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Polyadization of Algebraic Structures

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Abstract: A generalization of the semisimplicity concept for polyadic algebraic structures is proposed. If semisimple structures can be presented as block diagonal matrices (resulting in the Wedderburn decomposition), general forms of polyadic structures are given by block-shift matrices. We combine these forms to get a general shape of semisimple nonderived polyadic structures ("double" decomposition of two kinds). We then introduce the polyadization concept (a "polyadic constructor"), according to which one can construct a nonderived polyadic algebraic structure of any arity from a given binary structure. The polyadization of supersymmetric structures is also discussed. The "deformation" by shifts of operations on the direct power of binary structures is defined and used to obtain a nonderived polyadic multiplication. Illustrative concrete examples for the new constructions are given.

Keywords: direct product; direct power; polyadic semigroup; arity; polyadic ring; polyadic field

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1. Introduction

I am no poet, but if you think for yourselves, as I proceed, the facts will form a poem in your minds.
"The Life and Letters of Faraday" (1870) by Bence Jones Michael Faraday.

The concept of simple and semisimple rings, modules, and algebras (see, e.g., [1–4]) plays a crucial role in the investigation of Lie algebras and representation theory [5–7], as well as in category theory [8–10].

Here we first propose a generalization of this concept for polyadic algebraic structures [11], which can also be important, e.g., in the operad theory [12,13] and nonassociative structures [14,15]. If semisimple structures can be presented in block-diagonal matrix form (resulting to the Wedderburn decomposition [16–18]), corresponding general forms for polyadic rings can be decomposed to block-shift matrices [19]. We combine these forms and introduce a general shape of semisimple polyadic structures, which are nonderived in the sense that they cannot be obtained as successive compositions of binary operations, which can be treated as polyadic ("double") decomposition.

Second, going in the opposite direction, we define the polyadization concept ("polyadic constructor") according to which one can construct a nonderived polyadic algebraic structure of any arity from a given binary structure. Then, we briefly describe supersymmetric structure polyadization.

Third, we propose operations "deformed" by shifts to obtain a nonderived n-ary multiplication on the direct power of binary algebraic structures.

For these new constructions, some illustrative concrete examples are given.

2. Preliminaries

We use notation from [11,20]. In brief, a (one-set) *polyadic algebraic structure* $\mathcal{A}$ is a set A closed with respect to polyadic operations (or *n-ary multiplication*) $\mu^{[n]} : A^n \to A$ (*n-ary magma*). We denote *polyads* [21] by bold letters $\mathbf{a} = \mathbf{a}^{(n)} = (a_1, \ldots, a_n)$, $a_i \in A$. A *polyadic zero* is defined by $\mu^{[n]}\left[\mathbf{a}^{(n-1)}, z\right] = z$, $z \in A$, $\mathbf{a}^{(n-1)} \in A^{n-1}$, where z can be on any place. A (positive) *polyadic power* $\ell_\mu \in \mathbb{N}$ is $a^{\langle \ell_\mu \rangle} = \left(\mu^{[n]}\right)^{\circ \ell_\mu}\left[a^{\ell_\mu(n-1)+1}\right]$, $\quad a \in A$. An element of a polyadic algebraic structure a is called ℓ_μ*-nilpotent* (or simply *nilpotent* for $\ell_\mu = 1$), if there exist ℓ_μ such that $a^{\langle \ell_\mu \rangle} = z$. A *polyadic (or n-ary) identity* (or neutral element) is defined by $\mu^{[n]}\left[a, e^{n-1}\right] = a, \quad \forall a \in A$, where a can be on any place in the left-hand side. A one-set polyadic algebraic structure $\left\langle A \mid \mu^{[n]} \right\rangle$ is *totally associative* if $\left(\mu^{[n]}\right)^{\circ 2}[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mu^{[n]}\left[\mathbf{a}, \mu^{[n]}[\mathbf{b}], \mathbf{c}\right] = invariant$, with respect to placement of the internal multiplication on any of the n places, and $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are polyads of the necessary sizes [22,23]. A *polyadic semigroup* $\mathcal{S}^{(n)}$ is a one-set and one-operation structure in which $\mu^{[n]}$ is totally associative. A polyadic structure is *commutative*, if $\mu^{[n]} = \mu^{[n]} \circ \sigma$, or $\mu^{[n]}[\mathbf{a}] = \mu^{[n]}[\sigma \circ \mathbf{a}]$, $\mathbf{a} \in A^n$, for all $\sigma \in S_n$.

