

Four Equivalent Versions of Non-Abelian Gerbes

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Abstract

We recall and partially expand four versions of smooth, non-abelian gerbes: Čech cocycles, classifying maps, bundle gerbes, and principal 2-bundles. We prove that all these four versions are equivalent, and so establish new relations between interesting recent developments. Prominent partial results we prove are a bijection between continuous and smooth non-abelian cohomology, and an explicit equivalence between bundle gerbes and principal 2-bundles as 2-stacks.

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1 Introduction

Let G be a Lie group and M be a smooth manifold. There are (among others) the following four ways to say what a “smooth G -bundle” over M is:

- (1) *Čech 1-Cocycles*: an open cover $\{U_i\}$ of M , and for each non-empty intersection $U_i \cap U_j$ a smooth map $g_{ij} : U_i \cap U_j \rightarrow G$ satisfying the cocycle condition

$$g_{ij} \cdot g_{jk} = g_{ik}.$$

- (2) *Classifying maps*: a continuous map

$$f : M \rightarrow \mathfrak{B}G$$

to the classifying space $\mathfrak{B}G$ of the group G .

- (3) *Bundle 0-gerbes*: a surjective submersion $\pi : Y \rightarrow M$ and a smooth map $g : Y \times_M Y \rightarrow G$ satisfying

$$\pi_{12}^* g \cdot \pi_{23}^* g = \pi_{13}^* g,$$

where $\pi_{ij} : Y \times_M Y \times_M Y \rightarrow Y \times_M Y$ denotes the projection to the i th and the j th factor.

- (4) *Principal bundles*: a surjective submersion $\pi : P \rightarrow M$ with a smooth action of G on P that preserves π , such that the map

$$P \times G \rightarrow P \times_M P : (p, g) \mapsto (p, p.g)$$

is a diffeomorphism.

It is well-known that these four versions of “smooth G -bundles” are all equivalent. Indeed, (1) and (2) form the smooth and continuous G -valued Čech cohomology in degree one, respectively, and these coincide. Further, (3) and (4) form equivalent categories; and isomorphism classes of the objects (3) are in bijection with equivalence classes of the cocycles (1).

In this article we provide an analogous picture for “smooth Γ -gerbes”, where Γ is a strict Lie 2-group. In particular, Γ can be the automorphism 2-group of an ordinary Lie group G , in which case the term “non-abelian G -gerbe” is commonly used. We compare the following four versions:

Version I: *Smooth, non-abelian Čech Γ -cocycles* (Definition 3.6). These form the classical, smooth groupoid-valued cohomology $\check{H}^1(M, \Gamma)$ in the sense of Giraud [Gir71] and Breen [Bre94], [Bre90, Ch. 4].

Version II: *Classifying maps* (Definition 4.4). These are continuous maps $f : M \rightarrow \mathfrak{B}|\Gamma|$ to the classifying space of the geometric realization of Γ ; such maps have been introduced and studied by Baez and Stevenson [BS09].

Version III: *Γ -bundle gerbes* (Definition 5.1.1). These have been developed by Aschieri, Cantini and Jurco [ACJ05] as a generalization of the abelian bundle gerbes of Murray [Mur96]. Here we present an equivalent but more conceptual definition by applying a higher categorical version [NS11] of Grothendieck’s stackification construction to the monoidal pre-2-stack of principal Γ -bundles.

Version IV: *Principal Γ -2-bundles* (Definition 6.1.5). We give a new definition which is stricter than the original one of Bartels [Bar04], but more general than the one of Wockel [Woc]. In more detail, our principal 2-bundles carry *strict* Γ -actions, whereas local trivializations are so-called anafunctors.

Apart from improving the existing definitions of Versions III and IV, the main contribution of this article is to prove that all four versions listed above are equivalent. We follow the same line of arguments as in the case of G -bundles outlined before:

- Baez and Stevenson have shown that homotopy classes of classifying maps of Version II are in bijection with the continuous groupoid-valued Čech cohomology $\check{H}_c^1(M, \Gamma)$. We prove

(Proposition 4.1) that the inclusion of *smooth* into *continuous* Čech Γ -cocycles induces a bijection $\check{H}_c^1(M, \Gamma) \cong \check{H}^1(M, \Gamma)$. These two results establish the equivalence between our Versions I and II (Theorem 4.6).

- Γ -bundle gerbes and principal Γ -2-bundles over M form bicategories. We prove (Theorem 7.1) that these bicategories are equivalent, and so establish the equivalence between Versions III and IV in the strongest possible sense. Our proof provides explicit 2-functors in both directions.
- We prove the equivalence between Versions I and III by showing that non-abelian Γ -bundle gerbes are classified by the non-abelian cohomology $\check{H}^1(M, \Gamma)$ (Theorem 5.3.2).

The aim of this paper is to simplify and clarify the notion of a non-abelian gerbe, and to make it possible to compare and transfer available results between the various versions. As an example, we use Theorem 7.1 – the equivalence between Γ -bundle gerbes and principal Γ -2-bundles – in order to carry two facts about Γ -bundle gerbes over to principal Γ -2-bundles. We prove:

1. Principal Γ -2-bundles form a 2-stack over smooth manifolds (Theorem 6.2.1). This is a new and evidently important result, since it explains precisely in which way one can *glue* 2-bundles from local patches.
2. If Γ and Ω are weakly equivalent Lie 2-groups, the 2-stacks of principal Γ -2-bundles and principal Ω -2-bundles are equivalent (Theorem 6.2.2). This is another new result that generalizes the well-known fact that principal G -bundles and principal H -bundles form equivalent stacks, whenever G and H are isomorphic Lie groups.

The two facts about Γ -bundle gerbes (Theorems 5.1.5 and 5.2.2) on which these results are based are proved in an outmost abstract way: the first is a mere consequence of the definition of Γ -bundle gerbes we give, namely via a 2-stackification procedure for principal Γ -bundles. The second follows from the fact that principal Γ -bundles and principal Ω -bundles form equivalent monoidal pre-2-stacks, which is in turn a simple corollary of their description by anafunctors that we frequently use.

The present paper is part of a larger program. In a forthcoming paper, we address the discussion of non-abelian lifting problems, in particular string structures. In a second forthcoming paper we will present the picture of four equivalent versions in a setting *with connections*, based on the results of the present paper. Our motivation is to understand the role of 2-bundles with connections in higher gauge theories, where they serve as “B-fields”. Here, two (non-abelian) 2-groups are especially important, namely the string group [BCSS07] and the Jandl group [NS11]. More precisely, string-2-bundles appear in supersymmetric sigma models that describe fermionic string theories [Bun]; while Jandl-2-bundles appear in unoriented sigma models that describe e.g. bosonic type I string theories [SSW07].

This paper is organized as follows. In Section 2 we recall and summarize the theory of principal groupoid bundles and their description byanafunctors. The rest of the paper is based on this theory. In Sections 3, 4, 5 and 6 we introduce our four versions of smooth Γ -gerbes, and establish all but one equivalence. The remaining equivalence, the one between bundle gerbes and principal 2-bundles, is discussed in Section 7.

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2 Preliminaries

There is no claim of originality in this section. Our sources are Lerman [Ler], Metzler [Met], Heinloth [Hei04] and Moerdijk-Mrčun [MM03]. A slightly different but equivalent approach is developed in [MRS].

2.1 Lie Groupoids and Groupoid Actions on Manifolds

We assume that the reader is familiar with the notions of Lie groupoids, smooth functors and smooth natural transformations. In this paper, the following examples of Lie groupoids appear:

Example 2.1.1.

- (a) Every smooth manifold M defines a Lie groupoid denoted M_{dis} whose objects and morphisms are M , and all whose structure maps are identities.
- (b) Every Lie group G defines a Lie groupoid denoted $\mathcal{B}G$, with one object, with G as its smooth manifold of morphisms, and with the composition $g_2 \circ g_1 := g_2 g_1$.
- (c) Suppose X is a smooth manifold and $\rho : H \times X \rightarrow X$ is a smooth left action of a Lie group H on X . Then, a Lie groupoid $X//H$ is defined with objects X and morphisms $H \times X$, and with

$$s(h, x) := x \quad , \quad t(h, x) := \rho(h, x) \quad \text{and} \quad \text{id}_x := (1, x).$$

The composition is

$$(h_2, x_2) \circ (h_1, x_1) := (h_2 h_1, x_1),$$

where $x_2 = \rho(h_1, x_1)$. The Lie groupoid $X//H$ is called the *action groupoid* of the action of H on X .

(d) Let $t : H \rightarrow G$ be a homomorphism of Lie groups. Then,

$$\rho : H \times G \rightarrow G : (h, g) \mapsto (t(h)g)$$

defines a smooth left action of H on G . Thus, we have a Lie groupoid $G//H$.

(e) To every Lie groupoid Γ one can associate an *opposite Lie groupoid* Γ^{op} which has the source and the target map exchanged.

We say that a *right action* of a Lie groupoid Γ on a smooth manifold M is a pair (α, ρ) consisting of smooth maps $\alpha : M \rightarrow \Gamma_0$ and $\rho : M \times_{\alpha} \Gamma_1 \rightarrow M$ such that

$$\rho(\rho(x, g), h) = \rho(x, g \circ h) \quad , \quad \rho(x, \text{id}_{\alpha(x)}) = x \quad \text{and} \quad \alpha(\rho(x, g)) = s(g)$$

for all possible $g, h \in \Gamma_1, p \in \Gamma_0$ and $x \in M$. The map α is called *anchor*. Later on we will replace the letter ρ for the action by the symbol \circ that denotes the composition of the groupoid. A *left action* of Γ on M is a right action of the opposite Lie groupoid Γ^{op} . A smooth map $f : M \rightarrow M'$ between Γ -spaces with actions (α, ρ) and (α', ρ') is called Γ -*equivariant*, if

$$\alpha' \circ f = \alpha \quad \text{and} \quad f(\rho(x, g)) = \rho'(f(x), g).$$

Example 2.1.2.

- (a) Let Γ be a Lie groupoid. Then, Γ acts on right on its morphisms Γ_1 by $\alpha := s$ and $\rho := \circ$. It acts on the left on its morphisms by $\alpha := t$ and $\rho := \circ$.
- (b) Let G be a Lie group. Then, a right/left action of the Lie groupoid $\mathcal{B}G$ (see Example 2.1.1 (b)) on M is the same as an ordinary smooth right/left action of G on M .
- (c) Let X be a smooth manifold. A right/left action of X_{dis} (see Example 2.1.1 (a)) on M is the same as a smooth map $\alpha : M \rightarrow X$.

2.2 Principal Groupoid Bundles

We give the definition of a principal bundle in exactly the same way as we are going to define principal \mathcal{L} -bundles in Section 6.

Definition 2.2.1. *Let M be a smooth manifold, and let Γ be a Lie groupoid.*

1. A principal Γ -bundle over M is a smooth manifold P with a surjective submersion $\pi : P \rightarrow M$ and a right Γ -action (α, ρ) that respects the projection π , such that

$$\tau : P \times_{\alpha} \Gamma_1 \rightarrow P \times_M P : (p, g) \mapsto (p, \rho(p, g))$$

is a diffeomorphism.

2. Let P_1 and P_2 be principal Γ -bundles over M . A morphism $\varphi : P_1 \rightarrow P_2$ is a Γ -equivariant smooth map that respects the projections to M .

Principal Γ -bundles over M form a category $\mathcal{B}un_\Gamma(M)$. In fact, this category is a groupoid, i.e. all morphisms between principal Γ -bundles are invertible. There is an evident notion of a pullback f^*P of a principal Γ -bundle P over M along a smooth map $f : X \rightarrow M$, and similarly, morphisms between principal Γ -bundles pull back. These define a functor

$$f^* : \mathcal{B}un_\Gamma(M) \rightarrow \mathcal{B}un_\Gamma(X).$$

These functors make principal Γ -bundles a prestack over smooth manifolds. One can easily show that this prestack is a stack (for the Grothendieck topology of surjective submersions).

Example 2.2.2 (Ordinary principal bundles). For G a Lie group, we have an equality of categories

$$\mathcal{B}un_{\mathcal{B}G}(M) = \mathcal{B}un_G(M),$$

i.e. Definition 2.2.1 reduces consistently to the definition of an ordinary principal G -bundle.

Example 2.2.3 (Trivial principal groupoid bundles). For M a smooth manifold and $f : M \rightarrow \Gamma_0$ a smooth map, $P := M \times_t \Gamma_1$ and $\pi(m, g) := m$ define a surjective submersion, and $\alpha(m, g) := s(g)$ and $\rho((m, g), h) := (m, g \circ h)$ define a right action of Γ on P that preserves the fibers. The map τ we have to look at has the inverse

$$\tau^{-1} : P \times_M P \rightarrow P \times_t \Gamma_1 : ((m, g_1), (m, g_2)) \mapsto ((m, g_1), g_1^{-1} \circ g_2),$$

which is smooth. Thus we have defined a principal Γ -bundle, which is denoted \mathbf{I}_f and called the *trivial bundle for the map f* . Any bundle that is isomorphic to a trivial bundle is called *trivializable*.

Example 2.2.4 (Discrete structure groupoids). For X a smooth manifold, we have an equivalence of categories

$$\mathcal{B}un_{X_{dis}}(M) \cong C^\infty(M, X)_{dis}.$$

Indeed, for a given principal X_{dis} -bundle P one observes that the anchor $\alpha : P \rightarrow X$ descends along the bundle projection to a smooth map $f : M \rightarrow X$, and that isomorphic bundles determine the same map. Conversely, one associates to a smooth map $f : M \rightarrow X$ the trivial principal X_{dis} -bundle \mathbf{I}_f over M .

Example 2.2.5 (Exact sequences). Let

$$1 \longrightarrow H \xrightarrow{t} G \xrightarrow{p} K \longrightarrow 1 \tag{2.2.1}$$

be an exact sequence of Lie groups, and let $\Gamma := G//H$ be the action groupoid associated to the Lie group homomorphism $t : H \rightarrow G$ as explained in Example 2.1.1 (d). In this situation, $p : G \rightarrow K$

is a surjective submersion, and

$$\alpha : G \longrightarrow \Gamma_0 : g \longmapsto g \quad \text{and} \quad \rho : G_\alpha \times_t \Gamma_1 \longrightarrow G : (g, (h, g')) \longmapsto g'$$

define a smooth right action of Γ on G that preserves p . The inverse of the map τ is

$$\tau^{-1} : G \times_K G \longrightarrow G_\alpha \times_t \Gamma_1 : (g_1, g_2) \longmapsto (g_1, (t^{-1}(g_1 g_2^{-1}), g_2)),$$

which is smooth because t is an embedding. Thus, G is a principal Γ -bundle over K .

Next we provide some elementary statements about trivial principal Γ -bundles.

Lemma 2.2.6. *A principal Γ -bundle over M is trivializable if and only if it has a smooth section.*

Proof. A trivial bundle \mathbf{I}_f has the section

$$s_f : M \longrightarrow \mathbf{I}_f : x \longmapsto (x, \text{id}_{f(x)});$$

and so any trivializable bundle has a section. Conversely, suppose a principal Γ -bundle P has a smooth section $s : M \longrightarrow P$. Then, with $f := \alpha \circ s$,

$$\varphi : \mathbf{I}_f \longrightarrow P : (m, g) \longmapsto \rho(s(m), g)$$

is an isomorphism. □

The following consequence shows that principal Γ -bundles of Definition 2.2.1 are locally trivializable in the usual sense.

Corollary 2.2.7. *Let P be a principal Γ -bundle over M . Then, every point $x \in M$ has an open neighborhood U over which P has a trivialization: a smooth map $f : U \longrightarrow \Gamma_0$ and a morphism $\varphi : \mathbf{I}_f \longrightarrow P|_U$.*

Proof. One can choose U such that the surjective submersion $\pi : U \longrightarrow P$ has a smooth section. Then, Lemma 2.2.6 applies to the restriction $P|_U$. □

We determine the Hom-set $\mathcal{H}om(\mathbf{I}_{f_1}, \mathbf{I}_{f_2})$ between trivial principal Γ -bundles defined by smooth maps $f_1, f_2 : M \longrightarrow \Gamma_0$. To a bundle morphism $\varphi : \mathbf{I}_{f_1} \longrightarrow \mathbf{I}_{f_2}$ one associates the smooth function $g : M \longrightarrow \Gamma_1$ which is uniquely defined by the condition

$$(\varphi \circ s_{f_1})(x) = s_{f_2}(x) \circ g(x).$$

for all $x \in M$. It is straightforward to see that

Lemma 2.2.8. *The above construction defines a bijection*

$$\mathcal{H}om(\mathbf{I}_{f_1}, \mathbf{I}_{f_2}) \longrightarrow \{g \in C^\infty(M, \Gamma_1) \mid s \circ g = f_1 \text{ and } t \circ g = f_2\},$$

under which identity morphisms correspond to constant maps and the composition of bundle morphisms corresponds to the point-wise composition of functions.

Finally, we consider the case of principal bundles for action groupoids.

Lemma 2.2.9. *Suppose $X//H$ is a smooth action groupoid. The category $\mathcal{Bun}_{X//H}(M)$ is equivalent to a category with*

Objects: principal H -bundles P_H over M together with a smooth, H -anti-equivariant map $f : P_H \rightarrow X$, i.e. $f(p \cdot h) = h^{-1}f(p)$.

Morphisms: bundle morphisms $\varphi_H : P_H \rightarrow P'_H$ that respect the maps f and f' .

Proof. For a principal $X//H$ -bundle (P, α, ρ) we set $P_H := P$ with the given projection to M . The action of H on P_H is defined by

$$p \star h := \rho(p, (h, h^{-1} \cdot \alpha(p))).$$

This action is smooth, and it follows from the axioms of the principal bundle P that it is principal. The map $f : P_H \rightarrow X$ is the anchor α . The remaining steps are straightforward and left as an exercise. \square

2.3 Anafunctors

An anafunctor is a generalization of a smooth functor between Lie groupoids, similar to a Morita equivalence, and also known as a Hilsun-Skandalis morphism. The idea goes back to Benabou [Bén73], also see [Joh77]. The references for the following definitions are [Ler, Met].

Definition 2.3.1. *Let \mathcal{X} and \mathcal{Y} be Lie groupoids.*

1. *An anafunctor $F : \mathcal{X} \rightarrow \mathcal{Y}$ is a smooth manifold F , a left action (α_l, ρ_l) of \mathcal{X} on F , and a right action (α_r, ρ_r) of \mathcal{Y} on F such that the actions commute and $\alpha_l : F \rightarrow \mathcal{X}_0$ is a principal \mathcal{Y} -bundle over \mathcal{X}_0 .*
2. *A transformation between anafunctors $f : F \Rightarrow F'$ is a smooth map $f : F \rightarrow F'$ which is \mathcal{X} -equivariant, \mathcal{Y} -equivariant, and satisfies $\alpha'_l \circ f = \alpha_l$ and $\alpha'_r \circ f = \alpha_r$.*

The smooth manifold F of an anafunctor is called its *total space*. Notice that the condition that the two actions on F commute implies that each respects the anchor of the other. For fixed Lie groupoids \mathcal{X} and \mathcal{Y} , anafunctors $F : \mathcal{X} \rightarrow \mathcal{Y}$ and transformations form a category $\mathcal{Ana}^\infty(\mathcal{X}, \mathcal{Y})$. Since transformations are in particular morphisms between principal \mathcal{Y} -bundles, every transformation is invertible so that $\mathcal{Ana}^\infty(\mathcal{X}, \mathcal{Y})$ is in fact a groupoid.

Example 2.3.2 (Anafunctors from ordinary functors). Given a smooth functor $\phi : \mathcal{X} \rightarrow \mathcal{Y}$, we obtain an anafunctor in the following way. We set $F := \mathcal{X}_0 \times_{\phi} \times_t \mathcal{Y}_1$ with anchors $\alpha_l : F \rightarrow \mathcal{X}_0$ and

$\alpha_r : F \rightarrow \mathcal{Y}_0$ defined by $\alpha_l(x, g) := x$ and $\alpha_r(x, g) := s(g)$, and actions

$$\rho_l : \mathcal{X}_1 \times_{s \circ \alpha_l} F \rightarrow F \quad \text{and} \quad \rho_r : F \times_{\alpha_r} \mathcal{Y}_1 \rightarrow F$$

defined by $\rho_l(f, (x, g)) := (t(f), \phi(f) \circ g)$ and $\rho_r((x, g), f) := (x, g \circ f)$. In the same way, a smooth natural transformation $\eta : \phi \Rightarrow \phi'$ defines a transformation $f_\eta : F \Rightarrow F'$ by $f_\eta(x, g) := (x, \eta(x) \circ g)$. Conversely, one can show that an anafunctor comes from a smooth functor, if its principal Γ -bundle has a smooth section.

Example 2.3.3 (Anafunctors with discrete source). For M a smooth manifold and Γ a Lie groupoid, we have an equality of categories

$$\mathcal{Bun}_\Gamma(M) = \mathcal{Ana}^\infty(M_{dis}, \Gamma).$$

Further, *trivial* principal Γ -bundles correspond to smooth *functors*. In particular, with Example 2.2.2 we have,

- (a) For G a Lie group and M a smooth manifold, an anafunctor $F : M_{dis} \rightarrow \mathcal{B}G$ is the same as an ordinary principal G -bundle over M .
- (b) For M and X smooth manifolds, an anafunctor $F : M_{dis} \rightarrow X_{dis}$ is the same as a smooth map.

Example 2.3.4 (Anafunctors with discrete target). For Γ a Lie groupoid and M a smooth manifold, we have an equivalence of categories

$$\mathcal{Ana}^\infty(\Gamma, M) \cong C^\infty(\Gamma_0, M)_{dis}.$$

In order to see this, we insert the groupoid $\mathcal{Bun}_{M_{dis}}(\Gamma_0)$ in the middle, which is equivalent to $C^\infty(\Gamma_0, M)_{dis}$ according to Example 2.2.4. On the other side, the obvious functor

$$\mathcal{Ana}^\infty(\Gamma, M) \rightarrow \mathcal{Bun}_{M_{dis}}(\Gamma_0)$$

that forgets the Γ -action on the total space of an anafunctor, is an equivalence of categories.

For the following definition, we suppose \mathcal{X} , \mathcal{Y} and \mathcal{Z} are Lie groupoids, and $F : \mathcal{X} \rightarrow \mathcal{Y}$ and $G : \mathcal{Y} \rightarrow \mathcal{Z}$ are anafunctors given by $F = (F, \alpha_l, \rho_l, \alpha_r, \rho_r)$ and $G = (G, \beta_l, \tau_l, \beta_r, \tau_r)$.

Definition 2.3.5. The composition $G \circ F : \mathcal{X} \rightarrow \mathcal{Z}$ is the anafunctor defined in the following way:

1. Its total space is

$$E := (F \times_{\alpha_r \times \beta_l} G) / \sim$$

where $(f, \tau_l(h, g)) \sim (\rho_r(f, h), g)$ for all $h \in \mathcal{Y}_1$ with $\alpha_r(f) = t(h)$ and $\beta_l(g) = s(h)$.

2. The anchors are $(f, g) \mapsto \alpha_l(f)$ and $(f, g) \mapsto \beta_r(g)$.

3. The actions $\mathcal{X}_{1s \times_\alpha} E \rightarrow E$ and $E_{\beta \times_t} \mathcal{Z}_1 \rightarrow E$ are given, respectively, by

$$(\gamma, (f, g)) \mapsto (\rho_l(\gamma, f), g) \quad \text{and} \quad ((f, g), \gamma) \mapsto (f, \tau_r(g, \gamma)).$$

Remark 2.3.6. Lie groupoids, anafunctors and transformations form a bicategory. This bicategory is equivalent to the bicategory of differentiable stacks [Pro96].

In this article, anafunctors serve for two purposes. The first is that one can use conveniently the composition of anafunctors to define *extensions* of principal groupoid bundles:

Definition 2.3.7. If $P : M_{dis} \rightarrow \Gamma$ is a principal Γ -bundle over M , and $\Lambda : \Gamma \rightarrow \Omega$ is an anafunctor, then the principal Ω -bundle

$$\Lambda P := \Lambda \circ P : M_{dis} \rightarrow \Omega$$

is called the extension of P along Λ .

Unwinding this definition, the principal Ω -bundle ΛP has the total space

$$\Lambda P = (P \times_{\alpha} \Lambda) / \sim \tag{2.3.1}$$

where $(p, \rho_l(\gamma, \lambda)) \sim (\rho(p, \gamma), \lambda)$ for all $p \in P$, $\lambda \in \Lambda$ and $\gamma \in \Gamma_1$ with $\alpha(p) = t(\gamma)$ and $\alpha_l(\lambda) = s(\gamma)$. Here α is the anchor and ρ is the action of P , and $\Lambda = (\Lambda, \alpha_l, \alpha_r, \rho_l, \rho_r)$. The bundle projection is $(p, \lambda) \mapsto \pi(p)$, where π is the bundle projection of P , the anchor is $(p, \lambda) \mapsto \alpha_r(\lambda)$, and the action is $(p, \lambda) \circ \omega = (p, \rho_r(\lambda, \omega))$.

Extensions of bundles are accompanied by extensions of bundle morphisms. If $\varphi : P_1 \rightarrow P_2$ is a morphism between Γ -bundles, a morphism $\Lambda\varphi : \Lambda P_1 \rightarrow \Lambda P_2$ is defined by $\Lambda\varphi(p_1, \lambda) := (\varphi(p_1), \lambda)$ in terms of (2.3.1). Summarizing, we have

Lemma 2.3.8. Let M be a smooth manifold and $\Lambda : \Gamma \rightarrow \Omega$ be an anafunctor. Then, extension along Λ is a functor

$$\Lambda : \mathcal{Bun}_\Gamma(M) \rightarrow \mathcal{Bun}_\Omega(M).$$

Moreover, it commutes with pullbacks and so extends to a morphism between stacks.

Next we suppose that $t : H \rightarrow G$ is a Lie group homomorphism, and $G//H$ is the associated action groupoid of Example 2.1.1 (d). We look at the functor $\Theta : G//H \rightarrow \mathcal{B}H$ which is defined by $\Theta(h, g) := h$. Combining Lemma 2.2.9 with the extension along Θ , we obtain

Lemma 2.3.9. The category $\mathcal{Bun}_{G//H}(M)$ of principal $G//H$ -bundles over a smooth manifold M is equivalent to a category with

Objects: principal H -bundles P_H over M together with a section of ΘP_H .

Morphisms: morphisms φ of H -bundles so that $\Theta\varphi$ preserves the sections.

The second motivation for introducing anafunctors is that they provide the inverses to certain smooth functors which are not necessarily equivalences of categories.

