Delay Fokker-Planck equations, Novikov's theorem, and Boltzmann distributions as small delay approximations

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We study time-delayed stochastic systems that can be described by means of so-called delay Fokker-Planck equations. Using Novikov's theorem, we first show that the theory of delay Fokker-Planck equations is on an equal footing with the theory of ordinary Fokker-Planck equations. Subsequently, we derive stationary distributions in the case of small time delays. In the case of additive noise systems, these distributions can be cast into the form of Boltzmann distributions involving effective potential functions.

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I. INTRODUCTION

Many complex systems in inanimate and animate nature can be regarded as self-regulated systems, where the process of self-regulation involves a finite computation time. These systems can often be described in terms of dynamical systems involving time-delayed feedback loops. Prominent examples of such time-delayed systems are laser systems with optical feedback [1–3] related to the Ikeda and the Lang-Kobayashi equations [4], vertical-cavity surface-emitting lasers with time-delayed feedback control [5], and self-regulated voltage-controlled oscillators [6,7]. Furthermore, hydrodynamic problems [8,9], network systems [10], and biological systems [11] have been discussed in the context of time-delayed self-regulated systems. Moreover, the application of nonlinear time series analysis [12] to time-delayed systems and the reconstruction of the dynamics of time-delayed systems [13] are current challenges in physics and related disciplines.

This variety of applications calls for a theory of time-delayed systems. While for time-delayed deterministic and even chaotic systems many helpful theoretical results are available in the literature and are often related to the so-called extended phase space approach [2,3,6,9,14,15], theoretical tools to deal with time-delayed stochastic systems can hardly be found. However, in many of the aforementioned systems fluctuating forces play important roles. This holds in particular for biological systems. Therefore, in biophysics researchers are often concerned not only with time-delayed systems but with time-delayed stochastic systems (see, e.g., some recent studies on balancing tasks [16], movement control [17], the pupil light reflex [18], and postural sway [19]). So far, analytical studies have been focused on linear time-delayed stochastic systems [20–26], time-delayed bistable systems exhibiting weak fluctuating forces [5,27], nonlinear time-delayed stochastic systems that can be mapped to linear ones by means of appropriate variable transformations [23,26], and time-delayed stochastic systems with small time delays [28]. In this context, the delay Fokker-Planck equation that has been introduced by Guillouzic *et al.* a few years ago [22] has turned out to be a very useful tool to derive analytical results [23,24,28,29]. However, the delay Fokker-Planck equation approach is still in its infancy. Therefore, at issue is to deepen our understanding of the theory of delay Fokker-Planck equations and, in doing so, to gain further insights into the nature of time-delayed stochastic systems. In detail, first, we will show that delay Fokker-Planck equations can be derived in a way similar to ordinary Fokker-Planck equations: by exploiting Novikov's theorem. Second, we will determine the stationary distributions of time-delayed stochastic systems involving small time delays. Such distributions have been discussed in a previous study [28]. In this study distributions that differ from Boltzmann distributions have been found. In contrast, our approach will yield Boltzmann distributions that involve delay-dependent effective potential functions.

II. DELAY FOKKER-PLANCK EQUATIONS

A. Derivation by means of Novikov's theorem

Let $X(t) \in \mathbb{R}$ denote a random variable that describes the state of a time-delayed stochastic system subjected to natural boundary conditions [30]. We assume that the system dynamics is defined by the stochastic delay differential equation

$$\frac{d}{dt}X(t) = h(X(t),X(t-\tau)) + g(X(t),X(t-\tau))\Gamma(t) \qquad (1)$$

for $t \geq 0$, where $\tau \geq 0$ denotes the time delay of the system and g corresponds to a (state-dependent) noise amplitude. We assume that the initial condition can be written in terms of a function $\varphi(z)$ which gives us $X(z)=\varphi(z)$ for $z \in [-\tau,0]$. In Eq. (1) the function $\Gamma(t)$ corresponds to a Langevin force with $\langle\Gamma(t)\rangle=0$ and $\langle\Gamma(t)\Gamma(t')\rangle=\delta(t-t')$. More precisely, let $C_y[\chi]$ denote the characteristic functional of a random variable $y(t)$ and test function $\chi(t)$ defined by $C_y[\chi] =\langle\exp\{i\int_{-\infty}^{\infty}y(t)\chi(t)dt\}\rangle_y$. Then, the Langevin force $\Gamma(t)$ is completely defined by the characteristic functional [31]

$$C_\Gamma[\chi] = \exp\left(-\frac{1}{2}\int_{-\infty}^{\infty}\chi^2(t)dt\right). \qquad (2)$$

