Identifying noise sources of time-delayed feedback systems

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Abstract

We propose a novel method to identify noise sources of stochastic systems with time delays. In particular, we demonstrate how to distinguish between additive and multiplicative noise sources and to determine the structure of parametric and multiplicative noise sources of time-delayed human motor control systems.

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1 Introduction

Dynamical systems that involve time delays have been studied in various disciplines ranging from laser physics to biophysics [1–6]. In particular, many biological systems exhibit time-delayed control mechanisms that arise due to the transport of matter, energy, and information with finite propagation velocities [7–14] (for a recent review see [15]). Delay systems in general and

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biological systems in particular are typically subjected to noise sources. Different types of noise sources have been considered. Stochastic resonance has been found in delay systems with additive noise [5]. Multiplicative noise can lead to noise-induced drifts in time-delayed systems [16,17]. Multiplicative and parametric noise is assumed to be an intrinsic property of neural control mechanisms in general [18]. In particular, there is evidence that the pupil light reflex [11,19], pointing movements [12] and balancing movements [14] involve time-delayed control mechanisms with parametric and multiplicative noise. It is important to realize that multiplicative and parametric noise may support the functioning of biological systems (e.g. increased sensitivity due to stochastic resonance, noise-induced corrective movements [14], minimization of the accuracy-flexibility trade-off [20]). That is, noise sources may play a constructive role in biological systems. In view of these considerations, we can only obtain a complete understanding of stochastic delay systems if we are able to identify both their deterministic and stochastic constituents. Specifically, the challenge is to determine the evolution equations of experimentally observed stochastic delay systems. In contrast to several useful methods that reconstruct deterministic evolution equations of delay systems [21–23], we will show here how to determine the type and the structure of noise sources of delay systems. To this end, we will exploit, on the one hand, the Fokker-Planck approach to stochastic delay systems introduced by Guillouzic et al. [17,24–26] and, on the other hand, a recently developed data analysis method for Markov diffusion processes [27–30].

Let us consider stochastic delay systems described by a state variable X that satisfies the stochastic differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = h(X, X_\tau) + g(X, X_\tau)\Gamma(t) \tag{1}$$

for $t \geq 0$. Here, X_τ denotes the time-delayed state variable $X_\tau(t) = X(t-\tau)$. For $t \in [-\tau, 0]$ the state variable $X(t)$ is given by $X(t) = \phi(t)$. The variable Γ

denotes a Langevin force with $\langle\Gamma(t)\Gamma(t')\rangle = \delta(t-t')$ [31,32]. Accordingly, the whole expression $Y = g(X, X_\tau)\Gamma(t)$ describes the noise source of the system under study. For $g =$ const. we deal with additive noise. If g depends on X or X_τ, we deal with multiplicative noise. Systems with parametric noise will be studied in the approximation of multiplicative noise systems [14,23], which is well-known from ordinary stochastic systems [32] (for alternative approaches, e.g., the evaluation of expressions like Γ^2, see [33]). The noise term Y can be interpreted according to the Ito and Stratonovich calculus [16,17,26] — just as in the case of systems without delay [31,32]. In what follows, we will use the Ito calculus. Using the method of steps, Eq. (1) can be solved iteratively in time intervals of width τ. However, the time variable can be treated in either absolute or relative time.

First, let us discuss the method of steps with respect to an absolute time frame. Then, the time variable of the original problem (1) is used and Eq. (1) is solved iteratively for $t \in [T, T+\tau)$ with $T = N\tau$ and $N = 0, 1, 2, \ldots$ [34]. It is clear that at the interfaces between two intervals the limiting case

$$\lim_{\epsilon\to 0} P(x, T+\epsilon | x', T-\epsilon) = \delta(x - x') \tag{2}$$

holds, where $P(x, t|x', t')$ denotes a conditional probability density.

