Short-time correlations of many-body systems described by nonlinear Fokker-Planck equations and Vlasov-Fokker-Planck equations

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Abstract

Analytical expressions for short-time correlation functions, diffusion coefficients, mean square displacement, and second order statistics of many-body systems are derived using a mean field approach in terms of nonlinear Fokker-Planck equations and Vlasov-Fokker-Planck equations. The results are illustrated for the Desai-Zwanzig model, the nonlinear diffusion equation related to the Tsallis statistics, and a Vlasov-Fokker-Planck equation describing bunch particles in particle accelerator storage rings.

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1 Introduction

Mean field theory is a fundamental approach to study stochastic properties of many-body systems. In particular, using mean field theory first order statis-

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tics of many-body systems can be determined in terms of stationary and transient distribution functions. Famous examples of dynamical mean field models are the Boltzmann equation [1], Vlasov equations [2–6], and nonlinear Fokker-Planck equations [7–26]. These equations have in common that they are nonlinear with respect to density measures. The nonlinearities reflect the interactions between the subsystems of the corresponding many-body systems.

While the first order statistics of dynamical mean field models has extensively been studied, comparatively little is known about the second order statistics and time correlation functions. In this context, Vlasov-Fokker-Planck equations [27–36], and nonlinear Fokker-Planck equations are of particular interest because they allow for an interpretation in terms of generalized self-consistent Langevin equations [23,37–41]. That is, trajectories of Brownian particles can be computed for this kind of dynamical mean field models. Consequently, as far as Vlasov-Fokker-Planck equations and nonlinear Fokker-Planck equations are concerned, second order statistics and time correlation functions can at least be determined numerically by solving appropriately defined Brownian dynamics models.

Analytical expressions for time correlation functions have been derived previously in the context of generalized fluctuation-dissipation theorems [42–44]. In addition, a hierarchy of differential equations has been derived involving time correlation functions of different order. In some cases this hierarchy is closed and can be solved, in others it cannot [45]. Finally, there are some special cases in which exact expressions for two time-point joint probability densities of dynamical mean field models can be derived, that is, in which analytical expressions for the second order statistics of dynamical mean field models can be found [14,46]. However, a general analytical discussion of the second order statistics of mean field models in terms of nonlinear Fokker-Planck equations and Vlasov-Fokker-Planck equations has not been carried out so far.

In the present study, we will derive analytical expressions for second order statistics and time correlation functions for short time differences. In doing so, we will also determine diffusion coefficients relevant on short time scales. The focus will be on dynamical mean field models given in terms of nonlinear Fokker-Planck equations (sections 2.1 and 2.2) and Vlasov-Fokker-Planck equations (sections 2.3 and 2.4).

2 Short-time correlation functions and second order statistics

2.1 Nonlinear Fokker-Planck equations: general case

In line with mean field theory, we describe many-body systems in terms of the behavior of a single subsystem (μ-space description). Accordingly, let $X(t) \in \Omega$ denote a time-dependent state variable of a subsystem of a many-body systems, where $X(t)$ is defined on a single subsystem phase space Ω at every time point t. We assume that $X(t)$ has distribution $u(x)$ at an initial time $t = t_0$ and denote that single subsystem probability density by $P(x, t; u) = \langle \delta(x - X(t)) \rangle$, where $\langle \cdot \rangle$ is an ensemble average and $\delta(\cdot)$ is the delta function. We further assume that $P(x, t; u)$ satisfies a nonlinear Fokker-Planck equation of the form

$$\frac{\partial}{\partial t} P(x, t; u) = -\frac{\partial}{\partial x} D_1(x, P) P + \frac{\partial^2}{\partial x^2} D_2(x, P) P \ . \quad (1)$$

In what follows, the focus will be on many-body systems that can be described in terms of strongly nonlinear Fokker-Planck equations [14,41,47], which means that the transition probability density $P(x, t|x', t'; u)$ of $X(t)$ satisfies

$$\frac{\partial}{\partial t} P(x, t|x', t'; u) = \\ -\frac{\partial}{\partial x} D_1(x, P(x, t)) P(x, t|x', t'; u) + \frac{\partial^2}{\partial x^2} D_2(x, P(x, t)) P(x, t|x', t') \ . \quad (2)$$

In particular, in the stationary case, we have $P(x,t;u) = P_{\rm st}(x)$ and

$$\frac{\partial}{\partial t} P_{\rm st}(x,t|x',t';u=P_{\rm st}) = \left[-\frac{\partial}{\partial x} D_1(x,P_{\rm st}(x)) + \frac{\partial^2}{\partial x^2} D_2(x,P_{\rm st}(x)) \right] P_{\rm st}(x,t|x',t';P_{\rm st}) \ . \qquad (3)$$

2.1.1 *Natural boundary conditions*

Assuming natural boundary condition and $\Omega = \mathbb{R}$, the transition probability density $P_{\rm st}(x,t'+\Delta t|x',t';P_{\rm st})$ for small time differences Δt is determined by the short-time propagator $P^{(0)}(x,\Delta t|x')$ like $P_{\rm st}(x,t'+\Delta t|x',t';P_{\rm st}) = P^{(0)}(x,\Delta t|x') + O(\Delta t^2)$ with [48–50]

$$P^{(0)}(x,\Delta t|x') = \sqrt{\frac{1}{4\pi D_2(x',P_{\rm st}(x'))\Delta t}} \exp\left\{ -\frac{[x-x'-D_1(x',P_{\rm st}(x'))\Delta t]^2}{4D_2(x',P_{\rm st}(x'))\Delta t} \right\} \ . \qquad (4)$$

Consequently, the second order statistics of the many-body systems under consideration can be computed from

$$P_{\rm st}(x,t'+\Delta t;x',t';P_{\rm st}) = P^{(0)}(x,\Delta t|x')\, P_{\rm st}(x') + O(\Delta t^2) \ . \qquad (5)$$