Finally, note that we will interpret the noise term $g\Gamma(t)$ in terms of the Stratonovich calculus. In order to derive the delay Fokker-Planck equation of Eq. (1) we proceed as in the case of nondelayed systems (see, e.g., [31,32]). Let $P(x,t)=\langle\delta(x-X(t))\rangle$ denote the probability density of the stochastic process defined by Eq. (1). Differentiating P with respect to time and using Eq. (1), we get

$$\frac{\partial}{\partial t}P(x,t) = -\left\langle \frac{\partial}{\partial x}\delta(x-X(t))\frac{d}{dt}X(t)\right\rangle = -\frac{\partial}{\partial x}\underbrace{\langle h(X(t),X(t-\tau))\delta(x-X(t))\rangle}_{A} - \frac{\partial}{\partial x}\underbrace{\langle g(X(t),X(t-\tau))\Gamma(t)\delta(x-X(t))\rangle}_{B}. \tag{3}$$

Note that the stochastic process $X(t)$ has the joint probability density $P(x,t;x',t')$. Consequently, the A term of Eq. (3) can be written as

$$A = \int h(y,y_\tau)P(y,t;y_\tau,t-\tau)\delta(x-y)dy\, dy_\tau. \tag{4}$$

Integrating with respect to y and changing the notation y_τ into x_τ yields

$$A = \int h(x,x_\tau)P(x,t;x_\tau,t-\tau)dx_\tau. \tag{5}$$

If we substitute the factor 1 in terms of $\int\delta(x_\tau-X(t-\tau))dx_\tau$ into the B term occurring in Eq. (3), the B term can be equivalently expressed as

$$B = \int g(x,x_\tau)\underbrace{\langle\Gamma(t)\delta(x-X(t))\delta(x_\tau-X(t-\tau))\rangle}_{B'}dx_\tau. \tag{6}$$

In order to evaluate the B' term, we use Novikov's theorem [31,32], which states that for a Langevin force with characteristic functional (2) and an operator $\hat{C}[\Gamma]$ of the Langevin force $\Gamma(t)$ we have

$$\langle\Gamma(t)\hat{C}[\Gamma]\rangle = \left\langle \frac{\delta\hat{C}[\Gamma]}{\delta\Gamma(t)}\right\rangle, \tag{7}$$

where the δ symbols denote the variational derivative of $\hat{C}$ with respect to Γ. Next, we regard $X(t)$ and $X(t-\tau)$ as expressions depending on $\Gamma(t)$. We put $\hat{C}[\Gamma]=\delta(x-X(t))\delta(x_\tau-X(t-\tau))$. Then, the left hand side of Eq. (7) corresponds to the B' term of Eq. (6). The right hand side of Eq. (7) can be written as

$$\left\langle \frac{\delta\hat{C}[\Gamma]}{\delta\Gamma(t)}\right\rangle = \left\langle \frac{\delta\hat{C}}{\delta X(t)}\frac{\delta X(t)}{\delta\Gamma(t)}\right\rangle + \left\langle \frac{\delta\hat{C}}{\delta X(t-\tau)}\frac{\delta X(t-\tau)}{\delta\Gamma(t)}\right\rangle. \tag{8}$$

Due to causality, we have $\delta X(t-\tau)/\delta\Gamma(t)=0$ for $\tau>0$. That is, the second term on the right hand side of Eq. (8) vanishes. Let us discuss next the first term. It is clear that we have

$$\frac{\delta\hat{C}}{\delta X(t)} = -\frac{\partial}{\partial x}\delta(x-X(t))\delta(x_\tau-X(t-\tau)). \tag{9}$$

Furthermore, using Eq. (1) in terms of

$$X(t) = X(0) + \int_0^t h(X(s),X(s-\tau))ds + \int_0^t g(X(s),X(s-\tau))\Gamma(s)ds, \tag{10}$$

we find that

$$\frac{\delta X(t)}{\delta\Gamma(t)} = \frac{1}{2}g(X(t),X(t-\tau)). \tag{11}$$

Taking Eqs. (7)–(11) together, we obtain

$$B' = \left\langle \frac{\delta\hat{C}}{\delta X(t)}\frac{\delta X(t)}{\delta\Gamma(t)}\right\rangle = -\frac{1}{2}\frac{\partial}{\partial x}g(x,x_\tau)\langle\delta(x-X(t))\delta(x_\tau-X(t-\tau))\rangle = -\frac{1}{2}\frac{\partial}{\partial x}g(x,x_\tau)P(x,t;x_\tau,t-\tau). \tag{12}$$

Substituting Eqs. (5), (6), and (12), into Eq. (3), we obtain the delay Fokker-Planck equation

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}\int h(x,x_\tau)P(x,t;x_\tau,t-\tau)dx_\tau + \frac{1}{2}\int\frac{\partial}{\partial x}g(x,x_\tau)\frac{\partial}{\partial x}g(x,x_\tau)P(x,t;x_\tau,t-\tau)dx_\tau. \tag{13}$$

In order to obtain a delay Fokker-Planck equation for the stochastic delay differential equation (1) when interpreting the noise term in Eq. (1) according to the Ito calculus, we exploit the equivalence [29]

$$\underbrace{g\,\Gamma(t)}_{\text{Stratonovich}} = \frac{g}{2}\frac{dg}{dx} + \underbrace{g\,\Gamma(t)}_{\text{Ito}}. \tag{14}$$

That is, we write the Ito form (1) into a Stratonovich form (1) by replacing h by $h-2^{-1}g\,dg/dx$. Replacing h by $h-2^{-1}g\,dg/dx$ in Eq. (13) gives us

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}\int h(x,x_\tau)P(x,t;x_\tau,t-\tau)dx_\tau + \frac{1}{2}\int\frac{\partial^2}{\partial x^2}g^2(x,x_\tau)P(x,t;x_\tau,t-\tau)dx_\tau, \tag{15}$$

which is the delay Fokker-Planck equation related to the Ito interpretation of Eq. (1).