A polyadic structure is *solvable*, if for all polyads $\mathbf{b}$, $\mathbf{c}$ and an element x, one can (uniquely) resolve the equation (with respect to h) for $\mu^{[n]}[\mathbf{b}, x, \mathbf{c}] = a$, where x can be on any place, and $\mathbf{b}, \mathbf{c}$ are polyads of the needed lengths. A solvable polyadic structure is called a *polyadic quasigroup* [24]. An associative polyadic quasigroup is called a n*-ary* (or *polyadic*) *group* [25]. In an n-ary group the only solution of

$$\mu^{[n]}[\mathbf{b}, \bar{a}] = a, \quad a, \bar{a} \in A, \quad \mathbf{b} \in A^{n-1} \tag{1}$$

is called a *querelement* of a and denoted by $\bar{a}$ [26], where $\bar{a}$ can be on any place. Any idempotent a coincides with its querelement $\bar{a} = a$. The relation (1) can be considered as a definition of the unary *queroperation* $\bar{\mu}^{(1)}[a] = \bar{a}$ [27]. For further details and definitions, see [11].

3. Polyadic Semisimplicity

In general, simple algebraic structures are building blocks (direct summands) for the semisimple ones satisfying special conditions (see, e.g., [1,3]).

3.1. Simple Polyadic Structures

According to the Wedderburn–Artin theorem (see, e.g., [17,18,28]), a ring which is simple (having no two-sided ideals, except zero and the ring itself) and Artinian (having minimal right ideals) $\mathcal{R}_{simple}$ is isomorphic to a full $d \times d$ matrix ring

$$\mathcal{R}_{simple} \cong Mat^{full}_{d \times d}(\mathcal{D}) \tag{2}$$

over a division ring $\mathcal{D}$. As a corollary,

$$\mathcal{R}_{simple} \cong \mathrm{Hom}_{\mathcal{D}}(V(d \mid \mathcal{D}), V(d \mid \mathcal{D})) \equiv \mathrm{End}_{\mathcal{D}}(V(d \mid \mathcal{D})), \tag{3}$$

where $V(d \mid \mathcal{D})$ is a d-finite-dimensional vector space (left module) over $\mathcal{D}$. In the same way, a finite-dimensional simple associative algebra $\mathcal{A}$ over an algebraically closed field $\mathcal{F}$ is

$$\mathcal{A} \cong Mat^{full}_{d \times d}(\mathcal{F}). \tag{4}$$

In the polyadic case, the structure of a simple Artinian $[2, n]$-ring $\mathcal{R}^{[2,n]}_{simple}$ (with binary addition and n-ary multiplication $\mu^{[n]}$) was obtained in [19], where the Wedderburn–Artin theorem for $[2, n]$-rings was proved. Thus, instead of one vector space $V(d \mid \mathcal{D})$, one should consider a direct sum of $(n-1)$ vector spaces (over the same division ring $\mathcal{D}$), that is,

$$V_1(d_1 \mid \mathcal{D}) \oplus V_2(d_2 \mid \mathcal{D}) \oplus \ldots\ldots \oplus V_{n-1}(d_{n-1} \mid \mathcal{D}), \tag{5}$$

where $V_i(d_i \mid \mathcal{D})$ is a d_i-dimensional polyadic vector space [23], $i = 1, \dots, n-1$. Then, instead of (3) we have the cyclic direct sum of homomorphisms

$$\begin{aligned}\mathcal{R}^{[2,n]}_{simple} &\cong \mathrm{Hom}_{\mathcal{D}}(V_1(d_1 \mid \mathcal{D}), V_2(d_2 \mid \mathcal{D})) \oplus \mathrm{Hom}_{\mathcal{D}}(V_2(d_2 \mid \mathcal{D}), V_3(d_3 \mid \mathcal{D})) \oplus \dots \\ &\dots \oplus \mathrm{Hom}_{\mathcal{D}}(V_{n-1}(d_{n-1} \mid \mathcal{D}), V_1(d_1 \mid \mathcal{D})).\end{aligned} \tag{6}$$

