Single particle dynamics of many-body systems described by Vlasov-Fokker-Planck equations

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Abstract

Using Langevin equations we describe the random walk of single particles that belong to particle systems satisfying Vlasov-Fokker-Planck equations. In doing so, we show that Haissinski distributions of bunched particles in electron storage rings can be derived from a particle dynamics model.

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1 Introduction

Particle beams of electron storage rings can be bunched. A particle bunch is a group of particles that move together and have roughly the same energy. Bunches of charged particles produce wakefields that act back on the bunch particles [1]. There is a mean field theory of the wakefields which is similar to the Debye theory of polarization and leads to Vlasov equations for particle distributions. In order to take damping and diffusion due to synchrotron

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radiation into account, one may supplement the Vlasov equation with collision terms of the Fokker-Planck type [2–11]. Vlasov-Fokker-Planck equations thus obtained are of general interest because they also offer approximative descriptions of plasma diffusion (see e.g. [12–16]).

In view of these applications of Vlasov-Fokker-Planck equations, the question arises if stochastic single particle descriptions can be derived from them. For Vlasov equations without collision terms, Hamiltonian particle descriptions can be obtained [1,10]. For ordinary Fokker-Planck equations single particle descriptions are available in terms of Langevin equations [17]. As we will shown below, for Vlasov-Fokker-Planck equations Langevin equations can be defined as well. To this end, we will elaborate on recent developments in the theory of nonlinear Fokker-Planck equations [18–24].

2 Single particle motions defined by generalized Langevin equations

2.1 General case

Let $\rho(p,q,t)$ denote a particle density of the generalized coordinate q and momentum p with p and q defined on a phase space Ω. We assume that ρ is normalized to $M_0 = \int_\Omega \rho(p,q,t)\,\mathrm{d}p\,\mathrm{d}q$ and that the Hamiltonian H of the many-body system under consideration is given by

$$H = H_0(p,q) + \int_\Omega H_{\mathrm{MF}}(q-q')\rho(p',q')\,\mathrm{d}p'\,\mathrm{d}q' \ , \tag{1}$$

where H_0 describes the single particle Hamiltonian and the integral reflects particle-particle interactions in a self-consistent mean field fashion. Then, the

Liouville equation for $\rho(p,q,t)$ reads

$$\frac{\partial}{\partial t}\rho(p,q,t) + \frac{\partial H}{\partial p}\frac{\partial \rho}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial \rho}{\partial p} = 0 \tag{2}$$

and is of the type of a Vlasov equation because of the integral in Eq. (1). Adding a Fokker-Planck collision term [12–15], we obtain

$$\frac{\partial}{\partial t}\rho(p,q,t) + \frac{\partial H}{\partial p}\frac{\partial \rho}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial \rho}{\partial p} = \gamma\frac{\partial}{\partial p}(p\rho) + D\frac{\partial^2}{\partial p^2}\rho \ , \tag{3}$$

where $\gamma > 0$ is a damping constant and $D > 0$ is a diffusion coefficient. We will refer to Eq. (3) as a Vlasov-Fokker-Planck equation. Aiming at a stochastic descriptions in terms of Langevin equations, we transform Eq. (3) in an evolution equation for a probability density $P(p,q,t) = \rho(p,q,t)/M_0$ normalized to unity:

$$\frac{\partial}{\partial t}P(p,q,t) + \frac{\partial H'}{\partial p}\frac{\partial P}{\partial q} - \frac{\partial H'}{\partial q}\frac{\partial P}{\partial p} = \gamma\frac{\partial}{\partial p}(pP) + D\frac{\partial^2}{\partial p^2}P \ , \tag{4}$$

with

$$H' = H_0(p,q) + M_0\int\limits_{\Omega} H_{\mathrm{MF}}(q-q')P(p',q')\,\mathrm{d}p'\,\mathrm{d}q' \ , \tag{5}$$

We assume that $H_{\mathrm{MF}}(z)$ describes a symmetric function. Then, Eq. (4) can be interpreted as a free energy Fokker-Planck equation of a canonical-dissipative system [22]. To see this, we define the internal energy $U[P]$ as

$$U[P] = \langle H_0\rangle + \frac{M_0}{2}\int\limits_{\Omega}\int\limits_{\Omega} H_{\mathrm{MF}}(q-q')P(p,q)P(p',q')\,\mathrm{d}p\,\mathrm{d}q\ \mathrm{d}p'\,\mathrm{d}q' \tag{6}$$

where $\langle\cdot\rangle$ is defined as average with respect to P. Furthermore, we define the entropy $S[P] = -\int_\Omega P\ln P\,\mathrm{d}p\,\mathrm{d}q$ and the conservative drift forces

$$I_p = -\frac{\partial}{\partial q}\frac{\delta U}{\delta P} \ , \quad I_q = \frac{\partial}{\partial p}\frac{\delta U}{\delta P} \tag{7}$$