Note that from this result for arbitrary functions $f(x,y)$ the correlation functions $\langle f(X(t+\Delta t),X(t))\rangle$ can be determined up to terms of order Δt^2. In particular, autocorrelation functions such as $C_{mn} = \langle X^n(t+\Delta t)X^m(t)\rangle_{\rm st}$ can be computed from Eq. (5) for Δt small. Let us dwell on the case $m = n = 1$ with

$$C(\Delta t) = \langle X(t+\Delta t)X(t)\rangle_{\rm st} = \int_\Omega \int_\Omega xx' P_{\rm st}(x,t'+\Delta t;x',t';P_{\rm st})\, {\rm d}x\, {\rm d}x' \ . \qquad (6)$$

First, we note that the relation $\int_\Omega x\, P^{(0)}(x,\Delta t|x')\, {\rm d}x = x' + D_1(x,P_{\rm st})\Delta t$ holds because the short-time propagator is a Gaussian function. Consequently, we get $C(\Delta t) = \langle X^2 \rangle_{\rm st} + \Delta t \int_\Omega x D_1(x, P_{\rm st}) P_{\rm st}(x)\,{\rm d}x + O(\Delta t^2)$. Multiplying Eq. (1) in the stationary case with x^2 and integrating with respect to x, gives us

$$\int_\Omega x D_1(x, P_{\rm st}) P_{\rm st}(x)\,{\rm d}x = -\int_\Omega D_2(x, P_{\rm st}) P_{\rm st}(x)\,{\rm d}x \ . \tag{7}$$

This implies that

$$C(\Delta t) = C(0) - \langle D_2 \rangle_{\rm st} \Delta t + O(\Delta t^2) \ . \tag{8}$$

Accordingly, the short-time autocorrelation function $C^{(0)}(\Delta t)$ is simply given by $C^{(0)}(\Delta t) = C(0) - \langle D_2 \rangle_{\rm st} \Delta t$.

Of particular interest is the statistics of increments $\delta X(\Delta t) = X(t+\Delta t) - X(t)$. In general, the mean square displacement $\langle \delta X(\Delta t)^2 \rangle_{\rm st} = \langle [X(t+\Delta t) - X(t)]^2 \rangle_{\rm st}$ is related to the autocorrelation function $C(\Delta t)$ like $\langle \delta X(\Delta t)^2 \rangle_{\rm st} = 2[C(0) - C(\Delta t)]$. Consequently, from Eq. (8) it follows that

$$\left\langle \delta X(\Delta t)^2 \right\rangle_{\rm st} = 2 \langle D_2 \rangle_{\rm st} \Delta t + O(\Delta t^2) \tag{9}$$

We may define the time-dependent (or running) diffusion coefficient $D^{(z)}$ on the basis of the mean square displacement $\langle \delta X(\Delta t)^2 \rangle$ like $\langle \delta X(\Delta t)^2 \rangle = \int_0^{\Delta t} D^{(z)}\,{\rm d}z$ [51,52], which implies that the short-time diffusion coefficient is given by $D^{(0)} = \frac{1}{2} \lim_{\Delta t \to 0} \langle \delta X(\Delta t)^2 \rangle_{\rm st} / \Delta t$. Then, we find

$$D^{(0)} = \langle D_2(x, P_{\rm st}) \rangle_{\rm st} \ . \tag{10}$$

2.1.2 *Special case: additive noise*

For the special case $D_2 = Q$ the results derived above can be obtained in an alternative fashion. We would like to mention this second approach because it is this second approach that can also be generalized in order to treat

Vlasov-Fokker-Planck equations (see Sec. 2.3). Our departure point is the self-consistent Langevin

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = D_1(x, P_{\mathrm{st}})|_{x=X(t)} + \sqrt{Q}\,\Gamma(t) \tag{11}$$

corresponding to the strongly nonlinear Fokker-Planck equation (1) [14,41,47]. Here, $\Gamma(t)$ denotes a Langevin force [48] normalized with respect to the delta function like $\langle \Gamma(t)\Gamma(t')\rangle = 2\delta(t-t')$. Note that the mean of $\Gamma(t)$ vanishes. Moreover, $\Gamma(t)$ is an additive fluctuating force which implies that if we average over realizations of $\Gamma(t)$ under the condition that the realization of $X(t)$ satisfy some constraints, then the average vanishes as well. In short, due to the additivity of $\Gamma(t)$ all conditional averages of $\Gamma(t)$ involving constraints with respect to $X(t)$ vanish [53]. Now let $\langle\cdot\rangle|_{X(t)=x}$ denote the average over all realization of a statistical ensemble given that $X(t)$ equals x at time t. Then, since $\langle\Gamma(t)\rangle|_{X(t)=x} = 0$ vanishes as discussed before, from Eq. (11) it follows that

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}X(t)\right\rangle\bigg|_{X(t)=x} = D_1(x, P_{\mathrm{st}}) \ . \tag{12}$$

Next, recall that a differential equation of the form $\mathrm{d}A(t)/\mathrm{d}t = f(t)$ can be transformed into $A(t+\Delta t) = A(t) + \Delta t f(t) + O(\Delta t^2)$. Consequently, Eq. (12) can be transformed into

$$\langle X(t+\Delta t)\rangle|_{X(t)=x} = x + D_1(x, P_{\mathrm{st}}(x))\Delta t + O(\Delta t^2) \ . \tag{13}$$

Multiplying this relation with $xP_{\mathrm{st}}(x)$ and integrating with respect to x, gives us $C(\Delta t) = \langle X^2\rangle_{\mathrm{st}} + \Delta t \int_\Omega x D_1(x, P_{\mathrm{st}}) P_{\mathrm{st}}(x)\,\mathrm{d}x + O(\Delta t^2)$ again because of $\langle X(t+\Delta t)X(t)\rangle_{\mathrm{st}} = \int_\Omega x\,\langle X(t+\Delta t)\rangle|_{X(t)=x}\,P_{\mathrm{st}}(x)\,\mathrm{d}x$. As argued in Sec. 2.1.1 this leads to Eqs. (8,...,10).

