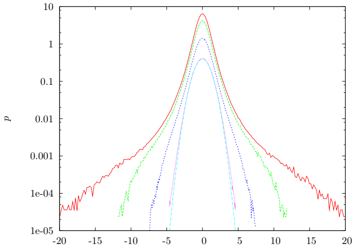
The most famous observable in basic turbulence research (except from the energy spectrum !?)



- Euler: $\delta \mathbf{u}_e(\mathbf{r}) = \mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})$ (longitudinal, transversal,...)
- Lagrange: $\delta \mathbf{u}_l(\boldsymbol{\tau}) = \mathbf{u}(\mathbf{t}+\boldsymbol{\tau}) - \mathbf{u}(\mathbf{t})$

Velocity increment statistics

Example: forced 2D turbulence ⇒ Eulerian pdfs are self-similar and close to gaussian / Lagrangian pdfs are not self-similar

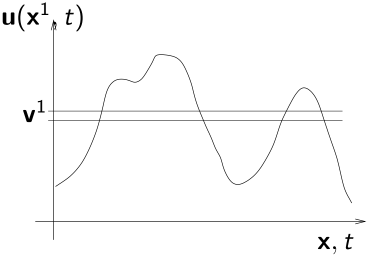


Other examples: 3D turbulence, MHD turbulence

Relating Eulerian and Lagrangian two point pdfs

Notation of fine grained pdfs

$$\hat{f}^1(\mathbf{v}^1;\mathbf{x}^1,t) := \delta(\mathbf{u}(\mathbf{x}^1,t)-\mathbf{v}^1)$$

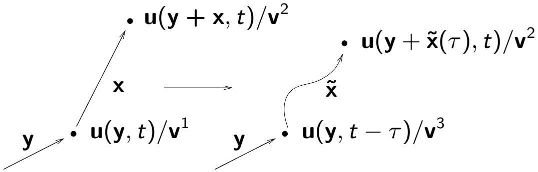


realization of random number: $\mathbf{u}(\mathbf{x}^1,t)$
sample space variable (bin number): $\mathbf{v}^1$

$$f^1(\mathbf{v}^1;\mathbf{x}^1,t) = \langle\delta(\mathbf{u}(\mathbf{x}^1,t)-\mathbf{v}^1)\rangle$$

Two point pdfs

$$\hat{f}^1(\mathbf{v}^1, \mathbf{v}^2; \mathbf{x}^1, \mathbf{x}^2, t^1, t^2) := \delta(\mathbf{u}(\mathbf{x}^1, t^1) - \mathbf{v}^1)\ \delta(\mathbf{u}(\mathbf{x}^2, t^2) - \mathbf{v}^2)$$

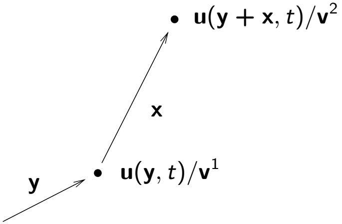


$$\hat{f}_e^{12}(\mathbf{v}^1, \mathbf{v}^2; \mathbf{y}, \mathbf{x}, t) = \delta(\mathbf{u}(\mathbf{y}, t) - \mathbf{v}^1)\ \delta(\mathbf{u}(\mathbf{y} + \mathbf{x}, t) - \mathbf{v}^2)$$

$$\hat{f}_l^{23}(\mathbf{v}^3, \mathbf{v}^2; \mathbf{y}, t, \tau) = \delta(\mathbf{u}(\mathbf{y}, t - \tau) - \mathbf{v}^3)\ \delta(\mathbf{u}(\mathbf{y} + \tilde{\mathbf{x}}(\tau), t) - \mathbf{v}^2)$$

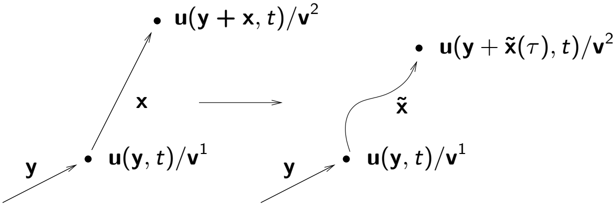
The first step

$$\hat{f}_e^{12} = \delta(\mathbf{u}(\mathbf{y},t) - \mathbf{v}^1)\ \delta(\mathbf{u}(\mathbf{y}+\mathbf{x},t) - \mathbf{v}^2)$$



The first step

$$\hat{f}_n^{12} = \int dx\ \delta(\mathbf{x} - \tilde{\mathbf{x}}(\tau))\ \delta(\mathbf{u}(\mathbf{y},t) - \mathbf{v}^1)\ \delta(\mathbf{u}(\mathbf{y}+\mathbf{x},t) - \mathbf{v}^2)$$