This means that after choosing a suitable basis in terms of matrices (when the ring multiplication $\mu^{[n]}$ coincides with the product of n matrices) we have

Theorem 1. *The simple polyadic ring $\mathcal{R}^{[2,n]}_{simple}$ is isomorphic to the $d \times d$ matrix ring (cf. (2)):*

$$\mathcal{R}^{[2,n]}_{simple} \cong Mat^{shift(n)}_{d\times d}(\mathcal{D}) = \left\{ \mathrm{M}^{shift(n)}(d \times d) \mid \nu^{[2]}, \mu^{[n]} \right\}, \tag{7}$$

where $\nu^{[2]}$ and $\mu^{[n]}$ are the binary addition and ordinary product of n matrices, $\mathrm{M}^{shift}_{d\times d}$ is the block-shift (traceless) matrix over $\mathcal{D}$ of the form (which follows from (6))

$$\begin{aligned}&\mathrm{M}^{shift(n)}(d \times d) \\ &= \begin{pmatrix} 0 & \mathrm{B}_1(d_1 \times d_2) & \dots & 0 & 0 \\ 0 & 0 & \mathrm{B}_2(d_2 \times d_3) & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & \mathrm{B}_{n-2}(d_{n-2} \times d_{n-1}) \\ \mathrm{B}_{n-1}(d_{n-1} \times d_1) & 0 & \dots & 0 & 0 \end{pmatrix},\end{aligned} \tag{8}$$

where $(n-1)$ blocks are nonsquare matrices $\mathrm{B}_i(d' \times d'') \in Mat^{full}_{d'\times d''}(\mathcal{D})$ over the division ring $\mathcal{D}$, and $d = d_1 + d_2 + \dots + d_{n-1}$.

Remark 1. *The set of the fixed size blocks $\{\mathrm{B}_i(d' \times d'')\}$ does not form a binary ring, because $d' \neq d''$.*

Assertion 1. *The block-shift matrices of the form (8) are closed with respect to n-ary multiplication and binary addition, and we call them* n-ary matrices.

Taking distributivity into account, we arrive at the polyadic ring structure (7).

Corollary 1. *In the limiting case $n = 2$, we have*

$$\mathrm{M}^{shift(n=2)}(d \times d) = \mathrm{B}_1(d_1 \times d_1) \tag{9}$$

and $d = d_1$, giving a binary ring (2).

Assertion 2. *A finite-dimensional simple associative n-ary algebra $\mathcal{A}^{(n)}$ over an algebraically closed field $\mathcal{F}$ [29] is isomorphic to the block-shift n-ary matrix (8) over $\mathcal{F}$*

$$\mathcal{A}^{(n)} \cong Mat^{shift(n)}_{d\times d}(\mathcal{F}). \tag{10}$$

3.2. Semisimple Polyadic Structures

The Wedderburn–Artin theorem for semisimple Artinian rings $\mathcal{R}_{semispl}$ states that $\mathcal{R}_{semispl}$ is a finite direct product of k simple rings, each of which has the form (2). Using (3) for each component, we decompose the d-finite-dimensional vector space (left module) into a direct sum of length k

$$V(d) = W^{(1)}\left(q^{(1)} \mid \mathcal{D}^{(1)}\right) \oplus W^{(2)}\left(q^{(2)} \mid \mathcal{D}^{(2)}\right) \oplus \dots \oplus W^{(k)}\left(q^{(k)} \mid \mathcal{D}^{(k)}\right), \tag{11}$$

where $d = q^{(1)} + q^{(2)} + \ldots + q^{(k)}$. Then, instead of (3) we have the following isomorphism. (We enumerate simple components by an upper index in round brackets (k) and block-shift components by lower index without brackets, and the arity is an upper index in square brackets $[n]$.)

