Localized structures in reaction-diffusion systems

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Outline

1. Introduction
   - Reaction-Diffusion Systems
   - Experimental system in question
   - Possible Model Systems

2. Localized structures in 3KRD system
   - Phenomenological 3KRD model
   - Stationary Turing structures
   - Localized structures and their stability

Reaction-Diffusion Systems

$$\partial_t \mathbf{u} = \mathbf{D}\nabla^2 \mathbf{u} + \mathbf{R}(\mathbf{u})$$

- $\mathbf{u} = \mathbf{u}(\mathbf{r}, \mathbf{t}) = (u_1, u_2, \ldots, u_n)^T$–a vector of concentration variables, $\mathbf{r} \subset \mathbb{R}^m$, $m = 1, 2, 3$;
- ∇^2– the Laplace operator;
- $\mathbf{R}(\cdot)$– a local reaction kinetics;
- $\mathbf{D}$– a diagonal diffusion coefficient matrix;

Reaction-Diffusion Systems: 1K systems

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- **R. Fisher** *The wave of advance of advantageous genes*;
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$$\partial_t u = d^2 u_{xx} + u(1-u)(u-\alpha),\ \alpha \in (0,1)$$

Reaction-Diffusion Systems: 2K systems

1952:

- **A. M. Turing** *The chemical basis of morphogenesis*

$$\partial_t u = D_u \nabla^2 u + f(u,v),$$
$$\partial_t v = D_v \nabla^2 v + g(u,v)$$

Reaction-Diffusion Systems: 2K systems

1952:

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Belousov-Zhabotinsky reaction, other autocatalytic and oscillatory chemical reactions, different musters on animal's skin, nerve pulse transmission, bacteria growth processes, structures, observed in semiconductors or gas-discharge systems, ...

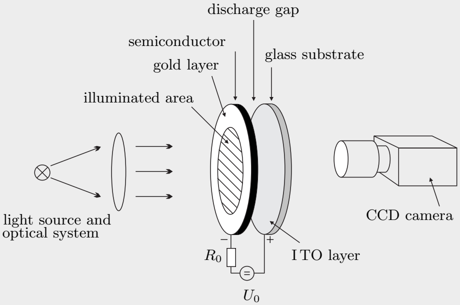
Reaction-Diffusion Systems: 3K and more

$$\partial_t u_\alpha = D_\alpha \nabla^2 u_\alpha + R_\alpha(u)$$

a model of blood clotting, population dynamics, ecology, photosensitive Belousov-Zhabotinsky reaction ,glycolysis, a model of CO oxidation on Pt(110), a model of Dictyostelium amoebae, patterns arising in gas-discharge system......

Experimental set-up

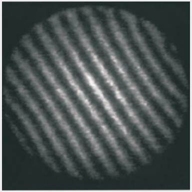
Planar dc gas-discharge system with high-ohmic electrode:



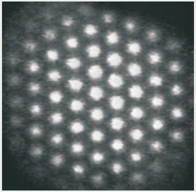
- $d = 0.1 - 2$ mm;
- Gases: N_2, He, Ar;
- $p \approx 30 - 400$ hPa;
- Semiconductor: GaAs
- $d_c = 0.5 - 1.5$ mm;
- $\rho_{SC} \approx 10^7 - 10^8\, \Omega \cdot$ cm;
- $U_0 = 1 - 5$ kV

Examples of observed patterns

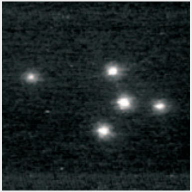
stripes



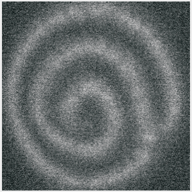
hexagons



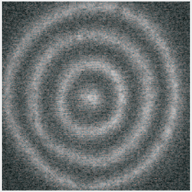
spots



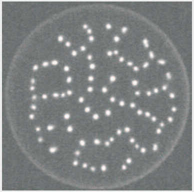
spirals



targets



chains



Possible Model Systems

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an appropriate reduced discharge models should be developed or qualitative models should be used