2.1.3 Reflective and mixed boundary conditions

In order to emphasize the relationship (10) between short-time diffusion coefficients $D^{(0)}$ and intrinsic diffusion coefficients D_2 of many-body systems, we consider now many-body systems involving reflective boundary conditions with $\Omega = [a, b]$ (and $a < b$) and mixed boundary conditions with $\Omega = [a, \infty]$ or $\Omega = [-\infty, b]$, where a and b correspond to reflective boundaries, respectively.

First, we realize that Eq. (3) holds for joint probability density $P_{\rm st}(x, t; x', t'; P_{\rm st})$ in form of

$$\frac{\partial}{\partial \Delta t} P_{\rm st}(x, t' + \Delta t; x', t'; P_{\rm st}) = \\ \left[-\frac{\partial}{\partial x} D_1(x, P_{\rm st}) + \frac{\partial^2}{\partial x^2} D_2(x, P_{\rm st}) \right] P_{\rm st}(x, t' + \Delta t; x', t'; P_{\rm st}) \tag{14}$$

If we multiply Eq. (14) with x and x' and integrate with respect to x and x', we obtain

$$\frac{\rm d}{{\rm d}\Delta t} C(\Delta t) = \int_\Omega \int_\Omega x' D_1(x, P_{\rm st}) P_{\rm st}(x, t' + \Delta t; x', t'; P_{\rm st})\, {\rm d}x\, {\rm d}x' \tag{15}$$

for $\Delta t > 0$. In the limit $\Delta t \to 0+$ (limit from above), we get

$$\left. \frac{\rm d}{{\rm d}\Delta t} C(\Delta t) \right|_{\Delta t \to 0+} = \int_\Omega x D_1(x, P_{\rm st}) P_{\rm st}(x)\, {\rm d}x \ . \tag{16}$$

Since Eq. (7) also holds for the aforementioned reflective and mixed boundary conditions, Eq. (16) can be written as

$$\left. \frac{\rm d}{{\rm d}\Delta t} C(\Delta t) \right|_{\Delta t \to 0+} = - \langle D_2 \rangle_{\rm st} \tag{17}$$

(see also [54]). From Eq. (17) we conclude again that Eqs. (8,...,10) hold.

2.2.1 Desai-Zwanzig model

We first illustrate our results for the Desai-Zwanzig model [55]. Accordingly, we consider a many-body systems with a single subsystem probability densities that is defined on $\Omega = \mathbb{R}$ and satisfies

$$\frac{\partial}{\partial t} P(x,t;u) = -\frac{\partial}{\partial x}\left(ax - bx^3 - \kappa(x - \langle X \rangle)\right) P + Q\frac{\partial^2}{\partial x^2} P \tag{18}$$

for $a, b, \kappa, Q > 0$. The drift and diffusion coefficient read $D_1(x, \langle X \rangle) = ax - bx^3 - \kappa(x - \langle X \rangle)$ and $D_2 = Q$, respectively. The stationary probability density is given by

$$P_{\mathrm{st}}(x) = \frac{1}{Z} \exp\left\{-\frac{bx^4/4 - (a-\kappa)x^2/2 - \kappa x m}{Q}\right\} \ , \tag{19}$$

where Z is normalization constant and m is the order parameter $m = \langle X \rangle_{\mathrm{st}}$. The order parameter m can be determined by means of the self-consistency equation $m = \int_\Omega x P_{\mathrm{st}}(x)\,\mathrm{d}x$. For κ/Q larger than a critical value the system is bistable and one finds $\langle X \rangle_{\mathrm{st}} = m$ with $m \neq 0$ [55].

Using Eqs. (4), (5), and (19), the second order statistic is found as

$$P_{\mathrm{st}}(x, t' + \Delta t; x', t'; m) = \frac{\exp\left\{-\frac{[x - x' + \Delta t(bx'^3 - (a-\kappa)x' - \kappa m)]^2 + \Delta t[bx'^4 - 2(a-\kappa)x'^2 - 4\kappa m x']}{4Q\Delta t}\right\}}{Z\sqrt{4\pi Q \Delta t}} + O(\Delta t^2) \ . \tag{20}$$

Eq. (20) provides us with an analytical expression for the short-time second order statistics of order Δt. Since we are dealing with a many-body system driven by additive noise, the short-time diffusion coefficient $D^{(0)}$ reads $D^{(0)} = Q$ (see Eq. (10)). Consequently, $\langle \delta X^2(\Delta t) \rangle$ is given by $\langle \delta X^2(\Delta t) \rangle \approx 2Q\Delta t$

for small Δt. Fig. 1 illustrates $\langle \delta X^2(\Delta t) \rangle$ as function of Δt as obtained from $\langle \delta X^2(\Delta t) \rangle = 2Q\Delta t$ and as obtained from a numerical simulation of the Desai-Zwanzig model (18).

