Turing instability of electrical discharges

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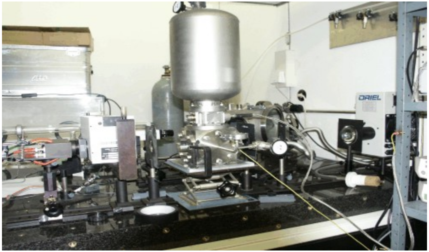


- Johannes Laven
- Weifeng Shang
- Lars Stollenwerk
- Hans-Georg Purwins
- Hendrik Bödeker
- Svetlana Gurevich



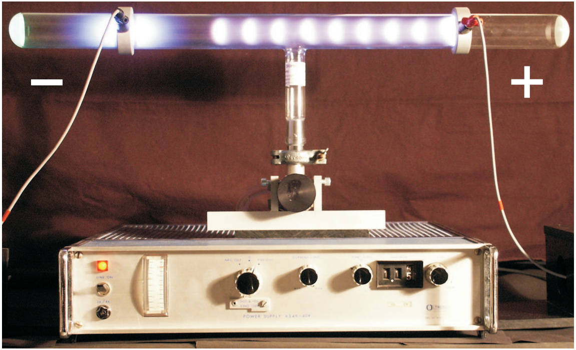



100 % Theory

100 % Experiment

Outline

Part I. Experimental data

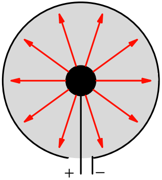


There are more things between cathode and anode than are dreamt in your philosophy.

H. Raether

Rubens, 1940

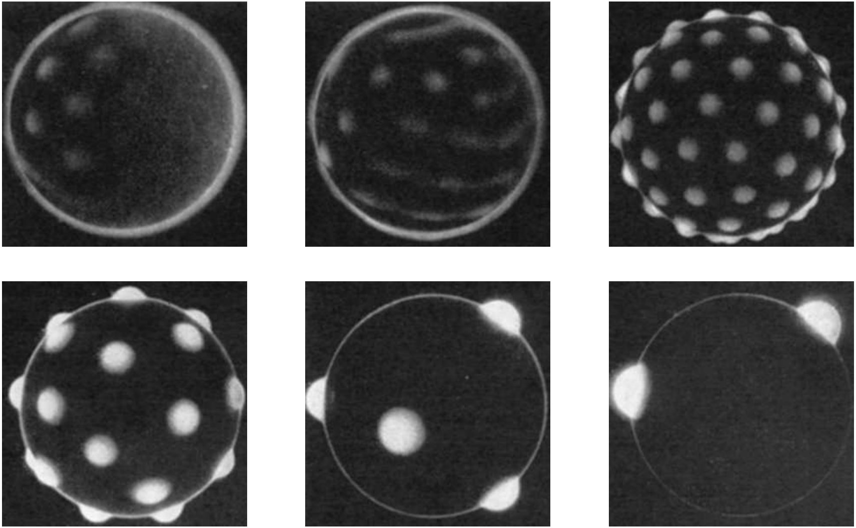
Formation of the anode spots in the glow discharge mode.



- Spherical cathode $R = 26.7$ cm;
- Spherical anode $r = 5.1$ cm;
- Nitrogen $p \sim 1$ Torr;
- Current $I \sim 500$ mA.

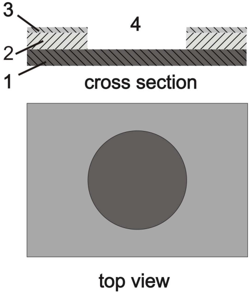
A homogeneous anode glow is unstable and evolves to the symmetric arrangements of current spots.