(where $\delta U/\delta P$ is the variational derivative of U) related to the conservative drift vector $\mathbf{I} = (I_p, I_q)$. Using $\nabla = (\partial/\partial p, \partial/\partial q)$, Eq. (4) can be written as

$$\frac{\partial}{\partial t}P(p,q,t) = -\nabla \cdot (\mathbf{I}P) + \gamma \frac{\partial}{\partial p} P \frac{\partial}{\partial p} \frac{\delta F}{\delta P} \ , \tag{8}$$

where Q is a noise amplitude $Q = D/\gamma$ and F the free energy $F = U - QS$. Eq. (8) corresponds to a free energy Fokker-Planck equation of a canonical-dissipative systems and can formally be derived from linear nonequilibrium thermodynamics [22]. Stationary distributions can be obtained from the free energy principle

$$\frac{\delta F}{\delta P} = \mu \tag{9}$$

where μ is a constant because $\mathbf{I}$ corresponds to a conservative force and satisfies $\nabla \cdot \mathbf{I} = 0$ and $\mathbf{I} \cdot \nabla \delta U/\delta P$ (see [22] for details). From Eq. (9) we obtain the implicit equation

$$P_{\rm st}(p,q) = \frac{1}{Z} \exp\left\{ -\frac{H'(p,q,P_{\rm st})}{Q} \right\} \ . \tag{10}$$

From solutions of Eq. (10) we then obtain the stationary particle density distributions $\rho_{\rm st}(p,q) = M_0 P_{\rm st}(p,q)$. Let us write Eq. (4) as

$$\frac{\partial}{\partial t}P(p,q,t) = -\nabla \cdot (\mathbf{h}P) + D\frac{\partial^2}{\partial p^2}P \tag{11}$$

with the probability-dependent drift vector

$$\mathbf{h}(p,q,P) = \mathbf{I}(p,q,P) + \gamma p \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ . \tag{12}$$

We assume that for solutions of Eq. (4), the time-dependent drift vector $\mathbf{h}'$ defined by $\mathbf{h}'(p,q,t) = \mathbf{h}(p,q,P)$ corresponds to the first Kramers-Moyal coefficients of a Markov diffusion process. Then, Eq. (4) is called a strong nonlinear

Fokker-Planck equation and the transition probability density $P(p,q,t|p',q',t')$ for $t > t'$ of the Markov diffusion process given by Eq. (4) satisfies [24]

$$\frac{\partial}{\partial t}P(p,q,t|p',q',t) = \\ -\nabla\cdot[\mathbf{h}(p,q,P(p,q,t))P(p,q,t|p',q',t)] + D\frac{\partial^2}{\partial p^2}P(p,q,t|p',q',t)\ . \qquad (13)$$

Note that the evolution equation for the transition probability is linear with respect to $P(p,q,t|p',q',t')$ because $\mathbf{h}$ depends on $P(p,q,t)$ and does not depend on $P(p,q,t|p',q',t')$. While the Vlasov-Fokker-Planck equation (11) only defines the evolution of $P(p,q,t)$ and does not define a stochastic process [21], Eqs. (11,...,13) indeed define a stochastic process. This process is given by the hierarchy of N-point joint probability densities

$$P(p_N,q_N,t_N;\ldots;p_1,q_1,t_1) = \\ P(p_N,q_N,t_N|p_{N-1},q_{N-1},t_{N-1})\cdots P(p_2,q_2,t_2|p_1,q_1,t_1)P(p_1,q_1,t_1) \qquad (14)$$

known from Markov processes. The N-point joint probability densities in general and $P(p,q,t)$ in particular can also be obtained from an appropriately defined Langevin equation. Let $p(t)$ and $q(t)$ define the random variables corresponding to $P(p,q,t) = \langle\delta(p-p(t))\delta(q-q(t))\rangle$, where $\delta(\cdot)$ is the Dirac delta function and $\langle\cdot\rangle$ denotes an ensemble average. Then, $p(t)$ and $q(t)$ satisfy the stochastic differential equation [21,24]