Definition 2.3.10. A smooth functor or anafunctor $F : \mathcal{X} \rightarrow \mathcal{Y}$ is called a *weak equivalence*, if there exists an anafunctor $G : \mathcal{Y} \rightarrow \mathcal{X}$ together with transformations $G \circ F \cong \text{id}_{\mathcal{X}}$ and $F \circ G \cong \text{id}_{\mathcal{Y}}$.

We have the following immediate consequence for the stack morphisms of Lemma 2.3.8.

Corollary 2.3.11. Let $\Lambda : \Gamma \rightarrow \Omega$ be a weak equivalence between Lie groupoids. Then, extension of principal bundles along Λ is an equivalence $\Lambda : \mathcal{Bun}_{\Gamma}(M) \rightarrow \mathcal{Bun}_{\Omega}(M)$ of categories. Moreover, these define an equivalence between the stacks \mathcal{Bun}_{Γ} and \mathcal{Bun}_{Ω} .

Concerning the claimed generalization of invertibility, we have the following well-known theorem, see [Ler, Lemma 3.34], [Met, Proposition 60].

Theorem 2.3.12. A smooth functor $F : \mathcal{X} \rightarrow \mathcal{Y}$ is a weak equivalence if and only if the following two conditions are satisfied:

(a) it is smoothly essentially surjective: the map

$$s \circ \text{pr}_2 : \mathcal{X}_0 \times_{F_0} \times_t \mathcal{Y}_1 \rightarrow \mathcal{Y}_0$$

is a surjective submersion.

(b) it is smoothly fully faithful: the diagram

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{F} & \mathcal{Y}_1 \\ s \times t \downarrow & & \downarrow s \times t \\ \mathcal{X}_0 \times \mathcal{X}_0 & \xrightarrow{F \times F} & \mathcal{Y}_0 \times \mathcal{Y}_0 \end{array}$$

is a pullback diagram.

Remark 2.3.13. It is worth noting that any smooth functor $F : \mathcal{X} \rightarrow \mathcal{Y}$ that is a weak equivalence actually has a *canonical* inverse anafunctor.

2.4 Lie 2-Groups and crossed Modules

A (strict) *Lie 2-group* is a Lie groupoid Γ whose objects and morphisms are Lie groups, and all whose structure maps are Lie group homomorphisms. One can conveniently bundle the multiplications and the inversions into smooth functors

$$m : \Gamma \times \Gamma \longrightarrow \Gamma \quad \text{and} \quad i : \Gamma \longrightarrow \Gamma.$$

Example 2.4.1. For A an abelian Lie group, the Lie groupoid $\mathcal{B}A$ from Example 2.1.1 (b) is a Lie 2-group. The condition that A is abelian is necessary.

Example 2.4.2. Let $t : H \longrightarrow G$ be a homomorphism of Lie groups, and let $G//H$ be the corresponding Lie groupoid from Example 2.1.1 (d). This Lie groupoid becomes a Lie 2-group if the following structure is given: a smooth left action of G on H by Lie group homomorphisms, denoted $(g, h) \mapsto {}^g h$, satisfying

$$t({}^g h) = gt(h)g^{-1} \quad \text{and} \quad t({}^t(h)x) = hxh^{-1}$$

for all $g \in G$ and $h, x \in H$. Indeed, the objects G of $G//H$ already form a Lie group, and the multiplication on the morphisms $H \times G$ of $G//H$ is the semi-direct product

$$(h_2, g_2) \cdot (h_1, g_1) = (h_2 {}^{g_2} h_1, g_2 g_1). \tag{2.4.1}$$

The homomorphism $t : H \longrightarrow G$ together with the action of G on H is called a *smooth crossed module*. Summarizing, every smooth crossed module defines a Lie 2-group.

Remark 2.4.3. Every Lie 2-group Γ can be obtained from a smooth crossed module. Indeed, one puts $G := \Gamma_0$ and $H := \ker(s)$, equipped with the Lie group structures defined by the multiplication functor m of Γ . The homomorphism $t : H \longrightarrow G$ is the target map $t : \Gamma_1 \longrightarrow \Gamma_0$, and the action of G on H is given by the formula ${}^g \gamma := \text{id}_g \cdot \gamma \cdot \text{id}_{g^{-1}}$ for $g \in \Gamma_0$ and $\gamma \in \ker(s)$. These two constructions are inverse to each other (up to canonical Lie group isomorphisms and strict Lie 2-group isomorphisms, respectively).

Example 2.4.4. Consider a connected Lie group H , so that its automorphism group $\text{Aut}(H)$ is again a Lie group [OV91]. Then, we have a smooth crossed module $(\text{Aut}(H), H, i, \text{ev})$, where $i : H \longrightarrow \text{Aut}(H)$ is the assignment of inner automorphisms to group elements, and $\text{ev} : \text{Aut}(H) \times H \longrightarrow H$ is the evaluation action. The associated Lie 2-group is denoted $\text{AUT}(H)$ and is called the *automorphism 2-group* of H .

Example 2.4.5. Let

$$1 \longrightarrow H \xrightarrow{t} G \xrightarrow{p} K \longrightarrow 1$$

be an exact sequence of Lie groups, i.e. an exact sequence in which p is a submersion and t is an embedding. The homomorphisms $t : H \longrightarrow G$ and $p : G \longrightarrow K$ define action groupoids $G//H$ and

$K//G$ as explained in Example 2.1.1. The first one is even a Lie 2-group: the action of G on H is defined by ${}^g h := t^{-1}(gt(h)g^{-1})$. This is well-defined: since

$$p(gt(h)g^{-1}) = p(g)p(t(h))p(g^{-1}) = p(g)p(g)^{-1} = 1,$$

the element $gt(h)g^{-1}$ lies in the image of t , and has a unique preimage. The action is smooth because t is an embedding. The axioms of a crossed module are obviously satisfied.

If a Lie groupoid Γ is a Lie 2-group in virtue of a multiplication functor $m : \Gamma \times \Gamma \rightarrow \Gamma$, then the category $\mathcal{Bun}_\Gamma(M)$ of principal Γ -bundles over a smooth manifold M is monoidal:

Definition 2.4.6. *Let $P : M_{dis} \rightarrow \Gamma$ and $Q : M_{dis} \rightarrow \Gamma$ be principal Γ -bundles. The tensor product $P \otimes Q$ is the anafunctor*

$$M_{dis} \xrightarrow{\text{diag}} M_{dis} \times M_{dis} \xrightarrow{P \times Q} \Gamma \times \Gamma \xrightarrow{m} \Gamma.$$

Example 2.4.7.

- (a) Since trivial principal Γ -bundles \mathbf{I}_f correspond to *smooth* functors $f : M_{dis} \rightarrow \Gamma$ (Example 2.3.3), it is clear that $\mathbf{I}_f \otimes \mathbf{I}_g = \mathbf{I}_{fg}$.
- (b) Unwinding Definition 2.4.6 in the general case, the tensor product of two principal Γ -bundles P_1 and P_2 with anchors α_1 and α_2 , respectively, and actions ρ_1 and ρ_2 , respectively, is given by

$$P_1 \otimes P_2 = ((P_1 \times_M P_2)_{m \circ (\alpha_1 \times \alpha_2)} \times_t \Gamma_1) / \sim, \quad (2.4.2)$$

where

$$(p_1, p_2, m(\gamma_1, \gamma_2) \circ \gamma) \sim (\rho_1(p_1, \gamma_1), \rho_2(p_2, \gamma_2), \gamma) \quad (2.4.3)$$

for all $p_1 \in P_1$, $p_2 \in P_2$ and morphisms $\gamma, \gamma_1, \gamma_2 \in \Gamma_1$ satisfying $t(\gamma_i) = \alpha_i(p_i)$ for $i = 1, 2$ and $s(\gamma_1)s(\gamma_2) = t(\gamma)$. The bundle projection is $\tilde{\pi}(p_1, p_2, \gamma) := \pi_1(p_1) = \pi_2(p_2)$, the anchor is $\tilde{\alpha}(p_1, p_2, \gamma) := s(\gamma)$, and the Γ -action is given by $\tilde{\rho}((p_1, p_2, \gamma), \gamma') := (p_1, p_2, \gamma \circ \gamma')$.

As a consequence of Lemma 2.3.8 and the fact that the composition of anafunctors is associative up to coherent transformations, we have

Proposition 2.4.8. *For M a smooth manifold and Γ a Lie 2-group, the tensor product*

$$\otimes : \mathcal{Bun}_\Gamma(M) \times \mathcal{Bun}_\Gamma(M) \rightarrow \mathcal{Bun}_\Gamma(M)$$

equips the groupoid of principal Γ -bundles over M with a monoidal structure. Moreover, it turns the stack \mathcal{Bun}_Γ into a monoidal stack.

Notice that the tensor unit of the monoidal groupoid $\mathcal{B}un_\Gamma(M)$ is the trivial principal Γ -bundle \mathbf{I}_1 associated to the constant map $1 : M \rightarrow \Gamma_0$, or, in terms of anafunctors, the one associated to the constant functor $1 : M \rightarrow \Gamma$.

A (weak) *Lie 2-group homomorphism* between Lie 2-groups (Γ, m_Γ) and (Ω, m_Ω) is an anafunctor $\Lambda : \Gamma \rightarrow \Omega$ together with a transformation

$$\begin{array}{ccc}
 \Gamma \times \Gamma & \xrightarrow{m_\Gamma} & \Gamma \\
 \Lambda \times \Lambda \downarrow & \eta \swarrow & \downarrow \Lambda \\
 \Omega \times \Omega & \xrightarrow{m_\Omega} & \Omega
 \end{array} \tag{2.4.4}$$

A Lie 2-group homomorphism is called *weak equivalence*, if the anafunctor Λ is a weak equivalence. Since extensions and tensor products are both defined via composition of anafunctors, we immediately obtain

Proposition 2.4.9. *Extension along a Lie 2-group homomorphism $\Lambda : \Gamma \rightarrow \Omega$ between Lie 2-groups is a monoidal functor*

$$\Lambda : \mathcal{B}un_\Gamma(M) \rightarrow \mathcal{B}un_\Omega(M)$$

between monoidal categories. Moreover, these form a monoidal morphism between monoidal stacks.

Since a monoidal functor is an equivalence of monoidal categories if it is an equivalence of the underlying categories, Corollary 2.3.11 implies:

Corollary 2.4.10. *For $\Lambda : \Gamma \rightarrow \Omega$ a weak equivalence between Lie 2-groups, the monoidal functor of Proposition 2.4.9 is an equivalence of monoidal categories. Moreover, these form a monoidal equivalence between monoidal stacks.*

If we represent the Lie 2-group Γ by a smooth crossed module $t : H \rightarrow G$ as described in Example 2.4.2, we want to determine explicitly how the tensor product looks like under the correspondence of $G//H$ -bundles and principal H -bundles with anti-equivariant maps to G , see Lemma 2.2.9.

Lemma 2.4.11. *Let $t : H \rightarrow G$ be a crossed module and let P and Q be $G//H$ -bundles over M . Let (P_H, f) and (Q_H, g) be the principal H -bundles together with their H -anti-equivariant maps that belong to P and Q , respectively, under the equivalence of Lemma 2.2.9. Then, the principal H -bundle that corresponds to the tensor product $P \otimes Q$ is given by*

$$(P \otimes Q)_H = (P \times_M Q) / \sim \quad \text{where} \quad (p \star h, q) \sim (p, q \star (f(p)^{-1}h)).$$

The action of H on $(P \otimes Q)_H$ is $[(p, q)] \star h = [(p \star h, q)]$, and the H -anti-equivariant map of $(P \otimes Q)_H$ is $[(p, q)] \mapsto f(p) \cdot g(q)$.

Similar to the tensor product of principal Γ -bundles, the dual P^\vee of a principal Γ -bundle P over M is the extension of P along the inversion $i : \Gamma \rightarrow \Gamma$ of the 2-group, $P^\vee := i(P)$. The equality $m \circ (\text{id}, i) = 1$ of functors $M \rightarrow \Gamma$ induces a “death map” $d : P \otimes P^\vee \rightarrow \mathbf{I}_1$. We are going to use this bundle morphism in Section 5.2, but omit a further systematical treatment of duals for the sake of brevity.

3 Version I: Groupoid-valued Cohomology

We have already mentioned group valued Čech 1-cocycles in the introduction. They consist of an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of M and smooth functions $g_{ij} : U_i \cap U_j \rightarrow G$ satisfying the cocycle condition $g_{ij} \cdot g_{jk} = g_{ik}$. Segal realized [Seg68] that this is the same as a smooth functor

$$g : \check{\mathcal{C}}(\mathcal{U}) \rightarrow \mathcal{B}G$$

where $\mathcal{B}G$ denotes the one-object groupoid introduced in Example 2.1.1 (b) and $\check{\mathcal{C}}(\mathcal{U})$ denotes the Čech groupoid corresponding to the cover \mathcal{U} . It has objects $\bigsqcup_{i \in I} U_i$ and morphisms $\bigsqcup_{i, j \in I} U_i \cap U_j$, and its structure maps are

$$s(x, i, j) = (x, i) \quad , \quad t(x, i, j) = (x, j) \quad , \quad \text{id}_{(x, i)} = (x, i, i) \quad \text{and} \quad (x, j, k) \circ (x, i, j) = (x, i, k).$$

Analogously, smooth natural transformations between smooth functors $\check{\mathcal{C}}(\mathcal{U}) \rightarrow \mathcal{B}G$ give rise to Čech coboundaries. Thus the set $[\check{\mathcal{C}}(\mathcal{U}), \mathcal{B}G]$ of equivalence classes of smooth functors equals the usual first Čech cohomology with respect to the cover \mathcal{U} . The classical first Čech-cohomology $\check{H}^1(M, G)$ of M is hence given by the colimit over all open covers \mathcal{U} of M

$$\check{H}^1(M, G) = \varinjlim_{\mathcal{U}} [\check{\mathcal{C}}(\mathcal{U}), \mathcal{B}G].$$

We use this coincidence in order to define the 0-th Čech cohomology with coefficients in a general Lie groupoid Γ :

Definition 3.1. *If Γ is a Lie groupoid we set*

$$\check{H}^0(M, \Gamma) := \varinjlim_{\mathcal{U}} [\check{\mathcal{C}}(\mathcal{U}), \Gamma]$$

where the colimit is taken over all covers \mathcal{U} of M and $[\check{\mathcal{C}}(\mathcal{U}), \Gamma]$ denotes the set of equivalence classes of smooth functors $\check{\mathcal{C}}(\mathcal{U}) \rightarrow \Gamma$.

Remark 3.2. The choice of the degree is such that $\check{H}^0(M, \Gamma)$ agrees in the case $\Gamma = G_{dis}$ (Example 2.1.1 (a)) with the classical 0-th Čech-cohomology $\check{H}^0(M, G)$ of M with values in G .

The geometrical meaning of the set is given in the following well-known theorem, which can be proved e.g. using Lemma 2.2.8.

Theorem 3.3. *There is a bijection*

$$\check{H}^0(M, \Gamma) \cong \left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \text{principal } \Gamma\text{-bundles over } M \end{array} \right\}.$$

If Γ is not only a Lie 2-groupoid but a Lie 2-group one can also define a *first* cohomology group $\check{H}^1(M, \Gamma)$. Indeed, consider the Lie 2-groupoid $\mathcal{B}\Gamma$ with one object, morphisms Γ_0 and 2-morphisms Γ_1 . Multiplication in Γ gives the composition of morphisms in $\mathcal{B}\Gamma$. Let $[\check{\mathcal{C}}(\mathcal{U}), \mathcal{B}\Gamma]$ denote the set of equivalence classes of smooth, weak 2-functors from the Čech-groupoid $\check{\mathcal{C}}(\mathcal{U})$ to the Lie 2-groupoid $\mathcal{B}\Gamma$. For the definition of weak functors see [Bén67] – below we will determine this set explicitly.

Definition 3.4. *For a 2-group Γ we set*

$$\check{H}^1(M, \Gamma) := \varinjlim_{\mathcal{U}} [\check{\mathcal{C}}(\mathcal{U}), \mathcal{B}\Gamma].$$

Remark 3.5. This agrees for $\Gamma = G_{dis}$ with the classical $\check{H}^1(M, G)$. Furthermore, for an abelian Lie group A the Lie groupoid $\mathcal{B}A$ is even a 2-group and $\check{H}^1(M, \mathcal{B}A)$ agrees with the classical Čech-cohomology $\check{H}^2(M, A)$.

Unwinding the above definition, we get Version I of smooth Γ -gerbes:

Definition 3.6. *Let Γ be a Lie 2-group, and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of M .*

1. A Γ -1-cocycle with respect to \mathcal{U} is a pair $(f_{\alpha\beta}, g_{\alpha\beta\gamma})$ of smooth maps

$$f_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \Gamma_0 \quad \text{and} \quad g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \longrightarrow \Gamma_1$$

satisfying $s(g_{\alpha\beta\gamma}) = f_{\beta\gamma} \cdot f_{\alpha\beta}$ and $t(g_{\alpha\beta\gamma}) = f_{\alpha\gamma}$, and

$$g_{\alpha\beta\delta} \circ (g_{\beta\gamma\delta} \cdot \text{id}_{f_{\alpha\beta}}) = g_{\alpha\gamma\delta} \circ (\text{id}_{f_{\beta\delta}} \cdot g_{\alpha\beta\gamma}). \quad (3.1)$$

Here, the symbols \circ and \cdot stand for the composition and multiplication of Γ , respectively.

2. Two Γ -1-cocycles $(f_{\alpha\beta}, g_{\alpha\beta\gamma})$ and $(f'_{\alpha\beta}, g'_{\alpha\beta\gamma})$ are equivalent, if there exist smooth maps $h_\alpha : U_\alpha \longrightarrow \Gamma_0$ and $s_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \Gamma_1$ with

$$\begin{aligned} s(s_{\alpha\beta}) &= g'_{\alpha\beta} \cdot h_\alpha \quad , \quad t(s_{\alpha\beta}) = h_\beta \cdot g_{\alpha\beta} \\ \text{and} \quad &(\text{id}_{h_\gamma} \cdot g_{\alpha\beta\gamma}) \circ (s_{\beta\gamma} \cdot \text{id}_{f_{\alpha\beta}}) \circ (\text{id}_{f_{\beta\gamma}} \cdot s_{\alpha\beta}) = s_{\alpha\gamma} \circ (g'_{\alpha\beta\gamma} \cdot \text{id}_{h_\alpha}). \end{aligned}$$

Remark 3.7. For a crossed module $t : G \rightarrow H$ and $\Gamma := G//H$ the associated Lie 2-group (Example 2.4.2) one can reduce Γ -1-cocycles to pairs

$$\tilde{f}_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow G \quad \text{and} \quad \tilde{g}_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \rightarrow H$$

which satisfies then a cocycle condition similar to (3.1). Analogously, coboundaries can be reduced to pairs

$$\tilde{h}_\alpha : U_\alpha \rightarrow G \quad \text{and} \quad \tilde{s}_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow H.$$

This yields the common definition of non-abelian cocycles, which can for example be found in [Bre90] or [BS09].

4 Version II: Classifying Maps

It is well known that for a Lie group G the *smooth* Čech-cohomology $\check{H}^1(M, G)$ and the *continuous* Čech-cohomology $\check{H}_c^1(M, G)$ agree if M is a smooth manifold (in particular paracompact). This can e.g. be shown by locally approximating continuous cocycles by smooth ones without changing the cohomology class – see [MW09] (even for G infinite-dimensional). Below we generalize this fact to certain Lie groupoids Γ . Here the continuous Čech-cohomology $\check{H}_c^1(M, \Gamma)$ is defined in the same way as the smooth one (Definition 3.4) but with all maps continuous instead of smooth. A Lie groupoid Γ is called *smoothly separable*, if the set $\pi_0\Gamma$ of isomorphism classes of objects is a smooth manifold for which the projection $\Gamma_0 \rightarrow \pi_0\Gamma$ is a submersion.

Proposition 4.1. *Let M be a smooth manifold and let Γ be a smoothly separable Lie 2-group. Then, the inclusion*

$$\check{H}^1(M, \Gamma) \rightarrow \check{H}_c^1(M, \Gamma)$$

of smooth into continuous Čech cohomology is a bijection.

Remark 4.2. It is possible that the assumption of being smoothly separable is not necessary, but a proof not assuming this would certainly be more involved than ours. Moreover, *all* Lie 2-groups we are interested in are smoothly separable.

Proof of Proposition 4.1. We denote by $\pi_1\Gamma$ the Lie subgroup of Γ_1 consisting of automorphisms of $1 \in \Gamma_0$. Since it has two commuting group structures – composition and multiplication – it is abelian. The idea of the proof is to reduce the statement via long exact sequences to statements proved in [MW09]. The exact sequence we need can be found in [Bre90]:

$$\check{H}^0(M, (\pi_0\Gamma)_{dis}) \rightarrow \check{H}^1(M, \mathcal{B}\pi_1\Gamma) \rightarrow \check{H}^1(M, \Gamma) \rightarrow \check{H}^1(M, (\pi_0\Gamma)_{dis}) \rightarrow \check{H}^2(M, \mathcal{B}\pi_1\Gamma).$$

Note that $\check{H}^1(M, \Gamma)$ and $\check{H}^1(M, (\pi_0\Gamma)_{dis})$ do not have group structures, hence exactness is only meant as exactness of pointed sets. But we actually have more structure, namely an action of

$\check{H}^1(M, \mathcal{B}_{\pi_1}\Gamma)$ on $\check{H}^1(M, \Gamma)$. This action factors to an action of

$$C := \text{coker}\left(\check{H}^0(M, (\pi_0\Gamma)_{dis}) \longrightarrow \check{H}^1(M, \mathcal{B}_{\pi_1}\Gamma)\right).$$

In fact on the non-empty fibres of the morphism $\check{H}^1(M, \Gamma) \longrightarrow \check{H}^1(M, (\pi_0\Gamma)_{dis})$ this action is simply transitive. In other words: $\check{H}^1(M, \Gamma)$ is a C -Torsor over

$$K := \ker\left(\check{H}^1(M, (\pi_0\Gamma)_{dis}) \longrightarrow \check{H}^2(M, \mathcal{B}_{\pi_1}\Gamma)\right).$$

The same type of sequence also exists in continuous cohomology

$$\check{H}_c^0(M, (\pi_0\Gamma)_{dis}) \longrightarrow \check{H}_c^1(M, \mathcal{B}_{\pi_1}\Gamma) \longrightarrow \check{H}_c^1(M, \Gamma) \longrightarrow \check{H}_c^1(M, (\pi_0\Gamma)_{dis}) \longrightarrow \check{H}_c^2(M, \mathcal{B}_{\pi_1}\Gamma).$$

With

$$\begin{aligned} C' &:= \text{coker}\left(\check{H}_c^0(M, (\pi_0\Gamma)_{dis}) \longrightarrow \check{H}_c^1(M, \mathcal{B}_{\pi_1}\Gamma)\right) \\ K' &:= \ker\left(\check{H}_c^1(M, (\pi_0\Gamma)_{dis}) \longrightarrow \check{H}_c^2(M, \mathcal{B}_{\pi_1}\Gamma)\right), \end{aligned}$$

we exhibit $\check{H}_c^1(M, \Gamma)$ as a C' -Torsor over K' .

The natural inclusions of smooth into continuous cohomology form a chain map between the two sequences. From [MW09] we know that they are isomorphisms on the second, fourth and fifth factor. In particular we have an induced isomorphism $K \xrightarrow{\sim} K'$. Lemma 4.3 below additionally shows that also the induced morphism $C \longrightarrow C'$ is an isomorphism. Using these isomorphisms we see that $\check{H}^1(M, \Gamma)$ and $\check{H}_c^1(M, \Gamma)$ are both C -torsors over K and that the natural map

$$\check{H}^1(M, \Gamma) \longrightarrow \check{H}_c^1(M, \Gamma)$$

is a morphism of torsors. But each morphism of group torsors is bijective, which concludes the proof. \square

Lemma 4.3. *The images of*

$$f : \check{H}^0(M, (\pi_0\Gamma)_{dis}) \longrightarrow \check{H}^1(M, \mathcal{B}_{\pi_1}\Gamma) \quad \text{and} \quad f' : \check{H}_c^0(M, (\pi_0\Gamma)_{dis}) \longrightarrow \check{H}_c^1(M, \mathcal{B}_{\pi_1}\Gamma)$$

are isomorphic.

Proof. $\check{H}^0(M, (\pi_0\Gamma)_{dis})$ is the group of smooth maps $s : M \longrightarrow \pi_0\Gamma$ and $\check{H}_c^0(M, (\pi_0\Gamma)_{dis})$ is the group of continuous maps $t : M \longrightarrow \pi_0\Gamma$. The groups $\check{H}^1(M, \mathcal{B}_{\pi_1}\Gamma) = \check{H}^2(M, \pi_1\Gamma)$ and $\check{H}_c^1(M, \mathcal{B}_{\pi_1}\Gamma) = \check{H}_c^2(M, \pi_1\Gamma)$ are isomorphic by the result of [MW09]. Under the connecting homomorphism

$$\check{H}^0(\pi_0\Gamma, (\pi_0\Gamma)_{dis}) \longrightarrow \check{H}^1(\pi_0\Gamma, \mathcal{B}_{\pi_1}\Gamma)$$

the identity $\text{id}_{\pi_0\Gamma}$ is sent to a class ξ_Γ with the property that $f(s) = s^*\xi_\Gamma$ and $f'(t) = t^*\xi_\Gamma$. Hence it suffices to show that for each continuous map $t : M \longrightarrow \pi_0\Gamma$ there is a smooth map $s : M \longrightarrow \pi_0\Gamma$

with $s^*\xi_\Gamma = t^*\xi_\Gamma$. It is well known that for each continuous map t between smooth manifolds a homotopic smooth map s exists. It remains to show that the pullback $\check{H}^1(\pi_0\Gamma, \mathcal{B}\pi_1\Gamma) \rightarrow \check{H}^1(M, \mathcal{B}\pi_1\Gamma)$ along smooth maps is homotopy invariant. This can e.g. be seen by choosing smooth (abelian) $\mathcal{B}\pi_1\Gamma$ -bundle gerbes as representatives, in which case the homotopy invariance can be deduced from the existence of connections. \square

It is a standard result in topology that the continuous G -valued Čech cohomology of paracompact spaces is in bijection with homotopy classes of maps to the classifying space $\mathfrak{B}G$ of the group G . A model for the classifying space $\mathfrak{B}G$ is for example the geometric realization of the nerve of the groupoid $\mathcal{B}G$, or Milnor’s join construction [Mil56].

