# Seminar on Condensed Mathematics

## Talk 1: Condensed Sets

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## Motivation:

**Question:** How to do algebra when rings/modules groups/spectra carry a topology?

**Examples**

(1) Representations of top groups ($GL_n(\mathbb{R})$, $GL_n(\mathbb{Q}_p)$, ...) on top. vector space.

(2) Continuous group cohomology $H^*_{cont}(G, M)$

(3) Algebraic / analytic geometry over $\mathbb{R}$ or $\mathbb{Q}_p$.

(4) The action of the Morava stabiliser group $G_n$ on Morava E-Theory $E_n$

$$\pi_*(E_n) = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][u^{\pm}]$$

$[n=1,\ G_1 = \mathbb{Z}_p^\times,\ E_1 = KU_p^\wedge]$

(5) Pontryagin duality, $G$ locally cpt grp

$\hat{G} = \mathrm{Hom}(G, \mathbb{T})$

$G \xrightarrow{\sim} \hat{\hat{G}}$ isomorphism

**Problems**

(1) top abelian groups / vector spaces do not form an abelian category

$$(\mathbb{R}, \text{discrete}) \xrightarrow{id} (\mathbb{R}, \text{canonical})$$

not an iso, but kernel and cokernel

(2) Short exact sequences of top $G$-modules

do not induces long exact sequences
in continuous group cohomology

(3) Quasi-coherent sheaves do not make sense,
for example if $K$ top field, $A \to B$ cont
map of top $K$-algebras
What is the base change of a continuous $A$-module
$M$ supposed to be?

$$M \hat{\otimes}_A B$$

(4) How does $G_n$ act continuously on $E_n$?
What does it mean that $E_n$ carries a topology?

($L_{K(n)} \mathbb{S} \longrightarrow E_n$ is a $G_n$-Galois ext!?)

(5) For a general top group $G$

$$G \longrightarrow \hat{G} \text{ is not continuous.}$$

Solutions (in this seminar)

(1) Introduce a bigger category Cond(Ab) of condensed abelian groups containing top. ab groups as a full subcategory and which is abelian.

(2) In this world: Continuous coh = sheaf coh.

(3) There is a notion of complete modules in this world called <u>solid</u>,

$$\text{Solid}(Ab) \subseteq \text{Cond}(Ab)$$

(4) Condensed spectra $\Rightarrow$ $G_n$ acts continuously on $E_n$.

(5) Cond(Ab) and Solid(Ab) are Cartesian closed.

## Coherent duality and six functor formalism

A six functor formalism consists of

$$X \text{ scheme} \longmapsto D(X) \text{ closed sym. mon, stable } \infty\text{-category}$$

$$X \to Y \text{ map} \longmapsto f^*: D(Y) \rightleftarrows D(X): f_*$$

$$X \to Y \text{ qcqs, separated, finite type} \longmapsto f_!: D(X) \rightleftarrows D(Y): f^!$$

such that:
- $f$ proper $f_! = f_*$
- $f$ open immersion $\Rightarrow f^! = f^*$ i.e. $f_! \dashv f^*$.

$\Rightarrow$ uniquely determine $f_!$

- Projection formula etc.
- <u>Duality</u>: $M \in D(X)$

$$\mathrm{Hom}_{D(Y)}(f_! M, \mathbb{1}) \cong R\mathrm{Hom}_{D(X)}(M, f^!(\mathbb{1}))$$

$f$ smooth of rel dim $d \Rightarrow$

$$f^!(\mathbb{1}) \simeq \Lambda^d(\Omega^1_{X/Y})[d].$$

<u>Problem:</u> For $D_{qcoh}(\mathcal{O}_X)$ this does not work! Still have $\otimes$, Hom, $f^* \dashv f_*$.

Example: $\mathrm{Spec}(A[1/f]) \hookrightarrow \mathrm{Spec}(A)$

then

$$i^* : D_{qcoh}(\mathcal{O}_{\mathrm{Spec}(A)}) \simeq D(A) \longrightarrow D(A[1/f]) = D_{qcoh}(\mathcal{O}_{\mathrm{Spec}(A[1/f])})$$
$$M \longrightarrow M[1/f]$$

Does not admit a left adjoint $i_!$.

Solution (in this case): Embedd $D_{qcoh}(\mathcal{O}_X)$ into a bigger category

$D(\mathcal{O}_{X,\blacksquare})$ : solid derived category.

There is a six functor formalism for that! in particular the functor

$$f_! : D(\mathcal{O}_{X,\blacksquare}) \longrightarrow D(\mathcal{O}_{Y,\blacksquare})$$

exists, but it does not preserve discrete objects in general, but it does for $f$ proper!