Outline

Phenomenological three-component RD model

$$\partial_t u = D_u \Delta u + f(u) - \kappa_3 v + \kappa_1 - \frac{\kappa_2}{\|\Omega\|} \int_\Omega u dr,$$

$$\tau \partial_t v = D_v \Delta v + u - v;$$

$u = u(\mathbf{r}, t)$–is related to the avalanche multiplication of charged carriers in the discharge gap;
$v = v(\mathbf{r}, t)$–the voltage drop at the semiconductor plate;
κ_1– is connected with the normalized applied voltage;
κ_2– describes a normalized internal resistance of the voltage source.

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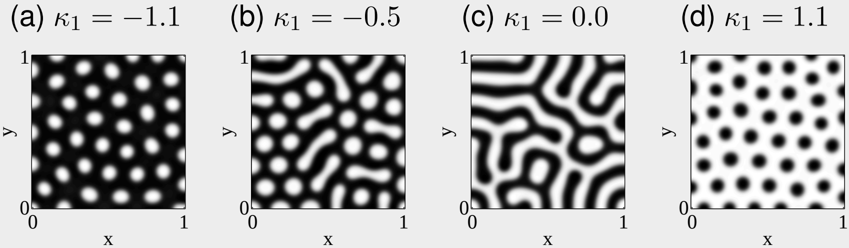
$u = u(\mathbf{r}, t)$, $v = v(\mathbf{r}, t)$, $w = w(\mathbf{r}, t)$, $\mathbf{r} \subset \mathbb{R}^2$, $f(u) = \lambda u - u^3$,
$D_u, D_v, D_w, \lambda, \tau, \theta, \kappa_3, \kappa_4 \geq 0$

Turing instability

The idea (`Turing, 1952`):

- The homogeneous solution of the system is stable in absence of diffusion;
- A difference in diffusion constants of components could be enough to destabilize the homogeneous solution;
- Another control parameter can be used (in our case we choose κ_1 as the control parameter)

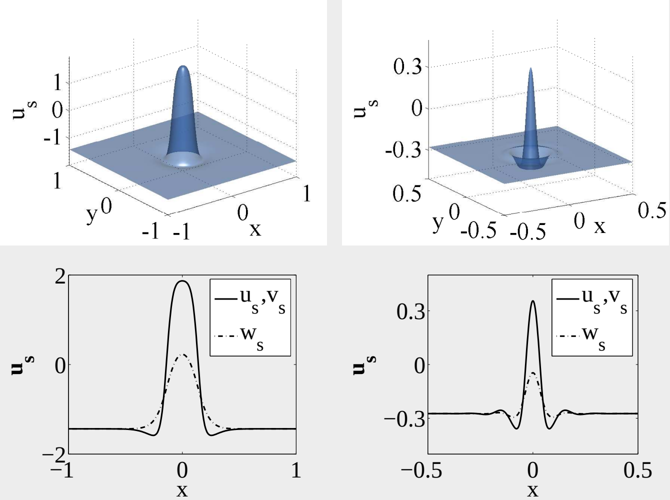
Stationary Turing patterns



Stationary solutions

(a)

(b)



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- Eigenvalue problem: $\mathfrak{L}'(\mathbf{u_s})\mathcal{F} = \lambda\mathcal{F}$;

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$$\mathbf{u_s}(\mathbf{r} + \epsilon) = \mathbf{u_s}(\mathbf{r}) + \epsilon \frac{\partial \mathbf{u_s}}{\partial \mathbf{r}} + O(\epsilon^2).$$