$$\mathcal{R}_{semispl} \cong \mathrm{End}_{\mathcal{D}^{(1)}} W^{(1)}\left(q^{(1)} \mid \mathcal{D}^{(1)}\right) \\ \oplus \mathrm{End}_{\mathcal{D}^{(2)}} W^{(2)}\left(q^{(2)} \mid \mathcal{D}^{(2)}\right) \oplus \ldots \oplus \mathrm{End}_{\mathcal{D}^{(k)}} W^{(k)}\left(q^{(k)} \mid \mathcal{D}^{(k)}\right). \tag{12}$$

With a suitable basis, the Wedderburn–Artin theorem follows.

Theorem 2. *A semisimple Artinian (binary) ring $\mathcal{R}_{semispl}$ is isomorphic to the $d \times d$ matrix ring*

$$\mathcal{R}_{semispl} \cong Mat^{diag(k)}_{q^{(j)} \times q^{(j)}}(\mathcal{D}) = \left\{\mathrm{M}^{diag(k)}(d \times d) \mid \nu^{[2]}, \mu^{[2]}\right\}, \tag{13}$$

where $\nu^{[2]}$ and $\mu^{[2]}$ are binary addition and binary product of matrices, and $\mathrm{M}^{diag(k)}(d \times d)$ are block-diagonal matrices of the form (which follows from (12))

$$\mathrm{M}^{diag(k)}(d \times d) = \begin{pmatrix} \mathrm{A}^{(1)}\left(q^{(1)} \times q^{(1)}\right) & 0 & \ldots & 0 \\ 0 & \mathrm{A}^{(2)}\left(q^{(2)} \times q^{(2)}\right) & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \mathrm{A}^{(k)}\left(q^{(k)} \times q^{(k)}\right) \end{pmatrix}, \tag{14}$$

where k square blocks are full matrix rings over division rings $\mathcal{D}^{(j)}$

$$\mathrm{A}^{(j)}\left(q^{(j)} \times q^{(j)}\right) \in Mat^{full}_{q^{(j)} \times q^{(j)}}\left(\mathcal{D}^{(j)}\right), \;\; j = 1, \ldots, k, \;\; d = q^{(1)} + q^{(2)} + \ldots + q^{(k)}. \tag{15}$$

The same matrix structure has a finite-dimensional semisimple associative algebra $\mathcal{A}$ over an algebraically closed field $\mathcal{F}$ (see (4)). For further details, see, e.g., [17,18,28].

General properties of semisimple Artinian $[2, n]$-rings were considered in [19] (for ternary rings, see [30,31]). Here we propose a new manifest matrix structure for them.

Thus, our task is to decompose each of the $V_i(d_i)$, in (5) into components as in (11)

$$V_i(d_i) = W_i^{(1)}\left(q_i^{(1)} \mid \mathcal{D}^{(1)}\right) \oplus W_i^{(2)}\left(q_i^{(2)} \mid \mathcal{D}^{(2)}\right) \oplus \ldots \oplus W_i^{(k)}\left(q_i^{(k)} \mid \mathcal{D}^{(k)}\right), \quad i = 1, \ldots, n-1. \tag{16}$$

In matrix language, this means that each block $\mathrm{B}_{d' \times d''}$ from the polyadic ring (8) should have the semisimple decomposition (14), i.e., be a block-diagonal square matrix of the same size $p \times p$, where $p = d_1 = d_2 = \ldots = d_{n-1}$ and the total matrix size becomes $d = (n-1)p$. Moreover, all blocks B should have diagonal blocks A of the same size, and therefore $q^{(j)} \equiv q_1^{(j)} = q_2^{(j)} = \ldots = q_{n-1}^{(j)}$ for all $j = 1, \ldots, k$ and $p = q^{(1)} + q^{(2)} + \ldots + q^{(k)}$, where k is the number of semisimple components. In this way, the cyclic direct sum of homomorphisms for the semisimple polyadic rings becomes (we use different division rings for each semisimple component as in (15))