Anode spots (Rubens, 1940)



Schoenbach, 2004

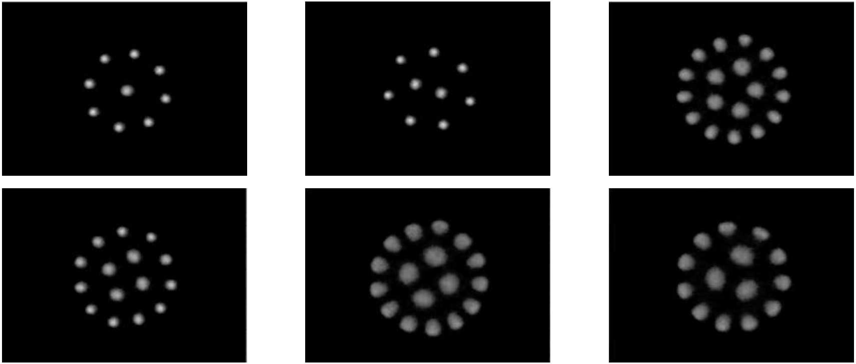
Formation of the cathode spots in the glow discharge mode.



1. Cathode $d = 100$ μm;
2. Spacer 250 μm;
3. Hollow anode $D = 1.5$ mm;
4. Xenon $p \sim 200$ Torr.

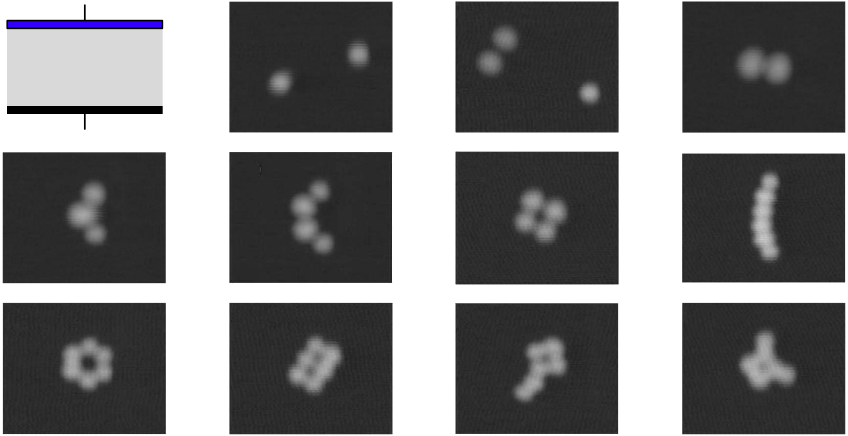
A homogeneous cathode glow is unstable and evolves to the symmetric arrangements of current spots.

Cathode spots (Schoenbach, 2004)



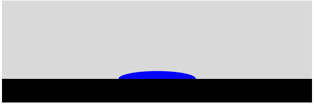
Theory: Benilov
1986, 2007

Cathode vs anode spots, Nasuno, 2003



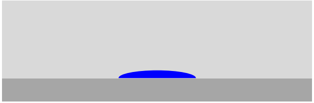
Nitrogen, $d \sim 100\ \mu$m, $p \sim 40$ Torr, sandwich-like geometry.