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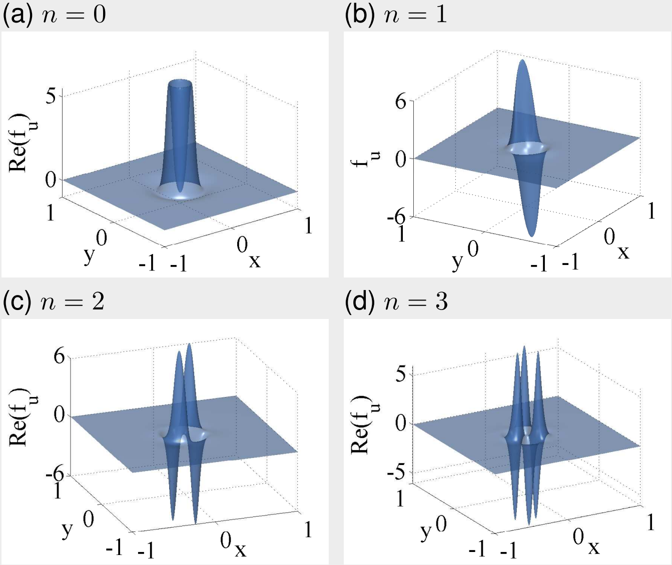
- $\widetilde{\mathbf{u}} \sim \widetilde{\mathbf{u}}_{\mathbf{n}} e^{in\phi}$:

$$\lambda_n \widetilde{\mathbf{u}}_n = \mathfrak{L}'_p \widetilde{\mathbf{u}}_n,$$

Critical modes



Properties of the operator $\mathfrak{L}'(\mathbf{u_s})$

$$\mathfrak{L}'(\mathbf{u_s}) = \begin{pmatrix} D_u\triangle + \lambda - 3\bar{u}^2 & -\kappa_3 & -\kappa_4 \\ \frac{1}{\tau} & \frac{D_v\triangle - 1}{\tau} & 0 \\ \frac{1}{\theta} & 0 & \frac{D_w\triangle - 1}{\theta} \end{pmatrix}$$

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$$\mathfrak{L}'(\mathbf{u_s}) = ML(\mathbf{u_s}),$$

where

$$M = M^{\dagger} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/\kappa_3\tau & 0 \\ 0 & 0 & -1/\kappa_4\theta \end{pmatrix},$$

and

$$L = L^{\dagger} = \begin{pmatrix} D_u\triangle + \lambda - 3u_s^2 & -\kappa_3 & -\kappa_4 \\ -\kappa_3 & -\kappa_3 D_v\triangle + \kappa_3 & 0 \\ -\kappa_4 & 0 & -\kappa_4 D_w\triangle + \kappa_4 \end{pmatrix}.$$

Properties of the operator $\mathfrak{L}'(\mathbf{u_s})$

- Two eigenvalue problems:

$$\mathfrak{L}'(\mathbf{u_s})\mathcal{F} = \lambda\mathcal{F}, \qquad \mathfrak{L}'^{\dagger}(\mathbf{u_s})\mathcal{F}^* = \bar{\lambda}\mathcal{F}^*,$$
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$$\langle\overline{\mathcal{F}}^*|\mathcal{F}\rangle = 0 \quad \text{if} \quad \lambda \neq \bar{\lambda}, \qquad \overline{\mathcal{F}}^* = M^{-1}\mathcal{F},$$
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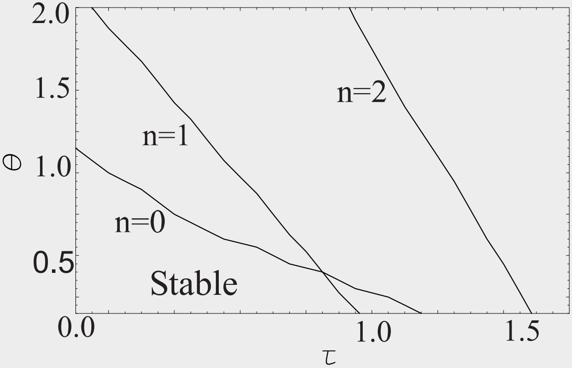
- If λ is real (e.g., $\lambda = 0$):

$$\mathcal{G}_r^* = M^{-1}\mathcal{G} = \begin{pmatrix} \frac{\partial \mathbf{u_s}}{\partial \mathbf{r}} \\ -\kappa_3\tau\frac{\partial \mathbf{v_s}}{\partial \mathbf{r}} \\ -\kappa_4\theta\frac{\partial \mathbf{w_s}}{\partial \mathbf{r}} \end{pmatrix}.$$

However, in this case $\langle\mathcal{G}_r^*|\mathcal{G}\rangle \neq 0$.