$$
\begin{aligned}
\mathcal{R}^{[2,n]}_{semispl} \cong\ & \mathrm{Hom}_{\mathcal{D}^{(1)}}\left(W_1^{(1)}\left(q^{(1)} \mid \mathcal{D}^{(1)}\right), W_2^{(1)}\left(q^{(1)} \mid \mathcal{D}^{(1)}\right)\right) \\
& \oplus \mathrm{Hom}_{\mathcal{D}^{(2)}}\left(W_1^{(2)}\left(q^{(2)} \mid \mathcal{D}^{(2)}\right), W_2^{(2)}\left(q^{(2)} \mid \mathcal{D}^{(2)}\right)\right) \oplus \ldots \\
& \ldots \oplus \mathrm{Hom}_{\mathcal{D}^{(k)}}\left(W_1^{(k)}\left(q^{(k)} \mid \mathcal{D}^{(k)}\right), W_2^{(k)}\left(q^{(k)} \mid \mathcal{D}^{(k)}\right)\right) \\
& \oplus \mathrm{Hom}_{\mathcal{D}^{(1)}}\left(W_2^{(1)}\left(q^{(1)} \mid \mathcal{D}^{(1)}\right), W_3^{(1)}\left(q^{(1)} \mid \mathcal{D}^{(1)}\right)\right) \\
& \oplus \mathrm{Hom}_{\mathcal{D}^{(2)}}\left(W_2^{(2)}\left(q^{(2)} \mid \mathcal{D}^{(2)}\right), W_3^{(2)}\left(q^{(2)} \mid \mathcal{D}^{(2)}\right)\right) \oplus \ldots \\
& \ldots \oplus \mathrm{Hom}_{\mathcal{D}^{(k)}}\left(W_2^{(k)}\left(q^{(k)} \mid \mathcal{D}^{(k)}\right), W_3^{(k)}\left(q^{(k)} \mid \mathcal{D}^{(k)}\right)\right) \\
& \vdots \\
& \oplus \mathrm{Hom}_{\mathcal{D}^{(1)}}\left(W_{n-2}^{(1)}\left(q^{(1)} \mid \mathcal{D}^{(1)}\right), W_{n-1}^{(1)}\left(q^{(1)} \mid \mathcal{D}^{(1)}\right)\right) \\
& \oplus \mathrm{Hom}_{\mathcal{D}^{(2)}}\left(W_{n-2}^{(2)}\left(q^{(2)} \mid \mathcal{D}^{(2)}\right), W_{n-1}^{(2)}\left(q^{(2)} \mid \mathcal{D}^{(2)}\right)\right) \oplus \ldots \\
& \ldots \oplus \mathrm{Hom}_{\mathcal{D}^{(k)}}\left(W_{n-2}^{(k)}\left(q^{(k)} \mid \mathcal{D}^{(k)}\right), W_{n-1}^{(k)}\left(q^{(k)} \mid \mathcal{D}^{(k)}\right)\right) \\
& \oplus \mathrm{Hom}_{\mathcal{D}^{(1)}}\left(W_{n-1}^{(1)}\left(q^{(1)} \mid \mathcal{D}^{(1)}\right), W_1^{(1)}\left(q^{(1)} \mid \mathcal{D}^{(1)}\right)\right) \\
& \oplus \mathrm{Hom}_{\mathcal{D}^{(2)}}\left(W_{n-1}^{(2)}\left(q^{(2)} \mid \mathcal{D}^{(2)}\right), W_1^{(2)}\left(q^{(2)} \mid \mathcal{D}^{(2)}\right)\right) \oplus \ldots \\
& \ldots \oplus \mathrm{Hom}_{\mathcal{D}^{(k)}}\left(W_{n-1}^{(k)}\left(q^{(k)} \mid \mathcal{D}^{(k)}\right), W_1^{(k)}\left(q^{(k)} \mid \mathcal{D}^{(k)}\right)\right).
\end{aligned} \tag{17}
$$

After choosing a suitable basis, we obtained a polyadic analog of the Wedderburn–Artin theorem for semisimple Artinian $[2,n]$-rings $\mathcal{R}^{[2,n]}_{semispl}$, which can be called as the *double decomposition* (of the *first kind* or *shift-diagonal*).