Maxwell time



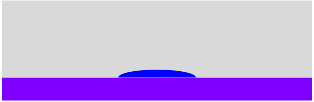
Metal

$$\tau_m \longrightarrow 0$$



Dielectric

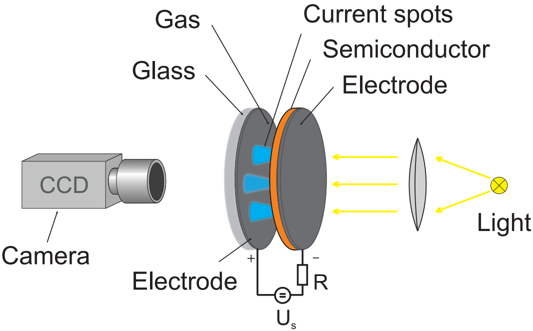
$$\tau_m \longrightarrow \infty$$



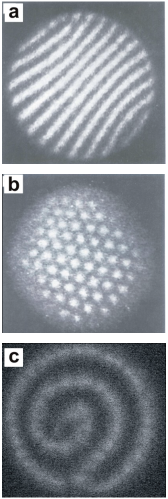
High-ohmic barrier

$$\tau_m = \epsilon_0 \rho \sim \tau_i$$

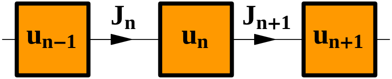
Purwins group at Münster



Gas, e.g.:	N_2	0.5 mm	100 Torr
Cathode:	GaAs	0.5 mm	$10^7\ \Omega\,\mathrm{cm}$



Par t II. RD systems



$$\frac{du_n}{dt} = Q_n + J_n - J_{n+1}$$

$$Q_n = f(u_n)$$

$$J_n \sim u_{n-1} - u_n$$

$$J_{n+1} \sim u_n - u_{n+1}$$

A discrete RD equation

$$\frac{du_n}{dt} = f(u_n) + u_{n-1} - 2u_n + u_{n+1}$$

One component RD systems

Fisher, Kolmogorov 1937

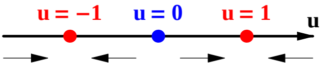
$$\partial_t u = D\partial_x^2 u + f(u)$$

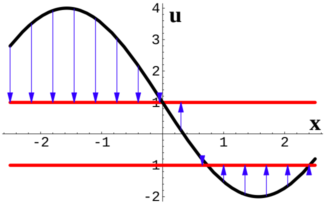
e.g.,

$$f(u) = u - u^3$$

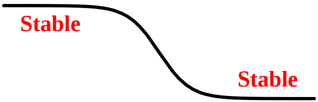
Local evolution

$$\partial_t u(x,t) = f(u)$$





Free energy



- Connected stable states;
- Fronts propagation;
- Phase transitions, flames, etc.

Free energy always reduces

$$\partial_t u = -\frac{\delta \mathfrak{F}}{\delta u} \qquad \mathfrak{F}[u] = \int \left[\frac{D(\partial_x u)^2}{2} + F(u) \right] d^n x$$

with

$$F'(u) = -f(u) \qquad \partial_t \mathfrak{F} < 0$$

“The chemical basis of morphogenesis”

Turing, 1952

$$\partial_t u = D_u \partial_x^2 u + f(u, v)$$

$$\partial_t v = D_v \partial_x^2 v + g(u, v)$$

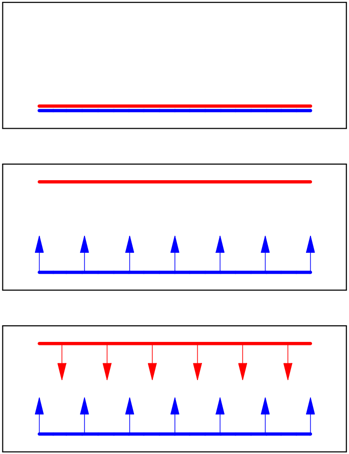
Reaction part

$$\frac{du}{dt} = f(u, v) \qquad \text{and} \qquad \frac{dv}{dt} = g(u, v)$$

Problem

How stability of a stationary state is affected by diffusion?

Prey-Predator Model



Linearized uniform system

$$\frac{du}{dt} = \alpha u - \beta v$$

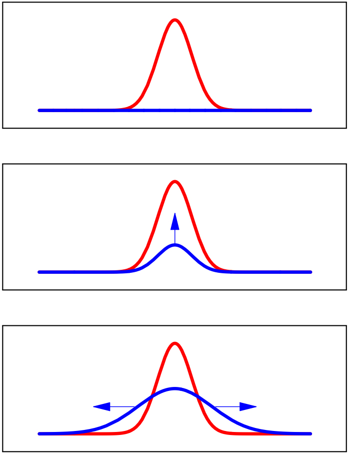
$$\frac{dv}{dt} = \gamma u - \delta v$$

Stability conditions

$$\alpha < \delta$$

$$\alpha\delta < \beta\gamma$$

Slow preys, fast predators



Linearized full system

$$\frac{\partial u}{\partial t} = D_u \partial_x^2 u + \alpha u - \beta v$$

$$\frac{\partial v}{\partial t} = D_v \partial_x^2 v + \gamma u - \delta v$$

The most important instability condition

$$D_u < D_v$$

(to be revised).