B. Small delay approximations of stationary distributions

We will consider two approaches. The first approach is based on an approximation of probability density distributions. The second approach has been proposed in an earlier work [22] and is based on an approximation of the stochastic delay differential equation (1).

1. Probability density approach: General case

Let us turn first to the small delay approximation of stationary distributions of Eq. (1) with $g(x,x_\tau)=g(x)$. For the sake of simplicity, we interpret Eq. (1) by means of the Ito calculus which means that our departure point will be the delay Fokker-Planck equation (15). Note that in order to discuss small delay approximations for Eq. (1) involving Stratonovich calculus, one needs to transform in a preanalytical step Eq. (1) into the corresponding Ito form using Eq. (14). First, note that for $\tau=0$ Eq. (1) with $g(x,x_\tau)=g(x)$ reduces to

$$\frac{d}{dt}X(t) = h^{(0)}(X) + g(X)\Gamma(t) \tag{16}$$

with $h^{(0)}(x)=h(x,x)$. Next, let us write Eq. (1) as

$$\frac{d}{dt}X(t) = h^{(0)}(X) + \underbrace{h(X(t),X(t-\tau)) - h^{(0)}(X)}_{R(X(t),X(t-\tau))} + g(X)\Gamma(t). \tag{17}$$

For small time delays τ the expression R corresponds to a perturbation of the unperturbed system (16). The perturbation term R is of the order τ. In what follows, we treat the perturbed system (17) by means of a result that has previously been derived for stochastic systems with perturbations given by time-delayed feedback loops [33]. Accordingly, we start off with the delay Fokker-Planck equation (13) in the stationary case given by

$$\left[h^{(0)}(x) + \int R(x,x_\tau)P_{\rm st}(x_\tau,t-\tau|x,t)dx_\tau\right]P_{\rm st}(x) = \frac{1}{2}\frac{d}{dx}g^2(x)P_{\rm st}(x), \tag{18}$$

where $P_{\rm st}(x)$ and $P_{\rm st}(x_\tau,t-\tau|x,t)$ denote the stationary distribution and the stationary conditional distribution of $X(t)$, respectively. As shown in [33], the zeroth order stationary distribution and the correction term of first order in τ can be combined to give a distribution $P_{\rm st}^{(1)}(x)$ normalized to unity that satisfies

$$P_{\rm st}(x) = P_{\rm st}^{(1)}(x) + O(\tau^2). \tag{19}$$

In short, $P_{\rm st}^{(1)}(x)$ is a first order approximation of the stationary distribution $P_{\rm st}(x)$. It can be shown that $P_{\rm st}^{(1)}(x)$ is defined by Eq. (18) when we replace in Eq. (18) $P_{\rm st}(x)$ by $P_{\rm st}^{(1)}(x)$ and $P_{\rm st}(x_\tau,t-\tau|x,t)$ by means of its zeroth order approximation $P_{\rm st}^{(0)}(x_\tau,t-\tau|x,t)$ [33]. Thus, we obtain

$$\left[h^{(0)}(x) + \int R(x,x_\tau)P_{\rm st}^{(0)}(x_\tau,t-\tau|x,t)dx_\tau\right]P_{\rm st}^{(1)}(x) = \frac{1}{2}\frac{d}{dx}g^2(x)P_{\rm st}^{(1)}(x). \tag{20}$$

At this stage, we can exploit the definition of R as indicated in Eq. (17) in order to simplify Eq. (20):

$$\int h(x,x_\tau)P_{\rm st}^{(0)}(x_\tau,t-\tau|x,t)dx_\tau P_{\rm st}^{(1)}(x) = \frac{1}{2}\frac{d}{dx}g^2(x)P_{\rm st}^{(1)}(x). \tag{21}$$