Now let Γ be a Lie 2-group, and let $|\Gamma|$ denote the geometric realization of the nerve of Γ . Since the nerve is a simplicial topological group, $|\Gamma|$ is a topological group. Version II for smooth Γ -gerbes is:

Definition 4.4 ([BS09]). *A classifying map for a smooth Γ -gerbe is a continuous map*

$$f : M \rightarrow \mathfrak{B}|\Gamma|.$$

We denote by $[M, \mathfrak{B}|\Gamma|]$ the set of homotopy classes of classifying maps.

Proposition 4.5 ([BS09, Theorem 1]). *Let Γ be a Lie 2-group. Then there is a bijection*

$$\check{H}_c^1(M, \Gamma) \cong [M, \mathfrak{B}|\Gamma|]$$

where the topological group $|\Gamma|$ is the geometric realization of the nerve of Γ .

Note that the assumption of [BS09, Theorem 1] that Γ is well-pointed is automatically satisfied because Lie groups are well-pointed. Propositions 4.1 and 4.5 imply the following equivalence theorem between Version I and Version II.

Theorem 4.6. *For M a smooth manifold and Γ a smoothly separable Lie 2-group, there is a bijection*

$$\check{H}^1(M, \Gamma) \cong [M, \mathfrak{B}|\Gamma|].$$

Remark 4.7. Baez and Stevenson argue in [BS09, Section 5.2.] that the space $\mathfrak{B}|\Gamma|$ is homotopy equivalent to a certain geometric realization of the Lie 2-groupoid $|\mathcal{B}\Gamma|$ from Section 3. Baas, Böstedt and Kro have shown [BBK] that $|\mathcal{B}\Gamma|$ classifies concordance classes of “charted Γ -2-bundles”. In particular, charted Γ -2-bundles are a further equivalent version of smooth Γ -gerbes.

5 Version III: Groupoid Bundle Gerbes

Several definitions of “non-abelian bundle gerbes” have appeared in literature so far [ACJ05, Jur, MRS]. The approach we give here not only shows a conceptually clear way to define non-abelian bundle gerbes, but also produces systematically a whole bicategory. Moreover, these bicategories form a 2-stack over smooth manifolds (with the Grothendieck topology of surjective submersions).

5.1 Definition via the Plus Construction

Recall that the stack $\mathcal{B}un_\Gamma$ of principal Γ -bundles is monoidal if Γ is a Lie 2-group (Proposition 2.4.8). Associated to the monoidal stack $\mathcal{B}un_\Gamma$ we have a pre-2-stack

$$\mathit{TrivGrb}_\Gamma := \mathcal{B}(\mathcal{B}un_\Gamma)$$

of *trivial* Γ -gerbes. Explicitly, there is one trivial Γ -gerbe \mathcal{I} over every smooth manifold M . The 1-morphisms from \mathcal{I} to \mathcal{I} are principal Γ -bundles over M , and the 2-morphisms between those are morphisms of principal Γ -bundles. Horizontal composition is given by the tensor product of principal Γ -bundles, and vertical composition is the ordinary composition of Γ -bundle morphisms.

Now we apply the *plus construction* of [NS11] in order to stackify this pre-2-stack. The resulting 2-stack is by definition the *2-stack of Γ -bundle gerbes*, i.e.

$$\mathit{Grb}_\Gamma := (\mathit{TrivGrb}_\Gamma)^+.$$

Unwinding the details of the plus construction, we obtain the following definitions:

Definition 5.1.1. *Let M be a smooth manifold. A Γ -bundle gerbe over M is a surjective submersion $\pi : Y \rightarrow M$, a principal Γ -bundle P over Y ^[2] and an associative morphism*

$$\mu : \pi_{23}^* P \otimes \pi_{12}^* P \rightarrow \pi_{13}^* P$$

of Γ -bundles over Y ^[3].

The morphism μ is called the *bundle gerbe product*. Its associativity is the evident condition for bundle morphisms over Y ^[4].

In order to proceed with the 1-morphisms, we say that a *common refinement* of two surjective submersions $\pi_1 : Y_1 \rightarrow M$ and $\pi_2 : Y_2 \rightarrow M$ is a smooth manifold Z together with surjective submersions $Z \rightarrow Y_1$ and $Z \rightarrow Y_2$ such that the diagram

$$\begin{array}{ccc} & Z & \\ & \swarrow & \searrow \\ Y_1 & & Y_2 \\ & \searrow^{\pi_1} & \swarrow_{\pi_2} \\ & M & \end{array}$$

is commutative.

We fix the following convention: suppose P_1 and P_2 are Γ -bundles over surjective submersions U_1 and U_2 , respectively, and V is a common refinement of U_1 and U_2 . Then, a bundle morphism $\varphi : P_1 \rightarrow P_2$ is understood to be a bundle morphism between the pullbacks of P_1 and P_2 to the common refinement V . For example, in the following definition this convention applies to $U_1 = Y_1^{[2]}$, $U_2 = Y_2^{[2]}$ and $V = Z^{[2]}$.

Definition 5.1.2. Let \mathcal{G}_1 and \mathcal{G}_2 be Γ -bundle gerbes over M . A 1-morphism $\mathcal{A} : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ is a common refinement Z of the surjective submersions of \mathcal{G}_1 and \mathcal{G}_2 together with a principal Γ -bundle Q over Z and a morphism

$$\beta : P_2 \otimes \zeta_1^* Q \rightarrow \zeta_2^* Q \otimes P_1$$

of Γ -bundles over $Z^{[2]}$, where $\zeta_1, \zeta_2 : Z^{[2]} \rightarrow Z$ are the two projections, such that α is compatible with the bundle gerbe products μ_1 and μ_2 .

The compatibility of α with μ_1 and μ_2 means that the diagram

$$\begin{array}{ccc}
\pi_{23}^* P_2 \otimes \pi_{12}^* P_2 \otimes \zeta_1^* Q & \xrightarrow{\mu_2 \otimes \text{id}} & \pi_{13}^* P_2 \otimes \zeta_1^* Q \\
\downarrow \text{id} \otimes \zeta_{12}^* \beta & & \downarrow \zeta_{13}^* \beta \\
\pi_{23}^* P_2 \otimes \zeta_2^* Q \otimes \pi_{12}^* P_1 & & \\
\downarrow \zeta_{23}^* \beta \otimes \text{id} & & \\
\zeta_3^* Q \otimes \pi_{23}^* P_1 \otimes \pi_{12}^* P_1 & \xrightarrow{\text{id} \otimes \mu_1} & \zeta_3^* Q \otimes \pi_{13}^* P_1
\end{array} \tag{5.1.1}$$

of morphisms of Γ -bundles over $Z^{[3]}$ is commutative.

If $\mathcal{A}_{12} : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ and $\mathcal{A}_{23} : \mathcal{G}_2 \rightarrow \mathcal{G}_3$ are 1-morphisms between bundle gerbes over M , the composition $\mathcal{A}_{23} \circ \mathcal{A}_{12} : \mathcal{G}_1 \rightarrow \mathcal{G}_3$ is given by the fibre product $Z := Z_{23} \times_{Y_2} Z_{12}$, the principal Γ -bundle $Q := Q_{23} \otimes Q_{12}$ over Z , and the morphism

$$P_3 \otimes \zeta_1^* Q \xrightarrow{\beta_{23} \otimes \text{id}} \zeta_2^* Q_{23} \otimes P_2 \otimes \zeta_1^* Q_{12} \xrightarrow{\text{id} \otimes \beta_{12}} \zeta_2^* Q \otimes P_1.$$

The identity 1-morphism $\text{id}_{\mathcal{G}}$ associated to a Γ -bundle gerbe \mathcal{G} is given by Y regarded as a common refinement of $\pi : Y \rightarrow M$ with itself, the trivial Γ -bundle \mathbf{I}_1 (the tensor unit of $\mathcal{B}un_{\Gamma}(Y)$), and the evident morphism $\mathbf{I}_1 \otimes P \rightarrow P \otimes \mathbf{I}_1$.

In order to define 2-morphisms, suppose that $\pi_1 : Y_1 \rightarrow M$ and $\pi_2 : Y_2 \rightarrow M$ are surjective submersions, and that Z and Z' are common refinements of π_1 and π_2 . Let W be a common refinement of Z and Z' with surjective submersions $r : W \rightarrow Z$ and $r' : W \rightarrow Z'$. We obtain

two maps

$$s_1 : W \xrightarrow{r} Z \longrightarrow Y_1 \quad \text{and} \quad t_1 : W \xrightarrow{r'} Z' \longrightarrow Y_1,$$

and analogously, two maps $s_2, t_2 : W \rightarrow Y_2$. These patch together to maps

$$x_W := (s_1, t_1) : W \rightarrow Y_1 \times_M Y_1 \quad \text{and} \quad y_W := (s_2, t_2) : W \rightarrow Y_2 \times_M Y_2.$$

Definition 5.1.3. Let \mathcal{G}_1 and \mathcal{G}_2 be Γ -bundle gerbes over M , and let $\mathcal{A}, \mathcal{A}' : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ be 1-morphisms. A 2-morphism $\varphi : \mathcal{A} \Rightarrow \mathcal{A}'$ is a common refinement W of the common refinements Z and Z' , together with a morphism

$$\varphi : y_W^* P_2 \otimes r^* Q \rightarrow r'^* Q' \otimes x_W^* P_1$$

of Γ -bundles over W that is compatible with the morphisms β and β' .

The compatibility means that a certain diagram over $W^{[2]}$ commutes. Fibrewise over a point $(w, w') \in W \times_M W$ this diagram looks as follows:

$$\begin{array}{ccc}
P_2|_{s_2(w'), t_2(w')} \otimes P_2|_{s_2(w), s_2(w')} \otimes Q|_{r(w)} & \xrightarrow{\text{id} \otimes \beta} & P_2|_{s_2(w'), t_2(w')} \otimes Q|_{r(w')} \otimes P_1|_{s_1(w), s_1(w')} \\
\downarrow \mu_2 \otimes \text{id} & & \downarrow \varphi \otimes \text{id} \\
P_2|_{s_2(w), t_2(w')} \otimes Q|_{r(w)} & & Q'|_{r'(w')} \otimes P_1|_{s_1(w'), t_1(w')} \otimes P_1|_{s_1(w), s_1(w')} \\
\downarrow \mu_2^{-1} \otimes \text{id} & & \downarrow \text{id} \otimes \mu_1 \\
P_2|_{t_2(w), t_2(w')} \otimes P_2|_{s_2(w), t_2(w)} \otimes Q|_{r(w)} & & Q'|_{r'(w')} \otimes P_1|_{s_1(w), t_1(w')} \\
\downarrow \text{id} \otimes \varphi & & \downarrow \text{id} \otimes \mu_1^{-1} \\
P_2|_{t_2(w), t_2(w')} \otimes Q'|_{r'(w)} \otimes P_1|_{s_1(w), t_1(w)} & \xrightarrow{\beta' \otimes \text{id}} & Q'|_{r'(w')} \otimes P_1|_{t_1(w), t_1(w')} \otimes P_1|_{s_1(w), t_1(w)}
\end{array} \tag{5.1.2}$$

Finally we identify two 2-morphisms $(W_1, r_1, r'_1, \varphi_1)$ and $(W_2, r_2, r'_2, \varphi_2)$ if the pullbacks of φ_1 and φ_2 to $W \times_{Z \times_{\times} Z'} W'$ agree. Explicitly, this condition means that for all $w_1 \in W_1$ and $w_2 \in W_2$ with $r_1(w_1) = r_2(w_2)$ and $r'_1(w_1) = r'_2(w_2)$, and for all $p_2 \in y_{W_1}^* P_2 = y_{W_2}^* P_2$ and $q \in r_1^* Q = r_2^* Q$ we have $\varphi_1(p_2, q) = \varphi_2(p_2, q)$.

Remark 5.1.4.

- In the above situation of a common refinement W of two common refinements Z, Z' of sur-

jective submersions Y_1, Y_2 , the diagram

$$\begin{array}{ccc}
 & Z & \\
 & \swarrow & \searrow \\
 Y_1 & & Y_2 \\
 & \nwarrow & \nearrow \\
 & W & \\
 & \downarrow r' & \\
 & Z' &
 \end{array}
 \tag{5.1.3}$$

is *not* necessarily commutative. In fact, diagram (5.1.3) commutes if and only if the two maps $x_W : W \rightarrow Y_1 \times_M Y_1$ and $y_W : W \rightarrow Y_2 \times_M Y_2$ factor through the diagonal maps $Y_1 \rightarrow Y_1 \times_M Y_1$ and $Y_2 \rightarrow Y_2 \times_M Y_2$, respectively.

- In the case that a 2-morphism φ is defined on a common refinement Z for which diagram (5.1.3) *does* commute, Definition 5.1.3 can be simplified. As remarked before, the two maps x_W and y_W factor through the diagonals, over which the bundles P_1 and P_2 have canonical trivializations (see Corollary 5.2.6). Under these trivializations, φ can be identified with a bundle morphism

$$\varphi : Q \rightarrow Q'.$$

Furthermore, the compatibility diagram (5.1.2) simplifies to the diagram

$$\begin{array}{ccc}
 P_2 \otimes \eta_1^* Q & \xrightarrow{\beta} & \eta_2^* Q \otimes P_1 \\
 \text{id} \otimes \eta_1^* \varphi \downarrow & & \downarrow \eta_2^* \varphi \otimes \text{id} \\
 P_2 \otimes \eta_1^* Q' & \xrightarrow{\beta'} & \eta_2^* Q' \otimes P_1.
 \end{array}
 \tag{5.1.4}$$

Next we define the vertical composition $\varphi_{23} \bullet \varphi_{12} : \mathcal{A}_1 \rightrightarrows \mathcal{A}_3$ of 2-morphisms $\varphi_{12} : \mathcal{A}_1 \rightrightarrows \mathcal{A}_2$ and $\varphi_{23} : \mathcal{A}_2 \rightrightarrows \mathcal{A}_3$. The refinement is the fibre product $W := W_{12} \times_{Z_2} W_{23}$ of the covers of φ_{12} and φ_{23} . The bundle gerbe products induce isomorphisms

$$x_W^* P_1 \cong x_{W_{23}}^* P_1 \otimes x_{W_{12}}^* P_1 \quad \text{and} \quad y_W^* P_2 \cong y_{W_{23}}^* P_2 \otimes y_{W_{12}}^* P_2$$

over W . Under these identifications, the morphism $y_W^* P_2 \otimes Q_1 \rightarrow Q_3 \otimes x_W^* P_1$ for the 2-morphism $\varphi_{23} \bullet \varphi_{12}$ is defined as

$$y_{W_{23}}^* P_2 \otimes y_{W_{12}}^* P_2 \otimes Q_1 \xrightarrow{\text{id} \otimes \varphi_{12}} y_{W_{23}}^* P_2 \otimes Q_2 \otimes x_{W_{12}}^* P_1 \xrightarrow{\varphi_{23} \otimes \text{id}} Q_3 \otimes x_{W_{23}}^* P_1 \otimes x_{W_{12}}^* P_1.$$

The identity for vertical composition is just the identity refinement and the identity morphism. Finally we come to the horizontal composition

$$\varphi_{23} \circ \varphi_{12} : \mathcal{A}_{23} \circ \mathcal{A}_{12} \rightrightarrows \mathcal{A}'_{23} \circ \mathcal{A}'_{12}$$

of 2-morphisms $\varphi_{12} : \mathcal{A}_{12} \rightrightarrows \mathcal{A}'_{12}$ and $\varphi_{23} : \mathcal{A}_{23} \rightrightarrows \mathcal{A}'_{23}$: its refinement W is given by $W_{12} \times_{(Y_2 \times Y_2)} W_{23}$. We look at the three relevant maps $x_W : W \rightarrow Y_1 \times_M Y_1$, $y_W : W \rightarrow Y_2 \times_M Y_2$ and $z_W : W \rightarrow Y_3 \times_M Y_3$. The morphism φ of the 2-morphism $\varphi_{23} \circ \varphi_{12}$ is defined as the composition

$$z_W^* P_3 \otimes Q_{23} \otimes Q_{12} \xrightarrow{\varphi_{23} \otimes \text{id}} Q'_{23} \otimes y_W^* P_2 \otimes Q_{12} \xrightarrow{\text{id} \otimes \varphi_{12}} Q'_{23} \otimes Q'_{12} \otimes x_W^* P_1.$$

It follows from the properties of the plus construction [NS11] that (a) these definitions fit together into a bicategory $\mathcal{G}rb_\Gamma(M)$, and that (b) these form a pre-2-stack $\mathcal{G}rb_\Gamma$ over smooth manifolds. That means, there are *pullback 2-functors*

$$f^* : \mathcal{G}rb_\Gamma(N) \rightarrow \mathcal{G}rb_\Gamma(M)$$

associated to smooth maps $f : M \rightarrow N$, and that these are compatible with the composition of smooth maps. Pullbacks of Γ -bundle gerbes, 1-morphisms, and 2-morphisms are obtained by just taking the pullbacks of all involved data. Finally, the plus construction implies (c):

Theorem 5.1.5 ([NS11, Theorem 3.3]). *The pre-2-stack $\mathcal{G}rb_\Gamma$ of Γ -bundle gerbes is a 2-stack.*

In the remainder of this section we give some examples and describe relations between the definitions given here and existing ones.

Example 5.1.6. Let A be an abelian Lie group, for instance $U(1)$. Then, $\mathcal{B}A$ -bundle gerbes are the same as the well-known A -bundle gerbes [Mur96].

Example 5.1.7. Let (G, H, t, α) be a smooth crossed module, and let $G//H$ the associated action groupoid. Then, a $(G//H)$ -bundle gerbe is the same as a *crossed module bundle gerbe* in the sense of Jurco [Jur]. The equivalence relation “stably isomorphic” of [Jur] is given by “1-isomorphic” in terms of the bicategory constructed here. These coincidences come from the equivalence between $(G//H)$ -bundles and so-called G - H -bundles used in [Jur, ACJ05] expressed by Lemma 2.3.9. In particular, in case of the automorphism 2-group $\text{AUT}(H)$ of a connected Lie group H , a $\text{AUT}(H)$ -bundle gerbe is the same as a H -bibundle gerbe in the sense of Aschieri, Cantini and Jurco [ACJ05].

Example 5.1.8. Let G be a Lie group, so that G_{dis} is a Lie 2-group. Then, there is an equivalence of 2-categories

$$\mathcal{G}rb_{G_{dis}}(M) \cong \mathcal{B}un_{\mathcal{B}G}(M)_{dis}.$$

Indeed, if \mathcal{G} is a G_{dis} -bundle gerbe over M , its principal G_{dis} -bundle over $Y^{[2]}$ is by Example 2.2.4 just a smooth map $\alpha : Y^{[2]} \rightarrow G$, and its bundle gerbe product degenerates to an equality $\pi_{23}^* \alpha \cdot \pi_{12}^* \alpha = \pi_{13}^* \alpha$ for functions on $Y^{[3]}$. In other words, a G_{dis} -bundle gerbe is the same as a so-called “ G -bundle 0-gerbe”. These form a category that is equivalent to the one of ordinary principal G -bundles, as pointed out in Section 1.

Remark 5.1.9. There are two differences between the definitions given here (for $\Gamma = \mathcal{BA}$) and the ones of Murray et al. [Mur96, MS00, Ste00]. Firstly, we have a slightly different ordering of tensor products of bundles. These orderings are not essential in the case of abelian groups because the tensor category of ordinary A -bundles is symmetric. In the non-abelian case, a consistent theory requires the conventions we have chosen here. Secondly, the definitions of 1-morphisms and 2-morphisms have been generalized step by step:

1. In [Mur96], 1-morphisms did not include a *common* refinement, but rather required that the surjective submersion of one bundle gerbe refines the other. This definition is too restrictive in the sense that e.g. $U(1)$ -bundle gerbes are not classified by $H^3(M, \mathbb{Z})$, as desired.
2. In [MS00], 1-morphisms were defined on the canonical refinement $Z := Y_1 \times_M Y_2$ of the surjective submersions of the bundle gerbes. This definition solves the previous problems concerning the classification of bundle gerbes, but makes the composition of 1-morphisms quite involved [Ste00].
3. In [Wal07], 1-morphisms were defined on refinements $\zeta : Z \rightarrow Y_1 \times_M Y_2$. This generalization allows the same elegant definition of composition we have given here, and results in the same isomorphism classes of bundle gerbes. Moreover, 2-morphisms are defined with *commutative* diagrams (5.1.3) – this makes the structure of the bicategory outmost simple (see Remark 5.1.4).
4. In the present article we have allowed for a yet more general refinement in the definition of 1-morphisms. Its achievement is that bundle gerbes come out as an example of a more general concept – the plus construction – and we get e.g. Theorem 5.1.5 for free.

Despite of these different definitions of 1-morphisms and 2-morphisms, the resulting bicategories of \mathcal{BA} -bundle gerbes in 2., 3. and 4. are all equivalent (see [Wal07, Theorem 1], [NS11, Remark 4.5] and Lemma 5.2.8 below).

5.2 Properties of Groupoid Bundle Gerbes

We recall that a homomorphism $\Lambda : \Gamma \rightarrow \Omega$ between Lie 2-groups is an anafunctor and a transformation (2.4.4) describing its compatibility with the multiplications. We recall further from Proposition 2.4.9 that extension along Λ is a 1-morphism

$$\Lambda : \mathcal{Bun}_\Gamma \rightarrow \mathcal{Bun}_\Omega$$

between monoidal stacks over smooth manifolds. That is, extension along Λ is compatible with pull-backs, tensor products, and morphisms between principal Γ -bundles. Applying it to the principal Γ -bundle P of a Γ -bundle gerbe \mathcal{G} , and also to the bundle gerbe product μ , we obtain immediately an Ω -bundle gerbe $\Lambda\mathcal{G}$. The same is evidently true for morphisms and 2-morphisms. Summarizing, we get:

Proposition 5.2.1. *Extension of bundle gerbes along a homomorphism $\Lambda : \Gamma \rightarrow \Omega$ between Lie 2-groups defines a 1-morphism*

$$\Lambda : \mathcal{G}rb_{\Gamma} \rightarrow \mathcal{G}rb_{\Omega}$$

of 2-stacks over smooth manifolds.

We recall that a weak equivalence between Lie 2-groups is a homomorphism $\Lambda : \Gamma \rightarrow \Omega$ that is a weak equivalence (see Definition 2.3.10). We have:

Theorem 5.2.2. *Suppose $\Lambda : \Gamma \rightarrow \Omega$ is a weak equivalence between Lie 2-groups. Then, the 1-morphism $\Lambda : \mathcal{G}rb_{\Gamma} \rightarrow \mathcal{G}rb_{\Omega}$ of Proposition 5.2.1 is an equivalence of 2-stacks.*

Proof. The monoidal equivalence $\Lambda : \mathcal{B}un_{\Gamma} \rightarrow \mathcal{B}un_{\Omega}$ between the monoidal stacks (Corollary 2.4.10) induces an equivalence $\mathit{TrivGrb}_{\Gamma}(M) \rightarrow \mathit{TrivGrb}_{\Lambda}(M)$ between pre-2-stacks. Since the plus construction is functorial, this induces in turn the claimed equivalence of 2-stacks. \square

Next we generalize a couple of well-known result from abelian to non-abelian bundle gerbes. We say a *refinement* of a surjective submersion $\pi : Y \rightarrow M$ is another surjective submersion $\omega : W \rightarrow M$ together with a smooth map $f : W \rightarrow Y$ such that $\zeta = \pi \circ f$. Notice that such a refinement induces smooth maps $f_k : W^{[k]} \rightarrow Y^{[k]}$ that commute with the various projections $\omega_{i_1 \dots i_k}$ and $\pi_{i_1 \dots i_k}$.

Lemma 5.2.3. *Suppose $\mathcal{G}_1 = (Y_1, P_1, \mu_1)$ and $\mathcal{G}_2 = (Y_2, P_2, \mu_2)$ are Γ -bundle gerbes over M , $f : Y_1 \rightarrow Y_2$ is a refinement of surjective submersions, and $\varphi : f_2^* P_2 \rightarrow P_1$ is an isomorphism of Γ -bundles over $Y_1^{[2]}$ that is compatible with the bundle gerbe products μ_1 and μ_2 in the sense that the diagram*

$$\begin{array}{ccc} \pi_{23}^* f_2^* P_2 \otimes \pi_{12}^* f_2^* P_2 & \xrightarrow{f_3^* \mu} & \pi_{13}^* f_2^* P_2 \\ \downarrow \pi_{23}^* \varphi \otimes \pi_{12}^* \varphi & & \downarrow \pi_{13}^* \varphi \\ \pi_{23}^* P_1 \otimes \pi_{12}^* P_1 & \xrightarrow{\mu} & \pi_{13}^* P_1 \end{array}$$

is commutative. Then, \mathcal{G}_1 and \mathcal{G}_2 are isomorphic.

The proof works just the same way as in the abelian case: one constructs the 1-isomorphism over the common refinement $Z := Y_1 \times_M Y_2$ in a straightforward way. As a consequence of Lemma 5.2.3 we have

Proposition 5.2.4. *Let $\mathcal{G} = (Y, P, \mu)$ be a Γ -bundle gerbe over M , and let $f : W \rightarrow Y$ be a refinement of its surjective submersion $\pi : Y \rightarrow M$. Then, $(W, f_2^* P, f_3^* \mu)$ is a Γ -bundle gerbe over M , and is isomorphic to \mathcal{G} .*

Lemma 5.2.5. *Let $\mathcal{G} = (Y, P, \mu)$ be a Γ -bundle gerbe over M . Then, there exist unique smooth maps $i : P \rightarrow P$ and $t : Y \rightarrow P$ such that*

(i) *the diagrams*

$$\begin{array}{ccc} P & \xrightarrow{i} & P \\ \chi \downarrow & & \downarrow \chi \\ Y^{[2]} & \xrightarrow{\text{flip}} & Y^{[2]} \end{array} \quad \text{and} \quad \begin{array}{ccc} & & P \\ & \nearrow t & \downarrow \chi \\ Y & \xrightarrow{\text{diag}} & Y^{[2]} \end{array}$$

are commutative.

(ii) *the map t is neutral with respect to the bundle gerbe product μ , i.e.*

$$\mu(t(y_2), p) = p = \mu(p, t(y_1)).$$

for all $p \in P$ with $\chi(p) = (y_1, y_2)$.

(iii) *the map i provides inverses with respect to the bundle gerbe product μ , i.e.*

$$\mu(i(p), p) = t(y_1) \quad \text{and} \quad \mu(p, i(p)) = t(y_2)$$

for all $p \in P$ with $\chi(p) = (y_1, y_2)$.