Stability analysis

Linearized full system is considered for $u \sim e^{ikx}, \quad v \sim e^{ikx}$

$$\frac{du}{dt} = \alpha u - \beta v \qquad \frac{du_k}{dt} = (\alpha - D_u k^2) u_k - \beta v_k$$

$$\frac{dv}{dt} = \gamma u - \delta v \qquad \frac{dv_k}{dt} = \gamma u_k - (\delta + D_v k^2) v_k$$

Stability conditions

$$\alpha < \delta \qquad \alpha - D_u k^2 < \delta + D_u k^2$$

$$\alpha\delta < \beta\gamma \qquad (\alpha - D_u k^2)(\delta + D_u k^2) < \beta\gamma$$

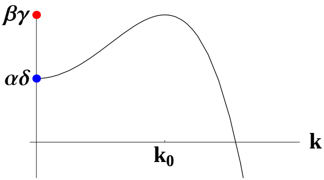
Turing instability

$$\alpha\delta < \beta\gamma, \qquad \text{but} \qquad (\alpha - D_u k^2)(\delta + D_v k^2) > \beta\gamma$$

$$\max_k \left[\alpha\delta + (\alpha D_v - \delta D_u)k^2 - (D_u k^2)(D_v k^2) \right] > \beta\gamma$$

A necessary condition

$$\alpha D_v > \delta D_u$$



A finite space scale

Resume

Parameters

$$\alpha, \quad \beta, \quad \gamma, \quad \delta, \quad D_u, \quad D_v$$

Citeria

$\alpha < \delta$	"Sp" criterion
$\alpha\delta < \beta\gamma$	"Det" criterion
$\alpha\delta + D_u D_v k_0^2 > \beta\gamma$	No simple interpretation
$\alpha D_v > \delta D_u$	Slow preys, fast predators

The most dangerous mode: $$k_0^2 = \frac{\alpha D_v - \delta D_u}{2 D_u D_v}$$

An example

Equations

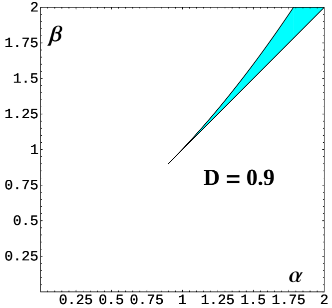
$$\partial_t u = D\partial_x^2 u + \alpha u - \beta v$$

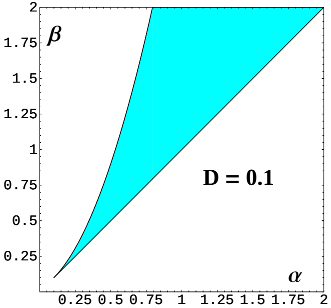
$$\partial_t v = \partial_x^2 v + u - v$$

Criteria

$$D < \alpha < 1$$

$$\alpha < \beta < \alpha + \frac{(\alpha - D)^2}{4D}$$





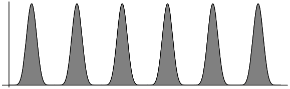
Nonlinear stage of instability

$$\partial_t u = D\partial_x^2 u + \alpha u - \beta v \underbrace{- u^3}_{\text{stab.}}$$

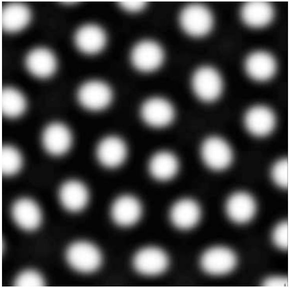
$$\partial_t v = \partial_x^2 v + u - v$$