The first step

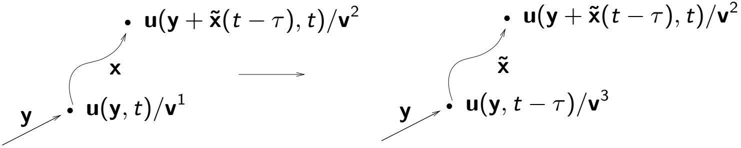
$$\hat{f}_n^{12} = \int dx\ \delta(\mathbf{x} - \tilde{\mathbf{x}}(\tau))\ \delta(\mathbf{u}(\mathbf{y}, t) - \mathbf{v}^1)\ \delta(\mathbf{u}(\mathbf{y} + \mathbf{x}, t) - \mathbf{v}^2)$$

$$\hat{f}_n^{12} = \delta(\mathbf{u}(\mathbf{y}, t) - \mathbf{v}^1)\ \delta(\mathbf{u}(\mathbf{y} + \tilde{\mathbf{x}}(\tau), t) - \mathbf{v}^2)$$

The second step

$$\hat{f}_m^{23}(\mathbf{v}^3, \mathbf{v}^2; \mathbf{y}, t, \tau)$$

$$= \int d\mathbf{v}^1 \; \delta(\mathbf{u}(\mathbf{y}, t-\tau) - \mathbf{v}^3) \; \delta(\mathbf{u}(\mathbf{y}, t) - \mathbf{v}^1) \; \delta(\mathbf{u}(\mathbf{y} + \tilde{\mathbf{x}}(\tau), t) - \mathbf{v}^2)$$



The second step

$$\hat{f}_m^{23}(\mathbf{v}^3, \mathbf{v}^2; \mathbf{y}, t, \tau)$$
$$= \int d\mathbf{v}^1 \; \delta(\mathbf{u}(\mathbf{y}, t-\tau) - \mathbf{v}^3) \; \delta(\mathbf{u}(\mathbf{y}, t) - \mathbf{v}^1) \; \delta(\mathbf{u}(\mathbf{y} + \tilde{\mathbf{x}}(\tau), t) - \mathbf{v}^2),$$

Again performing the integration ...

$$= \delta(\mathbf{u}(\mathbf{y}, t-\tau) - \mathbf{v}^3) \; \delta(\mathbf{u}(\mathbf{y} + \tilde{\mathbf{x}}(\tau), t) - \mathbf{v}^2)$$
$$= \hat{f}_l^{23}(\mathbf{v}^3, \mathbf{v}^2; \mathbf{y}, t, \tau)$$

Now we arrived at the fine grained Lagrangian distribution!!

Both steps together

Combining the two steps leads to

$$\hat{f}_l^{23}(\mathbf{v}^3, \mathbf{v}^2; \mathbf{y}, t, \tau)$$
$$= \int d\mathbf{v}^1 \; \delta(\mathbf{u}(\mathbf{y}, t-\tau) - \mathbf{v}^3) \int dx \; \delta(\mathbf{x} - \tilde{\mathbf{x}}(\tau)) \; \delta(\mathbf{u}(\mathbf{y}, t) - \mathbf{v}^1) \; \delta(\mathbf{u}(\mathbf{y}+\mathbf{x}, t) - \mathbf{v}^2)$$

Averaging the fine grained pdfs

An ensemble average leads to

$$f_l^{23}(\mathbf{v}^3, \mathbf{v}^2; \mathbf{y}, t, \tau)$$
$$= \left\langle \int d\mathbf{v}^1 \; \delta(\mathbf{u}(\mathbf{y}, t-\tau) - \mathbf{v}^3) \int dx \; \delta(\mathbf{x} - \tilde{\mathbf{x}}(\tau)) \; \delta(\mathbf{u}(\mathbf{y}, t) - \mathbf{v}^1) \; \delta(\mathbf{u}(\mathbf{y}+\mathbf{x}, t) - \mathbf{v}^2) \right\rangle$$

The Eulerian pdf is hidden in the expectation value!

How to extract the Eulerian pdf?

Decomposition of the expectation value using conditional pdfs

$$\begin{aligned}\langle \hat{f}(\mathbf{v}^1;\mathbf{x},t)\hat{f}(\mathbf{v}^2;\mathbf{x},t)\rangle &= \langle \delta(\mathbf{v}^1-\mathbf{u}^1(\mathbf{x},t))\delta(\mathbf{v}^2-\mathbf{u}^2(\mathbf{x},t))\rangle.\\ &= \langle \delta(\mathbf{v}^1-\mathbf{u}^1(\mathbf{x},t))|\mathbf{v}^2\rangle\langle \delta(\mathbf{v}^2-\mathbf{u}^2(\mathbf{x},t))\rangle\\ &= f(\mathbf{v}^1|\mathbf{v}^2;\mathbf{x},t)f(\mathbf{v}^2;\mathbf{x},t)\end{aligned}$$

Simply $p(a,b)=p(a|b)p(b)$!

$$\left\langle \prod_{i=1}^{n}\hat{f}(\mathbf{v}^i;\mathbf{x},t)\right\rangle = \langle \hat{f}(\mathbf{v}^n;\mathbf{x},t)\rangle \prod_{i=1}^{n-1}\langle \hat{f}(\mathbf{v}^i|\mathbf{v}^{i+1},\ldots,\mathbf{v}^{n-1};\mathbf{x},t)\rangle$$