Stability diagramm



Order of bifurcations

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- In the limit case the operator $\mathfrak{L}'(\mathbf{u_s})$ is Hermitian. All λ's are real.

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$$\left(\mu - \frac{\kappa_3 \tau \mu}{\tau \mu + 1}\right) f_u = D_u \triangle f_u + (f'(u_s) - \kappa_3) f_u - \kappa_4 f_w$$
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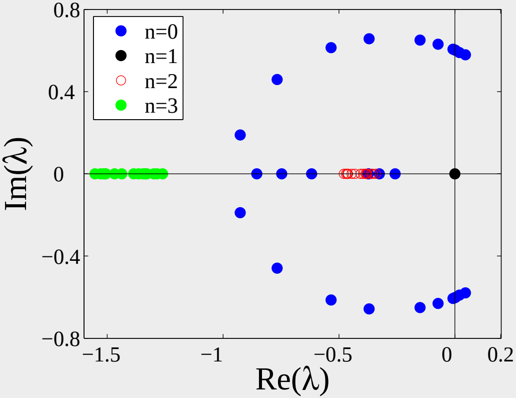
- Propagator mode: Generalized eigenfunction

$$\mathfrak{L}'(\mathbf{u_s})\mathfrak{L}'(\mathbf{u_s})\mathcal{P}_\mathbf{r} = 0 \Rightarrow \ \mathfrak{L}'(\mathbf{u_s})\mathcal{P}_\mathbf{r} = \mathcal{G}_\mathbf{r}$$

Breathing DSs

- Destabilization via the mode $n = 0$:



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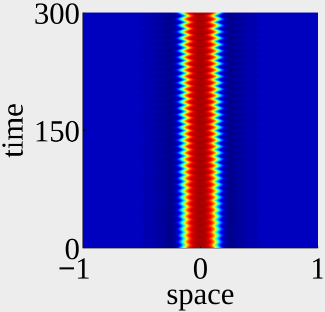
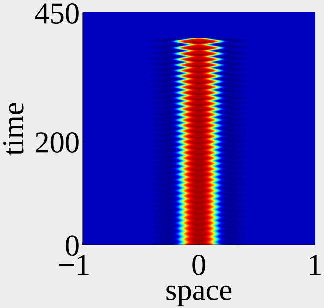




Amplitude equation

The idea:

- Two-scale expansion in the vicinity of bifurcation point,

$$\theta = \theta_c\text{:}\ \widetilde{\mathbf{u}} = Ae^{i\omega t}\mathcal{F}_c + c.c. + r$$

$$\theta = \theta_c + \varepsilon\text{:}\ \widetilde{\mathbf{u}} = A(t)e^{i\omega t}(\mathcal{F}_c + \varepsilon\mathcal{F}_\varepsilon) + c.c. + r;$$