Two space dimensions

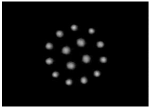
- Spatial structure
- Different symmetries
- Space scale
- Dissipative solitons



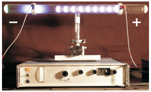
Turing pattern

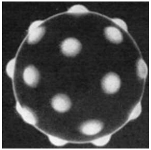


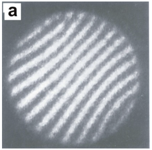
Part III. Applications

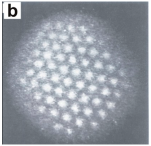


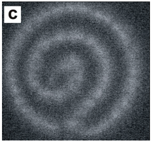
To which extent the concept of Turing patterns can be used to explain self-organization of glows?







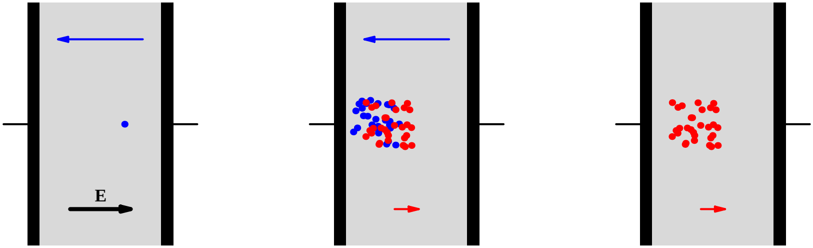




Key processes

- Particle drift
- Ionization
- Gamma process

E.g., 10^6 avalanches permanently coexist in the gas gap.

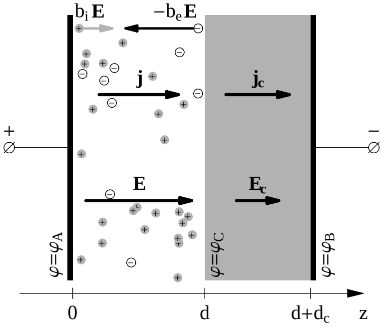


A seed electron | Ionization | Gamma process

Transport equations

Continuity equation

$$\partial_t n + \nabla \Gamma = Q$$



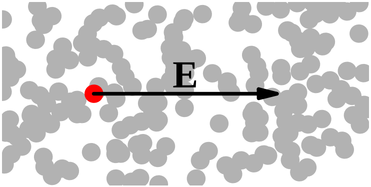
- Flux model

  $\Gamma = \text{drift} + \text{diffusion}$

- Source model

  $Q = \text{ionization}$

Physical ideas



- Average velocity

$$\mathbf{v}_{\text{drift}} = b\,\mathbf{E}$$

- Particle flux

$$\Gamma = n\,\mathbf{v}_{\text{drift}} - D\,\nabla n$$

- Impact ionization

$$\left(\frac{dn}{dt}\right)_{\text{ionization}} = \alpha(E)\,\Gamma_e$$

Mathematical model for gas

- Particles

$$\partial_t n_e + \nabla\Gamma_e = Q_e$$
$$\partial_t n_i + \nabla\Gamma_i = Q_i$$

- Source

$$Q_e = \alpha(E)\,\Gamma_e$$
$$Q_i = \alpha(E)\,\Gamma_e$$

- Fluxes

$$\Gamma_e = -n_e b_e \mathbf{E} - D_e \nabla n_e$$
$$\Gamma_i = +n_i b_i \mathbf{E} - D_i \nabla n_i$$

- Field

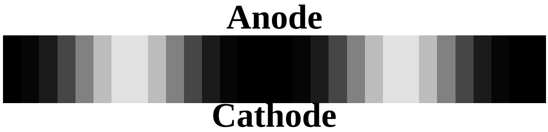
$$\mathbf{E} = -\nabla\varphi$$
$$\epsilon_0 \nabla^2 \varphi = -q(n_i - n_e)$$

Drift dominates over diffusion!