Theorem 3. *The semisimple polyadic Artinian ring $\mathcal{R}^{[2,n]}_{semispl}$ (of the first kind) is isomorphic to the $d \times d$ matrix ring*

$$
\mathcal{R}^{[2,n]}_{semispl} \cong Mat^{shift\text{-}diag(n,k)}_{d\times d}(\mathcal{D}) = \left\langle \left\{ \mathrm{N}^{shift\text{-}diag(n,k)}(d\times d)\right\} \mid \nu^{[2]}, \mu^{[n]} \right\rangle, \tag{18}
$$

where $\nu^{[2]}, \mu^{[n]}$ are the binary addition and ordinary product of n matrices, $\mathrm{N}^{shift\text{-}diag(n,k)}_{d\times d}$ (n is the arity of N*s and k is number of simple components of* N*'s) are the block-shift n-ary matrices with block-diagonal square blocks (which follows from (17))*

$$
\mathrm{N}^{shift\text{-}diag(n,k)}(d\times d) = \begin{pmatrix}
0 & \mathrm{B}_1^{(k)}(p\times p) & \ldots & 0 & 0 \\
0 & 0 & \mathrm{B}_2^{(k)}(p\times p) & \ldots & 0 \\
0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & 0 & \mathrm{B}_{n-2}^{(k)}(p\times p) \\
\mathrm{B}_{n-1}^{(k)}(p\times p) & 0 & \ldots & 0 & 0
\end{pmatrix}, \tag{19}
$$

$$
\mathrm{B}_i^{(k)}(p\times p) = \begin{pmatrix}
\mathrm{A}_i^{(1)}\left(q^{(1)}\times q^{(1)}\right) & 0 & \ldots & 0 \\
0 & \mathrm{A}_i^{(2)}\left(q^{(2)}\times q^{(2)}\right) & \ddots & \vdots \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & \ldots & \mathrm{A}_i^{(k)}\left(q^{(k)}\times q^{(k)}\right)
\end{pmatrix}, \tag{20}
$$

where

$$\begin{aligned} d &= (n-1)p, && (21) \\ p &= q^{(1)} + q^{(2)} + \ldots + q^{(k)}, && (22) \end{aligned}$$

and the k square blocks A *are full matrix rings over the division rings* $\mathcal{D}^{(j)}$

$$\mathrm{A}_i^{(j)}\left(q^{(j)} \times q^{(j)}\right) \in Mat^{full}_{q^{(j)}\times q^{(j)}}\left(\mathcal{D}^{(j)}\right), \ j = 1,\ldots,k, \ i = 1,\ldots,n-1. \tag{23}$$

Remark 2. *By analogy with (9), in the limiting case $n = 2$, we have in (19) one block* $\mathrm{B}_1^{(k)}(p \times p)$ *only, and (20) gives its standard (binary) semisimple ring decomposition.*

This allows us to introduce another possible double decomposition in the opposite sequence to (19) and (20); we call it *the second kind* or *reverse*, or *diagonal-shift*. Indeed, for a suitable basis, we first provide the standard block-diagonal decomposition (14), and then each block obeys the block-shift decomposition (8). Here we do not write the "reverse" analog of (17) and arrive directly at

Theorem 4. *The semisimple polyadic Artinian ring* $\hat{\mathcal{R}}^{[2,n]}_{semispl}$ *(of the second kind) is isomorphic to the $d \times d$ matrix ring*

$$\hat{\mathcal{R}}^{[2,n]}_{semispl} \cong Mat^{diag\text{-}shift(n,k)}_{d\times d}(\mathcal{D}) = \left\langle \left\{ \hat{\mathrm{N}}^{diag\text{-}shift(n,k)}(d\times d) \right\} \mid \nu^{[2]}, \mu^{[n]} \right\rangle, \tag{24}$$

where $\nu^{[2]}, \mu^{[n]}$ *are the binary addition and ordinary product of n matrices,* $\hat{\mathrm{N}}^{diag\text{-}shift(n,k)}_{d\times d}$ *(n is arity of* $\hat{\mathrm{N}}$*'s and k is number of simple components of* $\hat{\mathrm{N}}$*'s) are the block-diagonal n-ary matrices with block-shift nonsquare blocks*