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}q(t) &= \frac{\partial H_0(p,q)}{\partial p} \\ \frac{\mathrm{d}}{\mathrm{d}t}p(t) &= -\frac{\partial H_0(p,q)}{\partial q} - M_0\frac{\partial}{\partial q}\int_\Omega H_{\mathrm{MF}}(q-q')P(p',q',t)\,\mathrm{d}p'\,\mathrm{d}q' \\ &\quad -\gamma p + \sqrt{D}\,\Gamma(t)\ , \end{aligned} \qquad (15)$$

where $\Gamma(t)$ denotes a Langevin force with $\langle\Gamma(t)\Gamma(t')\rangle = 2\delta(t-t')$. The probability density $P(p,q,t)$ in Eq. (15) is either computed from Eq. (4) or from $P(p,q,t) = \langle\delta(p-p(t))\delta(q-q(t))\rangle$. In the former case, we have a two-layered description. In the latter case, we have a self-consistent Langevin equation [21].

2.2 Haissinski distributions of bunched particle beams

Particles in bunched longitudinal beams traveling through electron storage rings can be described in terms of their relative positions and rescaled energy deviations. Relative positions are defined with respect to moving frames related to the traveling bunches and will be denoted by q. Rescaled energies deviations will be denoted by p and are typically measured in terms of appropriately rescaled energy deviations from beam design energies. It then turns out that the bunch particle distribution $P(p,q,t)$ satisfies the Vlasov-Fokker-Planck equation (3) with H_0 defined by $H_0 = p^2/2 + q^2/2$ and $H_{\rm MF}$ given by the wakefield $H_{\rm W}$ of the particle bunch [5–11]. Following [25], we are interested to study qualitatively the collective phenomena that result from particle-particle interactions in particle bunches via wakefields and confine ourselves to model the impact of the wakefield $H_{\rm W}$ by a Dirac delta function: $H_{\rm MF} = H_{\rm W} = \kappa M_0^{-1}\delta(q-q')$. Then, Eq. (4) reads

$$\begin{aligned} &\frac{\partial}{\partial t}P(p,q,t) + p\frac{\partial P(p,q,t)}{\partial q} - \frac{\partial P(p,q,t)}{\partial p}\left[q + \kappa\frac{\partial}{\partial q}P(q,t)\right] \\ &= \gamma\frac{\partial}{\partial p}\left[pP(p,q,t)\right] + D\frac{\partial^2}{\partial p^2}P(p,q,t)\ , \end{aligned} \tag{16}$$

and Eq. (15) becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}q(t) &= p \\ \frac{\mathrm{d}}{\mathrm{d}t}p(t) &= -q - \kappa\frac{\partial P(q,t)}{\partial q} - \gamma p + \sqrt{D}\,\Gamma(t) \end{aligned} \tag{17}$$

with $P(q,t) = \int P(p,q,t)\,\mathrm{d}p = \langle \delta(q - q(t)) \rangle$ and $(p,q) \in \Omega = \mathbb{R}^2$. Stationary distributions can be found in form of $P_{\mathrm{st}}(p,q) = P_{\mathrm{st}}(p)P_{\mathrm{st}}(q)$ with

$$P_{\mathrm{st}}(p) = \frac{1}{\sqrt{2\pi Q}} \exp\left\{ -\frac{p^2}{2Q} \right\} \tag{18}$$

and $P_{\mathrm{st}}(q)$ given by

$$[Q + \kappa P_{\mathrm{st}}(q)] \frac{\mathrm{d}P_{\mathrm{st}}(q)}{\mathrm{d}q} = -q P_{\mathrm{st}}(q) \ . \tag{19}$$