Second, let us consider the method of steps with respect to a relative time frame. Then, a relative time variable $z \in [0, \tau)$ is used that is related to t by $z = t - N\tau$. By means of z, Eq. (1) can be written as the ordinary $(N+1)$-dimensional Ito-Langevin equation

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathbf{X}(z) = \mathbf{h}(\mathbf{X}(z)) + G(\mathbf{X}(z)) \cdot \mathbf{\Gamma}(z) \tag{3}$$

for $\mathbf{X} = (X_{N\tau}, \ldots, X_0)$, where the vectors $\mathbf{h}$ and $\mathbf{\Gamma}$ are defined by $h_k = h(X_{k\tau}, X_{(k-1)\tau})$ and $\mathbf{\Gamma} = (\Gamma_{N\tau}, \ldots, \Gamma_0)$ and G describes a $(N+1) \times (N+1)$-matrix with coefficients $G_{ik} = \delta_{ik} g(X_{k\tau}, X_{(k-1)\tau})$ [17,26,35]. The components

Γ_k satisfy $\langle \Gamma_{k\tau}(z)\Gamma_{k'\tau}(z')\rangle = \delta_{k\,k'}\delta(z-z')$ and initial distributions for $\mathbf{X}$ can be derived from the condition $\lim_{\epsilon\to 0} P(x_{k\tau},\epsilon|x_{k\tau-\tau},\tau-\epsilon) = \delta(x_{k\tau}-x_{(k-1)\tau})$, which is the counterpart of Eq. (2), and the requirement $X_{-\tau}(z) = \phi(z-\tau)$. Both methods are equivalent because we can put $X(t) = X_T(t-T)$ and $\Gamma(t) = \Gamma_T(t-T)$ for $t \in [T, T+\tau)$. From Eq. (3) it follows that the evolution of the transition probability density $P(\mathbf{x}, z|\mathbf{x}', z')$ is given by

$$\frac{\partial}{\partial z}P(\mathbf{x}, z|\mathbf{x}', z') = -\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{h}(\mathbf{x})P(\mathbf{x}, z|\mathbf{x}', z') + \frac{\partial}{\partial \mathbf{x}}\frac{\partial}{\partial \mathbf{x}} : D(\mathbf{x})P(\mathbf{x}, z|\mathbf{x}', z') \tag{4}$$

with $D_{ik} = \delta_{ik}\, g^2(x_{k\tau}, x_{(k-1)\tau})/2$ [31,32]. That is, univariate non-Markovian processes defined by Eq. (1) can equivalently be expressed in terms of multivariate Markov processes described by Eqs. (3) and (4). Note that this conclusion is in line with the well-known fact that there are non-Markovian processes that can be expressed in terms of Markov processes by introducing additional state variables [31]. The coefficients $\mathbf{h}$ and D can now be estimated from time series data if the data are regarded with respect to the relative time frame [27–30]. Using the backward transformation from the relative time frame to the absolute time frame, we find that the drift coefficient h and the diffusion coefficient $D(x, x_\tau) = g^2(x, x_\tau)/2$ are given by the ensemble averages

$$h(x, x_\tau) = \lim_{\Delta t\to 0} \frac{1}{\Delta t} \left.\langle X(t+\Delta t) - X(t)\rangle\right|_{X(t)=x, X_\tau(t)=x_\tau} , \tag{5}$$

$$D(x, x_\tau) = \lim_{\Delta t\to 0} \frac{1}{2\Delta t} \left.\left\langle [X(t+\Delta t) - X(t)]^2\right\rangle\right|_{X(t)=x, X_\tau(t)=x_\tau} . \tag{6}$$

If we deal with systems that are ergodic in the stationary case, the ensemble averages in Eqs. (5) and (6) can be computed by means of time averages. Let us study some generic cases in more detail.

Key features of human pointing movements can be understood from the stochastic delay equation $\mathrm{d}X/\mathrm{d}t = -a\tanh[cX_\tau + \sqrt{Q}\Gamma(t)]$ involving parametric noise, where a, c, and Q are positive constants [12]. For small noise amplitudes Q, a Taylor expansion with respect to the noise term gives us the multiplicative noise model $\mathrm{d}X/\mathrm{d}t = -a\tanh[cX_\tau] - a\sqrt{Q}\Gamma(t)\Big/[\cosh(cX_\tau)]^2$ [23,32]. This tanh-model belongs to a class of time-delayed systems for which the dynamics is dominated by time-delayed variables and can be described by $\mathrm{d}X/\mathrm{d}t = h(X_\tau) + g(X_\tau)\Gamma(t)$. For systems of this kind, Eqs. (5) and (6) reduce to $h(x_\tau) = \lim_{\Delta t\to 0}[\Delta t]^{-1}\,\langle X(t+\Delta t) - X(t)\rangle|_{X_\tau(t)=x_\tau}$ and $D(x_\tau) = \lim_{\Delta t\to 0}[2\Delta t]^{-1}\,\langle [X(t+\Delta t) - X(t)]^2\rangle|_{X_\tau(t)=x_\tau}$. We solved the tanh-model numerically for the multiplicative noise case and applied the proposed data analysis technique using a single stationary trajectory and time-averaging. As shown in Fig. 1, we recovered the drift and diffusion coefficients from the time series analysis. In particular, from Fig. 1 one can read off that the system under consideration exhibits a multiplicative noise source. Moreover, the explicit structure of the noise source can be read off. Next, let us consider two models involving both time-delayed and non-delayed variables.