Insert Fig. 1 about here

2.2.2 Plastino-Plastino model

A benchmark model in the theory of nonlinear Fokker-Planck equations is the model proposed by Plastino and Plastino [56] that is closely related to the generalized nonextensive entropy proposed by Tsallis [57–59] and to diffusion processes through porous media [38,60–62]. According to the Plastino-Plastino model the single subsystem probability density evolves like

$$\frac{\partial}{\partial t} P(x,t;u) = \frac{\partial}{\partial x} \gamma x P(x,t;u) + Q \frac{\partial^2}{\partial x^2} P^q \tag{21}$$

for $\gamma > 0$ and $\Omega = \mathbb{R}$. The stationary probability density reads

$$P_{\rm st}(x) = \frac{D_{\rm st}}{[1 + \gamma D_{\rm st}^{1-q}(1-q)x^2/(2qQ)]^{1/(1-q)}} \tag{22}$$

and corresponds to a classical solution with finite variance for $q \in (1/3,1)$ Here, $D_{\rm st}$ is a normalization constant defined by $D_{\rm st} = [\gamma/(2qQz_q^2)]^{1/(1+q)}$ and $z_q = \sqrt{\pi/(1-q)}\Gamma[0.5(1+q)/(1-q)]/\Gamma[1/(1-q)]$ [63]. In the stationary case, we have $D_1 = -\gamma x$, $D_2 = QD_{\rm st}^{q-1}/[1+\gamma D_{\rm st}^{1-q}(1-q)x^2/(2qQ)]^{1/(1-q)}$, and transition probability densities satisfy the multiplicative noise Fokker-Planck equation [38,45]

$$\begin{aligned} \frac{\partial}{\partial t} P_{\rm st}(x,t|x',t';P_{\rm st}) = & \frac{\partial}{\partial x} \gamma x P_{\rm st}(x,t|x',t';u) \\ & + QD_{\rm st}^{q-1} \frac{\partial^2}{\partial x^2} \left[1 + \frac{\gamma D_{\rm st}^{1-q}(1-q)x^2}{2qQ} \right] P_{\rm st}(x,t|x',t';P_{\rm st}) \ . \end{aligned} \tag{23}$$

Note that in the limit $q \to 1$ the Plastino-Plastino model given by Eqs. (21) and (23) reduces to the Fokker-Planck equation of an Ornstein-Uhlenbeck process and the stationary distribution (22) becomes a Gaussian distribution.

Using Eqs. (4), (5), and (22), the second order statistic of the Plastino-Plastino model can be computed from

$$P_{\rm st}(x,t'+\Delta t;x',t';P_{\rm st}) = \frac{\exp\left\{-\frac{[x-x'+\gamma x'\Delta t]^2}{4QD_{\rm st}^{q-1}[1+\gamma D_{\rm st}^{1-q}(1-q)[x']^2/(2qQ)]^{1/(1-q)}\Delta t}\right\}}{\sqrt{4\pi Q D_{\rm st}^{q+1}\Delta t}\left[1+\gamma D_{\rm st}^{1-q}(1-q)[x']^2/(2qQ)\right]^{(3-q)/[2(1-q)]}} + O(\Delta t^2)\ . \quad (24)$$

From Eq. (10) it follows that the short-time diffusion constant is given by

$$D^{(0)} = \langle D_2\rangle_{\rm st} = -\langle X D_1\rangle_{\rm st} = \gamma\left\langle X^2\right\rangle_{\rm st} = \gamma K_{\rm st}\ , \quad (25)$$

where $K_{\rm st}$ denotes the variance of the stationary distribution such that

$$\left\langle \delta X(\Delta t)^2\right\rangle = 2\gamma K_{\rm st}\Delta t + O(\Delta t^2)\ . \quad (26)$$

Here, $K_{\rm st}$ is defined by $K_{\rm st} = [2qQz_q^{(1-q)}/\gamma]^{2/(1+q)}/(3q-1)$ [45]. Let us compare this result with the exact expression for the mean square displacement. For the Plastino-Plastino model the autocorrelation function for arbitrary Δt can be found as $C(\Delta t) = K_{\rm st}\exp\{-\gamma\Delta t\}$ [45]. Consequently, we have

$$\left\langle \delta X(\Delta t)^2\right\rangle = 2K_{\rm st}(1-\exp\{-\gamma\Delta t\}) \quad (27)$$

We see that Eq. (26) indeed describes the correct linear approximation of the exact result (27) for small values of Δt.

2.3 Vlasov-Fokker-Planck equations: general case

Having discussed univariate nonlinear Fokker-Planck equations that can be cast into the generic form (1), we turn now to a generic class of multivariate nonlinear Fokker-Planck equations: nonlinear Fokker-Planck equations of the Vlasov-type. In order to grasp the essential features of Vlasov-Fokker-Planck equations, we assume that the dynamics of a single subsystem is defined on a two-dimensional phase space $\Omega = \mathbb{R}^2$ spanned by the generalized momentum p and position q of the subsystem. Let $\rho(p,q,t)$ denote the time-dependent subsystem density. The total single subsystem Hamiltonian $H = H_0 + H_{\rm MF}$ (energy per subsystem) is composed of the free single subsystem Hamiltonian $H_0(p,q)$ and a subsystem-subsystem interaction Hamiltonian $H_{\rm MF}(q) = \int_\Omega V(q-q')\rho(q')\mathrm{d}q'$, where V is the interaction potential and $\rho(q,t) = \int \rho(p,q,t)\,\mathrm{d}p$. The Vlasov equation for the evolution of ρ reads [2–6]

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial q}\frac{\partial H}{\partial p}\rho - \frac{\partial}{\partial p}\frac{\partial H}{\partial q}\rho = 0 \ . \tag{28}$$

In order to account for damping and fluctuations due to the contact of the many-body system with its environment, we add on the right-hand side of Eq. (28) a Fokker-Planck collision operator [27–36], which gives us the Vlasov-Fokker-Planck equation

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial q}\frac{\partial H}{\partial p}\rho - \frac{\partial}{\partial p}\frac{\partial H}{\partial q}\rho = \gamma\frac{\partial}{\partial p}p\rho + D\frac{\partial^2}{\partial p^2}\rho \tag{29}$$

with $\gamma, D > 0$. As shown in [37,47], we may interpret the Vlasov-Fokker-Planck equation in terms of the Brownian dynamics model

$$\frac{\partial}{\partial t}q = \frac{\partial H_0}{\partial p} \ , \tag{30}$$

$$\frac{\partial}{\partial t}p = -\frac{\partial H_0}{\partial q} - M_0\frac{\partial}{\partial q}\int V(q-q')P(q',t)\,\mathrm{d}p'\,\mathrm{d}q' - \gamma p + \sqrt{D}\,\Gamma(t) \ , \tag{31}$$

where $P(p,q,t) = \langle \delta(p-p(t))\delta(q-q(t))\rangle$, $P(q,t) = \langle \delta(q-q(t))\rangle$, and $\rho(p,q,t) = M_0 \langle \delta(p-p(t))\delta(q-q(t))\rangle$. That is, $M_0 = \int_\Omega \int_\Omega \rho(p,q,t)\,\mathrm{d}p\,\mathrm{d}q$ corresponds to the total mass of the many-body system under consideration.