Moreover, $\alpha(t(y)) = 1$ and $\alpha(i(p)) = \alpha(p)^{-1}$ for all $p \in P$ and $y \in Y$.

Proof. Concerning uniqueness, suppose (t, i) and (t', i') are pairs of maps satisfying (i), (ii) and (iii). Firstly, we have $t'(y) = \mu(t(y), t'(y)) = t(y)$ and so $t = t'$. Then, $\mu(i(p), p) = t(y_1) = t'(y_1) = \mu(i'(p), p)$ implies $i(p) = i'(p)$, and so $i = i'$. In order to see the existence of t and i , denote by $Q := \text{diag}^*P$ the pullback of P to Y , denote by Q^\vee the dual bundle and by $d : Q \otimes Q^\vee \rightarrow \mathbf{I}_1$ the death map. Consider the smooth map

$$Y \xrightarrow{s} \mathbf{I}_1 \xrightarrow{d^{-1}} Q \otimes Q^\vee \xrightarrow{\mu^{-1} \otimes \text{id}_{Q^\vee}} Q \otimes Q \otimes Q^\vee \xrightarrow{\text{id} \otimes d} \mathbf{I}_1 \otimes Q \cong Q \xrightarrow{\text{diag}} P$$

where $s : Y \rightarrow \mathbf{I}_1$ is the canonical section (see the proof of Lemma 2.2.6). It is straightforward to see that this satisfies the properties of the map t . Since all maps in the above sequence are (anchor-preserving) bundle morphisms, it is clear that $t \circ \alpha = 1$. \square

Corollary 5.2.6. *Let $\mathcal{G} = (Y, P, \mu)$ be a Γ -bundle gerbe over M , and let t and i be the unique maps of Lemma 5.2.5. Then,*

(i) *t is a section of diag^*P , and defines a trivialization $\text{diag}^*P \cong \mathbf{I}_1$.*

(ii) *i is a bundle isomorphism $i : P^\vee \rightarrow \text{flip}^*P$.*

(iii) $\mathcal{C}_0 := Y$ and $\mathcal{C}_1 := P$ define a Lie groupoid with source and target maps $\pi_1 \circ \chi$ and $\pi_2 \circ \chi$, respectively, composition μ , identities t and inversion i .

The following statement is well-known for abelian gerbes; the general version can be proved by a straightforward generalization of the constructions given in the proof of [Wal07, Proposition 3].

Lemma 5.2.7. *Every 1-morphism $\mathcal{A} : \mathcal{G} \rightarrow \mathcal{H}$ between Γ -bundle gerbes over M is invertible.*

The last statement of this section shows a way to bring 1-morphisms and 2-morphisms into a simpler form (see Remark 5.1.9). For bundle gerbes \mathcal{G}_1 and \mathcal{G}_2 with surjective submersions $\pi_1 : Y_1 \rightarrow M$ and $\pi_2 : Y_2 \rightarrow M$ we denote by $\mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)$ the Hom-category in the bicategory $\mathcal{G}rb_\Gamma(M)$, and by $\mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)^{FP}$ the category whose objects are those 1-morphisms whose common refinement is $Z := Y_1 \times_M Y_2$, and whose 2-morphisms are those 2-morphisms whose refinement is $W := Y_1 \times_M Y_2$ with the maps $r, r' : W \rightarrow Z$ the identity maps.

Lemma 5.2.8. *The inclusion $\mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)^{FP} \rightarrow \mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)$ is an equivalence of categories.*

Proof. First we show that it is essentially surjective. We assume $\mathcal{A} : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ is a general 1-morphism with a principal Γ -bundle Q over a common refinement Z of the surjective submersions $\pi_1 : Y_1 \rightarrow M$ and $\pi_2 : Y_2 \rightarrow M$ of the two bundle gerbes. We look at the principal Γ -bundle

$$\tilde{Q} := \kappa_2^* P_2 \otimes \text{pr}_2^* Q \otimes \kappa_1^* P_1$$

over $\tilde{Z} := Y_1 \times_M Z \times_M Y_2$, where

$$\kappa_1 : \tilde{Z} \rightarrow Y_1^{[2]} : (y_1, z, y_2) \mapsto (y_1, y_1(z)) \quad \text{and} \quad \kappa_2 : \tilde{Z} \rightarrow Y_2^{[2]} : (y_1, z, y_2) \mapsto (y_2(z), y_2).$$

The projection $\text{pr}_{13} : \tilde{Z} \rightarrow Y_1 \times_M Y_2$ is a surjective submersion, and over $\tilde{Z} \times_{Y_1 \times_M Y_2} \tilde{Z}$ we have a bundle morphism $\alpha : \text{pr}_1^* \tilde{Q} \rightarrow \text{pr}_2^* \tilde{Q}$ defined over a point (\tilde{z}, \tilde{z}') with $\tilde{z} = (y_1, z, y_2)$ and $\tilde{z}' = (y_1, z', y_2)$ by

$$\begin{aligned} \tilde{Q}_{\tilde{z}} & \xlongequal{\quad} P_2|_{y_2(z), y_2} \otimes Q_z \otimes P_1|_{y_1, y_1(z)} \\ & \quad \downarrow \mu_2^{-1} \otimes \text{id} \otimes \text{id} \\ & P_2|_{y_2(z'), y_2} \otimes P_2|_{y_2(z), y_2(z')} \otimes Q_z \otimes P_1|_{y_1, y_1(z)} \\ & \quad \downarrow \text{id} \otimes \beta \otimes \text{id} \\ & P_2|_{y_2(z'), y_2} \otimes Q_{z'} \otimes P_1|_{y_1(z), y_1(z')} \otimes P_1|_{y_1, y_1(z)} \\ & \quad \downarrow \text{id} \otimes \text{id} \otimes \mu_1 \\ & P_2|_{y_2(z'), y_2} \otimes Q_{z'} \otimes P_1|_{y_1, y_1(z')} \xlongequal{\quad} \tilde{Q}_{\tilde{z}'}. \end{aligned}$$

The compatibility condition (5.1.1) implies a cocycle condition for α over the three-fold fibre product of \tilde{Z} over $Y_1 \times_M Y_2$, and since principal Γ -bundles form a stack, the pair (\tilde{Q}, α) defines a principal Γ -bundle Q^{FP} over $Z^{FP} := Y_1 \times_M Y_2$. It is now straightforward to show that the bundle isomorphism

β itself descends to a bundle isomorphism β^{FP} over $Z^{FP} \times_M Z^{FP}$ in such a way that the triple $(Z^{FP}, Q^{FP}, \beta^{FP})$ forms a 1-morphism $\mathcal{A}^{FP} : \mathcal{G}_1 \rightarrow \mathcal{G}_2$.

In order to show that \mathcal{A}^{FP} is an essential preimage of \mathcal{A} , it remains to construct a 2-morphism $\varphi_{\mathcal{A}}^{FP} : \mathcal{A} \rightrightarrows \mathcal{A}^{FP}$. In the terminology of Definition 5.1.3, we choose $W = \tilde{Z}$ with $r := \text{pr}_2 : W \rightarrow Z$ and $r' := \text{pr}_{13} : W \rightarrow Z^{FP}$. Note that diagram (5.1.3) does not commute. The maps $x_W : W \rightarrow Y_1^{[2]}$ and $y_W : W \rightarrow Y_2^{[2]}$ are given by $x_W = s \circ \kappa_1$ and $y_W = \kappa_2$, where $s : Y_1^{[2]} \rightarrow Y_1^{[2]}$ switches the factors. Now, the bundle isomorphism of the 2-morphism $\varphi_{\mathcal{A}}^{FP}$ we want to construct is a bundle isomorphism

$$\varphi : y_W^* P_2 \otimes r^* Q \rightarrow \tilde{Q} \otimes x_W^* P_1$$

over W , and is fibrewise over a point $w = (y_1, z, y_2)$ given by

$$\begin{aligned} P_2|_{y_2(z), y_2} \otimes Q_z &\xrightarrow{\text{id} \otimes \text{id} \otimes t^{-1}} P_2|_{y_2(z), y_2} \otimes Q_z \otimes P_{y_1(z), y_1(z)} \\ &\quad \downarrow \text{id} \otimes \text{id} \otimes \mu_1^{-1} \\ P_2|_{y_2(z), y_2} \otimes Q_z \otimes P_1|_{y_1, y_1(z)} \otimes P_1|_{y_1(z), y_1} &= \tilde{Q}_w \otimes P_1|_{s(y_1, y_1(z))}, \end{aligned}$$

where t is the trivialization of $\text{diag}^* P$ of Corollary 5.2.6. The compatibility condition (5.1.2) is straightforward to check.

Now we show that the inclusion $\mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)^{FP} \rightarrow \mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)$ is full and faithful. Since it is clearly faithful, it only remains to show that it is full. Given a morphism $\mathcal{A} \rightarrow \mathcal{A}'$ in $\mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)$, i.e. a common refinement W of $Y_1 \times_M Y_2$ with itself and a bundle morphism φ , we have to find a morphism in $\mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)^{FP}$ such that the two morphisms are identified under the equivalence relation on bundle gerbe 2-morphisms. We denote the bundles over $Y_1 \times_M Y_2$ corresponding to \mathcal{A} and \mathcal{A}' by Q and Q' . The refinement maps are denoted as before by $r = (s_1, s_2) : W \rightarrow Y_1 \times_M Y_2$ and $r' = (t_1, t_2) : W \rightarrow Y_1 \times_M Y_2$. Then we obtain an isomorphism $r^* Q \rightarrow r'^* Q'$ fibrewise over a point $w \in W$ by

$$\begin{aligned} Q|_{s_1(w), s_2(w)} &\xrightarrow{d^{-1} \otimes \text{id}} P_2^\vee|_{s_2(w), t_2(w)} \otimes P_2|_{s_2(w), t_2(w)} \otimes Q|_{s_1(w), s_2(w)} \\ &\quad \downarrow \text{id} \otimes \varphi \\ P_2^\vee|_{s_2(w), t_2(w)} \otimes Q'|_{t_1(w), t_2(w)} \otimes P_1|_{s_1(w), t_1(w)} &\quad (5.2.1) \\ &\quad \downarrow \text{id} \otimes \beta'^{-1} \\ P_2^\vee|_{s_2(w), t_2(w)} \otimes P_2|_{s_2(w), t_2(w)} \otimes Q'|_{s_1(w), s_2(w)} &\xrightarrow{d \otimes \text{id}} Q'|_{s_1(w), s_2(w)} \end{aligned}$$

where $d : P_2^\vee|_{s_2(w), t_2(w)} \otimes P_2|_{s_2(w), t_2(w)} \rightarrow \mathbf{I}_1$ is the death map. One can use the compatibility condition for φ to show that this morphism descends to a morphism $\psi : Q \rightarrow Q'$ which is a morphism in $\mathcal{H}om(\mathcal{G}_1, \mathcal{G}_2)^{FP}$. The two morphisms (W, ψ) and $(Y_1 \times_M Y_2, \varphi)$ are identified if their pullbacks to

$$W \times_{(Y_1 \times_M Y_2 \times_M Y_1 \times_M Y_2)} (Y_1 \times_M Y_2) = \{w \in W \mid r(w) = r'(w)\} =: W_0$$

are equal. On the one side, the map $W_0 \rightarrow W$ is the inclusion and the map $W_0 \rightarrow Y_1 \times_M Y_2$ is equal to r . The pullback of ψ along r is by construction the map $r^*Q \rightarrow r^*Q'$ from (5.2.1). On the other side, bundles $x_W^*P_1$ and $y_W^*P_2$ over W_0 have canonical trivializations (Lemma 5.2.6 (i)) under which φ becomes also equal to the morphism (5.2.1). \square

5.3 Classification by Čech Cohomology

In this Section we prove that Versions I (Čech Γ -1-cocycles) and III (Γ -bundle gerbes) are equivalent. For this purpose, we extract a Čech cocycle from a Γ -bundle gerbe \mathcal{G} over M , and prove that this procedure defines a bijection on the level of equivalence classes (Theorem 5.3.2 below). First we have to assure the existence of appropriate open covers.

Lemma 5.3.1. *For every Γ -bundle gerbe $\mathcal{G} = (Y, P, \mu)$ over M there exists an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of M with sections $\sigma_i : U_i \rightarrow Y$, such that the principal Γ -bundles $(\sigma_i \times \sigma_j)^*P$ over $U_i \cap U_j$ are trivializable.*

Proof. One can choose an open cover such that the 2-fold intersections $U_i \cap U_j$ are contractible. Since every Lie 2-group is a crossed module $G//H$ (Remark 2.4.3), and $G//H$ -bundles are ordinary H -bundles (Lemma 2.2.9), these will admit sections over contractible smooth manifolds. But a section is enough to trivialize the original Γ -bundle (Lemma 2.2.6). \square

Let \mathcal{G} be a Γ -bundle gerbe over M , and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover with the properties of Lemma 5.3.1. We denote by $M_{\mathcal{U}}$ the disjoint union of all the open sets U_i , and by $\sigma : M_{\mathcal{U}} \rightarrow Y$ the union of the sections σ_i . Then, σ is a refinement of $\pi : Y \rightarrow M$, and we have a Γ -bundle gerbe $\mathcal{G}_{\mathcal{U}, \sigma}$ that is isomorphic to \mathcal{G} (Proposition 5.2.4).

The principal Γ -bundle P_{ij} of $\mathcal{G}_{\mathcal{U}, \sigma}$ over the component $U_i \cap U_j$ is by assumption trivializable. Thus there exists a trivialization $t_{ij} : P_{ij} \rightarrow \mathbf{I}_{f_{ij}}$ for smooth functions $f_{ij} : U_i \cap U_j \rightarrow \Gamma_0$. We define an isomorphism μ_{ijk} between trivial bundles such that the diagram

$$\begin{array}{ccc} P_{jk} \otimes P_{ij} & \xrightarrow{\mu} & P_{ik} \\ \downarrow t_{jk} \otimes t_{ij} & & \downarrow t_{ik} \\ \mathbf{I}_{f_{jk}} \otimes \mathbf{I}_{f_{ij}} & \xrightarrow{\mu_{ijk}} & \mathbf{I}_{f_{ik}} \end{array}$$

is commutative. Now we are in the situation of Lemma 5.2.3, which implies that the Γ -bundle gerbe $\mathcal{G}_{\mathcal{U}, \sigma, t} := (M_{\mathcal{U}}, \mathbf{I}_{f_{ij}}, \mu_{ijk})$ is still isomorphic to \mathcal{G} .

Combining Lemma 2.2.8 with Example 2.4.7 (a), we see that the isomorphisms μ_{ijk} correspond to smooth maps $g_{ijk} : U_i \cap U_j \cap U_k \rightarrow \Gamma_1$ such that $s(g_{ijk}) = f_{jk} \cdot f_{ij}$ and $t(g_{ijk}) = f_{ik}$. The

associativity condition for μ_{ijk} implies moreover that

$$g_{\alpha\gamma\delta} \circ (g_{\alpha\beta\gamma} \cdot \text{id}_{f_{\gamma\delta}}) = g_{\alpha\beta\delta} \circ (\text{id}_{f_{\alpha\beta}} \cdot g_{\beta\gamma\delta}).$$

Hence, the collection $\{f_{ij}, g_{ijk}\}$ is a Γ -1-cocycle on M with respect to the open cover \mathcal{U} .

Theorem 5.3.2. *Let M be a smooth manifold and let Γ be a Lie 2-group. The above construction defines a bijection*

$$\left\{ \begin{array}{l} \text{Isomorphism classes of} \\ \Gamma\text{-bundle gerbes over } M \end{array} \right\} \cong \check{H}^1(M, \Gamma).$$

Proof. In order to prove that we have a well-defined map, we make the claim that Γ -bundle gerbes $(M_{\mathcal{U}}, \mathbf{I}_{f_{ij}}, \mu_{ijk})$ and $(M_{\mathcal{V}}, \mathbf{I}_{h_{ij}}, \nu_{ijk})$ are isomorphic if and only if the corresponding Γ -1-cocycles are equivalent. This proves at the same time that the choices of open covers and sections we have made during the construction do not matter, that the resulting map is well-defined on isomorphism classes, and that this map is injective. Surjectivity follows by assigning to a Γ -1-cocycle (f_{ij}, g_{ijk}) with respect to some cover \mathcal{U} the Γ -bundle gerbe $(M_{\mathcal{U}}, \mathbf{I}_{f_{ij}}, \mu_{ijk})$ with μ_{ijk} determined by Lemma 2.2.8.

It remains to prove that claim. We assume $\mathcal{A} = (Z, Q, \alpha)$ is a 1-isomorphism between the Γ -bundle gerbes $(M_{\mathcal{U}}, \mathbf{I}_{f_{ij}}, \mu_{ijk})$ and $(M_{\mathcal{V}}, \mathbf{I}_{h_{ij}}, \nu_{ijk})$. Similarly to Lemma 5.3.1 one can show that there exists a cover \mathcal{W} of M by open sets W_i that refines both \mathcal{U} and \mathcal{V} , and that allows smooth sections $\omega_i : W_i \rightarrow Z$ for which the Γ -bundle $\omega_i^* Q$ is trivializable. In the terminology of the above construction, choosing a trivialization $t : \omega^* Q \rightarrow \mathbf{I}_{h_i}$ with smooth maps $h_i : W_i \rightarrow \Gamma_0$ over $M_{\mathcal{W}}$ converts the isomorphism α into smooth functions $s_{ij} : W_i \cap W_j \rightarrow \Gamma_1$ satisfying $s(s_{ij}) = g'_{ij} \cdot h_i$ and $t(s_{ij}) = h_j \cdot g_{ij}$. The compatibility diagram (5.1.1) implies the remaining condition that makes (h_i, s_{ij}) an equivalence between the Γ -2-cocycles (f_{ij}, g_{ijk}) and (f'_{ij}, g'_{ijk}) . \square

6 Version IV: Principal 2-Bundles

The basic idea of a smooth 2-bundle is that it gives for every point x in the base manifold M a Lie groupoid \mathcal{P}_x varying smoothly with x . Numerous different versions have appeared so far in the literature, e.g [Bar04, BS07, Woc, SP]. The main objective of *our* version of principal 2-bundles is to make the definition of the objects (i.e. the 2-bundles) *as simple as possible*, while keeping their isomorphism classes in bijection with non-abelian cohomology. Thus, our principal 2-bundles will be defined using *strict* actions of Lie 2-groups on Lie groupoids, and *not* using anafunctors. The necessary “weakness” will be pushed into the definition of 1-morphisms.

6.1 Definition of Principal 2-Bundles

As an important prerequisite for principal 2-bundles we have to discuss actions of Lie 2-groups on Lie groupoids, and equivariant anafunctors.

Definition 6.1.1. *Let \mathcal{P} be a Lie groupoid, and let Γ be a Lie 2-group. A smooth right action of Γ on \mathcal{P} is a smooth functor $R: \mathcal{P} \times \Gamma \rightarrow \mathcal{P}$ such that $R(p, 1) = p$ and $R(\rho, \text{id}_1) = \rho$ for all $p \in \mathcal{P}_0$ and $\rho \in \mathcal{P}_1$, and the diagram*

$$\begin{array}{ccc} \mathcal{P} \times \Gamma \times \Gamma & \xrightarrow{\text{id} \times m} & \mathcal{P} \times \Gamma \\ R \times \text{id} \downarrow & & \downarrow R \\ \mathcal{P} \times \Gamma & \xrightarrow{m} & \mathcal{P} \end{array}$$

of smooth functors is commutative (strictly, on the nose).

For example, every Lie 2-group acts on itself via multiplication. Note that due to strict commutativity, one has $R(R(p, g), g^{-1}) = p$ and $R(R(\rho, \gamma), i(\gamma)) = \rho$ for all $g \in \Gamma_0$, $p \in \mathcal{P}_0$, $\gamma \in \Gamma_1$ and $\rho \in \mathcal{P}_1$.

Remark 6.1.2. This definition could be weakened in two steps. First, one could allow a natural transformation in the above diagram instead of commutativity. Secondly, one could allow R to be an anafunctor instead of an ordinary functor. It turns out that for our purposes the above definition is sufficient.

Definition 6.1.3. *Let \mathcal{X} and \mathcal{Y} be Lie groupoids with smooth actions (R_1, ρ_1) , (R_2, ρ_2) of a Lie 2-group Γ . An equivariant structure on an anafunctor $F: \mathcal{X} \rightarrow \mathcal{Y}$ is a transformation*

$$\begin{array}{ccc} \mathcal{X} \times \Gamma & \xrightarrow{R_1} & \mathcal{X} \\ F \times \text{id} \downarrow & \lambda \swarrow & \downarrow F \\ \mathcal{Y} \times \Gamma & \xrightarrow{R_2} & \mathcal{Y} \end{array}$$

satisfying the following condition:

$$\begin{array}{ccc} \mathcal{X} \times \Gamma \times \Gamma & \xrightarrow{\text{id} \times m} & \mathcal{X} \times \Gamma \\ \downarrow F \times \text{id} \times \text{id} & \searrow R_1 \times \text{id} & \searrow R_1 \\ \mathcal{X} \times \Gamma & \xrightarrow{R_1} & \mathcal{X} \\ \lambda \times \text{id} \swarrow & \downarrow F \times \text{id} & \downarrow F \\ \mathcal{Y} \times \Gamma \times \Gamma & \xrightarrow{\text{id} \times m} & \mathcal{Y} \times \Gamma \\ \downarrow R_2 \times \text{id} & \searrow R_2 & \searrow R_2 \\ \mathcal{Y} \times \Gamma & \xrightarrow{R_2} & \mathcal{Y} \end{array} = \begin{array}{ccc} \mathcal{X} \times \Gamma \times \Gamma & \xrightarrow{\text{id} \times m} & \mathcal{X} \times \Gamma \\ \downarrow F \times \text{id} \times \text{id} & \searrow R_1 & \searrow R_1 \\ \mathcal{X} & & \mathcal{X} \\ \downarrow F & \swarrow \lambda & \downarrow F \\ \mathcal{Y} \times \Gamma \times \Gamma & \xrightarrow{\text{id} \times m} & \mathcal{Y} \times \Gamma \\ \downarrow R_2 \times \text{id} & \searrow R_2 & \searrow R_2 \\ \mathcal{Y} \times \Gamma & \xrightarrow{R_2} & \mathcal{Y} \end{array}$$

An anafunctor together with a Γ -equivariant structure is called Γ -equivariant anafunctor.

In Appendix A we translate this abstract (but evidently correct) definition of equivariance into more concrete terms involving a Γ_1 -action on the total space of the anafunctor.

Definition 6.1.4. If $(F, \lambda) : \mathcal{X} \rightarrow \mathcal{Y}$ and $(G, \gamma) : \mathcal{X} \rightarrow \mathcal{Y}$ are Γ -equivariant anafunctors, a transformation $\eta : F \Rightarrow G$ is called Γ -equivariant, if the following equality of transformation holds:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \mathcal{X} \times \Gamma & \xrightarrow{R_1} & \mathcal{X} \\
 \downarrow G \times \text{id} & \swarrow F \times \text{id} & \downarrow F \\
 \mathcal{Y} \times \Gamma & \xrightarrow{R_2} & \mathcal{Y} \\
 \uparrow \eta \times \text{id} & \nwarrow \lambda & \\
 \mathcal{X} \times \Gamma & &
 \end{array} & = &
 \begin{array}{ccc}
 \mathcal{X} \times \Gamma & \xrightarrow{R_1} & \mathcal{X} \\
 \downarrow G \times \text{id} & \swarrow \gamma & \downarrow G \\
 \mathcal{Y} \times \Gamma & \xrightarrow{R_2} & \mathcal{Y} \\
 \uparrow \eta & \nwarrow F & \\
 \mathcal{X} \times \Gamma & &
 \end{array}
 \end{array}$$

It follows from abstract nonsense in the bicategory of Lie groupoids, anafunctors and transformations that we have another bicategory with

- objects: Lie groupoids with smooth right Γ -actions.
- 1-morphisms: Γ -equivariant anafunctors.
- 2-morphisms: Γ -equivariant transformations.

We need three further notions for the definition of a principal 2-bundle. Let M be a smooth manifold, and let \mathcal{P} be a Lie groupoid. We say that a smooth functor $\pi : \mathcal{P} \rightarrow M_{dis}$ is a *surjective submersion functor*, if $\pi : \mathcal{P}_0 \rightarrow M$ is a surjective submersion. Let $\pi : \mathcal{P} \rightarrow M_{dis}$ be a surjective submersion functor, and let \mathcal{Q} be a Lie groupoid with some smooth functor $\chi : \mathcal{Q} \rightarrow M_{dis}$. Then, the fibre product $\mathcal{P} \times_M \mathcal{Q}$ is defined to be the full subcategory of $\mathcal{P} \times \mathcal{Q}$ over the submanifold $\mathcal{P}_0 \times_M \mathcal{Q}_0 \subset \mathcal{P}_0 \times \mathcal{Q}_0$.

Definition 6.1.5. Let M be a smooth manifold and let Γ be a Lie 2-group.

(a) A principal Γ -2-bundle over M is a Lie groupoid \mathcal{P} , a surjective submersion functor $\pi : \mathcal{P} \rightarrow M_{dis}$, and a smooth right action R of Γ on \mathcal{P} that preserves π , such that the smooth functor

$$\tau := (\text{pr}_1, R) : \mathcal{P} \times \Gamma \rightarrow \mathcal{P} \times_M \mathcal{P}$$

is a weak equivalence.

(b) A 1-morphism between principal Γ -2-bundles is a Γ -equivariant anafunctor

$$F : \mathcal{P}_1 \rightarrow \mathcal{P}_2$$

that respects the surjective submersion functors to M .

(c) A 2-morphism between 1-morphisms is a Γ -equivariant transformation between these.

Remark 6.1.6.

(a) The condition in (a) that the action R preserves the surjective submersion functor π means that the diagram of functors

$$\begin{array}{ccc} \mathcal{P} \times \Gamma & \xrightarrow{R} & \mathcal{P} \\ \text{pr}_1 \downarrow & & \downarrow \pi \\ \mathcal{P} & \xrightarrow{\pi} & M_{dis} \end{array}$$

is commutative.