This can be verified by substituting $P_{\mathrm{st}}(p,q)$ into Eq. (16) (see also [25]). Stationary solutions of this kind are called Haissinski solutions [1,26] and they are known to be stable provided they exist [25]. Alternatively, from the free energy principle we obtain Eq. (10) in form of

$$P_{\mathrm{st}}(p,q) = \frac{1}{Z} \exp\left\{ -\frac{p^2}{2Q} \right\} \exp\left\{ -\frac{q^2/2 + \kappa P_{\mathrm{st}}(q)}{Q} \right\} \ , \tag{20}$$

which gives us $P_{\mathrm{st}}(p,q) = P_{\mathrm{st}}(p)P_{\mathrm{st}}(q)$ and Eq. (18) as well as

$$P_{\mathrm{st}}(q) = \frac{1}{Z'} \exp\left\{ -\frac{q^2/2 + \kappa P_{\mathrm{st}}(q)}{Q} \right\} \ , \tag{21}$$

where Z' is a normalization constant. Eq. (21) can be evaluated using the concept of distortion functionals [27–30]. To this end, we transform Eq. (21) into

$$P_{\mathrm{st}}(q) \exp\left\{ \frac{\kappa}{Q} P_{\mathrm{st}}(q) \right\} = \frac{1}{Z''} W(q) \ , \quad W(q) = \frac{1}{\sqrt{2\pi Q}} \exp\left\{ -\frac{q^2}{2Q} \right\} \ , \tag{22}$$

where Z'' is another normalization constant. Next, we recall that the inverse of the function f given by $f(\mathrm{LW}) = \mathrm{LW} \exp\{\mathrm{LW}\}$ is Lambert's W-function (or Ω-function) denoted here by $\mathrm{LW}(f)$ [31]. Then, we introduce the distortion function $G(z) = z \exp\{\kappa z/Q\}$ and its inverse function $G^{-1}(z)$ given by $G^{-1}(z) = Q\,\mathrm{LW}(\kappa z/Q)/\kappa$. Thus, we get

$$P_{\mathrm{st}}(q) = \frac{Q}{\kappa} \mathrm{LW}\left(\frac{\kappa}{Z'' Q} W(q) \right) \ , \tag{23}$$

where Z'' is determined by the requirement $\int P_{\rm st}(q)\,{\rm d}q = 1$. In sum, Eqs. (18) and (23) define the stationary Haissinski solutions of the Vlasov-Fokker-Planck equation (16). Note that an expression similar to Eq. (23) has previously been derived [32]. For $\kappa = 0$ we have $LW(z) = z$, $Z'' = 1$ and $P_{\rm st}(q) = W(q)$. For $\kappa > 0$ we can read off from Eq. (17) that particles are driven away from regions of high density (because $P(q,t)$ acts as a potential) and, consequently, the distributions $P_{\rm st}(q)$ are broader than the Gaussian distribution $W(q)$. For $\kappa < 0$ particles are attracted by regions of high density and, consequently, we deal with squeezed distributions and the distributions $P_{\rm st}(q)$ are smaller than the Gaussian distribution $W(q)$, see Fig. 1.

Now let us simulate the self-consistent Langevin equation (17). In order to cope with the probability density occurring in Eq. (17), one may determine $P(q,t)$ in terms of a histogram [30]. Let us use an alternative method that has the advantage that there is no need to store a histogram. Recall that there are several representations of the Dirac delta function. For example, we may exploit the relation [33]

$$\delta(x-x') = \lim_{\Delta x\to 0} \frac{1}{\sqrt{2\pi}\Delta x} \exp\left\{-\frac{1}{2}\left[\frac{x-x'}{\Delta x}\right]^2\right\} . \qquad (24)$$

Using $P(q,t) = \langle \delta(q-q(t))\rangle$ and Eq. (24), we discretize Eq. (17) by means of the Euler forward scheme

$$\begin{aligned} q_{n+1} &= q_n + \Delta t p_n \\ p_{n+1} &= p_n \\ &-\Delta t\left[q_n - \frac{\kappa}{(L-1)\sqrt{2\pi}[\Delta x]^2}\sum_{l=1}^{L-1}\left([q_n - q_n^l]\exp\left\{-\frac{1}{2}\left[\frac{q_n-q_n^l}{\Delta x}\right]^2\right\}\right) + \gamma p_n\right] \\ &+\sqrt{D\Delta t}\, w_n\ , \end{aligned} \qquad (25)$$

where q_n and p_n describe the random variables $q(t_n)$ and $p(t_n)$ at time points $t_n = n\Delta t$. Δt and Δx should correspond to small numbers. The variables w_n are Gaussian distributed random numbers with $\langle w_n\rangle = 0$ and $\langle w_n w_{n'}\rangle =$