Since we are dealing with an additive noise source, we proceed now as in Sec. 2.1.2 by introducing the conditional average $\langle \cdot \rangle|_{p(t)=p_0,q(t)=q_0}$ (in short: $\langle \cdot \rangle|_{p_0,q_0}$). Then, from Eq. (30) it is clear that

$$\langle q(t+\Delta t)\rangle|_{p_0,q_0} = q_0 + \Delta t \frac{\partial H_0(p_0,q_0)}{\partial p} + O(\Delta t^2) \tag{32}$$

and

$$\begin{aligned}\langle p(t+\Delta t)\rangle|_{p_0,q_0} &= p_0 \\ +\Delta t &\left[-\frac{\partial H_0(p_0,q_0)}{\partial q} - M_0 \frac{\partial}{\partial q} \int V(q-q')P(q',t)\,\mathrm{d}q' \bigg|_{q_0} - \gamma p_0 \right] + O(\Delta t^2)\ .\end{aligned} \tag{33}$$

In the stationary case, we need to replace $P(q',t)$ in Eq. (33) by $P_{\mathrm{st}}(q')$. Multiplying Eqs. (32) and (33) with $q_0 P_{\mathrm{st}}(p_0,q_0)$ and $p_0 P_{\mathrm{st}}(p_0,q_0)$ and integrating with respect to p_0 and q_0, we obtain the correlation functions $\langle q(t+\Delta t)q(t)\rangle$, $\langle q(t+\Delta t)p(t)\rangle$, $\langle p(t+\Delta t)q(t)\rangle$, $\langle p(t+\Delta t)p(t)\rangle$ — just as in Sec. 2.1.2. The result is

$$\langle q(t+\Delta t)q(t)\rangle_{\mathrm{st}} = \left\langle q^2 \right\rangle_{\mathrm{st}} + \Delta t \left\langle q \frac{\partial H_0}{\partial p} \right\rangle_{\mathrm{st}} + O(\Delta t^2)\ , \tag{34}$$

$$\langle q(t+\Delta t)p(t)\rangle_{\mathrm{st}} = \langle p(t)q(t)\rangle_{\mathrm{st}} + \Delta t \left\langle p \frac{\partial H_0}{\partial p} \right\rangle_{\mathrm{st}} + O(\Delta t^2)\ , \tag{35}$$

$$\begin{aligned}\langle p(t+\Delta t)q(t)\rangle_{\mathrm{st}} &= \langle p(t)q(t)\rangle_{\mathrm{st}} - \Delta t \left[\left\langle q(t) \frac{\partial H'(p(t),q(t))}{\partial q} \right\rangle_{\mathrm{st}} + \gamma \langle p(t)q(t)\rangle_{\mathrm{st}} \right] \\ &\quad + O(\Delta t^2)\ ,\end{aligned} \tag{36}$$

$$\langle p(t+\Delta t)p(t)\rangle_{\mathrm{st}} = \left\langle p^2 \right\rangle_{\mathrm{st}} - \Delta t \left[\left\langle p(t) \frac{\partial H'(p(t),q(t))}{\partial q} \right\rangle_{\mathrm{st}} + \gamma \left\langle p^2 \right\rangle_{\mathrm{st}} \right] + O(\Delta t^2) \tag{37}$$

with $H' = H_0(p,q) + M_0 \int_\Omega V(q-q')P_{\rm st}(q')\,{\rm d}x$. In order to evaluate these relations, we first note that in the stationary case the Vlasov-Fokker-Planck equation (29) for $\rho_{\rm st}(p,q) = M_0 P_{\rm st}(p,q)$ reads

$$\frac{\partial}{\partial q}\frac{\partial H_0}{\partial p}P_{\rm st} - \frac{\partial}{\partial p}\frac{\partial H'}{\partial q}P_{\rm st} = \gamma\frac{\partial}{\partial p}pP_{\rm st} + D\frac{\partial^2}{\partial p^2}P_{\rm st}\ . \tag{38}$$

Multiplying Eq. (38) with q^2 and integrating with respect to p and q, we get $\langle q\partial H_0/\partial p\rangle = 0$ which implies

$$\langle q(t+\Delta t)q(t)\rangle_{\rm st} = \left\langle q^2\right\rangle_{\rm st} + O(\Delta t^2)\ . \tag{39}$$

Multiplying Eq. (38) with pq and integrating with respect to p and q, we get $\langle p\partial H_0/\partial p\rangle = \langle q\partial H'/\partial q\rangle + \gamma\langle pq\rangle_{\rm st}$ which implies that $\langle q(t+\Delta t)p(t)\rangle_{\rm st} = -\langle p(t+\Delta t)q(t)\rangle_{\rm st} + O(\Delta t^2)$. Note that the two correlation functions are not only equivalent up to terms of order Δt^2 but they are exactly the same functions. The reason for this is that p is an odd variable, whereas q is an even variable [48,49,64]. Therefore, if we reverse time the expression $\langle p(t)q(t')\rangle$ becomes $-\langle p(-t)q(-t')\rangle$. In the stationary case time reversal does not change the expectation value $\langle p(t)q(t')\rangle$, which means that $\langle p(t)q(t')\rangle_{\rm st} = -\langle p(-t)q(-t')\rangle_{\rm st}$. In particular, for $t = t'+\Delta t$ we have $\langle p(t'+\Delta t)q(t')\rangle_{\rm st} = \langle p(\Delta t)q(0)\rangle_{\rm st} = -\langle p(-\Delta t)q(0)\rangle_{\rm st} = -\langle p(0)q(\Delta t)\rangle_{\rm st} = -\langle p(t')q(t'+\Delta t)\rangle_{\rm st}$. As a result, we have

$$\langle q(t+\Delta t)p(t)\rangle_{\rm st} = -\langle p(t+\Delta t)q(t)\rangle_{\rm st} \tag{40}$$