Comparing Eqs. (21) and (15), we see that we used the intermediate steps given by Eqs. (16)–(20), to find a good approximation of the integral kernel occurring in Eq. (15). Since $P_{\rm st}^{(0)}(x_\tau,t-\tau|x,t)$ is the conditional probability density of the unperturbed problem (16), we are dealing with Markovian stationary transition probability density that only depends on a time differences (i.e., on τ) and is invariant under time inversion: $P_{\rm st}^{(0)}(x_\tau,t-\tau|x,t)=P_{\rm st}^{(0)}(x_\tau,t+\tau|x,t)$ [34]. Since we are interested only in small time delays, we can use the short time propagator of Eq. (16) given by [30]

$$P_{\rm st}^{(0)}(x_\tau,t+\tau|x,t) = \sqrt{\frac{1}{2\pi g^2(x)\tau}}\exp\left(-\frac{[x_\tau - x - h^{(0)}(x)\tau]^2}{2g^2(x)\tau}\right), \tag{22}$$

which (as mentioned before) also corresponds to $P_{\rm st}^{(0)}(x_\tau,t-\tau|x,t)$. Consequently, Eq. (21) becomes

$$h_{\rm eff}(x)P_{\rm st}^{(1)}(x) = \frac{1}{2}\frac{d}{dx}g^2(x)P_{\rm st}^{(1)}(x) \tag{23}$$

with

$$h_{\rm eff}(x) = \sqrt{\frac{1}{2\pi g^2(x)\tau}}\int_{-\infty}^{\infty} h(x,x_\tau) \times\exp\left(-\frac{[x_\tau - x - h^{(0)}(x)\tau]^2}{2g^2(x)\tau}\right)dx_\tau. \tag{24}$$

From Eq. (23) it follows that the stationary distribution $P_{\rm st}^{(1)}(x)$ reads

$$P_{\rm st}^{(1)}(x) = \frac{1}{Zg^2(x)}\exp\left(2\int^x \frac{h_{\rm eff}(x')}{g^2(x')}dx'\right), \tag{25}$$

where Z is a normalization constant. For stochastic delay differential equations that involve additive noise sources and can be cast into the form

$$\frac{d}{dt}X(t) = h(X(t),X(t-\tau)) + \sqrt{Q}\Gamma(t) \tag{26}$$

where $Q>0$ is the noise amplitude, we obtain

$$P_{\rm st}^{(1)}(x) = \frac{1}{Z}\exp\left(-\frac{2V_{\rm eff}(x)}{Q}\right), \tag{27}$$

where Z is again a normalization constant and the effective potential $V_{\rm eff}$ is given by

$$V_{\rm eff}(x) = \sqrt{\frac{1}{2\pi Q\tau}}\int^x dx'\int_{-\infty}^{\infty} dx_\tau h(x',x_\tau) \times\exp\left(-\frac{[x_\tau - x' - h^{(0)}(x')\tau]^2}{2Q\tau}\right). \tag{28}$$

That is, the small delay approximations $P_{\rm st}^{(1)}$ correspond to Boltzmann distributions.

For a stochastic delay differential equation (1) that exhibits a time-delayed variable in the amplitude function $g(x,x_\tau)$, we can proceed as in the previous case $g(x,x_\tau)=g(x)$. A detailed calculation (see Appendix A) yields the probability density

$$P_{\rm st}^{(1)}(x) = \frac{1}{ZD_{\rm eff}(x)}\exp\left(\int^x \frac{h_{\rm eff}(x')}{D_{\rm eff}(x')}dx'\right), \tag{29}$$

where Z is another normalization constant and the effective drift and diffusion coefficients $h_{\rm eff}(x)$ and $D_{\rm eff}(x)$ are described by

$$h_{\rm eff}(x) = \sqrt{\frac{1}{4\pi D^{(0)}(x)\tau}}\int_{-\infty}^{\infty} h(x,x_\tau) \times\exp\left(-\frac{[x_\tau - x - h^{(0)}(x)\tau]^2}{4D^{(0)}(x)\tau}\right)dx_\tau,$$

$$D_{\rm eff}(x) = \sqrt{\frac{1}{4\pi D^{(0)}(x)\tau}}\int_{-\infty}^{\infty} g(x,x_\tau) \times\exp\left(-\frac{[x_\tau - x - h^{(0)}(x)\tau]^2}{4D^{(0)}(x)\tau}\right)dx_\tau \tag{30}$$

with $D^{(0)}(x)=g^2(x,x)/2$. Next, let us consider an important special case, namely, stochastic systems that involve linear time-delayed feedback loops.

2. Probability density approach: Linear time-delayed feedback loops

We assume now that we are dealing with systems that can be described in terms of stochastic delay differential equations of the form

$$\frac{d}{dt}X(t) = \tilde{h}((X(t)) + \kappa X(t-\tau) + g(x)\Gamma(t). \tag{31}$$

Here, we have $h(x,x_\tau)=\tilde{h}(x)+\kappa x_\tau$ and $h^{(0)}(x)=\tilde{h}(x)+\kappa x$. The effective drift (24) reads

$$\begin{aligned} h_{\rm eff}(x) &= \sqrt{\frac{1}{2\pi g^2(x)\tau}}\int_{-\infty}^{\infty} dx_\tau h(x',x_\tau) \times\exp\left(-\frac{[x_\tau - x' - h^{(0)}(x')\tau]^2}{2g^2(x)\tau}\right) \\ &= \tilde{h}(x') + \kappa\sqrt{\frac{1}{2\pi g^2(x)\tau}}\int_{-\infty}^{\infty} dx_\tau x_\tau \times\exp\left(-\frac{[x_\tau - x' - h^{(0)}(x')\tau]^2}{2g^2(x)\tau}\right) \\ &= \tilde{h}(x') + \kappa[x' + h^{(0)}(x')\tau] = \tilde{h}(x') + \kappa x' + \kappa\tau h^{(0)}(x') \\ &= (1+\kappa\tau)h^{(0)}(x'). \end{aligned} \tag{32}$$