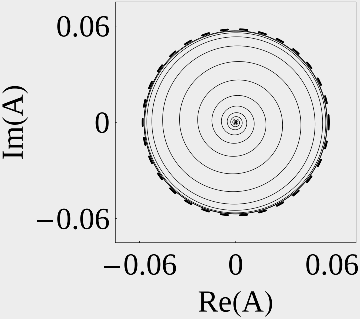
- Amplitude equation is a normal form of a Hopf bifurcation;
- Complex coefficients can be immediately evaluated if the solitary stationary solution is known;

$$\partial_t A = \varepsilon a_1 A + a_2 A|A|^2,$$

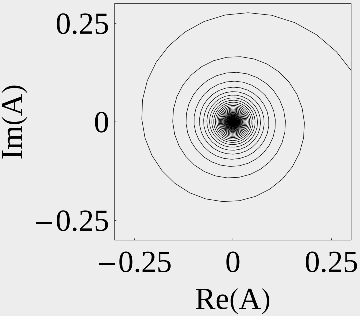
$$a_1 = \frac{\langle \mathfrak{L}'_\varepsilon(\mathbf{u_s})\mathcal{F}_c|\mathcal{F}_c^*\rangle}{\langle \mathcal{F}_c|\mathcal{F}_c^*\rangle}, \qquad a_2 = \frac{\langle \mathfrak{L}'''_c(\mathbf{u_s})\mathcal{F}_c\mathcal{F}_c\overline{\mathcal{F}}_c|\mathcal{F}_c^*\rangle}{2\langle \mathcal{F}_c|\mathcal{F}_c^*\rangle}.$$

Amplitude equation

Re$(a_1) < 0$, Re$(a_2) > 0$

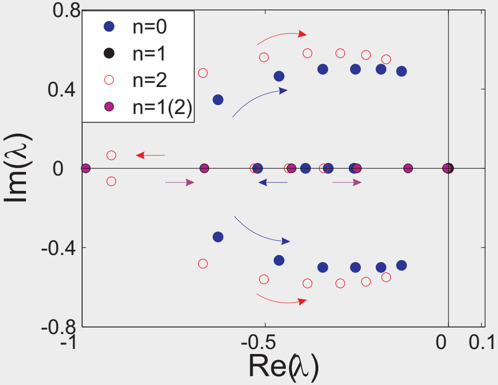


Re$(a_1) < 0$, Re$(a_2) < 0$



Moving DSs

- Destabilization via the mode $n = 1$:



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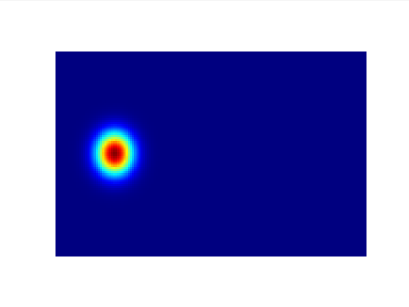

Moving DSs: Drift-Bifurcation

- Fredholm alternative:

$$\mathcal{L}'(\mathbf{u_s})\mathcal{P}_\mathbf{r} = \mathcal{G}_\mathbf{r} \Leftrightarrow \langle \mathcal{G}^*_\mathbf{r} | \mathcal{G}_\mathbf{r} \rangle = 0$$

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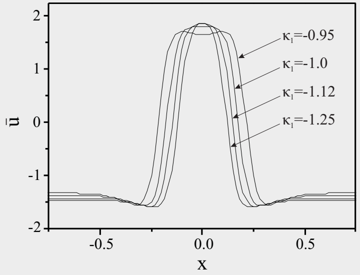
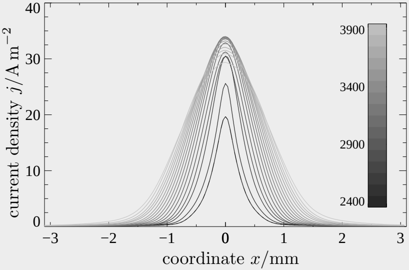
- Propagation velocity:

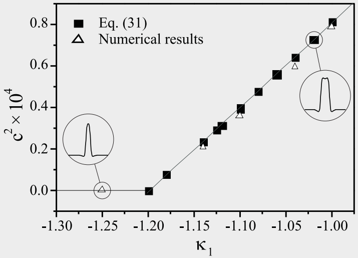
$$c^2 = \frac{\kappa_3}{\tau^2_{c,x}} \frac{\langle u^2_{s,x} \rangle}{\langle u^2_{s,xx} \rangle} \left(\tau - \tau_{c,x} \right)$$