$\Longrightarrow$ local version of coherent duality and much nicer identification of $f^!(1)$.

Recall: $\mathrm{Pro}(\mathrm{FinSet}) \simeq \{$ totally disconnected, cpt Hausdorff spaces $\}$

Definition (1) A condensed set is an accessible sheaf on the site $\mathrm{Pro}(\mathrm{FinSet})$ with surjective maps as covers, i.e. a functor

$$T : \mathrm{Pro}(\mathrm{FinSet})^{op} \longrightarrow \mathrm{Set}$$

s.t.
- $T(\emptyset) = pt$, $T(S_1 \amalg S_2) = T(S_1) \times T(S_2)$
- For a surjection $S' \twoheadrightarrow S$ of profinite sets we have that
$$T(S) \longrightarrow T(S') \rightrightarrows T(S' \times_S S')$$
is an equalizer diagram.
- $T$ preserves $\kappa$-filtered colimits for some regular cardinal $\kappa$

(2) For any (accessible, $\infty$-) category $\mathcal{C}$ we define $\mathrm{Cond}(\mathcal{C})$ as accessible (hyper) sheaves on $\mathrm{Pro}(\mathrm{FinSet})$ with values in $\mathcal{C}$.

$\boxed{\text{Example}}$ $T$ any top space, we have a condensed set

$$\underline{T} : S \longmapsto \mathrm{Hom}(S, T).$$

[This is clearly a sheaf but in general not quite accessible. But it is if $T$ is $T_1$-space]

Recall: A top space $X$ is called $\kappa$-compactly generated if for any other space $Y$ a map $X \to Y$ is continuous precisely if the composition $K \to X \to Y$ is continuous for any $\kappa$-small compact Hausdorff space $K$.

In other words: $X$ comes to quotient topology of the map $\coprod_{K \to X} K \longrightarrow X$.

Observation: Every compact Hausdorff space (of cardinality $< \kappa$) is a quotient of a profinite set (of cardinality $< \kappa$).

$$\underset{\text{Stone Čech compactification}}{\beta(K^\delta)} \longrightarrow K$$

Stone Čech compactification:

$$\mathrm{CHaus} \hookrightarrow \mathrm{Top}$$

$\beta$ left adjoint to the inclusion. ($\mathrm{CHaus}$: compact Hausdorff spaces)

If $S^\delta$ is discrete, then $\beta(S^\delta)$ can be described as

- the set of ultrafilters on $S^\delta$.
- As the right Kan extension of $\mathrm{FinSet} \to \mathrm{Sets}$ along itself

$$\begin{tikzcd} \mathrm{FinSet} \arrow[r] \arrow[d] & \mathrm{Top} \\ \mathrm{Set} \arrow[ur, dashed, "\beta"'] & \end{tikzcd}$$

$$\beta(S^0) \cong \varprojlim_{S^0 \to F} F \quad \text{it is a profinite set}$$

$F$ ← finite sets

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$\Rightarrow$ to test compactly generated spaces we can restrict to maps $K \to X$ where $K$ is profinite

<u>Proposition</u>: (1) The functor $X \longrightarrow \underline{X}$ from $T_1$-top spaces to condensed sets is faithful and full when restricted to the full subcategory of compactly generated top. spaces $X$.

(2) There is a functor $\mathrm{Cond}(Set) \longrightarrow Top$ which assigns to $T \in \mathrm{Cond}(Set)$ the space with und. set $T(*)$ and equipped with the quotient topology from $\coprod_{K \to T} K \longrightarrow T(*)$ where $K$ runs through all ($\kappa$-small) profinite sets for $\kappa$ suff. large. We have an iso

$$\mathrm{Hom}_{Top}(T(*), X) \cong \mathrm{Hom}_{\mathrm{Cond}(Set)}(T, \underline{X}).$$

$\square$

<u>Remark</u>: Part 1 holds also for top rings/abelian groups, ... Part 2 <u>not</u>.

**Example:** $(\mathbb{R}, \text{disc}) \longrightarrow (\mathbb{R}, \text{canonical})$ is injective in Cond(Ab) with cokernel

$$Q: \quad S \longmapsto \frac{\{\text{continuous maps } S \to \mathbb{R}\}}{\{\text{locally constant maps } S \to \mathbb{R}\}}$$

**Theorem** Cond(Ab) is an abelian category satisfying Grothendieck's axioms

AB3, AB3*, AB4, AB4*, AB5, AB6

(like abelian groups)