From Eq. (25) we read off that the stationary distributions $P_{\rm st}^{(1)}$ are given by

$$P_{\rm st}^{(1)}(x) = \frac{1}{Zg^2(x)}\exp\left(2(1+\kappa\tau)\int^x \frac{h^{(0)}(x')}{g^2(x')}dx'\right). \tag{33}$$

In particular, for systems with additive noise sources [i.e., for $g(x)=\sqrt{Q}$], we obtain Boltzmann distributions (27) with effective potentials

$$V_{\rm eff}(x) = -(1+\kappa\tau)\int^x h^{(0)}(x')dx'. \tag{34}$$

As a first example, one may consider the linear stochastic delay differential equation $dX(t)/dt=-aX(t)-bX(t-\tau)+\sqrt{Q}\Gamma(t)$ for which an exact analytical solution of $P_{\rm st}(x)$ exists. In Appendix B it is shown in detail how the perturbation theoretical approach can be applied to this kind of system and the analytical solution of $P_{\rm st}(x)$ is reproduced in the limit of small time delays.

Next, let us study systems that evolve in double-well potentials and are subjected to time-delayed linear feedback loops. Note that such systems have recently attracted considerable attention [5,27]. To begin with, we assume that the system dynamics is given by

$$\frac{d}{dt}X(t) = aX(t) - bX^3(t) + cX(t-\tau) + \sqrt{Q}\Gamma(t) \tag{35}$$

with $a>0, b>0$. The parameter c can assume arbitrary values. From Eq. (35) we read off that $h(x,x_\tau)=ax-bx^3+cx_\tau$ and $h^{(0)}(x)=(a+c)x-bx^3$. Substituting these functions into Eq. (28) we obtain $V_{\rm eff}$. Thus, the first order approximation $P_{\rm st}^{(1)}(x)$ for the bistable system (35) is given by (27) with

$$V_{\rm eff}(x) = (1+c\tau)\left(\frac{bx^4}{4} - (a+c)\frac{x^2}{2}\right). \tag{36}$$

Figure 1 shows $P_{\rm st}^{(1)}(x)$ as computed from Eqs. (27) and (36) and as obtained by solving the model (35) numerically.

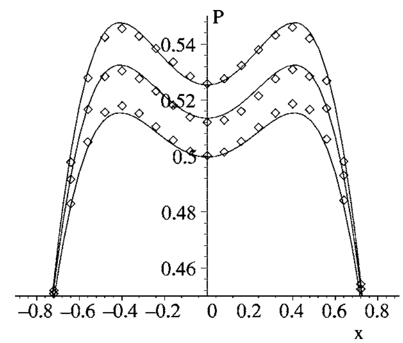

FIG. 1. Solid lines represent stationary probability densities $P_{\rm st}^{(1)}(x)$ of the double-well potential model (35) computed from Eqs. (27) and (36) for $\tau=0$, 0.05, and 0.1 (from top to bottom). Other parameters: $a=3.0, b=3.0, c=-2.5, Q=1.0$. Diamonds represent exact stationary distributions $P_{\rm st}(x)$ of Eq. (35) obtained by solving Eq. (35) numerically using an Euler forward scheme [30] in combination with a Box-Muller algorithm for the realizations of the Langevin force (ensemble size $N=1.3\times10^8$, single time step $\Delta t=0.01$).

3. Stochastic delay differential equation approach

As suggested in [22], for stochastic delay differential equations of the form (1) with $g(x,x_\tau)=g(x)$, one may derive approximative stochastic differential equations that describe Markov diffusion processes. In order to derive these approximative evolution equations, one needs to state two hypotheses. The first is that the Taylor approximation $X(t-\tau)=x(t)-\tau\, dX(t)/dt+O(\tau^2)$ holds for the stochastic systems under consideration. The second hypothesis is that for the nonlinear drift function $h(x,x_\tau)$ a similar Taylor expansion holds:

$$h(X(t),X(t-\tau)) = h^{(0)}(x) - \tau \left.\frac{\partial}{\partial x_\tau}h(X(t),x_\tau)\right|_{x_\tau=X(t)}\frac{d}{dt}X(t) + O(\tau^2). \tag{37}$$

The stationary distributions obtained assume the form

$$P_{\rm st}^{(1)}(x) = \frac{1}{Zg^2(x)[1-\tau\,\partial h(x,x)/\partial x_\tau]^2} \times\exp\left(2\int^x \frac{h^{(0)}(x')}{g^2(x')[1-\tau\,\partial h(x',x')/\partial x_\tau]}dx'\right). \tag{38}$$