$2\delta_{nn'}$ (see [17] for details). The variables q_n^l denote further realizations of the random variable $q(t)$. That is, we solve Eq. (25) for L realizations given by q_n and q_n^l, where L should correspond to a large number. Fig. 2 shows $P_{\rm st}(p)$ computed from the random walk model (25) for $\kappa > 0$ and shows for the sake of comparison the exact result given by the Gaussian distribution (18). Fig. 3 shows the Haissinski distribution $P_{\rm st}(q)$ as obtained from the simulation of Eq. (25) for the same parameter value κ and as computed from Eq. (23). Fig. 4 shows the Haissinski distribution $P_{\rm st}(q)$ in the case of $\kappa < 0$ as obtained from the simulation of the Langevin equation (17) via Eq. (25) and as computed from Eq. (23). We have also computed the distribution $P_{\rm st}(p)$ for this parameter value and have again obtained a distribution as shown in Fig. 2. Figures 2, 3, and 4 illustrate that we can describe the single particle dynamics of a particle ensemble satisfying the Vlasov-Fokker-Planck equations (16) by means of the corresponding Langevin equation (17) and its discretization (25).

2.3 Generalized Haissinski distributions

Our considerations can easily be generalized to systems with wakefields $H_{\rm W} = \kappa M_0^{-1}\delta(q-q')$ and single particle Hamiltonians $H_0 = p^2/2 + V(q)$, where $V(q)$ describes a potential with respect to the coordinate q. Then, Eq. (16) is replaced by

$$
\begin{aligned}
&\frac{\partial}{\partial t}P(p,q,t) + p\frac{\partial P(p,q,t)}{\partial q} - \frac{\partial P(p,q,t)}{\partial p}\left[\frac{\mathrm{d}V}{\mathrm{d}q} + \kappa\frac{\partial}{\partial q}P(q,t)\right] \\
&= \gamma\frac{\partial}{\partial p}\left[pP(p,q,t)\right] + D\frac{\partial^2}{\partial p^2}P(p,q,t) \ . \qquad (26)
\end{aligned}
$$

Likewise, we need to replace Eq. (17) by

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}q(t) &= p \\
\frac{\mathrm{d}}{\mathrm{d}t}p(t) &= -\frac{\mathrm{d}V}{\mathrm{d}q} - \kappa\frac{\partial P(q,t)}{\partial q} - \gamma p + \sqrt{D}\,\Gamma(t) \ . \qquad (27)
\end{aligned}
$$

Stationary solutions are given by $P_{\rm st}(p,q) = P_{\rm st}(p)P_{\rm st}(q)$ with $P_{\rm st}(p)$ defined by Eq. (18) and $P_{\rm st}(q)$ given by Eq. (23), where $W(q)$ now corresponds to the Boltzmann distribution

$$W(q) = \frac{\exp\{-V(q)/Q\}}{\int \exp\{-V(q)/Q\}\,{\rm d}q} \; . \qquad (28)$$

3 Conclusion

We have proposed a random walk model in terms of a Langevin equation for particles satisfying a Vlasov-Fokker-Planck equation. The advantage of the random walk model is that all quantities of interest can be computed from the model. For example, single point and joint probability densities as well as correlation functions can be derived. A second advantage is that the random walk model can be solved numerically with relatively little computational effort. As an example, we have studied possible impacts of wakefields on particle bunches in longitudinal particle beams. Using a relatively simple approximation of a wakefield, we have seen that particle-particle interactions can lead to a broadening or squeezing of Gaussian distributions that describe the spatial distribution of the particles in the absence of particle-particle interactions. These broadened and squeezed Gaussian distributions correspond to Haissinski distributions that have also been derived from the proposed random walk model. Having discussed the stationary case, future studies may be devoted to examine time-dependent solutions of the Vlasov-Fokker-Planck equation for bunched particles. Since analytical time-dependent solutions of nonlinear Vlasov-Fokker-Planck equations can hardly be obtained, most probably we will need to employ numerical methods in order to obtain time-dependent solutions. In this context, the proposed Langevin equation offers a promising departure point.