Insert Figure 1 about here.

First, we consider a model proposed by Tass et al. that describes periodic tracking movements under artificially delayed visual feedback [13]. We use the stochastic evolution equation (42) in reference [23] but replace the noise term $g = \sqrt{Q}\Gamma(t)$ by a multiplicative noise term $g(X, X_\tau)$ with $g(X, X_\tau) = \sqrt{Q}\Gamma(t)\;[1-\epsilon\cos(\pi X)]/\,[1-\epsilon]$ with $\epsilon \in [0, 1)$. The noise amplitude $g(X, X_\tau)$ depends only on the non-delayed variable and has a minimum at $X = 0$. Note that X is a re-scaled periodic random variable such that $X \in [-1, 1]$. We computed a stationary trajectory from the model by Tass et al. and evaluated the trajectory using Eq. (6) and the time-averaging procedure (see Fig. 2). Fig. 2 shows the diffusion coefficient $D(x, x_\tau) = g^2(x, x_\tau)/2$ as a function of x. We see that the proposed data analysis technique nicely reconstructs the multiplicative noise term. Fig. 2 also shows $D(x, x_\tau)$ as a function of x_τ for a fixed value of x. From this graph and similar graphs that can be obtained for different x-values, we conclude that the diffusion coefficient does not depend on the non-delayed variable. Note that by means of Eq. (5), we also extracted the drift term from the stochastic trajectory. Thus, we obtained the same figure as shown in reference [23] (Fig. 4).

Insert Figure 2 about here.

A common feature of simple dynamical systems is that they are destabilized by noise, on the one hand, and time-delays, on the other hand. For example, the variability of the random walk given by $\mathrm{d}X/\mathrm{d}t = -aX - bX_\tau + \sqrt{Q}\,\Gamma(t)$ for $a, b > 0$ increases monotonically with Q and τ [34,35]. As stated in the introduction, noise can also improve the functioning of systems. Such constructive impacts of noise have been found for example in balancing movements. In a study by Cabrera and Milton it has been shown that multiplicative noise results in corrective movements in stick balancing tasks on time scales shorter than the delay time [14]. We study here the Cabrera-Milton model in the regime of overdamped movements, which reads $\mathrm{d}X/\mathrm{d}t = q\sin(X) - R_0X_\tau - \sqrt{Q}X_\tau\Gamma(t)$ for the angular variable $X \in [0, 2\pi]$ and exhibits a parabolic diffusion coefficient $D(x, x_\tau) = Qx_\tau^2/2$ that depends on the time-delayed variable. Here, q, R_0, and Q are positive constants. In order to ease the presentation of our numerical results, we introduce again a re-scaled periodic variable defined on $[-1, 1]$. In addition, we add a small additive noise to the diffusion coefficient, which gives us $D(x, x_\tau) = Q(\epsilon + x_\tau^2)/2$. Then, the model equation for overdamped balancing movements reads $\mathrm{d}X/\mathrm{d}t = q\sin(\pi X) - R_0X_\tau - \sqrt{Q}\sqrt{\epsilon + X_\tau^2}\,\Gamma(t)$ and the amplitude of the additive noise is given by $Q\,\epsilon/2$. The stochastic differential equation was solved numerically to obtain a single stationary trajectory. The trajectory was evaluated using Eqs. (5) and (6) in combination with time-averaging (see Figs. 3 and 4). Fig. 3 shows the drift coefficient $h(x, x_\tau)$ as a function of x for fixed x_τ and as a function of x_τ for fixed x as obtained from the model equation (lines) and from the data analysis (diamonds). In Fig. 4 we have plotted the diffusion coefficient $D(x, x_\tau)$ as a function of x for $x_\tau = 0$ as obtained from the time series analysis. From this graph and similar graphs for other values of x_τ, we can read off that the diffusion coefficient does not depend on x. In addition, the amplitude of the additive noise source can be estimated. Fig. 4 also gives $D(x, x_\tau)$ as a function of x_τ for a fixed value of x. We see that the data analysis yields both qualitatively and quantitatively the correct result.