For systems involving linear time-delayed feedback loops, that is, for systems that are described by Eq. (31), we have $[1+\tau\partial h(x',x')/\partial x_\tau]=1+\kappa\tau$. Using $1/(1+\kappa\tau)=1-\tau\kappa+O(\tau^2)$, we see that Eq. (38) reduces to Eq. (33). That is, the probability density approach and the stochastic delay differential equation approach yield consistent results. However, for systems with drift functions $h(x,x_\tau)$ that involve nonlinear terms like $x^n x_\tau^m$ with $n\geqslant 1, m\geqslant 1$ we obtain different kinds of first order approximations. In particular, for an additive noise system (i.e., we have $g=\sqrt{Q}$) the probability density approach yields a Boltzmann distribution given by Eqs. (27) and (28). In contrast, for $g=\sqrt{Q}$ Eq. (38) reads

$$P_{\rm st}^{(1)}(x) = \frac{1}{Z[1-\tau\,\partial h(x,x)/\partial x_\tau]^2} \times\exp\left(\frac{2}{Q}\int^x \frac{h^{(0)}(x')}{[1-\tau\,\partial h(x',x')/\partial x_\tau]}dx'\right). \tag{39}$$

That is, a time-delayed system that exhibits additive noise is approximated by means of a system that does not involve a time delay but exhibits multiplicative noise. We may speculate where this discrepancy comes from. It is clear that the hypothesis (37) holds in the limit of vanishing noise, where $X(t)$ can be regarded as a continuously differentiable function. However, when the noise amplitude cannot be regarded as a small parameter then $X(t)$ is certainly not a continuously differentiable function and for nonlinear functions h we may have to add on the right hand side of Eq. (37) terms that are of the order τ and depend on the noise amplitude g^2. Let us illustrate this point for a drift term $h(x,x_\tau)=ax^n x_\tau^m$ with $n\geqslant 1, m\geqslant 1$. In this case, taking the average of Eq. (37) for the stationary case yields the relationship

$$\langle X^n(t)X^m(t-\tau)\rangle_{\rm st} = \langle x^{n+m}\rangle_{\rm st} + O(\tau^2). \tag{40}$$

The expectation value $\langle h(X(t),X(t-\tau))\rangle_{\rm st}$ is a continuously differentiable function with respect to τ. Therefore, in this case the Taylor expansion can be applied and yields

$$\langle X^n(t)X^m(t-\tau)\rangle_{\rm st} = \langle X^{n+m}(t)\rangle_{\rm st} + \underbrace{\tau \left.\frac{d}{dz}\langle X^n(t)X^m(t-z)\rangle_{\rm st}\right|_{z=0}}_{F} + O(\tau^2). \tag{41}$$

We see that the hypothesis (37) does not account for the linear term in τ proportional to F. Let us mention once again that this term will become small in the limit of vanishing noise amplitude [i.e., for $g^2\to 0$ we have $\tau F\propto O(2)$]. For finite noise amplitudes, however, the expression τF can make a significant contribution to the first order approximation. Let us illustrate this point by means of two examples.

First, let us consider the case $n=m=1$ addressed in [22]. As shown in [29(b)], we then obtain $F=-\langle g^2(X)\rangle_{\rm st}/2$. That is, we have

$$\langle X(t)X(t-\tau)\rangle_{\rm st} = \langle X^2\rangle_{\rm st} - \frac{1}{2}\tau\langle g^2(X)\rangle_{\rm st} + O(\tau^2). \tag{42}$$

In this case, the hypothesis (37) is inconsistent with (42). In fact, as shown in [22] the small delay approximation based on the stochastic delay differential equation becomes worse when the noise amplitude g^2 is increased. In sum, for stochastic delay differential equations involving a drift term $h\propto X(t)X(t-\tau)$, a more appropriate hypothesis would read

$$h(X(t),X(t-\tau)) = h^{(0)}(x) - \tau \left.\frac{\partial}{\partial x_\tau}h(X(t),x_\tau)\right|_{x_\tau=X(t)}\frac{d}{dt}X(t) + O(\tau g^2) + O(\tau^2). \tag{43}$$

Equation (43) is consistent with Eq. (42).

Second, let us consider the stochastic delay differential equation

$$\frac{d}{dt}X(t) = -\gamma X^2(t)X(t-\tau) + \sqrt{Q}\Gamma(t). \tag{44}$$

The probability density approach yields the Boltzmann distribution (27) with

$$V_{\rm eff}(x) = \frac{\gamma x^4}{4}\frac{1}{1+2\gamma\tau x^2/3}; \tag{45}$$

see Eq. (28). For $\tau=0$ the potential increases as x^4, whereas for $\tau\to\infty$ the potential behaves as x^2. As a result, when we increase τ, the potential becomes less attractive, the stationary distribution becomes wider, and the variance and all even moments of $P_{\rm st}^{(1)}$ increase as a function of τ. In contrast, the stochastic delay differential equation approach yields the stationary distribution