$$\hat{\mathrm{N}}^{diag\text{-}shift(n,k)}(d\times d) = \begin{pmatrix} \hat{\mathrm{A}}^{(1)}\left(q^{(1)}\times q^{(1)}\right) & 0 & \ldots & 0 \\ 0 & \hat{\mathrm{A}}^{(2)}\left(q^{(2)}\times q^{(2)}\right) & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \hat{\mathrm{A}}^{(k)}\left(q^{(k)}\times q^{(k)}\right) \end{pmatrix}, \tag{25}$$

$$\hat{\mathrm{A}}^{(j)}\left(q^{(j)}\times q^{(j)}\right) =$$

$$\begin{pmatrix} 0 & \hat{\mathrm{B}}_1^{(j)}\left(p_1^{(j)}\times p_2^{(j)}\right) & \ldots & 0 & 0 \\ 0 & 0 & \hat{\mathrm{B}}_2^{(j)}\left(p_2^{(j)}\times p_3^{(j)}\right) & \ldots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & \hat{\mathrm{B}}_{n-2}^{(j)}\left(p_{n-2}^{(j)}\times p_{n-1}^{(j)}\right) \\ \hat{\mathrm{B}}_{n-1}^{(j)}\left(p_{n-1}^{(j)}\times p_1^{(j)}\right) & 0 & \ldots & 0 & 0 \end{pmatrix}, \tag{26}$$

where

$$\begin{aligned} q^{(j)} &= p_1^{(j)} + p_2^{(j)} + \ldots + p_{n-1}^{(j)}, && (27) \\ d &= q^{(1)} + q^{(2)} + \ldots + q^{(k)}, && (28) \end{aligned}$$

and the $(n-1)k$ blocks $\hat{\mathrm{B}}$ *are nonsquare matrices over the division rings* $\mathcal{D}^{(j)}$

$$\hat{\mathrm{B}}_i^{(j)}\left(p_i^{(j)}\times p_{i+1}^{(j)}\right) \in Mat^{full}_{p_i^{(j)}\times p_{i+1}^{(j)}}\left(\mathcal{D}^{(j)}\right), \ j = 1,\ldots,k, \ i = 1,\ldots,n-1. \tag{29}$$

Definition 1. *The ring obeying the double decomposition of the first kind (19) and (20) (of the second kind (25) and (26)) is called a* polyadic ring of the first kind *(resp.* of the second kind*).*

Proposition 1. *The polyadic rings of the first and second kind are not isomorphic.*

Proof. This follows from the manifest forms (19), (20), and (25), (26). In addition, in general case, the $\hat{\mathrm{B}}$-matrices can be nonsquare (29). □

Thus, the two double decompositions introduced above can lead to a new classification for polyadic analogs of semisimple rings.

Example 1. *Let us consider the double decomposition of two kinds for ternary ($n = 3$) rings with two semisimple components ($k = 2$) and blocks as full $q \times q$ matrix rings over $\mathbb{C}$. Indeed, we have for the ternary nonderived rings $\mathcal{R}^{[2,3]}_{semispl}$ and $\hat{\mathcal{R}}^{[2,3]}_{semispl}$ of the first and second kind, respectively, the following block structures:*

$$\mathrm{N}^{shift\text{-}diag(3,2)}(4q \times 4q) = \begin{pmatrix} 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_2 \\ B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \end{pmatrix},$$
$$\hat{\mathrm{N}}^{diag\text{-}shift(3,2)}(4q \times 4q) = \begin{pmatrix} 0 & \hat{A}_1 & 0 & 0 \\ \hat{A}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \hat{B}_1 \\ 0 & 0 & \hat{B}_2 & 0 \end{pmatrix}, \tag{30}$$

where $A_i, B_i, \hat{A}_i, \hat{B}_i \in Mat^{full}_{q\times q}(\mathbb{C})$. In terms of component blocks, the ternary multiplications in the rings $\mathcal{R}^{[2,n]}_{semispl}$ and $\hat{\mathcal{R}}^{[2,3]}_{semispl}$ are