Note that in the three aforementioned examples we kept the number N of generated data points fixed. We chose N such that even for the models with very large recurrence times we obtained a good match between the reconstructed and the original functions h and D. More precisely, we used $N = 10^8$ and we observed that for the parameters used in our simulations (see captions of Figs. 1, 2, and 3) the Cabrera-Milton model exhibited the largest recurrence time of all three models. For the Cabrera-Milton model we obtained an average recurrence time of 2×10^4 with respect to the sampling rate and the most unlikely states at $(x, x_\tau) \approx (0, \pm 1)$ and $(x, x_\tau) \approx (\pm 1, 0)$. That is, these states were visited on the average only every 20000 data points. For the model by Tass et al. we found an average recurrence time of about 10^3 with respect to the sampling rate and the most unlikely events at $(x, x_\tau) \approx (0, \pm 1)$ and $(x, x_\tau) \approx (\pm 1, 0)$. Finally, for the tanh-model the average recurrence time was about 40 with respect to the sampling rate and the states at $x_\tau = \pm 1$ (this relatively small recurrence time seems to be due to the fact that here we are dealing with a one-dimensional phase space, whereas in the two other cases the phase spaces are two-dimensional).

Insert Figures 3 and 4 about here.

Having illustrated the power of the proposed data analysis method, the question arises whether or not we can find support for our hypothesis that we deal with stochastic delay systems described by delay Langevin equations of the form (1). Since Eq. (1) can equivalently be expressed in terms of Eq. (3), we need to show that the Markov property with respect to the relative time frame given by z holds. To a certain extent, this can be done [27]. First, one may show that the Chapman-Kolmogorov equation $P(\mathbf{x},z|\mathbf{x}'',z'') = \int P(\mathbf{x},z|\mathbf{x}',z')P(\mathbf{x}',z'|\mathbf{x}'',z'')\,\mathrm{d}^{N+1}x'$ holds for $z \geq z' \geq z''$ which implies that we deal with a Markov process. Second, one may compute Kramers-Moyal coefficients $D_{i_1,\ldots,i_n}(\mathbf{x},z)$ of order n given by

$$D_{i_1,\ldots,i_n}(\mathbf{x},z) = \frac{1}{n!}\lim_{\epsilon\to 0}\int_{\Omega}(y_{i_1\tau}-x_{i_1\tau})\cdots(y_{i_n\tau}-x_{i_n\tau})\,P(\mathbf{y},z+\epsilon|\mathbf{x},z)\,\mathrm{d}^{N+1}y\ , \tag{7}$$

where $i_1,\ldots,i_n$ are integers with $i_k \geq 0$. If the forth order coefficients vanish, it follows from the Pawula theorem that we are dealing with a Markov diffusion process [31,32].

We have shown how to determine the noise sources of time-delayed feedback systems that satisfy delay Langevin equations. In doing so, we are in the position to distinguish on the basis of experimental observations between different types of noise sources such as additive and multiplicative noise sources. Moreover, the functional dependencies of noise sources on time-delayed and non-delayed variables can be assessed. Finally, we have demonstrated that by means of a relative time frame we are able to prove that the systems under consideration can be described in terms of delay Langevin equations at all. This last issue is related to the Kramers-Moyal expansion of the Chapman-Kolmogorov equation. Since it is known from the theory of Markov processes that there are processes for which the Kramers-Moyal expansion fails, although the Chapman-Kolmogorov equation is satisfied (e.g. Lévy flights), there might be a much larger class of time-delayed feedback systems to which our data analysis method can be applied (see also [36]). In this context, the Markov property may be proven by showing that $P(\mathbf{x},z|\mathbf{x}',z';\mathbf{x}'',z'';\mathbf{x}''',z''';\ldots) = P(\mathbf{x},z|\mathbf{x}',z')$ holds for $z \geq z' \geq z'' \geq z''' \geq \ldots$ [29].