(b) The condition in (b) that the anafunctor F respects the surjective submersion functors means in the first place that there exists a transformation

$$\begin{array}{ccc} \mathcal{P}_1 & \xrightarrow{F} & \mathcal{P}_2 \\ & \searrow \pi_1 & \swarrow \pi_2 \\ & & M_{dis}. \end{array}$$

However, since the target of the anafunctors π_1 and $\pi_2 \circ F$ is the *discrete* groupoid M_{dis} , the equivalence of Example 2.3.4 applies, and implies that if such a transformation exists, it is unique. Indeed, it is easy to see that an anafunctor $F : \mathcal{P} \rightarrow \mathcal{Q}$ with anchors $\alpha_l : F \rightarrow \mathcal{P}_0$ and $\alpha_r : F \rightarrow \mathcal{Q}_0$ respects smooth functors $\pi : \mathcal{P} \rightarrow M_{dis}$ and $\chi : \mathcal{Q} \rightarrow M_{dis}$ if and only if $\pi \circ \alpha_l = \chi \circ \alpha_r$.

Example 6.1.7. The *trivial* Γ -2-bundle over M is defined by

$$\mathcal{P} := M_{dis} \times \Gamma \quad , \quad \pi := \text{pr}_1 \quad , \quad R := \text{id}_M \times m.$$

Here, the smooth functor τ even has a smooth inverse *functor*. In the following we denote the trivial Γ -2-bundle by \mathcal{I} .

6.2 Properties of Principal 2-Bundles

Principal Γ -2-bundles over M form a bicategory denoted $2\text{-}\mathcal{Bun}_\Gamma(M)$. There is an evident pullback 2-functor

$$f^* : 2\text{-}\mathcal{Bun}_\Gamma(N) \rightarrow 2\text{-}\mathcal{Bun}_\Gamma(M)$$

associated to smooth maps $f : M \rightarrow N$, and these make $2\text{-}\mathcal{Bun}_\Gamma$ a pre-2-stack over smooth manifolds. We deduce the following important two theorems about this pre-2-stack. The first asserts that it actually a 2-stack:

Theorem 6.2.1. *Principal Γ -2-bundles form a 2-stack 2-Bun_Γ over smooth manifolds.*

Proof. This follows from Theorem 5.1.5 (Γ -bundle gerbes form a 2-stack) and Theorem 7.1 (the equivalence $\mathcal{G}rb_\Gamma \cong 2\text{-Bun}_\Gamma$) we prove in Section 7. \square

The second concerns a homomorphism $\Lambda : \Gamma \rightarrow \Omega$ of Lie 2-groups, which induces the extension $\Lambda : \mathcal{G}rb_\Gamma \rightarrow \mathcal{G}rb_\Omega$ between 2-stacks of bundle gerbes (Proposition 5.2.1). Combined with the equivalence $\mathcal{G}rb_\Gamma \cong 2\text{-Bun}_\Gamma$ of Theorem 7.1, it defines a 1-morphism

$$\Lambda : 2\text{-Bun}_\Gamma \rightarrow 2\text{-Bun}_\Omega$$

between 2-stacks of principal 2-bundles. Now we get as a direct consequence of Theorem 5.2.2:

Theorem 6.2.2. *If $\Lambda : \Gamma \rightarrow \Omega$ is a weak equivalence between Lie 2-groups, then the 1-morphism $\Lambda : 2\text{-Bun}_\Gamma \rightarrow 2\text{-Bun}_\Omega$ is an equivalence of 2-stacks.*

A third consequence of the equivalence of Theorem 7.1 in combination with Lemma 5.2.7 is

Corollary 6.2.3. *Every 1-morphism $F : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ between principal Γ -2-bundles over M is invertible.*

The following discussion centers around *local trivializability* that is implicitly contained in Definition 6.1.5. A principal Γ -2-bundle that is isomorphic to the trivial Γ -2-bundle \mathcal{I} introduced in Example 6.1.7 is called *trivializable*. A *section* of a principal Γ -2-bundle \mathcal{P} over M is an anafunctor $S : M_{dis} \rightarrow \mathcal{P}$ such that $\pi \circ S = \text{id}_{M_{dis}}$ (recall that an anafunctor $\pi \circ S : M \rightarrow M$ is the same as a smooth map). One can show that every point $x \in M$ has an open neighborhood U together with a section $s : U_{dis} \rightarrow \mathcal{P}|_U$. Such sections can even be chosen to be smooth functors, rather than anafunctors, namely simply as ordinary sections of the surjective submersion $\pi : (\mathcal{P}|_U)_0 \rightarrow U_{dis}$.

Lemma 6.2.4. *A principal Γ -2-bundle over M is trivializable if and only if it has a smooth section.*

Proof. The trivial Γ -2-bundle \mathcal{I} has the section $S(m) := (m, 1)$, where 1 denotes the unit of Γ_0 . If \mathcal{P} is trivializable, and $F : \mathcal{I} \rightarrow \mathcal{P}$ is an isomorphism, then, $F \circ S$ is a section of \mathcal{P} . Conversely, suppose \mathcal{P} has a section $S : M_{dis} \rightarrow \mathcal{P}$. Then, we get the anafunctor

$$\mathcal{I} = M_{dis} \times \Gamma \xrightarrow{S \times \text{id}} \mathcal{P} \times \Gamma \xrightarrow{R} \mathcal{P}. \quad (6.2.1)$$

It has an evident Γ -equivariant structure and respects the projections to M . According to Corollary 6.2.3, this is sufficient to have a 1-isomorphism. \square

Corollary 6.2.5. *Every principal Γ -2-bundle is locally trivializable, i.e. every point $x \in M$ has an open neighborhood U and a 1-morphism $T : \mathcal{I} \rightarrow \mathcal{P}|_U$.*

Remark 6.2.6. In Wockel's version [Woc] of principal 2-bundles, local trivializations are required to be smooth functors and to be invertible as smooth functors, rather than allowing anafunctors. This version turns out to be too restrictive in the sense that the resulting bicategory receives no 2-functor from the bicategory $\mathcal{G}rb_\Gamma(M)$ of Γ -bundle gerbes that establishes an equivalence.

It is also possible to reformulate our definition of principal 2-bundles in terms of local trivializations. This reformulation gives us criteria which might be easier to check than the actual definition, similar to the case of ordinary principal bundles.

Proposition 6.2.7. *Let \mathcal{P} be a Lie groupoid, $\pi : \mathcal{P} \rightarrow M_{dis}$ be a surjective submersion functor, and R be a smooth right action of Γ on \mathcal{P} that preserves π . Suppose every point $x \in M$ has an open neighborhood U together with a Γ -equivariant anafunctor $T : \mathcal{I} \rightarrow \mathcal{P}|_U$ that respects the projections. Then, $\pi : \mathcal{P} \rightarrow M_{dis}$ is a principal Γ -2-bundle over M .*

Proof. We only have to prove that the functor τ is a weak equivalence, and we use Theorem 2.3.12. Since all morphisms of \mathcal{P} have source and target in the same fibre of $\pi : \mathcal{P} \rightarrow M_{dis}$, we may check the two conditions of Theorem 2.3.12 locally, i.e. for $\mathcal{P}|_{U_i}$ where U_i is an open cover of M . Using local trivializations $\mathcal{T}_i : \mathcal{I} \rightarrow \mathcal{P}|_{U_i}$, the smooth functor τ translates into the smooth functor $(\text{id}, \text{pr}_1, m) : M_{dis} \times \Gamma \times \Gamma \rightarrow (M_{dis} \times \Gamma) \times_M (M_{dis} \times \Gamma)$. This functor is an isomorphism of Lie groupoids, and hence essentially surjective and fully faithful. \square

7 Equivalence between Bundle Gerbes and 2-Bundles

In this section we show that Versions III and IV of smooth Γ -gerbes are equivalent in the strongest possible sense:

Theorem 7.1. *For M a smooth manifold and Γ a Lie 2-group, there is an equivalence of bicategories*

$$\mathcal{G}rb_\Gamma(M) \cong 2\text{-}\mathcal{B}un_\Gamma(M).$$

This equivalence is natural in M , i.e. it is an equivalence between pre-2-stacks.

The following outlines the proof of Theorem 7.1. In Section 7.1 we construct explicitly a 2-functor

$$\mathcal{E}_M : 2\text{-}\mathcal{B}un_\Gamma(M) \rightarrow \mathcal{G}rb_\Gamma(M).$$

We then use a general criterion assuring that \mathcal{E}_M is an equivalence of bicategories. This criterion is stated in Lemma B.1: it requires (A) that \mathcal{E}_M is fully faithful on Hom-categories, and (B) to choose certain preimages of objects and 1-morphisms under \mathcal{E}_M . Under these circumstances, Lemma B.1 constructs an inverse 2-functor \mathcal{R}_M together with the required pseudonatural transformations

assuring that \mathcal{E}_M and \mathcal{R}_M form an equivalence of bicategories. Condition (A) is proved as Lemma 7.1.9 in Section 7.1. The choices (B) are constructed in Section 7.2.

In order to prove that the 2-functors \mathcal{E}_M extend to the claimed equivalence between pre-2-stacks, we use another criterion stated in Lemma B.3. The only additional assumption of Lemma B.3 is that the given 2-functors \mathcal{E}_M form a 1-morphism of pre-2-stacks; this is proved in Proposition 7.1.10. Then, the inverse 2-functors \mathcal{R}_M obtained before form an inverse 1-morphism between pre-2-stacks.

7.1 From Principal 2-Bundles to Bundle Gerbes

In this section we define the 2-functor $\mathcal{E}_M : 2\text{-Bun}_\Gamma(M) \rightarrow \text{Grb}_\Gamma(M)$.

Definition of \mathcal{E}_M on objects

Let \mathcal{P} be a principal Γ -2-bundle over M , with projection $\pi : \mathcal{P} \rightarrow M$ and right action R of Γ on \mathcal{P} . The first ingredient of the Γ -bundle gerbe $\mathcal{E}_M(\mathcal{P})$ is the surjective submersion $\pi : \mathcal{P}_0 \rightarrow M$. The second ingredient is a principal Γ -bundle P over $\mathcal{P}_0^{[2]}$. We put

$$P := \mathcal{P}_1 \times \Gamma_0.$$

Bundle projection, anchor and Γ -action are given, respectively, by

$$\begin{aligned} \chi(\rho, g) &:= (t(\rho), R(s(\rho), g^{-1})) \quad , \quad \alpha(\rho, g) := g \\ &\text{and} \quad (\rho, g) \circ \gamma := (R(\rho, \text{id}_{g^{-1}} \cdot \gamma), s(\gamma)). \end{aligned} \quad (7.1.1)$$

These definitions are motivated by Remark 7.1.2 below.

Lemma 7.1.1. *This defines a principal Γ -bundle over $\mathcal{P}_0^{[2]}$.*

Proof. First we check that $\chi : P \rightarrow \mathcal{P}_0^{[2]}$ is a surjective submersion. Since the functor $\tau = (\text{id}, R)$ is a weak equivalence, we know from Theorem 2.3.12 that

$$f : (\mathcal{P}_0 \times \Gamma_0)_{\tau \times t \times t} \mathcal{P}_1^{[2]} \rightarrow \mathcal{P}_0^{[2]} : (p, g, \rho_1, \rho_2) \mapsto (s(\rho_1), s(\rho_2))$$

is a surjective submersion. Now consider the smooth surjective map

$$g : (\mathcal{P}_0 \times \Gamma_0)_{\tau \times t \times t} \mathcal{P}_1^{[2]} \rightarrow \mathcal{P}_1 \times \Gamma_0 : (p, g, \rho_1, \rho_2) \mapsto (\rho_1^{-1} \circ R(\rho_2, \text{id}_{g^{-1}}), g^{-1}).$$

We have $\chi \circ g = f$; thus, χ is a surjective submersion. Next we check that we have defined an action. Suppose $(\rho, g) \in P$ and $\gamma \in \Gamma_1$ such that $\alpha(\rho, g) = g = t(\gamma)$. Then, $\alpha((\rho, g) \circ \gamma) = s(\gamma)$. Moreover, suppose $\gamma_1, \gamma_2 \in \Gamma_1$ with $t(\gamma_1) = g$ and $t(\gamma_2) = s(\gamma_1)$. Then,

$$\begin{aligned} ((\rho, g) \circ \gamma_1) \circ \gamma_2 &= (R(\rho, \text{id}_{g^{-1}} \cdot \gamma_1), s(\gamma_1)) \circ \gamma_2 \\ &= (R(\rho, \text{id}_{g^{-1}} \cdot \gamma_1 \cdot \text{id}_{s(\gamma_1)^{-1}} \cdot \gamma_2), s(\gamma_2)) = (\rho, g) \circ (\gamma_1 \circ \gamma_2), \end{aligned}$$

where we have used that $\gamma_1 \circ \gamma_2 = \gamma_1 \cdot \text{id}_{s(\gamma_1)^{-1}} \cdot \gamma_2$ in any 2-group. It remains to check that the smooth map

$$\tilde{\tau} : P_\alpha \times_t \Gamma_1 \longrightarrow P_\chi \times_\chi P : ((\rho, g), \gamma) \longmapsto ((\rho, g), (\rho, g) \circ \gamma)$$

is a diffeomorphism. For this purpose, we consider the diagram

$$\begin{array}{ccc} & & \mathcal{P}_1^{[2]} \\ & & \downarrow s \times t \\ (\mathcal{P}_0 \times \Gamma_0) \times (\mathcal{P}_0 \times \Gamma_0) & \xrightarrow{\tau \times \tau} & \mathcal{P}_0^{[2]} \times \mathcal{P}_0^{[2]} \end{array} \quad (7.1.2)$$

and claim that (a) $N_1 := P_\alpha \times_t \Gamma_1$ is a pullback of (7.1.2), (b) $N_2 := P_\chi \times_\chi P$ is a pullback of (7.1.2), and (c) that the unique map $N_1 \longrightarrow N_2$ is $\tilde{\tau}$. Thus, $\tilde{\tau}$ is a diffeomorphism.

In order to prove claim (a) we use again that the functor $\tau = (\text{id}, R)$ is a weak equivalence, so that by Theorem 2.3.12 the triple $(\mathcal{P}_1 \times \Gamma_1, \tau, s \times t)$ is a pullback of (7.1.2). We consider the smooth map

$$\xi : N_1 \longrightarrow \mathcal{P}_1 \times \Gamma_1 : ((\rho, g), \gamma) \longmapsto (R(\rho, \text{id}_{g^{-1}}), \gamma)$$

which is a diffeomorphism because $(\rho, \gamma) \longmapsto ((R(\rho, \text{id}_{t(\gamma)}), t(\gamma)), \gamma)$ is a smooth map which is inverse to ξ . Thus, putting $f_1 := \tau \circ \xi$ and $g_1 := (s \times t) \circ \xi$ we see that (N_1, f_1, g_1) is a pullback of (7.1.2). In order to prove claim (b), we put

$$\begin{aligned} f_2((\rho_1, g_1), (\rho_2, g_2)) &:= (R(\rho_1, \text{id}_{g_1^{-1}}), \rho_2) \\ g_2((\rho_1, g_1), (\rho_2, g_2)) &:= (R(s(\rho), g_1^{-1}), g_2, R(t(\rho_1), g_1^{-1}), g_1), \end{aligned}$$

and it is straightforward to check that the cone (N_2, f_2, g_2) makes (7.1.2) commutative. The triple (N_2, f_2, g_2) is also universal: in order to see this suppose N' is any smooth manifold with smooth maps $f' : N' \longrightarrow \mathcal{P}_1^{[2]}$ and $g' : N' \longrightarrow (\mathcal{P}_0 \times \Gamma_0) \times (\mathcal{P}_0 \times \Gamma_0)$ so that (7.1.2) is commutative. For $n \in N'$, we write $f'(n) = (\rho_1, \rho_2)$ and $g'(n) = (p_1, g_1, p_2, g_2)$. Then, $\sigma(n) := ((R(\rho_1, \text{id}_{g_2^{-1}}), g_2), (\rho_2, g_1))$ defines a smooth map $\sigma : N' \longrightarrow P_\chi \times_\chi P$. One checks that $f_2 \circ \sigma = f'$ and $g_2 \circ \sigma = g'$, and that σ is the only smooth map satisfying these equations. This proves that (N_2, f_2, g_2) is a pullback. We are left with claim (c). Here one only has to check that $\tau : N_1 \longrightarrow N_2$ satisfies $f_2 = f_1 \circ \tau$ and $g_2 = g_1 \circ \tau$. \square

Remark 7.1.2. The smooth functor $\tau = (\text{id}, R) : \mathcal{P} \times \Gamma \longrightarrow \mathcal{P} \times_M \mathcal{P}$ is a weak equivalence, and so has a canonical inverse anafunctor τ^{-1} (Remark 2.3.13). The anafunctor

$$\mathcal{P}_0^{[2]} \xrightarrow{\iota} \mathcal{P} \times_M \mathcal{P} \xrightarrow{c} \mathcal{P} \times_M \mathcal{P} \xrightarrow{\tau^{-1}} \mathcal{P} \times \Gamma \xrightarrow{\text{pr}_2} \Gamma,$$

where c is the functor that switches the factors, corresponds to a principal Γ -bundle over $\mathcal{P}_0^{[2]}$ that is canonically isomorphic to the bundle P defined above.

It remains to provide the bundle gerbe product

$$\mu : \pi_{23}^* P \otimes \pi_{12}^* P \longrightarrow \pi_{13}^* P,$$

which we define by the formula

$$\mu((\rho_{23}, g_{23}), (\rho_{12}, g_{12})) := (\rho_{12} \circ R(\rho_{23}, \text{id}_{g_{12}}), g_{23}g_{12}). \quad (7.1.3)$$

Lemma 7.1.3. *(7.1.3) defines an isomorphism $\mu : \pi_{23}^* P \otimes \pi_{12}^* P \longrightarrow \pi_{13}^* P$ of principal Γ -bundles over $\mathcal{P}_0^{[3]}$.*

Proof. First of all, we recall from Example 2.4.7 (b) that an element in the tensor product $\pi_{23}^* P \otimes \pi_{12}^* P$ is represented by a triple (p_{23}, p_{12}, γ) where $p_{23}, p_{12} \in P$ with $\pi_1(\chi(p_{23})) = \pi_2(\chi(p_{12}))$, and $\alpha(p_{23}) \cdot \alpha(p_{12}) = t(\gamma)$. In (7.1.3) we refer to triples where $\gamma = \text{id}_{g_{23}g_{12}}$, and this definition extends to triples with general $\gamma \in \Gamma_1$ by employing the equivalence relation

$$(p_1, p_2, \gamma) \sim (p_1 \circ (\gamma \cdot \text{id}_{\alpha(p_2)^{-1}}), p_2, \text{id}_{s(\gamma)}). \quad (7.1.4)$$

The complete formula for μ is then

$$\mu((\rho_{23}, g_{23}), (\rho_{12}, g_{12}), \gamma) = (\rho_{12} \circ R(\rho_{23}, \text{id}_{g_{23}^{-1}} \cdot \gamma), s(\gamma)). \quad (7.1.5)$$

Next we check that (7.1.5) is well-defined under the equivalence relation (7.1.4):

$$\begin{aligned} \mu(((\rho_{23}, g_{23}), (\rho_{12}, g_{12}), \gamma)) &= (\rho_{12} \circ R(\rho_{23}, \text{id}_{g_{23}^{-1}} \cdot \gamma), s(\gamma)) \\ &= (\rho_{12} \circ R(\rho_{23} \circ R(\text{id}_{R(s(\rho_{23}), g_{23}^{-1})}, \gamma \cdot \text{id}_{g_{12}^{-1}}), \text{id}_{g_{12}}), s(\gamma)) \\ &= \mu((\rho_{23} \circ R(\text{id}_{R(s(\rho_{23}), g_{23}^{-1})}, \gamma \cdot \text{id}_{g_{12}^{-1}}), s(\gamma)g_{12}^{-1}), (\rho_{12}, g_{12}), \text{id}_{s(\gamma)}) \\ &= \mu(((\rho_{23}, g_{23}) \circ (\gamma \cdot \text{id}_{g_{12}^{-1}}), (\rho_{12}, g_{12}), \text{id}_{s(\gamma)})). \end{aligned}$$

Now we have shown that μ is a well-defined map from $\pi_{23}^* P \otimes \pi_{12}^* P$ to $\pi_{13}^* P$, and it remains to prove that it is a bundle morphism. Checking that it preserves fibres and anchors is straightforward. It remains to check that (7.1.5) preserves the Γ -action. We calculate

$$\begin{aligned} \mu(((\rho_{23}, g_{23}), (\rho_{12}, g_{12}), \gamma) \circ \tilde{\gamma}) &= \mu((\rho_{23}, g_{23}), (\rho_{12}, g_{12}), \gamma \circ \tilde{\gamma}) \\ &= (\rho_{23} \circ R(\rho_{12}, \text{id}_{g_{12}} \cdot i(\gamma \circ \tilde{\gamma})), s(\tilde{\gamma})) \\ &= (\rho_{23} \circ R(R(\rho_{12}, \text{id}_{g_{12}}), i(\gamma) \circ i(\tilde{\gamma})), s(\tilde{\gamma})) \\ &= (\rho_{23} \circ R(R(\rho_{12}, \text{id}_{g_{12}}), i(\gamma))) \circ R(\text{id}_{R(s(\rho_{12}), g)}, i(\tilde{\gamma})), s(\tilde{\gamma})) \\ &= (\rho_{23} \circ R(\rho_{12}, \text{id}_{g_{12}} \cdot i(\gamma)) \circ R(\text{id}_{R(s(\rho_{12}), g)}, i(\tilde{\gamma})), s(\tilde{\gamma})) \\ &= (\rho_{23} \circ R(\rho_{12}, \text{id}_{g_{12}} \cdot i(\gamma)), s(\gamma)) \circ \tilde{\gamma} \\ &= \mu((\rho_{23}, g_{23}), (\rho_{12}, g_{12}), \gamma) \circ \tilde{\gamma}. \end{aligned}$$

Summarizing, μ is a morphism of Γ -bundles over $\mathcal{P}_0^{[3]}$. □

In order to finish the definition of the Γ -bundle gerbe $\mathcal{E}_M(\mathcal{P})$ it remains to verify the following assertion, which merely uses the definitions and can be left for the reader.

Lemma 7.1.4. *The bundle morphism μ is associative.*

Definition of \mathcal{E}_M on 1-morphisms

We define a 1-morphism $\mathcal{E}_M(F) : \mathcal{E}_M(\mathcal{P}) \rightarrow \mathcal{E}_M(\mathcal{P}')$ between Γ -bundle gerbes from a 1-morphism $F : \mathcal{P} \rightarrow \mathcal{P}'$ between principal Γ -2-bundles. The refinement of the surjective submersions $\pi : \mathcal{P} \rightarrow M$ and $\pi' : \mathcal{P}' \rightarrow M$ is the fibre product $Z := \mathcal{P}_0 \times_M \mathcal{P}'_0$. Its principal Γ -bundle has the total space

$$Q := F \times \Gamma_0,$$

and its projection, anchor and Γ -action are given, respectively, by

$$\begin{aligned} \chi(f, g) &:= (\alpha_l(f), R(\alpha_r(f), g^{-1})) \quad , \quad \alpha(f, g) := g \\ &\text{and} \quad (f, g) \circ \gamma := (\rho(f, \text{id}_{g^{-1}} \cdot \gamma), s(\gamma)), \end{aligned} \quad (7.1.6)$$

where $\rho : F \times \Gamma_1 \rightarrow F$ denotes the Γ_1 -action on F that comes from the given Γ -equivariant structure on F (see Appendix A).

Lemma 7.1.5. *This defines a principal Γ -bundle Q over Z .*

Proof. We show first the the projection $\chi : Q \rightarrow Z$ is a surjective submersion. Since the functor $\tau' : \mathcal{P}' \times \Gamma \rightarrow \mathcal{P} \times_M \mathcal{P}$ is a weak equivalence, we have by Theorem 2.3.12 a pullback

$$\begin{array}{ccc} X & \longrightarrow & (\mathcal{P}'_0 \times \Gamma_0) \times_{R \times t} (\mathcal{P}'_1 \times_M \mathcal{P}'_1) \\ \xi \downarrow & & \downarrow \text{spr}_2 \\ F \times_{\pi' \circ \alpha_l(f)} \times_{\pi'} \mathcal{P}'_0 & \longrightarrow & \mathcal{P}'_0 \times_M \mathcal{P}'_0 \end{array}$$

along the bottom map $(f, p') \mapsto (\alpha_r(f), p')$, which is well-defined because the anafunctor F preserves the projections to M (see Remark 6.1.6 (b)). In particular, the map ξ is a surjective submersion. It is easy to see that the smooth map

$$k : X \rightarrow F \times \Gamma_0 : ((f, p'), (p'_0, g, \rho, \tilde{\rho})) \mapsto (f \circ \rho^{-1} \circ R(\tilde{\rho}, \text{id}_{g^{-1}}), g^{-1})$$

is surjective. Now we consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{k} & F \times \Gamma_0 \\ \xi \downarrow & & \downarrow \chi \\ F \times_{\pi' \circ \alpha_l(f)} \times_{\pi'} \mathcal{P}'_0 & \xrightarrow{\alpha_l \times \text{id}} & \mathcal{P}'_0 \times_M \mathcal{P}'_0. \end{array}$$

The surjectivity of k and the fact that ξ and $\alpha_l \times \text{id}$ are surjective submersions shows that χ is one, too.