(see also [48, Sec. 7.3]). Finally, multiplying Eq. (38) with p^2 and integrating with respect to p and q, we get $\langle p\partial H'/\partial q\rangle = -\gamma\langle p^2\rangle_{\rm st} + D$, which implies

$$\langle p(t+\Delta t)p(t)\rangle_{\rm st} = \left\langle p^2\right\rangle_{\rm st} - D\Delta t + O(\Delta t^2)\ . \tag{41}$$

Using Eqs. (39,...,41), one can show that the statistics of the increments $\delta p(\Delta t) = p(t+\Delta t) - p(t)$ and $\delta q(\Delta t) = q(t+\Delta t) - q(t)$ satisfies

$$\left\langle \delta q(\Delta t)^2 \right\rangle_{\rm st} = O(\Delta t^2) \ , \tag{42}$$

$$\langle \delta q(\Delta t) \delta p(\Delta t) \rangle_{\rm st} = O(\Delta t^2) \ , \tag{43}$$

$$\left\langle \delta p(\Delta t)^2 \right\rangle_{\rm st} = 2D\Delta t + O(\Delta t^2) \ . \tag{44}$$

Note that there is no linear term in Eq. (42) because of the aforementioned relationship $\langle q(t+\Delta t)p(t)\rangle_{\rm st} = -\langle p(t+\Delta t)q(t)\rangle_{\rm st}$.

2.4 Vlasov-Fokker-Planck equations: bunch-particle dynamics in storage rings

We apply the general results of the previous section to the particular case of the physical system represented by electron bunches in storage rings. Electron bunches induce electromagnetic fields, which act back on the particle bunches and are referred to as wakefields [65,66]. Using the Vlasov theory of many-body systems, these wakefields can be modeled in terms of subsystem-subsystem interaction Hamiltonians $H_{\rm MF}$. Taking damping due to radiation losses into account as well as fluctuations, we end up with a description of particle bunches in terms of Vlasov-Fokker-Planck equations (29) [27–36]. In what follows, we study a simple model for the bunch-particle dynamics in storage rings that has been discussed in more detail in [47]. Accordingly, we assume a quadratic free energy Hamiltonian $H_0(p,q) = p^2/2 + q^2/2$ and a short-ranged interaction potential $V(q-q') = \kappa M_0^{-1}\delta(q-q')$, where κ describes the strength of the impact of the wakefield and can assume both positive and negative values. In this case, the self-consistent Langevin equation given by Eqs. (30) and (31) becomes

$$\frac{\partial}{\partial t} q = p \ , \tag{45}$$

$$\frac{\partial}{\partial t} p = -q - \kappa \frac{\partial}{\partial q} P(q,t) - \gamma p + \sqrt{D}\,\Gamma(t) \ . \tag{46}$$

The stationary solution factorizes: $P_{\rm st}(p,q) = P_{\rm st}(p)P_{\rm st}(q)$. $P_{\rm st}(q)$ corresponds to a Haissinski distribution, whereas $P_{\rm st}(p)$ is a Gaussian distribution $P_{\rm st}(p) \propto \exp\{-\gamma p^2/(2D)\}$ with $\langle p \rangle_{\rm st} = 0$ and variance $\langle p^2 \rangle_{\rm st} = D/\gamma$ [47]. Note that for negative κ the Haissinski distribution $P_{\rm st}(q)$ does not necessarily exist. However, one can determine a boundary value $\kappa_c < 0$ such that for all $\kappa > \kappa_c$ the Haissinski distribution is well-defined [67,68]. Since we have $\langle p\partial H_0/\partial p \rangle_{\rm st} = \langle p^2 \rangle_{\rm st} = D/\gamma$ and $\langle pq \rangle_{\rm st} = 0$, from Eqs. (42,...,44) it follows that

$$\langle q(t+\Delta t)p(t) \rangle_{\rm st} = \frac{D}{\gamma}\Delta t + O(\Delta t^2) \ , \tag{47}$$

$$\langle p(t+\Delta t)p(t) \rangle_{\rm st} = \frac{D}{\gamma} - D\Delta t + O(\Delta t^2) \ , \tag{48}$$

$$\left\langle \delta p^2(\Delta t) \right\rangle_{\rm st} = 2D\Delta + O(\Delta t^2) \ . \tag{49}$$

As shown in Fig. 2 for small Δt the mean square displacement $\langle \delta p(\Delta t)^2 \rangle$ indeed behaves like $\langle \delta p(\Delta t)^2 \rangle \approx 2D\Delta t$.

Insert Fig. 2 about here

3 Conclusions

We have studied the second order statistics of many-body systems using mean field theory in terms of nonlinear diffusion equations such as nonlinear Fokker-Planck equations and Vlasov-Fokker-Planck equations. Analytical expressions for two-time point probability densities, time correlation functions, mean square displacements and diffusion coefficients have been derived for stationary systems on small time scales. In doing so, we have shown that for a large class of systems the mean square displacement on small time scales is proportional to the averaged diffusion coefficient of the relevant Brownian subsystem dynamics.

In particular, our analysis has shown that the mean square displacement of particle systems described by nonlinear diffusion equations increases linearly on small time scales. This is in contrast with the fact that the variance of intially delta-distributed particle distributions increase nonlinearly with respect to time in the case of systems described by the nonlinear diffusion equation [38,56,60–62]. In line with earlier considerations [41], we see from this example that in the context of the mean field approach to many-body systems we need to distinguish carefully between the transient behavior (as described by time-dependent single-time point probability densities) and the stationary transition dynamics (as described by stationary two-time point transition probability densities).