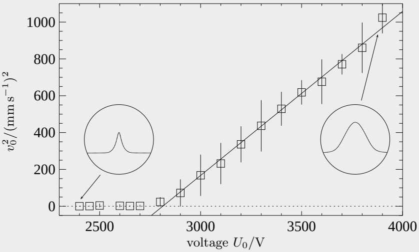
Drift-Bifurcation due to a change of shape



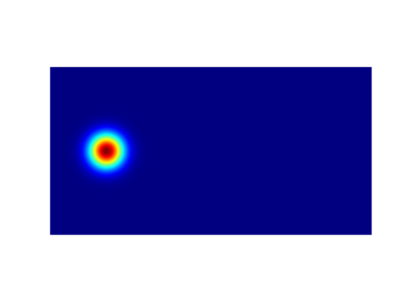


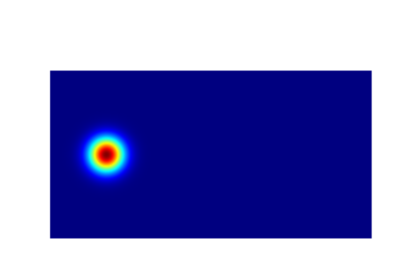
Drift-Bifurcation due to a change of shape



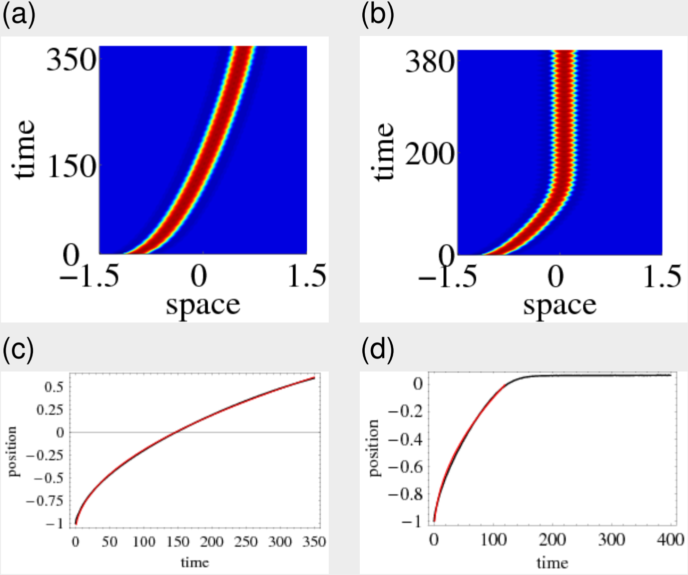


Breathing and moving DSs

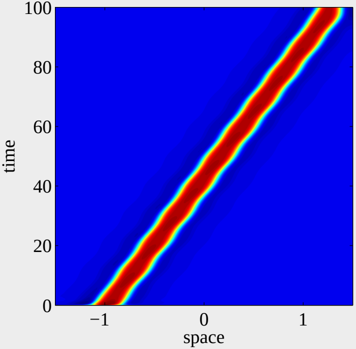




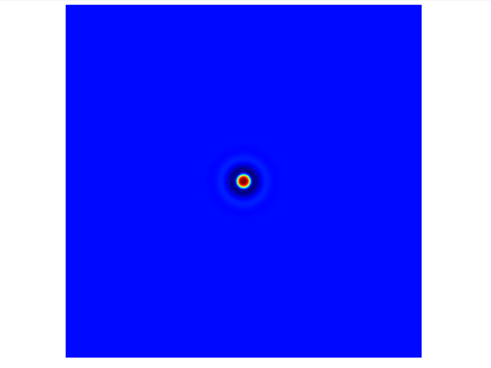
Breathing and moving DSs



Breathing and moving DSs



Breathing DSs with oscillatory tails



Breathing DSs with oscillatory tails