Next, one checks (similarly as in the proof of Lemma 7.1.1) that the Γ -action on Q defined above is well-defined and preserves the projection. Then it remains to check that the smooth map

$$\xi : Q_\alpha \times_t \Gamma_1 \longrightarrow Q \times_{\mathcal{P}_0 \times_M \mathcal{P}'_0} Q : (f, g, \gamma) \longmapsto (f, g, \rho(f, \text{id}_{g^{-1}} \cdot \gamma), s(\gamma))$$

is a diffeomorphism. An inverse map is given as follows. Given an element (f_1, g_1, f_2, g_2) on the right hand side, we have $\alpha_l(f_1) = \alpha_l(f_2)$, so that there exists a unique element $\rho' \in \mathcal{P}'_1$ such that $f_1 \circ \rho' = f_2$. One calculates that (ρ', g_2) and $(\text{id}_{\alpha_r(f_1)}, g_1)$ are elements of the principal Γ -bundle $\mathcal{P}' \times \Gamma_0$ over \mathcal{P}'_0 of Lemma 7.1.1. Thus, there exists a unique element $\gamma \in \Gamma_1$ such that $(\rho', g_2) = (\text{id}_{\alpha_r(f_1)}, g_1) \circ \gamma$. Clearly, $t(\gamma) = g_1$ and $s(\gamma) = g_2$, and we have $\rho' = R(\text{id}_{\alpha_r(f_1)}, \text{id}_{g_1^{-1}} \cdot \gamma)$. We define $\xi^{-1}(f_1, g_1, f_2, g_2) := (f_1, g_1, \gamma)$. The calculation that ξ^{-1} is an inverse for ξ uses property (ii) of Definition A.1 for the action ρ , and is left to the reader. \square

The next step in the definition of the 1-morphism $\mathcal{E}(F)$ is to define the bundle morphism

$$\beta : P' \otimes \zeta_1^* Q \longrightarrow \zeta_2^* Q \otimes P$$

over $Z \times_M Z$. We use the notation of Example 2.4.7 (b) for elements of tensor products of principal Γ -bundles; in this notation, the morphism β in the fibre over a point $((p_1, p'_1), (p_2, p'_2)) \in Z \times_M Z$ is given by

$$\beta : ((\rho', g'), (f, g), \gamma) \longmapsto ((\tilde{f}, g'gh), (\tilde{\rho}, h^{-1}), \gamma),$$

where $h \in \Gamma_0$ and $\tilde{\rho} \in \mathcal{P}'_1$ are chosen such that $s(\tilde{\rho}) = R(p_2, h^{-1})$ and $t(\tilde{\rho}) = p_1$, and

$$\tilde{f} := \rho(\tilde{\rho}^{-1} \circ f \circ R(\rho', \text{id}_g), \text{id}_h). \quad (7.1.7)$$

Lemma 7.1.6. *This defines an isomorphism between principal Γ -bundles.*

Proof. The existence of choices of $\tilde{\rho}, h$ follows because the functor $\tau' : \mathcal{P}' \times \Gamma \longrightarrow \mathcal{P}' \times_M \mathcal{P}'$ is smoothly essentially surjective (Theorem 2.3.12); in particular, one can choose them locally in a smooth way. We claim that the equivalence relation on $\zeta_2^* Q \otimes P$ identifies different choices; thus, we have a well-defined smooth map. In order to prove this claim, we assume other choices $\tilde{\rho}', h'$. The pairs $(\tilde{\rho}, h^{-1})$ and $(\tilde{\rho}', h'^{-1})$ are elements in the principal Γ -bundle P' over $\mathcal{P}'_0 \times_M \mathcal{P}'_0$ and sit over the same fibre; thus, there exists a unique $\tilde{\gamma} \in \Gamma_1$ such that $(\tilde{\rho}, h^{-1}) \circ \tilde{\gamma} = (\tilde{\rho}', h'^{-1})$, in particular, $R(\tilde{\rho}, \text{id}_h \cdot \tilde{\gamma}) = \tilde{\rho}'$. Now we have

$$\begin{aligned} ((\tilde{f}, g'gh), (\tilde{\rho}, h^{-1}), \gamma) &= ((\tilde{f}, g'gh), (\tilde{\rho}, h^{-1}), (\text{id}_{t(\gamma)} \cdot i(\tilde{\gamma}) \cdot \tilde{\gamma}) \circ \gamma) \\ &\sim ((\tilde{f}, g'gh) \circ (\text{id}_{t(\gamma)} \cdot i(\tilde{\gamma})), (\tilde{\rho}, h^{-1}) \circ \tilde{\gamma}, \gamma) \end{aligned}$$

so that it suffices to calculate

$$\begin{aligned}
(\tilde{f}, g'gh) \circ (\text{id}_{t(\tilde{\gamma})} \cdot i(\tilde{\gamma})) &= (\rho(\tilde{f}, \text{id}_{h^{-1}} \cdot i(\tilde{\gamma})), g'gh') \\
&= (\rho(\tilde{\rho}^{-1} \circ f \circ R(\rho', \text{id}_g), i(\tilde{\gamma})), g'gh') \\
&= (\rho(R(\tilde{\rho}^{-1}, i(\tilde{\gamma}) \cdot \text{id}_{h'^{-1}}) \circ f \circ R(\rho', \text{id}_g), \text{id}_{h'}), g'gh'),
\end{aligned}$$

where the last step uses the compatibility condition for ρ from Definition A.1 (ii). In any 2-group, we have $i(\tilde{\gamma}) \cdot \text{id}_{s(\tilde{\gamma})} = (\text{id}_{t(\tilde{\gamma})^{-1}} \cdot \tilde{\gamma})^{-1}$, in which case the last line is exactly the formula (7.1.7) for the pair $(\tilde{\rho}', h')$.

Next we check that β is well-defined under the equivalence relation on the tensor product $P' \otimes \zeta_1^* Q$. We have

$$x := ((\rho', g'), (f, g), (\gamma_1 \cdot \gamma_2) \circ \gamma) \sim ((\rho', g') \circ \gamma_1, (f, g) \circ \gamma_2, \gamma) =: x'$$

for $\gamma_1, \gamma_2 \in \Gamma_1$ such that $t(\gamma_1) = g'$, $t(\gamma_2) = g$ and $s(\gamma_1)s(\gamma_2) = t(\gamma)$. Taking advantage of the fact that we can make the same choice of $(\tilde{\rho}, h)$ for both representatives x and x' , it is straightforward to show that $\beta(x) = \beta(x')$. Finally, it is obvious from the definition of β that it is anchor-preserving and Γ -equivariant. \square

In order to show that the triple (Z, Q, β) defines a 1-morphism between bundle gerbes, it remained to verify the following fact, which is straightforward and left to the reader:

Lemma 7.1.7. *The bundle isomorphism β is compatible with the bundle gerbe products μ_1 and μ_2 in the sense of diagram (5.1.1).*

Definition of \mathcal{E}_M on 2-morphisms, compositors and unitors

Let $F_1, F_2 : \mathcal{P} \rightarrow \mathcal{P}'$ be 1-morphisms between principal Γ -bundles over M , and let $\eta : F \Rightarrow G$ be a 2-morphism. Between the Γ -bundles Q_1 and Q_2 , which live over the same common refinement $Z = \mathcal{P}_0 \times_M \mathcal{P}'_0$, we find immediately the smooth map

$$\eta : Q_1 \rightarrow Q_2 : (f_1, g) \mapsto (\eta(f_1), g)$$

which is easily verified to be a bundle morphism. Its compatibility with the bundle morphisms β_1 and β_2 in the sense of the simplified diagram (5.1.4) is also easy to check. Thus, we have defined a 2-morphism $\mathcal{E}_M(\eta) : \mathcal{E}_M(F_1) \Rightarrow \mathcal{E}_M(F_2)$.

The compositor for 1-morphisms $F_1 : \mathcal{P} \rightarrow \mathcal{P}'$ and $F_2 : \mathcal{P}' \rightarrow \mathcal{P}''$ is a bundle gerbe 2-morphism

$$c_{F_1, F_2} : \mathcal{E}_M(F_2 \circ F_1) \rightarrow \mathcal{E}_M(F_2) \circ \mathcal{E}_M(F_1).$$

Employing the above constructions, the 1-morphism $\mathcal{E}_M(F_2 \circ F_1)$ is defined on the common refinement $Z_{12} := \mathcal{P}_0 \times_M \mathcal{P}''_0$ and has the Γ -bundle $Q_{12} = (F_1 \times_{\mathcal{P}'_0} F_2) / \mathcal{P}'_1 \times \Gamma_0$, whereas the 1-morphism

$\mathcal{E}_M(F_2) \circ \mathcal{E}_M(F_1)$ is defined on the common refinement $Z := \mathcal{P}_0 \times_M \mathcal{P}'_0 \times_M \mathcal{P}''_0$ and has the Γ -bundle $Q_2 \otimes Q_1$ with $Q_k = F_k \times \Gamma_0$. The compositor c_{F_1, F_2} is defined over the refinement Z with the obvious refinement maps $\text{pr}_{13} : Z \rightarrow Z_{12}$ and $\text{id} : Z \rightarrow Z$ making diagram (5.1.3) commutative. It is thus a bundle morphism $c_{F_1, F_2} : \text{pr}_{13}^* Q_{12} \rightarrow Q_2 \otimes Q_1$. For elements in a tensor product of Γ -bundles we use the notation of Example 2.4.7 (b). Then, we define c_{F_1, F_2} by

$$((p, p', p''), (f_1, f_2, g)) \mapsto ((\rho_2(\tilde{\rho}^{-1} \circ f_2, \text{id}_h), gh), (f_1 \circ \tilde{\rho}, h^{-1}), \text{id}_g), \quad (7.1.8)$$

where $h \in \Gamma_0$ and $\tilde{\rho} : R(p', h^{-1}) \rightarrow \alpha_r(f_1) = \alpha_l(f_2)$ are chosen in the same way as in the proof of Lemma 7.1.6. The assignment (7.1.8) does not depend on the choices of h and $\tilde{\rho}$, and also not on the choice of the representative (f_1, f_2) in $(F_1 \times_{\mathcal{P}'_0} F_2) / \mathcal{P}'_1$. It is obvious that (7.1.8) is anchor-preserving, and its Γ -equivariance can be seen by choosing $(\tilde{\rho}, h)$ in order to compute $c_{F_1, F_2}((p, p', p''), (f_1, f_2, g))$ and $(\tilde{\rho}', h)$ with $\tilde{\rho}' := R(\tilde{\rho}, \text{id}_{g^{-1}} \cdot \gamma^{-1})$ in order to compute $c_{F_1, F_2}(((p, p', p''), (f_1, f_2, g)) \circ \gamma)$. In order to complete the construction of the bundle gerbe 2-morphism c_{F_1, F_2} we have to prove that the bundle morphism c_{F_1, F_2} is compatible with the isomorphisms β_{12} of $\mathcal{E}_M(F_2 \circ F_1)$ and $(\text{id} \otimes \beta_1) \circ (\beta_2 \otimes \text{id})$ of $\mathcal{E}_M(F_2) \circ \mathcal{E}_M(F_1)$ in the sense of diagram (5.1.4). We start with an element $((\rho'', g''), (f_{12}, g)) \in \mathcal{E}_M(\mathcal{P}'') \otimes \zeta_1^* Q_{12}$, where $f_{12} = (f_1, f_2)$. We have

$$\beta_{12}((\rho'', g''), (f_{12}, g)) = (\widetilde{f_{12}}, g'' gh, \tilde{\rho}, h^{-1})$$

upon choosing $(\tilde{\rho}, h)$ as required in the definition of $\mathcal{E}_M(F_2 \circ F_1)$. Writing $\widetilde{f_{12}} = (\tilde{f}_1, \tilde{f}_2)$ further we have

$$(\zeta_2^* c_{F_1, F_2} \otimes \text{id})((\widetilde{f_{12}}, g'' gh, \tilde{\rho}, h^{-1})) = (\rho_2(\tilde{\rho}_2^{-1} \circ \tilde{f}_2, \text{id}_{h_2}), g'' gh h_2, \tilde{f}_1 \circ \tilde{\rho}_2, h_2^{-1}, \tilde{\rho}, h^{-1}) \quad (7.1.9)$$

upon choosing appropriate $(\tilde{\rho}_2, h_2)$ as required in the definition of c_{F_1, F_2} . This is the result of the clockwise composition of diagram (5.1.4). Counter-clockwise, we first get

$$(\text{id} \otimes \zeta_1^* c_{F_1, F_2})((\rho'', g''), (f_{12}, g)) = (\rho'', g'', f'', gh_1, f', h_1^{-1})$$

for choices $(\tilde{\rho}_1, h_1)$, where $f'' := \rho_2(\tilde{\rho}_1^{-1} \circ f_2, \text{id}_{h_1})$ and $f' := f_1 \circ \tilde{\rho}_1$. Next we apply the isomorphism β_2 of $\mathcal{E}_M(F_2)$ and get

$$(\beta_2 \otimes \text{id})(\rho'', g'', f'', gh_1, f', h_1^{-1}) = (\widetilde{f''}, g'' gh h_2, \hat{\rho}, \hat{h}^{-1}, f', h_1^{-1})$$

where we have used the choices $(\hat{\rho}, \hat{h})$ defined by $\hat{\rho} := R(\tilde{\rho}_1^{-1}, h_1) \circ R(\tilde{\rho}_2, h^{-1} h_1)$ and $\hat{h} := h_1^{-1} h h_2$. The last step is to apply the isomorphism β_1 of $\mathcal{E}_M(F_2)$ which gives

$$(\text{id} \otimes \beta_1)(\widetilde{f''}, g'' gh h_2, \hat{\rho}, \hat{h}^{-1}, f', h_1^{-1}) = (\widetilde{f''}, g'' gh h_2, \tilde{f}', h_2^{-1}, \tilde{\rho}, h^{-1}), \quad (7.1.10)$$

where we have used the choices $(\tilde{\rho}, h)$ from above. Comparing (7.1.9) and (7.1.10), we have obviously coincidence in all but the first and the third component. For these remaining factors, coincidence follows from the definitions of the various variables.

Finally, we have to construct unitors. The unitor for a principal Γ -2-bundle \mathcal{P} over M is a bundle gerbe 2-morphism

$$u_{\mathcal{P}} : \mathcal{E}_M(\text{id}_{\mathcal{P}}) \rightrightarrows \text{id}_{\mathcal{E}_M(\mathcal{P})}.$$

Abstractly, one can associate to $\text{id}_{\mathcal{E}_M(\mathcal{P})}$ the 1-morphism $\text{id}_{\mathcal{E}_M(\mathcal{P})}^{FP}$ constructed in the proof of Lemma 5.2.8, and then notice that $\text{id}_{\mathcal{E}_M(\mathcal{P})}^{FP}$ and $\mathcal{E}_M(\text{id}_{\mathcal{P}})$ are canonically 2-isomorphic. In more concrete terms, the unitor $u_{\mathcal{P}}$ has the refinement $W := \mathcal{P}_0^{[3]}$ with the surjective submersions $r := \text{pr}_{12}$ and $r' := \text{pr}_3$ to the refinements $Z = \mathcal{P}_0^{[2]}$ and $Z' = \mathcal{P}_0$ of the 1-morphisms $\mathcal{E}_M(\text{id}_{\mathcal{P}})$ and $\text{id}_{\mathcal{E}_M(\mathcal{P})}$, respectively. The relevant maps x_W and y_W are pr_{13} and pr_{23} , respectively. The principal Γ -bundle of the 1-morphism $\text{id}_{\mathcal{E}_M(\mathcal{P})}$ is the trivial bundle $Q' = \mathbf{I}_1$. We claim that the principal Γ -bundle Q of $\mathcal{E}_M(\text{id}_{\mathcal{P}})$ is the bundle P of the bundle gerbe $\mathcal{E}_M(\mathcal{P})$. Indeed, the formulae (7.1.6) reduce for the identity anafunctor $\text{id}_{\mathcal{P}}$ to those of (7.1.1). Now, the bundle isomorphism of the unitor $u_{\mathcal{P}}$ is

$$y_W^* P \otimes r^* Q = \text{pr}_{23}^* P \otimes \text{pr}_{12}^* P \xrightarrow{\mu} \text{pr}_{13}^* P \cong r'^* Q' \otimes x_W^* P,$$

where μ is the bundle gerbe product of $\mathcal{E}_M(\mathcal{P})$. The commutativity of diagram (5.1.2) follows from the associativity of μ .

Proposition 7.1.8. *The assignments \mathcal{E}_M for objects, 1-morphisms and 2-morphisms, together with the compositors and unitors defined above, define a 2-functor*

$$\mathcal{E}_M : 2\text{-Bun}_{\Gamma}(M) \longrightarrow \text{Grb}_{\Gamma}(M).$$

Proof. A list of axioms for a 2-functor with the same conventions as we use here can be found in [SW, Appendix A]. The first axiom requires that the 2-functor \mathcal{E}_M respects the vertical composition of 2-morphisms – this follows immediately from the definition.

The second axiom requires that the compositors respect the horizontal composition of 2-morphisms. To see this, let $F_1, F'_1 : \mathcal{P} \rightarrow \mathcal{P}'$ and $F_2, F'_2 : \mathcal{P}' \rightarrow \mathcal{P}''$ be 1-morphisms between principal Γ -2-bundles, and let $\eta_1 : F_1 \rightrightarrows F'_1$ and $\eta_2 : F_2 \rightrightarrows F'_2$ be 2-morphisms. Then, the diagram

$$\begin{array}{ccc} \mathcal{E}_M(F_2 \circ F_1) & \xrightarrow{\mathcal{E}_M(\eta_1 \circ \eta_2)} & \mathcal{E}_M(F'_2 \circ F'_1) \\ \downarrow c_{F_1, F_2} & & \downarrow c_{F'_1, F'_2} \\ \mathcal{E}_M(F_2) \circ \mathcal{E}_M(F_1) & \xrightarrow{\mathcal{E}_M(\eta_1) \circ \mathcal{E}_M(\eta_2)} & \mathcal{E}_M(F'_2) \circ \mathcal{E}_M(F'_1) \end{array}$$

has to commute. Indeed, in order to compute c_{F_1, F_2} and $c_{F'_1, F'_2}$ one can make the same choice of $(\tilde{\rho}, h)$, because the transformations η_1 and η_2 preserve the anchors. Then, commutativity follows from the fact that η_1 and η_2 commute with the groupoid actions and the Γ_1 -action according to Definition A.1.

The third axiom describes the compatibility of the compositors with the composition of 1-morphisms in the sense that the diagram

$$\begin{array}{ccc}
\mathcal{E}_M(F_3 \circ F_2 \circ F_1) & \xrightarrow{c_{F_2 \circ F_1, F_3}} & \mathcal{E}_M(F_3) \circ \mathcal{E}_M(F_2 \circ F_1) \\
\downarrow c_{F_3 \circ F_2, F_1} & & \downarrow \text{id} \circ c_{F_2, F_1} \\
\mathcal{E}_M(F_3 \circ F_2) \circ \mathcal{E}_M(F_1) & \xrightarrow{c_{F_3, F_2} \circ \text{id}} & \mathcal{E}_M(F_3) \circ \mathcal{E}_M(F_2) \circ \mathcal{E}_M(F_1).
\end{array}$$

is commutative. In order to verify this, one starts with an element (f_1, f_2, f_3, g) in $\mathcal{E}_M(F_3 \circ F_2 \circ F_1)$. In order to go clockwise, one chooses pairs $(\tilde{\rho}_{12,3}, h_{12,3})$ and $(\tilde{\rho}_{1,2}, h_{1,2})$ and gets from the definitions

$$\text{CW} = ((\rho_3(\tilde{\rho}_{12,3}^{-1} \circ f_3, \text{id}_{h_{12,3}}), gh_{12,3}), (\rho_2(\tilde{\rho}_{1,2}^{-1} \circ f_2 \circ \tilde{\rho}_{12,3}, \text{id}_{h_{1,2}}), h_{12,3}^{-1} h_{1,2}), (f_1 \circ \tilde{\rho}_{1,2}, h_{1,2}^{-1})).$$

Counter-clockwise, one can choose firstly again the pair $(\tilde{\rho}_{1,2}, h_{1,2})$ and then the pair $(\tilde{\rho}_{2,3}, h_{2,3})$ with $\tilde{\rho}_{2,3} = R(\tilde{\rho}_{12,3}, \text{id}_{h_{1,2}})$ and $h_{2,3} = h_{1,2}^{-1} h_{12,3}$. Then, one gets

$$\begin{aligned}
\text{CCW} = & ((\rho_3(\tilde{\rho}_{2,3}^{-1} \circ \rho_3(f_3, \text{id}_{h_{1,2}}), \text{id}_{h_{2,3}}), gh_{1,2} h_{2,3}), \\
& (\rho_2(\tilde{\rho}_{1,2}^{-1} \circ f_2, \text{id}_{h_{1,2}}) \circ \tilde{\rho}_{2,3}, h_{2,3}^{-1}), (f_1 \circ \tilde{\rho}_{1,2}, h_{1,2}^{-1})),
\end{aligned}$$

where one has to use formula (A.2) for the Γ_1 -action on the composition of equivariant anafunctors. Using the definitions of $h_{2,3}$ and $\tilde{\rho}_{2,3}$ as well as the axiom of Definition A.1 (ii) one can show that $\text{CW} = \text{CCW}$.

The fourth and last axiom requires that compositors and unitors are compatible with each other in the sense that for each 1-morphism $F : \mathcal{P} \rightarrow \mathcal{P}'$ the 2-morphisms

$$\mathcal{E}_M(F) \cong \mathcal{E}_M(F \circ \text{id}_{\mathcal{P}}) \xrightarrow{c_{\text{id}_{\mathcal{P}}, F}} \mathcal{E}_M(F) \circ \mathcal{E}_M(\text{id}_{\mathcal{P}}) \xrightarrow{\text{id} \circ u_{\mathcal{P}}} \mathcal{E}_M(F) \circ \text{id}_{\mathcal{E}_M(\mathcal{P})} \cong \mathcal{E}_M(F)$$

and

$$\mathcal{E}_M(F) \cong \mathcal{E}_M(\text{id}_{\mathcal{P}'} \circ F) \xrightarrow{c_{F, \text{id}_{\mathcal{P}'}}} \mathcal{E}_M(\text{id}_{\mathcal{P}'}) \circ \mathcal{E}_M(F) \xrightarrow{u_{\mathcal{P}'} \circ \text{id}} \text{id}_{\mathcal{E}_M(\mathcal{P}')} \circ \mathcal{E}_M(F) \cong \mathcal{E}_M(F)$$

are the identity 2-morphisms. We prove this for the first one and leave the second as an exercise. Using the definitions, we see that the 2-morphism has the refinement $W := \mathcal{P}_0 \times_M \mathcal{P}_0 \times_M \mathcal{P}'_0$ with $r = \text{pr}_{13}$ and $r' = \text{pr}_{23}$. The maps $x_W : W \rightarrow \mathcal{P}_0 \times_M \mathcal{P}_0$ and $y_W : W \rightarrow \mathcal{P}'_0 \times_M \mathcal{P}'_0$ are pr_{12} and $\Delta \circ \text{pr}_3$, respectively, where Δ is the diagonal map. Its bundle morphism is a morphism

$$\varphi : \text{pr}_{13}^* Q \rightarrow \text{pr}_{23}^* Q \otimes \text{pr}_{12}^* P,$$

where $Q = F \times \Gamma_0$ is the principal Γ -bundle of $\mathcal{E}_M(F)$, and $P = \mathcal{P}_1 \times \Gamma_0$ is the principal Γ -bundle of $\mathcal{E}_M(\mathcal{P})$. Over a point (p_1, p_2, p') and $(f, g) \in \text{pr}_{13}^* Q$, i.e. $\alpha_l(f) = p_1$ and $R(\alpha_r(f), g^{-1}) = p'$, the bundle morphism φ is given by

$$(f, g) \mapsto (\rho(\tilde{\rho}^{-1} \circ f, \text{id}_h), gh, \tilde{\rho}, h^{-1}),$$

where $h \in \Gamma_0$, and $\tilde{\rho} \in \mathcal{P}_1$ with $s(\tilde{\rho}) = R(p_2, h^{-1})$ and $t(\tilde{\rho}) = \alpha_l(f)$. We have to compare (W, φ) with the identity 2-morphism of $\mathcal{E}_M(F)$, which has the refinement Z with $r = r' = \text{id}$ and the identity bundle morphism. According to the equivalence relation on bundle gerbe 2-morphisms we have to evaluate φ over a point $w \in W$ with $r(w) = r'(w)$, i.e. w is of the form $w = (p, p, p')$. Here we can choose $h = 1$ and $\tilde{\rho} = \text{id}_p$, in which case we have $\varphi(f, g) = ((f, g), (\text{id}_p, 1))$. This is indeed the identity on Q . \square

Properties of the 2-functor \mathcal{E}_M

For the proof of Theorem 7.1 we provide the following two statements.

Lemma 7.1.9. *The 2-functor \mathcal{E}_M is fully faithful on Hom-categories.*

Proof. Let $\mathcal{P}, \mathcal{P}'$ be principal Γ -2-bundles over M , let $F_1, F_2 : \mathcal{P} \rightarrow \mathcal{P}'$ be 1-morphisms. By Lemma 5.2.8 every 2-morphism $\eta : \mathcal{E}_M(F_1) \Rightarrow \mathcal{E}_M(F_2)$ can be represented by one whose refinement is $\mathcal{P}_0 \times_M \mathcal{P}'_0$, so that its bundle isomorphism is $\eta : Q_1 \rightarrow Q_2$, where $Q_k := F_k \times \Gamma$ for $k = 1, 2$. We can read of a map $\eta : F_1 \rightarrow F_2$, and it is easy to see that this is a 2-morphism $\eta : F_1 \Rightarrow F_2$. This procedure is clearly inverse to the 2-functor \mathcal{E}_M on 2-morphisms. \square

Proposition 7.1.10. *The 2-functors \mathcal{E}_M form a 1-morphism between pre-2-stacks.*

Proof. For a smooth map $f : M \rightarrow N$, we have to look at the diagram

$$\begin{array}{ccc} 2\text{-Bun}_\Gamma(N) & \xrightarrow{f^*} & 2\text{-Bun}_\Gamma(M) \\ \mathcal{E}_N \downarrow & & \downarrow \mathcal{E}_M \\ \text{Grb}_\Gamma(N) & \xrightarrow{f^*} & \text{Grb}_\Gamma(M) \end{array}$$

of 2-functors. For \mathcal{P} a principal Γ -2-bundle over N , the Γ -bundle gerbe $\mathcal{E}_M(f^*\mathcal{P})$ has the surjective submersion $\text{pr}_1 : Y := M \times_N \mathcal{P}_0 \rightarrow M$, the principal Γ -bundle $P := M \times_N \mathcal{P}_1 \times \Gamma_0$ over $Y^{[2]}$, and a bundle gerbe product μ defined as in (7.1.3) that ignores the M -factor. On the other hand, the Γ -bundle gerbe $f^*\mathcal{E}_N(\mathcal{P})$ has the same surjective submersion, and – up to canonical identifications between fibre products – the same Γ -bundle and the same bundle gerbe product. These canonical identifications make up a pseudonatural transformation that renders the above diagram commutative. \square

7.2 From Bundle Gerbes to Principal 2-Bundles

In this section we provide the data we will feed into Lemma B.1 in order to produce a 2-functor $\mathcal{R}_M : \mathcal{G}rb_\Gamma(M) \rightarrow 2\text{-}\mathcal{B}un_\Gamma(M)$ that is inverse to the 2-functor \mathcal{E}_M constructed in the previous section. These data are:

1. A principal Γ -2-bundle $\mathcal{R}_\mathcal{G}$ for each Γ -bundle gerbe \mathcal{G} over M .
2. A 1-isomorphism $\mathcal{A}_\mathcal{G} : \mathcal{G} \rightarrow \mathcal{E}_M(\mathcal{R}_\mathcal{G})$ for each Γ -bundle gerbe \mathcal{G} over M .
3. A 1-morphism $\mathcal{R}_\mathcal{A} : \mathcal{P} \rightarrow \mathcal{P}'$ for all principal Γ -2-bundles $\mathcal{P}, \mathcal{P}'$ over M and all bundle gerbe 1-morphisms $\mathcal{A} : \mathcal{E}_M(\mathcal{P}) \rightarrow \mathcal{E}_M(\mathcal{P}')$.