In conclusion, we would like to remark that the number of data points required by the proposed data analysis technique increases with the desired resolution of the functional dependencies of noise sources on delayed and non-delayed state variables. Roughly speaking, from long time series the structure of noise sources can be determined with a high resolution, whereas from short time series we can obtain at best hints about the type of noise sources involved in the systems being studied. More precisely, the crucial issue is how often on the average the systems visit the points in their phase spaces we are interested in (for the recurrence time problem see also the discussion above). Therefore, if we have only relatively short time series at our disposal, we may add a priori information about the system under study (such as symmetry properties) and assume that the noise sources are described by particular functions with unknown parameters. The parameters can then be fitted by the proposed data analysis technique as shown in [37].

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Figure caption:

Fig. 1: Drift function $h(x_\tau) = -a\tanh(cx_\tau)$ plotted versus x_τ (dashed line), diffusion coefficient $D(x_\tau) = 0.5Qa^2/[\cosh(cx_\tau)]^4$ plotted versus x_τ (solid line), and reconstructions of h and D by means of time series analysis (diamonds). Parameters: $\tau = 0.1$, $a = 2$, $c = \pi/2$, $Q = 1$ (number of data points $N = 10^8$, sampling period $\Delta t = 0.01$, spatial discretization of h and D with $\Delta x_\tau = 0.1$).

Fig. 2: Diffusion coefficient $D(x, x_\tau) = 0.5Q\{[1 - \epsilon\cos(\pi x)]/[1 - \epsilon]\}^2$ plotted versus x (dashed line), plotted versus x_τ for $x = 0$ (straight, solid line), and reconstructions of $D(x, 0)$ and $D(0, x_\tau)$ by means of time series analysis (diamonds). Parameters: $\tau = 0.2$, $\epsilon = 0.5$, $Q = 1$ ($N = 10^8$, $\Delta t = 0.02$, $\Delta x = \Delta x_\tau = 0.1$).

Fig. 3: The drift function $h(x, x_\tau) = q\sin(\pi x) - R_0 x_\tau$ is shown for $h(x, 0)$ (dashed line) and $h(0, x_\tau)$ (solid line). Diamonds indicate results obtained from the data analysis. Parameters: $\tau = 0.2$, $q = 0.5$, $R_0 = 2$, $Q = 1$, $\epsilon = 0.1$ ($N = 10^8$, $\Delta t = 0.02$, $\Delta x = \Delta x_\tau = 0.1$).

Fig. 4: Diffusion coefficient $D(x, x_\tau) = 0.5Q(\epsilon + x_\tau^2)$ as a function of x for $x_\tau = 0$ (straight, dashed line), as a function of x_τ (solid line), and reconstructions of $D(x, 0)$ and $D(0, x_\tau)$ by means of time series analysis (diamonds). Parameters see Fig. 3.

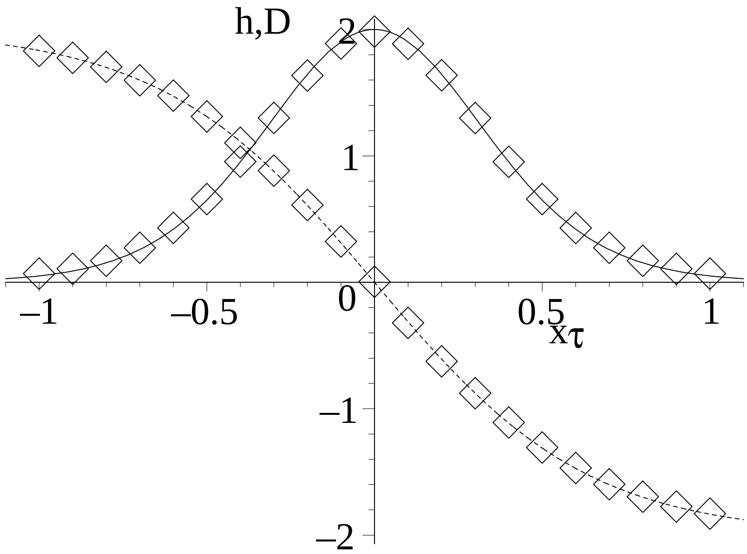
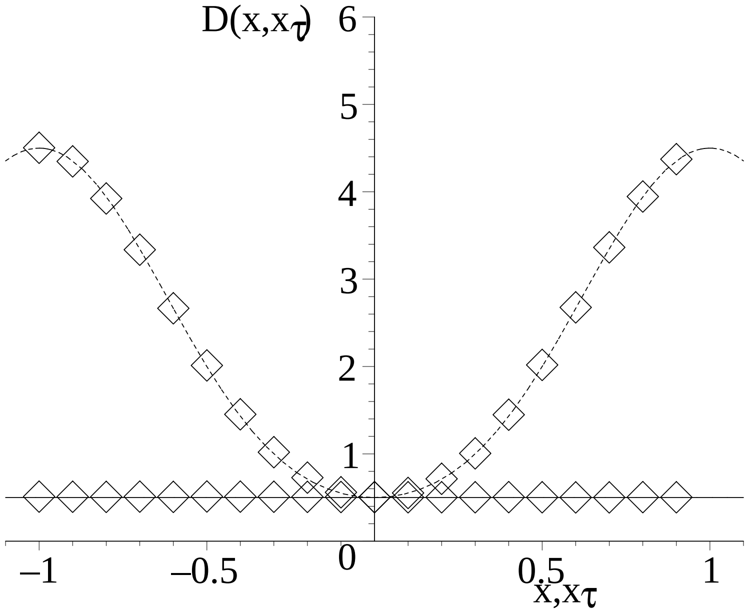