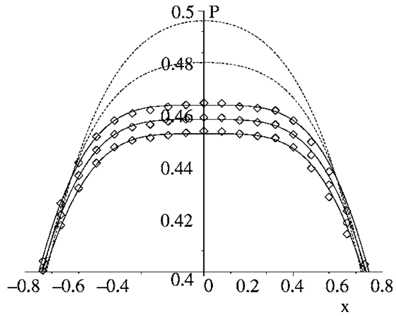

FIG. 2. Solid lines represent stationary probability densities $P_{\rm st}^{(1)}(x)$ of the model (44) computed from Eqs. (27) and (45) for $\tau=0$, 0.05, and 0.1 (from top to bottom). Dashed lines represent stationary probability densities $P_{\rm st}^{(1)}(x)$ of the model (44) computed from Eq. (46) for $\tau=0.05$ and 0.1 (from bottom to top). Other parameters: $\gamma=Q=1.0$. Diamonds represent exact stationary distributions $P_{\rm st}(x)$ obtained by solving Eq. (44) numerically (see also caption for Fig. 1).

$$P_{\rm st}^{(1)}(x)=\frac{1}{Z(1+\gamma\tau x^2)^2}\exp\left(-\frac{2V_{\rm eff}(x)}{Q}\right) \qquad (46)$$

with $V_{\rm eff}(x)$ given by Eq. (45) again [see Eq. (39)]. The question arises, how does the distribution (46) behave as a function of τ? To answer this question we consider the approximative Ito stochastic differential equation of Eq. (44), which reads

$$\frac{d}{dt}X(t)=-\gamma(1+\gamma\tau X^2)X^3+\sqrt{Q}(1+\gamma\tau X^2)\Gamma(t); \qquad (47)$$

see [22]. From the corresponding Fokker-Planck equation we see that in the stationary case the moments $\langle X^{2n}\rangle_{\rm st}$ satisfy

$$\langle X^4\rangle_{\rm st}+\gamma\tau\langle X^6\rangle_{\rm st}=\frac{Q}{2}\left(1+2\gamma\tau\langle X^2\rangle_{\rm st}+\gamma^2\tau^2\langle X^4\rangle_{\rm st}\right). \qquad (48)$$

The moments are functions of the parameter τ. Differentiating Eq. (48) with respect to τ at $\tau=0$, we get

$$\left.\frac{d}{d\tau}\langle X^4\rangle_{\rm st}\right|_{\tau=0}=\gamma[Q\langle X^2\rangle_{\rm st}-\langle X^6\rangle_{\rm st}]|_{\tau=0}. \qquad (49)$$

For $\gamma=Q=1$ we find $Q\langle X^2\rangle_{\rm st}-\langle X^6\rangle_{\rm st}<0$ for $\tau=0$. That is, the moment $\langle X^4\rangle_{\rm st}$ decreases when the delay is increased. In fact, the distribution predicted by the stochastic delay differential equation approach becomes smaller when τ is increased—as shown in Fig. 2. In sum, the probability density approach proposes that the distribution $P_{\rm st}^{(1)}$ becomes wider for increasing delay, whereas the stochastic delay differential equation approach proposes that the stationary distribution $P_{\rm st}^{(1)}$ becomes smaller. As shown in Fig. 2 the numerical solution of Eq. (47) clearly indicates that the distribution becomes wider with increasing delay. Moreover, there is a good match between the numerically and analytically obtained results when the probability density approach is applied.

TABLE I. Stochastic delay differential equation (SDDE) approach versus probability density function (PDF) approach.

h	g^2	SDDE	PDF
Linear in x_τ	$g^2(x)$	×	×
Nonlinear in x_τ	$g^2(x)\to 0$	×	×
Nonlinear in x_τ	$g^2(x)\propto O(0)$	?	×
Nonlinear in x_τ	$g^2(x,x_\tau)$		×

III. CONCLUSIONS

The objective of the present study was to improve our understanding of time-delayed stochastic systems, in general, and delay Fokker-Planck equations, in particular. In this regard, we have shown that delay Fokker-Planck equations can be derived by means of Novikov's theorem—just like ordinary Fokker-Planck equations. This result puts the theory of delay Fokker-Planck equations on an equal footing with the theory of ordinary Fokker-Planck equations. On the basis of delay Fokker-Planck equations, we have derived stationary distribution functions that describe the stationary states of time-delayed stochastic systems involving small time delays. For systems involving additive noise sources these stationary distributions correspond to Boltzmann distributions.