Construction of the principal Γ -2-bundle $\mathcal{R}_\mathcal{G}$

We assume that \mathcal{G} consists of a surjective submersion $\pi : Y \rightarrow M$, a principal Γ -bundle P over $Y^{[2]}$ and a bundle gerbe product μ . Let $\alpha : P \rightarrow \Gamma_0$ be the anchor of P , and let $\chi : P \rightarrow Y^{[2]}$ be the bundle projection.

The Lie groupoid \mathcal{P} of the principal 2-bundle $\mathcal{R}_\mathcal{G}$ is defined by

$$\mathcal{P}_0 := Y \times \Gamma_0 \quad \text{and} \quad \mathcal{P}_1 := P \times \Gamma_0;$$

source map, target maps, and composition are given by, respectively,

$$\begin{aligned} s(p, g) &:= (\pi_2(\chi(p)), g) \quad , \quad t(p, g) := (\pi_1(\chi(p)), \alpha(p)^{-1} \cdot g) \\ &\quad \text{and} \quad (p_2, g_2) \circ (p_1, g_1) := (\mu(p_1, p_2), g_1). \end{aligned} \quad (7.2.1)$$

The identity morphism of an object $(y, g) \in \mathcal{P}_0$ is $(t_y, g) \in \mathcal{P}_1$, where t_y denotes the unit element in P over the point (y, y) , see Lemma 5.2.5. The inverse of a morphism $(p, g) \in \mathcal{P}_1$ is $(i(p), \alpha(p)^{-1}g)$, where $i : P \rightarrow P$ is the map from Lemma 5.2.5. The bundle projection is $\pi(y, g) := \pi(y)$. The action is given on objects and morphisms by

$$R_0((y, g), g') := (y, gg') \quad \text{and} \quad R_1((p, g), \gamma) := \left(p \circ (\text{id}_g \cdot \gamma \cdot \text{id}_{t(\gamma)^{-1}g^{-1}\alpha(p)}), g \cdot s(\gamma) \right). \quad (7.2.2)$$

Lemma 7.2.1. *This defines a functor $R : \mathcal{P} \times \Gamma \rightarrow \mathcal{P}$, and R is an action of Γ on \mathcal{P} .*

Proof. We assume that $t : H \rightarrow G$ is a smooth crossed module, and that Γ is the Lie 2-group associated to it, see Example 2.4.2 and Remark 2.4.3. Then we use the correspondence between principal Γ -bundles and principal H -bundles with H -anti-equivariant maps to G of Lemma 2.2.9. Writing $\gamma = (h, g')$, we have

$$R_1((p, g), \gamma) = (p \star^g h, gg').$$

With this simple formula at hand it is straightforward to show that R respects source and target maps and satisfies the axiom of an action. For the composition, we assume composable $(p_2, g_2), (p_1, g_1) \in \mathcal{P}_1$, i.e. $g_2 = \alpha(p_1)^{-1}g_1$, and composable $(h_2, g'_2), (h_1, g'_1) \in \Gamma_1$, i.e. $g'_2 = t(h_1)g'_1$. Then we have

$$\begin{aligned}
R((p_2, g_2) \circ (p_1, g_1), (h_2, g'_2) \circ (h_1, g'_1)) &= R((\mu(p_1, p_2), g_1), (h_2 h_1, g'_1)) \\
&= (\mu(p_1, p_2) \star^{g_1}(h_2 h_1), g_1 g'_1) \\
&= (\mu(p_1 \star^{g_1} h_2, p_2) \star^{g_1} h_1, g_1 g'_1) \\
&= (\mu(p_1, p_2 \star^{g_2} h_2) \star^{g_1} h_1, g_1 g'_1) \\
&= (\mu(p_1 \star^{g_1} h_1, p_2 \star^{g_2} h_2), g_1 g'_1) \\
&= (p_2 \star^{g_2} h_2, g_2 g'_2) \circ (p_1 \star^{g_1} h_1, g_1 g'_1) \\
&= R((p_2, g_2), (h_2, g'_2)) \circ R((p_1, g_1), (h_1, g'_1)),
\end{aligned}$$

finishing the proof. \square

It is obvious that the action R preserves the projection π . Thus, in order to complete the construction of the principal 2-bundle $\mathcal{R}_{\mathcal{G}}$ it remains to show that the functor $\tau = (\text{pr}_1, R)$ is a weak equivalence. This is the content of the following two lemmata in connection with Theorem 2.3.12.

Lemma 7.2.2. *τ is smoothly essentially surjective.*

Proof. The condition we have to check is whether or not the map

$$(Y \times \Gamma_0 \times \Gamma_0)_{\tau \times t} ((P \times \Gamma_0) \times_M (P \times \Gamma_0)) \xrightarrow{(s \times s) \circ \text{pr}_2} (Y \times \Gamma_0) \times_M (Y \times \Gamma_0)$$

is a surjective submersion. The left hand side is diffeomorphic to $(P \times \Gamma_0)_{\pi_1 \times \pi_1} (P \times \Gamma_0)$ via pr_2 , so that this is equivalent to checking that

$$s \times s : (P \times \Gamma_0)_{\pi_1 \circ \chi \times \pi_1 \circ \chi} (P \times \Gamma_0) \longrightarrow (Y \times \Gamma_0) \times_M (Y \times \Gamma_0)$$

is a surjective submersion. Since the Γ_0 -factors are just spectators, this is in turn equivalent to checking that

$$(\pi_2 \times \pi_2) \circ (\chi \times \chi) : P_{\pi_1 \circ \chi \times \pi_1 \circ \chi} P \longrightarrow Y^{[2]}$$

is a surjective submersion. It fits into the pullback diagram

$$\begin{array}{ccc}
P & \xrightarrow{\pi_1 \circ \chi \times \pi_1 \circ \chi} & P \times P \\
\downarrow \chi \times \chi & & \downarrow \chi \times \chi \\
Y^{[2]} & \xrightarrow{\pi_1 \times \pi_1} & Y^{[2]} \times Y^{[2]} \\
\downarrow \pi_2 \times \pi_2 & & \downarrow \pi_2 \times \pi_2 \\
Y^{[2]} & \xrightarrow{\quad} & Y \times Y
\end{array}$$

which has a surjective submersion on the right hand side; hence, also the map on the left hand side must be a surjective submersion. \square

Lemma 7.2.3. τ is smoothly fully faithful.

Proof. We assume a smooth manifold N with two smooth maps

$$f : N \longrightarrow (\mathcal{P}_0 \times \Gamma_0) \times (\mathcal{P}_0 \times \Gamma_0) \quad \text{and} \quad g : N \longrightarrow \mathcal{P}_1 \times_M \mathcal{P}_1$$

such that the diagram

$$\begin{array}{ccc}
N & \xrightarrow{g} & \mathcal{P}_1 \times_M \mathcal{P}_1 \\
\downarrow f & & \downarrow s \times t \\
(\mathcal{P}_0 \times \Gamma_0) \times (\mathcal{P}_0 \times \Gamma_0) & \xrightarrow{\tau \times \tau} & (\mathcal{P}_0 \times_M \mathcal{P}_0) \times (\mathcal{P}_0 \times_M \mathcal{P}_0)
\end{array}$$

is commutative. For a fixed point $n \in N$ we put

$$((p_1, g_1), (p_2, g_2)) := g(n) \in (P \times \Gamma_0) \times_M (P \times \Gamma_0)$$

and

$$((y, g, \tilde{g}), (y', g', \tilde{g}')) := f(n) \in (Y \times \Gamma_0 \times \Gamma_0) \times (Y \times \Gamma_0 \times \Gamma_0).$$

The commutativity of the diagram implies $\chi(p_1) = \chi(p_2) = (y', y)$, so that there exists $\gamma' \in \Gamma_1$ with $p_2 = p_1 \circ \gamma'$. We define $\gamma := \text{id}_{g_1^{-1}} \cdot \gamma' \cdot \text{id}_{\alpha(p_2)^{-1}g_2}$, which yields a morphism $\gamma \in \Gamma_1$ satisfying $\tau(p_1, g_1, \gamma) = (p_1, g_1, p_2, g_2) = g(n)$. On the other hand, we check that

$$(s(p_1, g_1, \gamma), t(p_1, g_1, \gamma)) = (\pi_2(p_1), g_1, s(\gamma), \pi_1(p_1), \alpha(p_1)^{-1}g_1, t(\gamma)) = f(n),$$

using that $s(\gamma) = g_1^{-1}g_2$ and $t(\gamma) = g_1^{-1}\alpha(p_1)\alpha(p_2)^{-1}g_2$. Summarizing, we have defined a smooth map

$$\sigma : N \longrightarrow \mathcal{P}_1 \times \Gamma_1 : n \longmapsto (p_1, g_1, \gamma)$$

such that $\tau \circ \sigma = g$ and $(s \times t) \circ \sigma = f$. Now let $\sigma' : N \rightarrow \mathcal{P}_1 \times \Gamma_1$ be another such map, and let $\sigma'(n) =: (p'_1, g'_1, \gamma')$. The condition that $\tau(\sigma(n)) = g(n) = \tau(\sigma'(n))$ shows immediately that $p_1 = p'_1$ and $g_1 = g'_1$, and then that $p_1 \circ \gamma = p_1 \circ \gamma'$. But since the Γ -action on P is principal, we have $\gamma = \gamma'$. This shows $\sigma = \sigma'$. Summarizing, $\mathcal{P}_1 \times \Gamma_1$ is a pullback. \square

Example 7.2.4. Suppose $\Gamma = \mathcal{B}U(1)$, see Example 2.1.1 (b), and suppose \mathcal{G} is a Γ -bundle gerbe over M , also known as a $U(1)$ -bundle gerbe, see Example 5.1.6. Then, the associated principal $\mathcal{B}U(1)$ -2-bundle $\mathcal{R}_{\mathcal{G}}$ has the groupoid \mathcal{P} with $\mathcal{P}_0 = Y$ and $\mathcal{P}_1 = P$, source and target maps $s = \pi_2 \circ \chi$ and $t = \pi_1 \circ \chi$, and composition $p_2 \circ p_1 = \mu(p_1, p_2)$. The action of $\mathcal{B}U(1)$ on \mathcal{P} is trivial on the level of objects and the given $U(1)$ -action on P on the level of morphisms. The same applies for general abelian Lie groups A instead of $U(1)$.

Construction of the 1-isomorphism $\mathcal{A}_{\mathcal{G}} : \mathcal{G} \rightarrow \mathcal{E}_M(\mathcal{R}_{\mathcal{G}})$

The Γ -bundle gerbe $\mathcal{E}_M(\mathcal{R}_{\mathcal{G}})$ has the surjective submersion $\tilde{Y} := Y \times \Gamma_0$ with $\tilde{\pi}(y, g) := \pi(y)$. The total space of its Γ -bundle \tilde{P} is $\tilde{P} := P \times \Gamma_0 \times \Gamma_0$; it has the anchor $\alpha(p, g, h) = h$, the bundle projection

$$\tilde{\chi} : \tilde{P} \rightarrow \tilde{Y}^{[2]} : (p, g, h) \mapsto ((\pi_1(\chi(p)), \alpha(p)^{-1}g), (\pi_2(\chi(p)), gh^{-1})),$$

the Γ -action is

$$\begin{aligned} (p, g, h) \circ \gamma &\stackrel{(7.1.1)}{=} ((p, g) \circ R((t_{\pi_2(\chi(p))}, gh^{-1}), \gamma), s(\gamma)) \\ &\stackrel{(7.2.2)}{=} ((p, g) \circ (t_{\pi_2(\chi(p))} \circ (\text{id}_{gh^{-1}} \cdot \gamma \cdot \text{id}_{g^{-1}}), gh^{-1}s(\gamma)), s(\gamma)) \\ &\stackrel{(7.2.1)}{=} (\mu(t_{\pi_2(\chi(p))} \circ (\text{id}_{gh^{-1}} \cdot \gamma \cdot \text{id}_{g^{-1}}), p), gh^{-1}s(\gamma), s(\gamma)) \\ &\stackrel{(2.4.3)}{=} (p \circ (\text{id}_{gh^{-1}} \cdot \gamma \cdot \text{id}_{g^{-1}\alpha(p)}), gh^{-1}s(\gamma), s(\gamma)), \end{aligned}$$

and its bundle gerbe product $\tilde{\mu}$ is given by

$$\begin{aligned} \tilde{\mu}((p_{23}, g_{23}, h_{23}), (p_{12}, g_{12}, h_{12})) &\stackrel{(7.1.3)}{=} ((p_{12}, g_{12}) \circ R((p_{23}, g_{23}), \text{id}_{h_{12}}), h_{23}h_{12}) \\ &\stackrel{(7.2.2)}{=} ((p_{12}, g_{12}) \circ (p_{23}, g_{23}h_{12}), h_{23}h_{12}) \\ &\stackrel{(7.2.1)}{=} (\mu(p_{23}, p_{12}), g_{23}h_{12}, h_{23}h_{12}). \end{aligned}$$

In order to compare the bundle gerbes \mathcal{G} and $\mathcal{E}_M(\mathcal{R}_{\mathcal{G}})$ we consider the smooth maps $\sigma : Y \rightarrow Y \times \Gamma_0$ and $\tilde{\sigma} : P \rightarrow \tilde{P}$ that are defined by $\sigma(y) := (y, 1)$ and $\tilde{\sigma}(p) := (p, \alpha(p), \alpha(p))$.

Lemma 7.2.5. $\tilde{\sigma}$ defines an isomorphism $\tilde{\sigma} : P \rightarrow (\sigma \times \sigma)^* \tilde{P}$ of Γ -bundles over $Y^{[2]}$. Moreover,

the diagram

$$\begin{array}{ccc}
\pi_{23}^* P \otimes \pi_{12}^* P & \xrightarrow{\bar{\sigma} \otimes \bar{\sigma}} & \tilde{\pi}_{23}^* \tilde{P} \otimes \tilde{\pi}_{12}^* \tilde{P} \\
\downarrow \mu & & \downarrow \tilde{\mu} \\
\pi_{13}^* P & \xrightarrow{\bar{\sigma}} & \tilde{\pi}_{13}^* \tilde{P}
\end{array}$$

is commutative.

Proof. For the first part it suffices to prove that $\bar{\sigma}$ is Γ -equivariant, preserves the anchors, and that the diagram

$$\begin{array}{ccc}
P & \xrightarrow{\bar{\sigma}} & \tilde{P} \\
\downarrow \chi & & \downarrow \tilde{\chi} \\
Y^{[2]} & \xrightarrow{\sigma \times \sigma} & \tilde{Y}^{[2]}
\end{array}$$

is commutative. Indeed, the commutativity of the diagram is obvious, and also that the anchors are preserved. For the Γ -equivariance, we have

$$\bar{\sigma}(p \circ \gamma) = (p \circ \gamma, s(\gamma), s(\gamma)) = (p, \alpha(p), \alpha(p)) \circ \gamma = \bar{\sigma}(p) \circ \gamma.$$

Finally, we calculate

$$\begin{aligned}
\tilde{\mu}((p_{23}, \alpha(p_{23}), \alpha(p_{23})), (p_{12}, \alpha(p_{12}), \alpha(p_{12}))) &= (\mu(p_{23}, p_{12}), \alpha(p_{23})\alpha(p_{12}), \alpha(p_{23})\alpha(p_{12})) \\
&= (\mu(p_{23}, p_{12}), \alpha(\mu(p_{23}, p_{12})), \alpha(\mu(p_{23}, p_{12})))
\end{aligned}$$

which shows the commutativity of the diagram. \square

Via Lemma 5.2.7 the bundle morphism $\bar{\sigma}$ defines the required 1-morphism $\mathcal{A}_{\mathcal{G}}$, and Lemma 5.2.3 guarantees that $\mathcal{A}_{\mathcal{G}}$ is a 1-isomorphism.

Construction of the 1-morphism $\mathcal{R}_{\mathcal{A}} : \mathcal{P} \rightarrow \mathcal{P}'$

Let $\mathcal{A} : \mathcal{E}_M(\mathcal{P}) \rightarrow \mathcal{E}_M(\mathcal{P}')$ be a 1-morphism between Γ -bundle gerbes obtained from principal Γ -2-bundles \mathcal{P} and \mathcal{P}' over M . By Lemma 5.2.8 we can assume that \mathcal{A} consists of a principal Γ -bundle $\chi : Q \rightarrow Z$ with $Z = \mathcal{P}_0 \times_M \mathcal{P}'_0$, and some isomorphism β over $Z^{[2]}$. For preparation, we consider the fibre products $Z_r := \mathcal{P}_0 \times_M \mathcal{P}'_0^{[2]}$ and $Z_l := \mathcal{P}_0^{[2]} \times_M \mathcal{P}'_0$ with the obvious embeddings $\iota_l : Z_l \rightarrow Z$ and $\iota_r : Z_r \rightarrow Z$ obtained by doubling elements. Together with the trivialization of Corollary 5.2.6, the pullbacks of β along ι_l and ι_r yield bundle morphisms

$$\beta_l := \iota_l^* \beta : \text{pr}_{13}^* Q \rightarrow \text{pr}_{23}^* Q \otimes \text{pr}_{12}^* P \quad \text{and} \quad \beta_r := \iota_r^* \beta : \text{pr}_{23}^* P' \otimes \text{pr}_{12}^* Q \rightarrow \text{pr}_{13}^* Q,$$

where $P := \mathcal{P}_1 \times \Gamma_0$ and $P' := \mathcal{P}' \times \Gamma_0$ are the principal Γ -bundles of the Γ -bundle gerbes $\mathcal{E}_M(\mathcal{P})$ and $\mathcal{E}_M(\mathcal{P}')$, respectively.

Lemma 7.2.6. *The bundle morphisms β_l and β_r have the following properties:*

(i) *They commute with each other in these sense that the diagram*

$$\begin{array}{ccc}
 P'_{p'_1, p'_2} \otimes Q_{p_1, p'_1} & \xrightarrow{\beta_r} & Q_{p_1, p'_2} \\
 \text{id} \otimes \beta_l \downarrow & \searrow \beta & \downarrow \beta_l \\
 P'_{p'_1, p'_2} \otimes Q_{p_2, p'_1} \otimes P_{p_1, p_2} & \xrightarrow{\beta_r \otimes \text{id}} & Q_{p_2, p'_2} \otimes P_{p_1, p_2}
 \end{array}$$

is commutative for all $((p_1, p'_1), (p_2, p'_2)) \in Z^{[2]}$.

(ii) *β_l is compatible with the bundle gerbe product μ in the sense that*

$$\beta_l|_{p_1, p_3, p'} = (\text{id} \otimes \mu_{p_1, p_2, p_3}) \circ (\beta_l|_{p_2, p_3, p'} \otimes \text{id}) \circ \beta_l|_{p_1, p_2, p'}$$

for all $(p_1, p_2, p_3, p') \in \mathcal{P}_0^{[3]} \times \mathcal{P}'_0$.

(iii) *β_r is compatible with the bundle gerbe product μ' in the sense that*

$$\beta_r|_{p, p'_1, p'_3} \circ (\mu'_{p'_1, p'_2, p'_3} \otimes \text{id}) = \beta_r|_{p, p'_2, p'_3} \circ (\text{id} \otimes \beta_r|_{p, p'_1, p'_2})$$

for all $(p, p'_1, p'_2, p'_3) \in \mathcal{P}_0 \times \mathcal{P}'_0^{[3]}$.

Proof. The identities (ii) and (iii) follow by restricting the commutative diagram (5.1.1) to the submanifolds $\mathcal{P}_0^{[3]} \times \mathcal{P}'_0$ and $\mathcal{P}_0 \times \mathcal{P}'_0^{[3]}$ of $Z^{[3]}$, respectively. Similarly, the commutativity of the two triangular subdiagrams in (i) follows by restricting (5.1.1) along appropriate embeddings $Z^{[2]} \rightarrow Z^{[3]}$. \square

Now we are in position to define the anafunctor $\mathcal{R}_{\mathcal{A}}$. First, we consider the left action

$$\beta_0 : \Gamma_0 \times Q \rightarrow Q : (g, q) \mapsto \beta_r((\text{id}, g), q)$$

that satisfies $\alpha(\beta_0(g, q)) = g\alpha(q)$. The action β_0 is properly discontinuous and free because β_r is a bundle isomorphism. The quotient $F := Q/\Gamma_0$ is the total space of the anafunctor $\mathcal{R}_{\mathcal{A}}$ we want to construct. Left and right anchor of an element $q \in F$ with $\chi(q) = (p, p')$ are given by

$$\alpha_l(q) := p \quad \text{and} \quad \alpha_r(q) := R(p', \alpha(q)).$$

The actions are defined by

$$\rho_l(\rho, q) := \beta_l^{-1}(q, (\rho, 1)) \quad \text{and} \quad \rho_r(q, \rho') := \beta_r((R(\rho', \text{id}_{\alpha(q)^{-1}}), 1), q).$$

The left action is invariant under the action β_0 because of Lemma 7.2.6 (i). For the right action, invariance follows from Lemma 7.2.6 (ii) and the identity

$$\mu'((R(\rho', \text{id}_{\alpha(q)^{-1}g^{-1}}), 1), (\text{id}, g)) \stackrel{(7.1.3)}{=} \mu'((\text{id}, g), (R(\rho', \text{id}_{\alpha(q)^{-1}}), 1)).$$

Lemma 7.2.7. *The above formulas define an anafunctor $F : \mathcal{P} \rightarrow \mathcal{P}'$.*

Proof. The compatibility between anchors and actions is easy to check. The axiom for the actions ρ_l and ρ_r follows from Lemma 7.2.6 (ii) and (iii). Lemma 7.2.6 (i) shows that the actions commute. It remains to prove that $\alpha_l : F \rightarrow \mathcal{P}_0$ is a principal \mathcal{P}' -bundle. Since α_l is a composition of surjective submersions, we only have to show that the map

$$\tau : F_{\alpha_r} \times_t \mathcal{P}' \rightarrow F_{\alpha_l} \times_{\alpha_l} F : (q, \rho') \mapsto (q, \rho_r(q, \rho'))$$

is a diffeomorphism. We construct an inverse map τ^{-1} as follows. For (q_1, q_2) with $\chi(q_1) = (p, p')$ and $\chi(q_2) = (p, \tilde{p}')$, choose a representative

$$((\tilde{\rho}', g'), \tilde{q}) := \beta_r|_{p, p', \tilde{p}'}^{-1}(q_2).$$

Such choices can be made locally in a smooth way, and the result will not depend on them. We have $\chi(\tilde{q}) = (p, p')$ that there exists a unique $\gamma \in \Gamma_1$ such that $q_1 = \tilde{q} \circ \gamma$. Now we put

$$\tau^{-1}(q_1, q_2) := (q_1, R(\tilde{\rho}', \gamma^{-1})).$$

The calculation of $\tau^{-1} \circ \tau$ is straightforward. For the calculation of $(\tau \circ \tau^{-1})(q_1, q_2)$ we have to compute in the second component

$$\begin{aligned} \beta_r((R(\tilde{\rho}', \gamma^{-1} \cdot \text{id}_{\alpha(q_1)^{-1}}), 1), q_1) &= \beta_r((R(\tilde{\rho}', \gamma^{-1} \cdot \text{id}_{\alpha(q_1)^{-1}}), 1) \circ (\gamma \cdot \text{id}_{\alpha(\tilde{q})^{-1}}), \tilde{q}) \\ &= \beta_r((\tilde{\rho}', \alpha(q_1)\alpha(\tilde{q})^{-1}), \tilde{q}) \\ &= \beta_0(\alpha(q_1)\alpha(\tilde{q})^{-1}g'^{-1}, \beta_r((\tilde{\rho}', g'), \tilde{q})) \\ &= \beta_0(\alpha(q_1)\alpha(\tilde{q})^{-1}g'^{-1}, q_2), \end{aligned}$$

and this is equivalent to q_2 . □

In order to promote the anafunctor F to a 1-morphism between principal 2-bundles, we have to do two things: we have to check that F commutes with the projections of the bundle \mathcal{P}_1 and \mathcal{P}_2 , and we have to construct a Γ -equivariant structure on F . For the first point we use Remark 6.1.6 (b), whose criterion $\pi \circ \alpha_l = \pi \circ \alpha_r$ is clearly satisfied. For the second point we provide a smooth action $\rho : F \times \Gamma_1 \rightarrow F$ in the sense of Definition A.1 and use Lemma A.2, which provides a construction of a Γ -equivariant structure. The action is defined by

$$\rho(q, \gamma) := \beta_l^{-1}(q \circ (\text{id}_{\alpha(q)} \cdot \gamma \cdot \text{id}_{t(\gamma)^{-1}}), (\text{id}_{R(\alpha_l(q), t(\gamma))}, t(\gamma))). \quad (7.2.3)$$

Lemma 7.2.8. *This defines a smooth action of Γ_1 on F in the sense of Definition A.1.*

Proof. Smoothness is clear from the definition. The identity

$$\rho(\rho(q, \gamma_1), \gamma_2) = \beta_l^{-1}(q \circ (\text{id}_{\alpha(q)} \cdot \gamma_1 \cdot \gamma_2 \cdot \text{id}_{t(\gamma_2)^{-1}t(\gamma_1)^{-1}}), (\text{id}, t(\gamma_1 \cdot \gamma_2))) = \rho(q, \gamma_1 \cdot \gamma_2)$$

follows from the definition and the two identities

$$\begin{aligned} \alpha(\rho(q, \gamma)) &= \alpha(q)s(\gamma) \quad \text{and} \\ (\gamma_1 \cdot \text{id}_{t(\gamma_1)^{-1}}) \cdot (\text{id}_{s(\gamma_1)} \cdot \gamma_2 \cdot \text{id}_{t(\gamma_2)^{-1}t(\gamma_1)^{-1}}) &= \gamma_1 \cdot \gamma_2 \cdot \text{id}_{t(\gamma_2)^{-1}t(\gamma_1)^{-1}}. \end{aligned} \quad (7.2.4)$$

The latter can easily be verified upon substituting a crossed module for Γ . Checking condition (i) of Definition A.1 just uses the definitions. We check condition (ii) in two steps. First we prove the identity

$$\rho(\rho_l(\rho, q), \gamma_l \circ \gamma) = \rho_l(R(\rho, \gamma_l), \rho(q, \gamma)).$$