Reduction idea

Separation of scales:

$$d = 0.1\text{–}0.5\ \text{mm} \qquad \text{and} \qquad D \sim 1\ \text{mm}$$



- A locally 1D solution.
- A continuous set of 1D solutions is required.

- CVC

$$U_{\text{gas}} = U(I)$$

Procedure

General formulation:

$$\hat{L}_0(\partial_z)\mathfrak{u} = \hat{L}_1(\partial_x, \partial_y, \partial_t)\mathfrak{u}$$

1D solution

$$\hat{L}_0(\partial_z)\mathfrak{u} = 0 \qquad \rightarrow \qquad \mathfrak{u} = \mathfrak{u}_{\mathrm{eq}}(z, C)$$

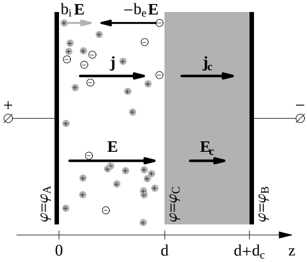
Perturbation

$$\mathfrak{u} = \mathfrak{u}_{\mathrm{eq}}\big(z, C(x, y, t)\big) + \tilde{\mathfrak{u}}(x, y, z, t)$$

Compatibility

$$\underbrace{\hat{L}_0'(\partial_z)\tilde{\mathfrak{u}} = \hat{L}_1(\partial_x, \partial_y, \partial_t)\mathfrak{u}_{\mathrm{eq}}}_{\textbf{A singular equation}} \quad \rightarrow \quad \langle\ \text{Projector}\ |\hat{L}_1\mathfrak{u}_{\mathrm{eq}}(z, C)\rangle = 0$$

Reduced equations



- "Prey"

$$j_z(x, y, t)$$

- "Predator"

$$\delta U = U_{gas}(x, y, t) - U_b$$

$$\partial_t j_z = D_e \frac{b_i}{b_e} \nabla^2 j_z + \frac{C_\gamma}{\tau_i} \left(\alpha_b' \delta U + \frac{\alpha_b'' \tau_i^2 d}{24 \epsilon_0^2} j_z^2 \right) j_z$$

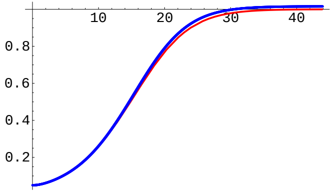
$$\partial_t \delta U = \frac{d_c^2}{3\tau_c} \nabla^2 \delta U + \frac{U_s - U_b - \delta U}{\tau_c} - \frac{j_z}{\mathfrak{c}}$$

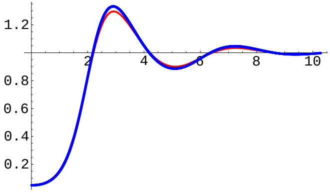
Normalized form

A two-component reaction-diffusion system:

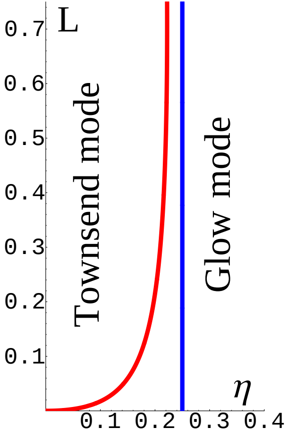
"current"	$u(x, y, t)$	$\tau \partial_t u = L^2 \nabla^2 u + (v + \eta u^2) u$
"voltage"	$v(x, y, t)$	$\partial_t v = \nabla^2 v + s - u - v$

E.g., for the transient oscillations (current vs time)





Results



An example for $\tau = 1$ and $s = 1$

$$\partial_t u = L^2 \nabla^2 u + (v + \eta u^2) u$$
$$\partial_t v = \nabla^2 v + 1 - u - v$$

- Mechanism
- Instability threshold
- Spatial scale
- Initial dynamics