Fig. 1.



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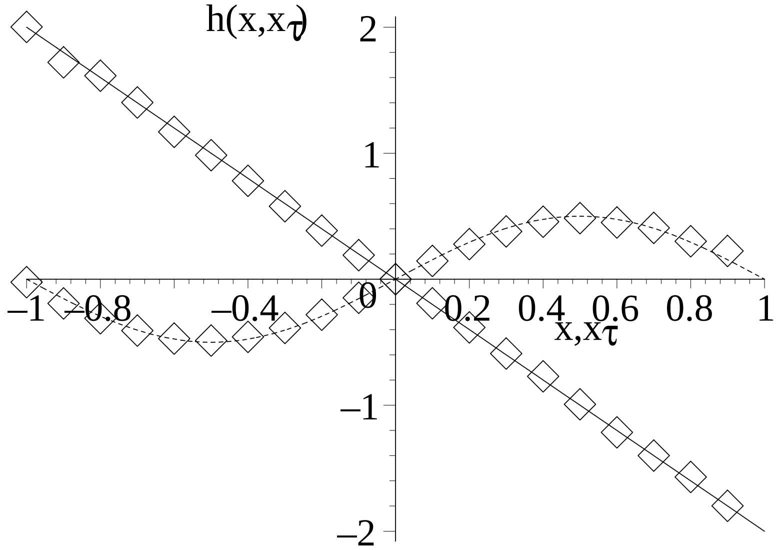

Fig. 3.

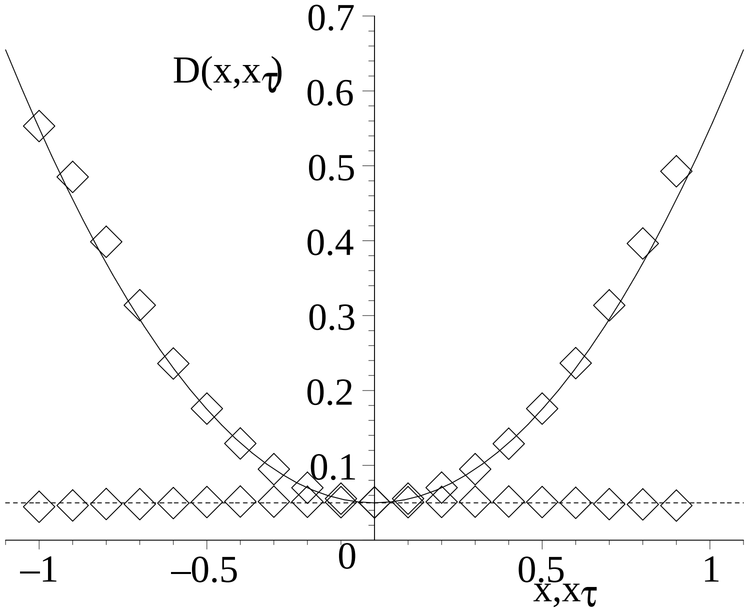

Fig. 4.