In a previous work, time-delayed stochastic systems have been studied in the small delay limit as well using Markov approximations of stochastic delay differential equations [22]. For systems involving linear time-delayed feedback loops the distributions derived by means of this alternative approach correspond to the distributions derived in our approach. However, if we deal with evolution equations with nonlinear terms of time-delayed variables, both approaches yield different results. We have argued that the reason for this discrepancy might be the fact that the Taylor expansion of nonlinear functions that is used in the alternative approach neglects particular first order terms. These first order terms scale with the noise amplitude and vanish in the limit of vanishing noise amplitudes. In two examples, we have explicitly demonstrated this issue. In addition, we would like to note that the approach via approximative stochastic differential equations cannot deal with noise sources that depend on time-delayed variables. The reason for this is that the approximative stochastic differential equations would exhibit products of Langevin forces [i.e., terms like $\Gamma(t)\Gamma(t)$], which are difficult to handle. In sum, we may compare the approaches by means of probability density functions and approximation of stochastic delay differential equations (see Table I). First of all, both approaches can successfully be applied for systems with linear time-delayed feedback loops and for systems in the weak noise limit. For nonlinear systems with noise sources that cannot be regarded as small quantities, it is not clear whether or not approximative delay differential equations yield correct results. In the case of systems with noise amplitudes that depend on time-delayed variables, approximative delay differential equations cannot be applied.

APPENDIX A: GENERAL CASE $g=g(x,x_\tau)$

For $D(x,x_\tau)=g^2(x,x_\tau)/2$ Eq. (15) becomes

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}\int h(x,x_\tau)P(x,t;x_\tau,t-\tau)dx_\tau + \int \frac{\partial^2}{\partial x^2}D(x,x_\tau)P(x,t;x_\tau,t-\tau)dx_\tau. \quad \text{(A1)}$$

The decompositions $h(x,x_\tau)=h^{(0)}(x)+R(x,x_\tau)$ and $D(x,x_\tau)=D^{(0)}(x)+R'(x,x_\tau)$ with $D^{(0)}(x)=D(x,x)$ can be used to identify for small delays τ terms of zeroth and first order: $h^{(0)},D^{(0)}\propto O(0),R,R'\propto O(\tau)$. Accordingly, the Fokker-Planck equation of the unperturbed system is given by

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}h^{(0)}(x)P + \frac{\partial^2}{\partial x^2}D^{(0)}(x)P(x,t). \quad \text{(A2)}$$

The short time propagator reads

$$P_{\rm st}^{(0)}(x_\tau,t+\tau|x,t) = \sqrt{\frac{1}{4\pi D^{(0)}(x)\tau}}\exp\left(-\frac{[x_\tau - x - h^{(0)}(x)\tau]^2}{4D^{(0)}(x)\tau}\right). \quad \text{(A3)}$$

Substituting $P_{\rm st}(x,t;x_\tau,t-\tau)=P_{\rm st}^{(0)}(x_\tau,t-\tau|x,t)P_{\rm st}^{(1)}(x)$ into the stationary version of Eq. (A1), we obtain

$$h_{\rm eff}(x)P_{\rm st}^{(1)}(x) = \frac{\partial}{\partial x}D_{\rm eff}(x)P_{\rm st}^{(1)}(x), \quad \text{(A4)}$$

where the effective drift and diffusion coefficients are described by Eq. (30). Solving Eq. (A4) for $P_{\rm st}^{(1)}(x)$ gives us Eq. (29).

APPENDIX B: LINEAR CASE

In the linear case we have

$$\frac{d}{dt}X(t) = -aX(t) - bX(t-\tau) + \sqrt{Q}\Gamma(t) \quad \text{(B1)}$$

with $h(x,x_\tau)=-ax-bx_\tau$ and $h^{(0)}(x)=-(a+b)x$. Equation (28) becomes

$$V_{\rm eff} = -(1-b\tau)(a+b)\frac{x^2}{2}. \quad \text{(B2)}$$

The stationary distribution (27) explicitly reads

$$P_{\rm st}^{(1)}(x) = \frac{1}{Z}\exp\left(-\frac{(1-b\tau)(a+b)}{Q}x^2\right) \quad \text{(B3)}$$

and can alternatively be expressed by

$$P_{\rm st}^{(1)}(x) = \frac{1}{\sqrt{2\pi\sigma^{2(1)}}}\exp\left(-\frac{x^2}{2\sigma^{2(1)}}\right), \quad \text{(B4)}$$

where $\sigma^{2(1)}$ denotes the variance of the first order approximation and is given by

$$\sigma^{2(1)} = \frac{Q}{2(1-b\tau)(a+b)} = \frac{(1+b\tau)Q}{2(a+b)} + O(\tau^2). \quad \text{(B5)}$$

Let us compare the first order approximation with the exact stationary distribution of Eq. (B1), which reads [20,24]

$$P_{\rm st}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{x^2}{2\sigma^2}\right). \quad \text{(B6)}$$

In order to write down the variance σ^2 of Eq. (B6), one needs to distinguish between the cases $b>a$, $a=b$, and $b>a$. The explicit expressions for σ^2 can be found in [20,24]. In the limit of small time delays, however, all three expressions reduce to

$$\sigma^2 = \frac{(1+b\tau)Q}{2(a+b)} + O(\tau^2). \quad \text{(B7)}$$

Comparing Eqs. (B4)–(B7), we see that $\sigma^{2(1)}$ and $P_{\rm st}^{(1)}$, respectively, correspond indeed to the correct first order approximations of the variance and the stationary distribution of the linear model (B1).

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