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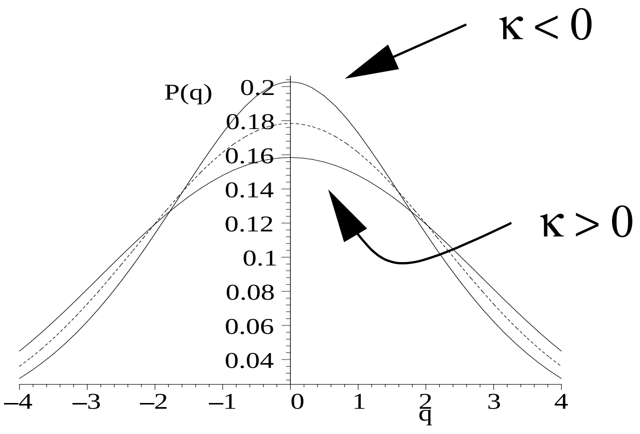
Figure captions:

Fig. 1: Haissinski distributions given by Eq. (23) for $Q = 5$ and several parameters of κ: broadened distribution for $\kappa = 15$ (solid line), Gaussian distribution for $\kappa = 0$ (dashed line), and squeezed distribution for $\kappa = -10$ (solid line)

Fig. 2: Gaussian stationary distribution $P_{\rm st}(p)$ of the Vlasov-Fokker-Planck equation (16) as obtained from the analytical expression (18) (solid line) and the simulation of the Langevin equation (17) via Eq. (25) (diamonds). Parameters: $Q = 5$, $\kappa = 15$, $\gamma = 1$ ($L = 25000$, $\Delta t = 0.03$, $2[\Delta x]^2 = 0.1$, $p_0^l = q_0^l = 0$, evaluation of the stationary case p_n, q_n at $n = 200$).

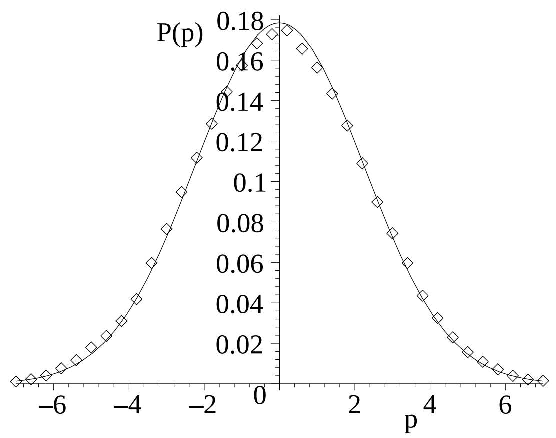
Fig. 3: Haissinski distribution for $\kappa = 15$ computed from Eq. (23) (solid line) and the Langevin equation (17) (diamonds) (parameters other than κ as in Fig. 2).

Fig. 4: Haissinski distribution for $\kappa = -10$ computed from Eq. (23) (solid line) and the Langevin equation (17) (diamonds) (parameters other than κ as in Fig. 2).



κ < 0
κ > 0
P(q)
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
–4
–3
–2
–1
0
1
2
3
4
q

Fig. 1.



P(p)
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
–6
–4
–2
0
2
4
6
p

Fig. 2.

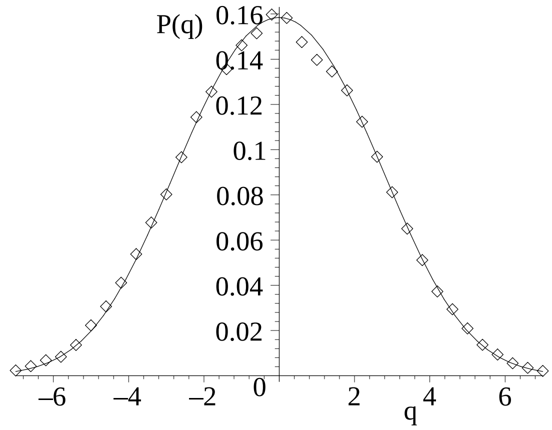

P(q)
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
–6
–4
–2
0
2
4
6
q

Fig. 3.

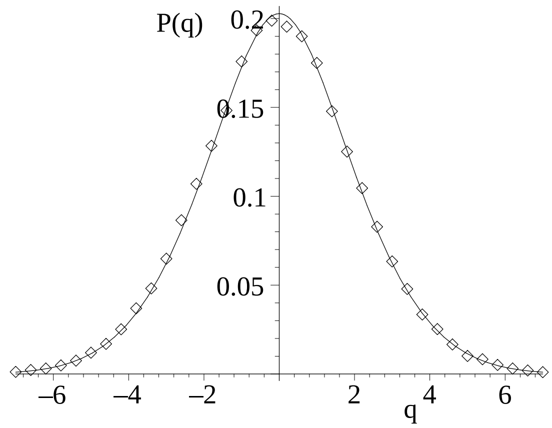

P(q)
0.2
0.15
0.1
0.05
–6
–4
–2
2
4
6
q

Fig. 4.