Splitting

Splitting up the expectation value leads to

$$f_l^{13}(\mathbf{v}^3, \mathbf{v}^1; \mathbf{y}, t, \tau)$$
$$= \int d\mathbf{v}^1 \int d\mathbf{x} \langle \hat{f}(\mathbf{v}^3|\mathbf{v}^1, \mathbf{v}^2, \mathbf{x}; \mathbf{y}, t, \tau)\rangle \langle \hat{f}(\mathbf{x}|\mathbf{v}^1, \mathbf{v}^2; \tau)\rangle \langle \hat{f}_e^{12}(\mathbf{v}^1, \mathbf{v}^2; \mathbf{x}, \mathbf{y}, t)\rangle.$$
$$= \int d\mathbf{v}^1 \int d\mathbf{x}\ p(\mathbf{v}^3|\mathbf{v}^1, \mathbf{v}^2, \mathbf{x}; \mathbf{y}, t, \tau)\ p(\mathbf{x}|\mathbf{v}^1, \mathbf{v}^2; \tau) f_e^{12}(\mathbf{v}^1, \mathbf{v}^2; \mathbf{x}, \mathbf{y}, t)$$

Averaging again

For an translational invariant and stationary flow we can average with respect to t and $\mathbf{y}$

$$
\begin{aligned}
& f_l^{13}(\mathbf{v}^3, \mathbf{v}^1; \tau) \\
& = \int d\mathbf{v}^1 \int d\mathbf{x} \langle \hat{f}(\mathbf{v}^3|\mathbf{v}^1, \mathbf{v}^2, \mathbf{x}; \tau)\rangle \langle \hat{f}(\mathbf{x}|\mathbf{v}^1, \mathbf{v}^2; \tau)\rangle \langle \hat{f}_e^{12}(\mathbf{v}^1, \mathbf{v}^2; \mathbf{x})\rangle \\
& = \int d\mathbf{v}^1 \int d\mathbf{x} \; p(\mathbf{v}^3|\mathbf{v}^1, \mathbf{v}^2, \mathbf{x}; \tau) \; p(\mathbf{x}|\mathbf{v}^1, \mathbf{v}^2; \tau) f_e^{12}(\mathbf{v}^1, \mathbf{v}^2; \mathbf{x}).
\end{aligned}
$$

The transition probabilities

The first transition pdf mixes Eulerian pdfs from diffrent scales

$$p(\mathbf{x}|\mathbf{v}^1, \mathbf{v}^2; \tau) \approx p(\mathbf{x}; \tau). \quad (1)$$

Corrsin approximation

The second transition pdf introduces time (and space) correlations of the velocity field

$$p(\mathbf{v}^3|\mathbf{v}^1, \mathbf{v}^2, \mathbf{x}; \tau) \quad (2)$$

First results

What are we doing

- pseudospectral simulation of forced 2D Navier-Stokes-equation (dealiased,1024^2 grid points,rk4)
- 10^5 tracer particles are immersed into the flow (bicubic interpolation)

Going back to step 1

The lagrangian two point pdf for equal times is

$$f_n^{12}(\mathbf{v}^1, \mathbf{v}^2; \tau) = \int d\mathbf{x}\ p(\mathbf{x}|\mathbf{v}^1, \mathbf{v}^2; \tau) f_e^{12}(\mathbf{v}^1, \mathbf{v}^2; \mathbf{x}).$$

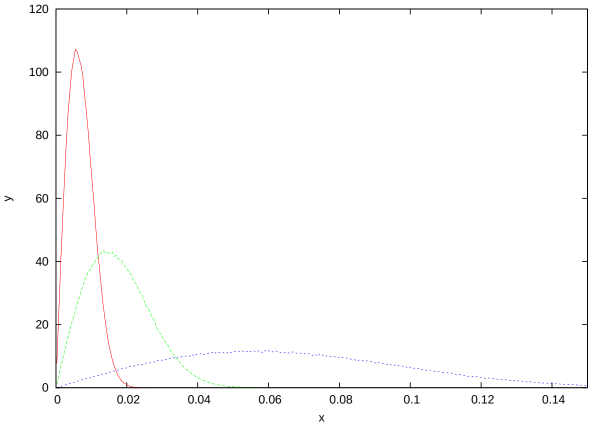
Usage of Corrsins approximation

$$f_n^{12}(\mathbf{v}^1, \mathbf{v}^2; \tau) \approx \int d\mathbf{x}\ p(\mathbf{x}; \tau) f_e^{12}(\mathbf{v}^1, \mathbf{v}^2; \mathbf{x}).$$

Particle distribution pdf

Assuming isotropy

$$p(\mathbf{x};\tau). \implies \tilde{p}(r;\tau)$$



Conclusion and Outlook

- The Eulerian and Lagrangian two point pdf's can be related via transition probabilities
- Now we can look for approximations
- ....