Moreover, we have seen that for short time differences subsystem-subsystem interactions in many-body systems described by Vlasov-Fokker-Planck equations do not determine the time correlation functions of momentum and position variables. We found that autocorrelation functions of momentum variables and cross-correlation functions of momentum and position variables are determined by damping and diffusion constants. That is, the dissipative part of the system dynamics dominates the stochastic behavior of the many-body systems in this time domain. Finally, we have found that autocorrelation functions of position variables are of second or higher order with respect to time-differences. Consequently, in this regard, many-body systems with interacting subsystems exhibit qualitatively the same behavior as many-body systems with non-interacting subsystems as described by the Kramers equation, where autocorrelation functions of position variables are related to time-differences Δt like Δt^3 [69].

References

[1] F. Reif, *Fundamentals of statistical and thermal physics* (McGraw-Hill Book Company, New York, 1965).

[2] R. A. Cairns, *Plasma physics* (Blackie, Philadelphia, 1985).

[3] Y. N. Dnestrovskii and D. P. Kostomarov, *Numerical simulations of plasmas* (Springer, Berlin, 1986).

[4] Y. L. Klimontovich, *Statistical physics* (Harwood Academic Publ., New York, 1986).

[5] D. R. Nicholson, *Introduction to plasma theory* (John Wiley and Sons, New York, 1983).

[6] F. M. Nekrasov, A. G. Elfimov, C. A. de Azevedo, and A. S. de Assis, Phys. Lett. A **251**, 44 (1999).

[7] L. Borland, Phys. Rev. Lett. **89**, 098701 (2002).

[8] L. L. Bonilla, J. Stat. Phys. **46**, 659 (1987).

[9] P. H. Chavanis, Phys. Rev. E **68**, 036108 (2003).

[10] P. H. Chavanis and C. Sire, Phys. Rev. E **69**, 016116 (2004).

[11] H. Daido, Phys. Rev. Lett. **87**, 048101 (2001).

[12] M. Doi and S. F. Edwards, *The theory of polymer dynamics* (Clarendon Press, Oxford, 1988).

[13] T. D. Frank, Physica D **189**, 199 (2004).

[14] T. D. Frank, Physica D **195**, 229 (2004).

[15] J. Garcia-Ojalvo and J. M. Sancho, *Noise in spatially extended systems* (Springer, New York, 1999).

[16] S. Hess, Z. Naturforschung A **31**, 1034 (1976).

[17] G. Kaniadakis and P. Quarati, Phys. Rev. E **48**, 4263 (1993).

[18] G. Kaniadakis, Phys. Lett. A **288**, 283 (2001).

[19] R. Kawai, X. Sailer, and L. Schimansky-Geier, Phys. Rev. E **69**, 051104 (2004).

[20] Y. Kuramoto, *Chemical oscillations, waves, and turbulence* (Springer, Berlin, 1984).

[21] M. Kostur, J. Luczka, and L. Schimansky-Geier, Phys. Rev. E **65**, 051115 (2002).

[22] J. H. Li and P. Hänggi, Phys. Rev. E **64**, 011106 (2001).

[23] H. C. Öttinger, *Stochastic processes in polymeric fluids* (Springer, Berlin, 1996).

[24] S. H. Strogatz, Physica D **143**, 1 (2000).

[25] P. A. Tass, *Phase resetting in medicine and biology - Stochastic modelling and data analysis* (Springer, Berlin, 1999).

[26] C. van den Broeck, J. M. R. Parrondo, and R. Toral, Phys. Rev. Lett. **73**, 3395 (1994).

[27] R. C. Davidson, H. Qin, and T. S. F. Wang, Phys. Lett. A **252**, 213 (1999).

[28] R. C. Davidson, H. Qin, and P. J. Channell, Phys. Lett. A **258**, 297 (1999).

[29] R. C. Davidson and H. Qin, *Physics of intense charged particle beam in high energy accelerator* (World Scientific, Singapore, 2001).

[30] Y. Shobuda and K. Hirata, Part. Accel. **62**, 165 (1999).

[31] Y. Shobuda and K. Hirata, Phys. Rev. E **64**, 67501 (2001).

[32] S. Heifets, Phys. Rev. E **54**, 2889 (1996).

[33] S. Heifets and B. Podobedov, Phys. Rev. ST-AB **2**, 044402 (1999).

[34] S. Heifets, Phys. Rev. ST-AB **4**, 044401 (2001).

[35] G. V. Stupakov, B. N. Breizman, and M. S. Pekker, Phys. Rev. E **55**, 5976 (1997).

[36] M. Venturini and R. Warnock, Phys. Rev. Lett. **89**, 224802 (2002).

[37] E. J. Allen and H. D. Victory Jr., Physica A **209**, 318 (1994).

[38] L. Borland, Phys. Rev. E **57**, 6634 (1998).

[39] A. N. Drozdov and M. Morillo, Phys. Rev. E **54**, 931 (1996).

[40] T. D. Frank, Physica A **301**, 52 (2001).

[41] T. D. Frank, Physica A **331**, 391 (2004).

[42] T. D. Frank, Phys. Lett. A **329**, 475 (2004).

[43] M. Hütter and H. C. Öttinger, Phys. Rev. E **54**, 2526 (1996).

[44] M. Shiino, Phys. Rev. A **36**, 2393 (1987).

[45] T. D. Frank, Eur. Phys. J. B **37**, 139 (2004).

[46] T. D. Frank, Physica A **340**, 251 (2004).

[47] T. D. Frank, Phys. Lett. A **319**, 173 (2003).

[48] H. Risken, *The Fokker-Planck equation — Methods of solution and applications* (Springer, Berlin, 1989).