Main ingredient is the decomposition

$$\text{id}_{\alpha(q)} \cdot (\gamma_l \circ \gamma) \cdot \text{id}_{t(\gamma_l)^{-1}} = (\text{id}_{\alpha(q)} \cdot \gamma \cdot \text{id}_{t(\gamma)^{-1}}) \circ (\text{id}_{\alpha(q)s(\gamma)t(\gamma)^{-1}} \cdot \gamma_l \cdot \text{id}_{t(\gamma_l)^{-1}}) \quad (7.2.5)$$

that can e.g. be verified in the crossed module language. Now we compute

$$\begin{aligned} \rho(\rho_l(\rho, q), \gamma_l \circ \gamma) &= \beta_l^{-1}(q \circ (\text{id}_{\alpha(q)} \cdot (\gamma_l \circ \gamma) \cdot \text{id}_{t(\gamma_l)^{-1}}), (R(\rho, t(\gamma_l)), t(\gamma_l))) \\ &\stackrel{(7.2.5)}{=} \beta_l^{-1}(q \circ (\text{id}_{\alpha(q)} \cdot \gamma \cdot \text{id}_{t(\gamma)^{-1}}), (R(\rho, \gamma_l), t(\gamma_l))) \\ &= \rho_l(R(\rho, \gamma_l), \rho(q, \gamma)). \end{aligned}$$

The second step is to show the identity

$$\rho(\rho_r(q, \rho'), \gamma \circ \gamma_r) = \rho_r(\rho(q, \gamma), R(\rho', \gamma_r)).$$

Here we use the decomposition

$$\text{id}_{\alpha(q)} \cdot (\gamma \circ \gamma_r) \cdot \text{id}_{t(\gamma)^{-1}} = (\text{id}_{\alpha(q)} \cdot \gamma \cdot \text{id}_{t(\gamma)^{-1}}) \circ (\text{id}_{\alpha(q)} \cdot \gamma_r \cdot \text{id}_{t(\gamma_r)^{-1}}). \quad (7.2.6)$$

Then we compute

$$\begin{aligned} \rho(\rho_r(q, \rho'), \gamma \circ \gamma_r) &= \beta_l^{-1}(\beta_r((R(\rho', \text{id}_{\alpha(q)^{-1}}), 1), q \circ (\text{id}_{\alpha(q)} \cdot (\gamma \circ \gamma_r) \cdot \text{id}_{t(\gamma)^{-1}})), (\text{id}, t(\gamma))) \\ &\stackrel{(7.2.6)}{=} \beta_l^{-1}(\beta_r((R(\rho', \gamma_r \cdot \text{id}_{s(\gamma)^{-1}\alpha(q)^{-1}}), 1), \\ &\quad \beta_0(\alpha(q)s(\gamma_r)s(\gamma)^{-1}\alpha(q)^{-1}, q \circ (\text{id}_{\alpha(q)} \cdot \gamma \cdot \text{id}_{t(\gamma)^{-1}}))), (\text{id}, t(\gamma))) \\ &\stackrel{(7.2.4)}{=} \beta_l^{-1}(\beta_r((R(\rho', \gamma_r \cdot \text{id}_{\alpha(\rho(q, \gamma))^{-1}}), q \circ (\text{id}_{\alpha(q)} \cdot \gamma \cdot \text{id}_{t(\gamma)^{-1}})), (\text{id}, t(\gamma))) \\ &= \rho_r(\rho(q, \gamma), R(\rho', \gamma_r)), \end{aligned}$$

where we have employed the equivalence relation on F that was generated by the action of β_0 . \square

Construction of a 2-isomorphism $\eta_{\mathcal{A}} : \mathcal{A} \Longrightarrow \mathcal{E}_M(\mathcal{R}_{\mathcal{A}})$

We may again assume that the common refinement of \mathcal{A} is the fibre product $\mathcal{P}_0 \times_M \mathcal{P}'_0$; otherwise, the proof of Lemma 5.2.8 provides a 2-isomorphism between \mathcal{A} and one of these. Now, \mathcal{A} and $\mathcal{E}_M(\mathcal{R}_{\mathcal{A}})$ have the same common refinement, and $\eta_{\mathcal{A}}$ is given by the map

$$\eta : Q \longrightarrow F \times \Gamma_0 : q \longmapsto (q, \alpha(q)).$$

This is obviously smooth and respects the projections to the base: if $\chi(q) = (p, p')$, then

$$\chi(q, \alpha(q)) \stackrel{(7.1.6)}{=} (\alpha_l(q), R(\alpha_r(q), \alpha(q)^{-1})) = (p, p').$$

Further, it respects the Γ -actions:

$$\eta(q \circ \gamma) = (q \circ \gamma, s(\gamma)) = \beta_l^{-1}(q \circ \gamma, (\text{id}, 1)) \stackrel{(7.2.3)}{=} (\rho(q, \text{id}_{\alpha(q)^{-1}} \cdot \gamma), s(\gamma)) \stackrel{(7.1.6)}{=} \eta(q) \circ \gamma,$$

so that η is a bundle morphism. It remains to verify the commutativity of the compatibility diagram (5.1.4). Let $((\rho', g'), q') \in P' \otimes \zeta_1^* Q$, and let $(q, (\rho, g)) \in \zeta_2^* Q \otimes P$ be a representative for $\beta((\rho', g'), q')$. In particular, we have $\alpha(q)g = g'\alpha(q')$, since β_r is anchor-preserving. Then, we get clockwise

$$(\eta \otimes \text{id})(\beta((\rho', g'), q')) = ((q, \alpha(q)), (\rho, g)). \quad (7.2.7)$$

Counter-clockwise, we have to use the isomorphism of Lemma 7.1.6 that we call $\tilde{\beta}$ here. Then,

$$\tilde{\beta}((\text{id} \otimes \eta)((\rho', g'), q')) = \tilde{\beta}((\rho', g'), (q', \alpha(q'))) = ((\tilde{q}, g'\alpha(q')g^{-1}), (\rho, g)) \quad (7.2.8)$$

where the choices $(\tilde{\rho}, h)$ we have to make for the definition of $\tilde{\beta}$ are here (ρ, g^{-1}) , and \tilde{q} is defined in (7.1.7), which gives here

$$\tilde{q} = \beta_l^{-1}(\beta_r((\rho', 1), q'), (R(\rho^{-1}, \text{id}_{g^{-1}}), g^{-1})).$$

Comparing (7.2.7) and (7.2.8) it remains to prove $q = \tilde{q}$ in F . As F was the quotient of Q by the action β_0 , it suffices to have

$$\begin{aligned} \beta_0(g', \tilde{q}) &\stackrel{(i)}{=} \beta_l^{-1}(\beta_r((\text{id}, g'), \beta_r((\rho', 1), q')), (R(\rho^{-1}, \text{id}_{g^{-1}}), g^{-1})) \\ &\stackrel{(iii)}{=} \beta_l^{-1}(\beta_r((\rho', g'), q'), (R(\rho^{-1}, \text{id}_{g^{-1}}), g^{-1})) \\ &= \beta_l^{-1}(\beta_l^{-1}(q, (\rho, g)), (R(\rho^{-1}, \text{id}_{g^{-1}}), g^{-1})) \\ &\stackrel{(ii)}{=} \beta_l^{-1}(q, (\text{id}, 1)) \\ &= q. \end{aligned}$$

This finishes the construction of the 2-isomorphism $\eta_{\mathcal{A}}$.

A Equivariant Anafunctors and Group Actions

In this section we are concerned with a Lie 2-group Γ and Lie groupoids \mathcal{X} and \mathcal{Y} with actions $R_1 : \mathcal{X} \times \Gamma \rightarrow \mathcal{X}$ and $R_2 : \mathcal{Y} \times \Gamma \rightarrow \mathcal{Y}$.

Definition A.1. An action of the 2-group Γ on an anafunctor $F : \mathcal{X} \rightarrow \mathcal{Y}$ is an ordinary smooth action $\rho : F \times \Gamma_1 \rightarrow F$ of the group Γ_1 on the total space F that

(i) preserves the anchors in the sense that the diagrams

$$\begin{array}{ccc} F \times \Gamma_1 & \xrightarrow{\rho} & F \\ \alpha_l \times t \downarrow & & \downarrow \alpha_l \\ \mathcal{X}_0 \times \Gamma_0 & \xrightarrow{R_1} & \mathcal{X}_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} F \times \Gamma_1 & \xrightarrow{\rho} & F \\ \alpha_r \times s \downarrow & & \downarrow \alpha_r \\ \mathcal{Y}_0 \times \Gamma_0 & \xrightarrow{R_2} & \mathcal{Y}_0 \end{array}$$

are commutative.

(ii) is compatible with the Γ -actions in the sense that the identity

$$\rho(\chi \circ f \circ \eta, \gamma_l \circ \gamma \circ \gamma_r) = R_1(\chi, \gamma_l) \circ \rho(f, \gamma) \circ R_2(\eta, \gamma_r)$$

holds for all appropriately composable $\chi \in \mathcal{X}_1$, $\eta \in \mathcal{Y}_1$, $f \in F$, and $\gamma_l, \gamma, \gamma_r \in \Gamma_1$.

If $F_1, F_2 : \mathcal{X} \rightarrow \mathcal{Y}$ are anafunctors with Γ -action, a transformation $\eta : F_1 \Rightarrow F_2$ is called Γ -equivariant if the map $\eta : F_1 \rightarrow F_2$ between total spaces is Γ_1 -equivariant in the ordinary sense.

Anafunctors $\mathcal{X} \rightarrow \mathcal{Y}$ with Γ -actions together with Γ -equivariant transformations form a groupoid $\text{Ana}_\Gamma^\infty(\mathcal{X}, \mathcal{Y})$. On the other hand, there is another groupoid $\Gamma\text{-Ana}^\infty(\mathcal{X}, \mathcal{Y})$ consisting of Γ -equivariant anafunctors (Definition 6.1.3) and Γ -equivariant transformations (Definition 6.1.4).

Lemma A.2. The categories $\text{Ana}_\Gamma^\infty(\mathcal{X}, \mathcal{Y})$ and $\Gamma\text{-Ana}^\infty(\mathcal{X}, \mathcal{Y})$ are canonically isomorphic.

Proof. We construct a functor

$$\mathcal{E} : \text{Ana}_\Gamma^\infty(\mathcal{X}, \mathcal{Y}) \rightarrow \Gamma\text{-Ana}^\infty(\mathcal{X}, \mathcal{Y}). \quad (\text{A.1})$$

Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be an anafunctor with Γ -action ρ . We shall define a transformation

$$\lambda_\rho : F \circ R_1 \Rightarrow R_2 \circ (F \times \text{id}).$$

First of all, the composite

$$\mathcal{X} \times \Gamma \xrightarrow{R_1} \mathcal{X} \xrightarrow{F} \mathcal{Y}$$

is given by the total space $(\mathcal{X}_0 \times \Gamma_0)_{R_1 \times \alpha_l} F$, left and right anchors send an element (x, g, f) to (x, g) and $\alpha_r(f)$, respectively, and the actions are

$$(\chi, \gamma) \circ (x, g, f) = (t(\chi), t(\gamma), R_1(\chi, \gamma) \circ f) \quad \text{and} \quad (x, g, f) \circ \eta = (x, g, f \circ \eta).$$

On the other hand, the composite

$$\mathcal{X} \times \Gamma \xrightarrow{F \times \text{id}} \mathcal{Y} \times \Gamma \xrightarrow{R_2} \mathcal{Y}$$

is given by the total space $((F \times \Gamma_1)_{R_2 \circ (\alpha_r \times s)} \times_t \mathcal{Y}_1) / \sim$ with the equivalence relation

$$(f \circ \eta', \gamma \circ \gamma', \eta) \sim (f, \gamma, R_2(\eta', \gamma') \circ \eta).$$

Left and right anchor send an element (f, γ, η) to $(\alpha_l(f), t(\gamma))$ and $s(\eta)$, respectively, and the actions are

$$(\chi, \gamma') \circ (f, \gamma, \eta) = (\chi \circ f, \gamma' \circ \gamma, \eta) \quad \text{and} \quad (f, \gamma, \eta) \circ \eta' = (f, \gamma, \eta \circ \eta').$$

The inverse of the following map will define the transformation λ :

$$(F \times \Gamma_1)_{R_2 \circ (\alpha_r \times s)} \times_t \mathcal{Y}_1 \longrightarrow (\mathcal{X}_0 \times \Gamma_0)_{R_1 \times \alpha_l} F : (f, \gamma, \eta) \mapsto (\alpha_l(f), t(\gamma), \rho(f, \gamma) \circ \eta).$$

Condition (i) assures that this map ends in the correct fibre product, and condition (ii) assures that it is well-defined under the equivalence relation \sim . The left anchors are automatically respected, and the right anchors require condition (i). Similarly, the left action is respected automatically, and the right actions due to condition (ii). The axiom for a transformation is satisfied because ρ is a group action. This defines the functor \mathcal{E} on objects. On morphisms, it is straightforward to check that the conditions on both hand sides coincide; in particular, \mathcal{E} is full and faithful.

In order to prove that the functor \mathcal{E} is an isomorphism, we start with a given Γ -equivariant structure λ on the anafunctor F . Then, an action $\rho : F \times \Gamma_1 \longrightarrow F$ is defined by

$$(f, \gamma) \mapsto \text{pr}_3(\lambda^{-1}(f, \gamma, \text{id}_{R_2(\alpha_r(f), s(\gamma))}))$$

with $\text{pr}_3 : (\mathcal{X}_0 \times \Gamma_0)_{R_1 \times \alpha_l} F \longrightarrow F$ the projection. The axiom for an action is satisfied due to the identity λ obeys. It is straightforward to verify conditions (i) and (ii) of Definition A.1. To close the proof it suffices to notice that the two procedures we have defined are (strictly) inverse to each other. \square

We are also concerned with the composition of anafunctors with Γ -action. Suppose that \mathcal{Z} is a third Lie groupoid with a Γ -action R_3 , and $F : \mathcal{X} \longrightarrow \mathcal{Y}$ and $G : \mathcal{Y} \longrightarrow \mathcal{Z}$ are anafunctors with Γ -actions $\rho : F \times \Gamma_1 \longrightarrow F$ and $\tau : G \times \Gamma_1 \longrightarrow G$. Then, the composition $G \circ F$ is equipped with the Γ -action defined by

$$(F \times_{\mathcal{Y}_0} G) \times \Gamma_1 \longrightarrow (F \times_{\mathcal{Y}_0} G) : ((f, g), \gamma) \mapsto (\rho(f, \gamma), \tau(g, \text{id}_{s(\gamma)})). \quad (\text{A.2})$$

We leave it to the reader to check

Lemma A.3. *Let \mathcal{X} , \mathcal{Y} and \mathcal{Z} be Lie groupoids with Γ -actions R_1 , R_2 and R_3 .*

- (a) *Let $F : \mathcal{X} \rightarrow \mathcal{Y}$ and $G : \mathcal{Y} \rightarrow \mathcal{Z}$ be Γ -equivariant anafunctors. If Γ -equivariant structures on F and G correspond to Γ_1 -actions under the isomorphism of Lemma A.2, then the Γ -equivariant structure on the composite $F \circ G$ corresponds to the Γ_1 -action defined above.*
- (b) *The isomorphism of Lemma A.2 identifies the trivial Γ -equivariant structure on the identity anafunctor $\text{id} : \mathcal{X} \rightarrow \mathcal{X}$ with the Γ_1 -action $R_1 : \mathcal{X}_1 \times \Gamma_1 \rightarrow \mathcal{X}_1$ on its total space \mathcal{X} .*

B Constructing Equivalences between 2-Stacks

Let \mathcal{C} be a bicategory (we assume that associators and unifiers are *invertible* 2-morphisms). We fix the following terminology: a 1-isomorphism $f : X_1 \rightarrow X_2$ in \mathcal{C} always includes the data of an inverse 1-morphism $\bar{f} : X_2 \rightarrow X_1$ and of 2-isomorphisms $i : \bar{f} \circ f \Rightarrow \text{id}$ and $j : \text{id} \Rightarrow f \circ \bar{f}$ satisfying the zigzag identities. Let \mathcal{D} be another bicategory. A 2-functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is assumed to have *invertible* compositors and unitors.

The following lemma is certainly “well-known”, although we have not been able to find a reference for exactly this statement.

Lemma B.1. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a 2-functor that is fully faithful on Hom-categories. Suppose one has chosen:*

1. *for every object $Y \in \mathcal{D}$ an object $G_Y \in \mathcal{C}$ and a 1-isomorphism $\xi_Y : Y \rightarrow F(G_Y)$.*
2. *for all objects $X_1, X_2 \in \mathcal{C}$ and all 1-morphisms $g : F(X_1) \rightarrow F(X_2)$, a 1-morphism $G_g : X_1 \rightarrow X_2$ in \mathcal{C} together with a 2-isomorphism $\eta_g : g \Rightarrow F(G_g)$.¹*

Then, there is a 2-functor $G : \mathcal{D} \rightarrow \mathcal{C}$ and pseudonatural equivalences

$$a : \text{id}_{\mathcal{D}} \Rightarrow F \circ G \quad \text{and} \quad b : G \circ F \Rightarrow \text{id}_{\mathcal{C}}.$$

In particular, F is an equivalence of bicategories.

Proof. We recall our convention concerning 1-isomorphisms: the 1-isomorphisms ξ_Y include choices of inverse 1-morphisms $\bar{\xi}_Y$ together with 2-isomorphisms $i_Y : \bar{\xi}_Y \circ \xi_Y \Rightarrow \text{id}$ and $j_Y : \text{id} \Rightarrow \xi_Y \circ \bar{\xi}_Y$ satisfying the zigzag identities.

First we explicitly construct the 2-functor G . On objects, we put $G(Y) := G_Y$. We use the notation $\tilde{g} := (\xi_{Y_2} \circ g) \circ \bar{\xi}_{Y_1}$ for all 1-morphisms $g : Y_1 \rightarrow Y_2$ in \mathcal{D} , and define $G(g) = G_{\tilde{g}}$. If

¹More accurately we should write $G_{X_1, X_2, g}$ and $\eta_{X_1, X_2, g}$, but we will suppress X_1 and X_2 in the notation.

$g, g' : Y_1 \rightarrow Y_2$ are 1-morphisms, and $\psi : g \Rightarrow g'$ is a 2-morphism, we consider the 2-morphism $\tilde{\psi}$ defined by

$$F(G_{\tilde{g}}) \xrightarrow{\eta_{\tilde{g}}^{-1}} (\xi_{Y_2} \circ g) \circ \bar{\xi}_{Y_1} \xrightarrow{(\text{id} \circ \psi) \circ \text{id}} (\xi_{Y_2} \circ g') \circ \bar{\xi}_{Y_1} \xrightarrow{\eta_{\tilde{g}'}} F(G_{\tilde{g}'}) .$$

Since F is fully faithful on 2-morphisms, we may choose the unique 2-morphism $G(\psi) : G(g) \Rightarrow G(g')$ such that $F(G(\psi)) = \tilde{\psi}$. In order to define the compositor of G we look at 1-morphisms $g_{12} : Y_1 \rightarrow Y_2$ and $g_{23} : Y_2 \rightarrow Y_3$. We consider the 2-morphism

$$\begin{aligned} F(G(g_{23}) \circ G(g_{12})) &\xrightarrow{c_{G(g_{12}), G(g_{23})}^{-1}} F(G_{\tilde{g}_{23}}) \circ F(G_{\tilde{g}_{12}}) \\ &\quad \downarrow \eta_{\tilde{g}_{23}}^{-1} \circ \eta_{\tilde{g}_{12}}^{-1} \\ &((\xi_{Y_3} \circ g_{23}) \circ \bar{\xi}_{Y_2}) \circ ((\xi_{Y_2} \circ g_{12}) \circ \bar{\xi}_{Y_1}) \\ &\quad \downarrow a, i_{Y_2} \\ &(\xi_{Y_3} \circ (g_{23} \circ g_{12})) \circ \bar{\xi}_{Y_1} \xrightarrow{\eta_{g_{23} \circ g_{12}}} F(G(g_{23} \circ g_{12})); \end{aligned}$$

its unique preimage under the 2-functor F is the compositor

$$c_{g_{12}, g_{23}} : G(g_{23}) \circ G(g_{12}) \Rightarrow G(g_{23} \circ g_{12}) .$$

In order to define the unitor of G we consider an object $Y \in \mathcal{D}$ and look at the 2-morphism

$$F(G(\text{id}_Y)) \xrightarrow{\eta_{\text{id}_Y}^{-1}} (\xi_Y \circ \text{id}_Y) \circ \bar{\xi}_Y \xrightarrow{l_{\xi_Y, j_Y^{-1}}} \text{id}_{F(G(Y))} \xrightarrow{u_{G(Y)}^{-1}} F(\text{id}_{G(Y)}) .$$

Its unique preimage under the 2-functor F is the unitor $u_Y : G(\text{id}_Y) \Rightarrow \text{id}_{G(Y)}$. The second step is to verify the axioms of a 2-functor. This is simple but extremely tedious and can only be left as an exercise. The third step is to construct the pseudonatural transformation

$$a : \text{id}_{\mathcal{D}} \Rightarrow F \circ G .$$

Its component at an object Y in \mathcal{D} is the 1-morphism $a(Y) := \xi_Y : Y \rightarrow F(G(Y))$. Its component at a 1-morphism $g : Y_1 \rightarrow Y_2$ is the 2-morphism $a(g)$ defined by

$$\begin{aligned} a(Y_2) \circ g & \xlongequal{\quad} \xi_{Y_2} \circ g \\ & \quad \downarrow \text{id} \circ l_{\xi_{Y_2} \circ g}^{-1} \\ & (\xi_{Y_2} \circ g) \circ \text{id} \\ & \quad \downarrow a, i_{Y_2}^{-1} \\ & ((\xi_{Y_2} \circ g) \circ \bar{\xi}_{Y_1}) \circ \xi_{Y_1} \\ & \quad \downarrow \eta_{\tilde{g}} \circ \text{id} \\ & F(G_{\tilde{g}}) \circ \xi_{Y_1} \xlongequal{\quad} F(G(g)) \circ a(Y_1) . \end{aligned}$$

There are two axioms a pseudonatural transformation has to satisfy, and their proofs are again left as an exercise. It is easy to see that a is a pseudonatural *equivalence*, with an inverse transformation given by $\bar{a}(Y) := \bar{\xi}_Y$. The fourth and last step is to construct the pseudonatural transformation

$$b : G \circ F \Longrightarrow \text{id}_{\mathcal{C}}.$$

Its component at an object X is $b(X) := G_{\bar{\xi}_{F(X)}} : G(F(X)) \rightarrow X$. Its component at a 1-morphism $f : X_2 \rightarrow X_1$ is the 2-morphism

$$b(f) : b(X_2) \circ G(F(f)) \Longrightarrow f \circ b(X_1)$$

given as the unique preimage under F of the 2-morphism

$$\begin{aligned} F(b(X_2) \circ G(F(f))) &\xrightarrow{c^{-1}} F(b(X_2)) \circ F(G(F(f))) \\ &\quad \eta_{\bar{\xi}_{F(X_2)}}^{-1} \circ \eta_{F(f)}^{-1} \Downarrow \\ &\quad \bar{\xi}_{F(X_2)} \circ ((\xi_{F(X_2)} \circ F(f)) \circ \xi_{F(X_1)}) \\ &\quad \quad \quad a, i_{F(X_2)}, r \Downarrow \\ &\quad \quad \quad F(f) \circ \bar{\xi}_{F(X_1)} \\ &\quad \quad \quad \text{id}_{F(f)} \circ \eta_{\bar{\xi}_{F(X_1)}} \Downarrow \\ &\quad \quad \quad F(f) \circ F(b(X_1)) \xrightarrow{c} F(f \circ b(X_1)). \end{aligned}$$

The proofs of the axioms are again left for the reader, and again it is easy to see that b is a pseudonatural *equivalence* with an inverse transformation given by $\bar{b}(X) := G_{\xi_{F(X)}}$. \square

As a consequence of Lemma B.1 we obtain the certainly well-known

Corollary B.2. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be essentially surjective, and an equivalence on all Hom-categories. Then, F is an equivalence of bicategories.*

Since we work with 2-stacks over manifolds, we need the following “stacky” extension of Lemma B.1. For a pre-2-stack \mathcal{C} , we denote by \mathcal{C}_M the 2-category it associates to a smooth manifold M , and by $\psi^* : \mathcal{C}_N \rightarrow \mathcal{C}_M$ the 2-functor it associates to a smooth map $\psi : M \rightarrow N$. The pseudonatural equivalences $\psi^* \circ \varphi^* \cong (\varphi \circ \psi)^*$ will be suppressed from the notation in the following. If \mathcal{C} and \mathcal{D} are pre-2-stacks, a 1-morphism $F : \mathcal{C} \rightarrow \mathcal{D}$ associates 2-functors $F_M : \mathcal{C}_M \rightarrow \mathcal{D}_M$ to a smooth manifold M , pseudonatural equivalences

$$F_\psi : \psi^* \circ F_N \rightarrow F_M \circ \psi^*$$

to smooth maps $\psi : M \rightarrow N$, and certain modifications $F_{\psi, \varphi}$ that control the relation between F_ψ and F_φ for composable maps ψ and φ .

Lemma B.3. *Suppose \mathcal{C} and \mathcal{D} are pre-2-stacks over smooth manifolds, and $F : \mathcal{C} \rightarrow \mathcal{D}$ is a 1-morphism. Suppose that for every smooth manifold M*

1. the assumptions of Lemma B.1 for the 2-functor F_M are satisfied and
2. the data (G_Y, ξ_Y) and (G_g, η_g) is chosen for all objects Y and 1-morphisms g in \mathcal{D}_M .

Then, there is a 1-morphism $G : \mathcal{D} \rightarrow \mathcal{C}$ of pre-2-stacks together with 2-isomorphisms

$$a : F \circ G \Rightarrow \text{id}_{\mathcal{D}} \quad \text{and} \quad b : G \circ F \Rightarrow \text{id}_{\mathcal{C}}$$

such that for every smooth manifold M the 2-functor G_M and the pseudonatural transformations a_M and b_M are the ones of Lemma B.1. In particular, F is an equivalence of pre-2-stacks.

For the proof one constructs the required pseudonatural equivalences G_ψ and the modifications $G_{\psi, \varphi}$ from the given ones, F_ψ and $F_{\psi, \varphi}$, respectively, in a similar way as explained in the proof of Lemma B.1. Since these constructions are straightforward to do but would consume many pages, and the statement of the lemma is not too surprising and certainly well-known to many people, we decided to leave these constructions for the interested reader.

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