[49] H. Haken, *Synergetics: Itroduction and Advanced topics* (Springer, Berlin, 2004).

[50] M. F. Wehner and W. G. Wolfer, Phys. Rev. A **27**, 2663 (1983).

[51] N. V. Brilliantov and T. Pöschel, Phys. Rev. E **61**, 1716 (2000).

[52] M. Vlad, F. Spineanu, J. H. Misguich, and R. Balescu, Phys. Rev. E **63**, 066304 (2001).

[53] T. D. Frank, R. Friedrich, and P. J. Beek, Stochastics and Dynamics, (2005), Time series analysis for multivariate time-delayed systems with noise: applications to laser physics and human movements, in press.

[54] T. D. Frank, Phys. Rev. E **69**, 061104 (2004).

[55] R. C. Desai and R. Zwanzig, J. Stat. Phys. **19**, 1 (1978).

[56] A. R. Plastino and A. Plastino, Physica A **222**, 347 (1995).

[57] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).

[58] C. Tsallis, Physica D **193**, 3 (2004).

[59] S. Abe and Y. Okamoto, *Nonextensive statistical mechanics and its applications* (Springer, Berlin, 2001).

[60] L. Borland, F. Pennini, A. R. Plastino, and A. Plastino, Eur. Phys. J. B **12**, 285 (1999).

[61] E. K. Lenzi, R. S. Mendes, and C. Tsallis, Phys. Rev. E **67**, 031104 (2003).

[62] C. Tsallis and D. J. Bukman, Phys. Rev. E **54**, R2197 (1996).

[63] T. D. Frank and A. Daffertshofer, Physica A **285**, 351 (2000).

[64] N. G. van Kampen, Physica **23**, 816 (1957).

[65] A. Chao, *Physics of collective instabilities in high energy accelerators* (John Wiley and Sons, New York, 1993).

[66] H. Wiedemann, *Particle accelerator physics II — Nonlinear and higher-order beam dynamics* (Springer, Berlin, 1993).

[67] M. Venturini, Phys. Rev. ST-AB **5**, 054403 (2002).

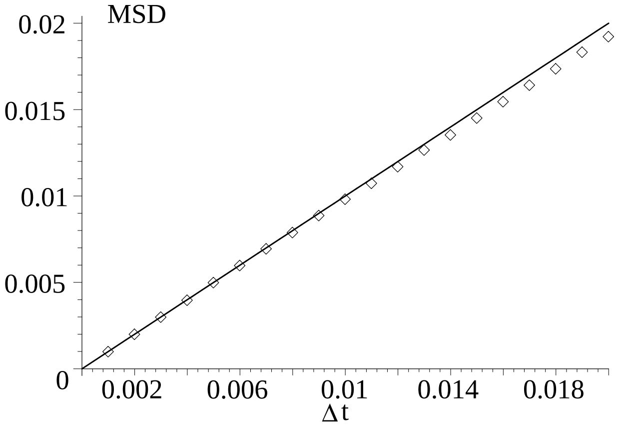
[68] C. Thomas, R. Bartonlini, J. I. M. Botman, G. Dattoli, L. Mezi, and M. Migliorati, Europhysics Letters **60**, 66 (2002).

[69] M. C. Wang and G. E. Uhlenbeck, Rev. Mod. Phys. **17**, 323 (1945).

Figure captions:

Fig. 1: Mean square displacement (MSD) $\langle \delta X(\Delta t)^2 \rangle_{\rm st}$ of the Desai-Zwanzig model (18) as a function of Δt in the bistable regime. Solid line: analytical expression $\langle \delta X(\Delta t)^2 \rangle_{\rm st} = 2Q\Delta t$ valid for short time differences Δt. Diamonds: exact result obtained by solving the Desai-Zwanzig model numerically by virtue of the corresponding self-consistent Langevin equation (11) using an Euler forward scheme (single time step 10^{-5}; number of realizations 30000; random numbers via Box-Muller). Parameters: $a = b = 1.0$, $\kappa = 2.0$, $Q = 0.5$. Initial distribution $u(x) = \delta(x-1)$. Before computing $\langle \delta X(\Delta t)^2 \rangle_{\rm st}$, the Euler forward scheme was iterated repeatedly until the system settled down in the stationary regime with order parameter $m \approx 0.7$.

Fig. 2: Mean square displacement $\langle \delta p(\Delta t)^2 \rangle_{\rm st}$ of the Brownian dynamics model given by Eqs. (45) and (46) as a function of Δt. Solid line: short-time approximation $\langle \delta p(\Delta t)^2 \rangle_{\rm st} = 2D\Delta t$. Diamonds: exact result obtained by solving the model numerically using the same simulation scheme as in [47] (single time step 0.005; number of realizations 5000). Parameters: $\gamma = 1.0$, $\kappa = 15.0$, $D = 5.0$.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0.01
0.005
0
0.002
0.006
0.01
0.014
0.018
Δt

Fig. 1.

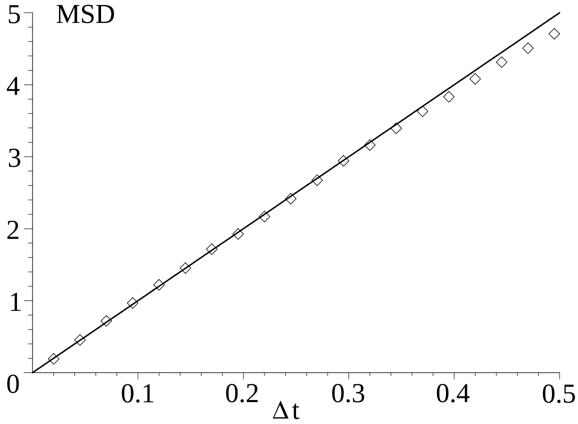
MSD
5
4
3
2
1
0
0.1
0.2
0.3
0.4
0.5
Δt

Fig. 2.