kind I:

$$A_1' B_1'' A_1''' = A_1, \quad A_2' B_2'' A_2''' = A_2, \tag{31}$$
$$B_1' A_1'' B_1''' = B_1, \quad B_2' A_2'' B_2''' = B_2. \tag{32}$$

kind II:

$$\hat{A}_1' \hat{A}_2'' \hat{A}_1''' = \hat{A}_1, \quad \hat{A}_2' \hat{A}_1'' \hat{A}_2''' = \hat{A}_2, \tag{33}$$
$$\hat{B}_1' \hat{B}_2'' \hat{B}_1''' = \hat{B}_1, \quad \hat{B}_2' \hat{B}_1'' \hat{B}_2''' = \hat{B}_2. \tag{34}$$

It follows from (31), (32), and (33), (34) that $\mathcal{R}^{[2,3]}_{semispl}$ and $\hat{\mathcal{R}}^{[2,3]}_{semispl}$ are not ternary isomorphic.

Note that the sum of the block structures (30) obeys nontrivial properties.

Remark 3. *Consider a binary sum of the block matrices of the first and second kind: (30)*

$$\begin{aligned} \mathrm{P}^{(3,2)}(4q \times 4q) &= \mathrm{N}^{shift\text{-}diag(3,2)}(4q \times 4q) + \hat{\mathrm{N}}^{diag\text{-}shift(3,2)}(4q \times 4q) \\ &= \begin{pmatrix} 0 & \hat{A}_1 & A_1 & 0 \\ \hat{A}_2 & 0 & 0 & A_2 \\ B_1 & 0 & 0 & \hat{B}_1 \\ 0 & B_2 & \hat{B}_2 & 0 \end{pmatrix}. \end{aligned} \tag{35}$$

The set of matrices (35) forms the nonderived $[2,3]$-ring $\mathcal{P}^{[2,3]}$ over $\mathbb{C}$

$$\mathcal{P}^{[2,3]} = \left\langle \left\{ \mathrm{P}^{(3,2)}(4q \times 4q) \right\} \mid \nu^{[2]}, \mu^{[3]} \right\rangle, \tag{36}$$

where $\nu^{[2]}$, $\mu^{[3]}$ are the binary addition and ordinary product of 3 *matrices (35).*

Notice that the P*-matrices (35) are the block-matrix versions of the circle matrices* M_{circ}*, which were studied in [32] in connection with* 8*-vertex solutions to the constant Yang–Baxter equation [33] and the corresponding braiding quantum gates [34,35].*

Supersymmetric Double Decomposition

Let us generalize the above double decomposition (of the first kind) to superrings and superalgebras. For that we first assume that the constituent vector spaces (entering in (17)) are super vector spaces ($\mathbb{Z}_2$-graded vector spaces) obeying the standard decomposition into even and odd parts:

$$W_i^{(j)}\left(q^{(j)} \mid \mathcal{D}^{(j)}\right) = W_i^{(j)}\left(q_{even}^{(j)} \mid \mathcal{D}^{(j)}\right)_{even} \oplus W_i^{(j)}\left(q_{odd}^{(j)} \mid \mathcal{D}^{(j)}\right)_{odd}, \; i = 1, \dots, n-1, \; j = 1, \dots, k, \tag{37}$$

where $q_{even}^{(j)}$ and $q_{odd}^{(j)}$ are dimensions of the even and odd spaces, respectively; $q^{(j)} = q_{even}^{(j)} + q_{odd}^{(j)}$.

The parity of a homogeneous element of the vector space $v \in W_i^{(j)}\left(q^{(j)} \mid \mathcal{D}^{(j)}\right)$ is defined by $|v| = \bar{0}$ (resp. $\bar{1}$), if $v \in W_i^{(j)}\left(q_{even}^{(j)} \mid \mathcal{D}^{(j)}\right)_{even}$ (resp. $W_i^{(j)}\left(q_{odd}^{(j)} \mid \mathcal{D}^{(j)}\right)_{odd}$), and $\bar{0}, \bar{1} \in \mathbb{Z}_2$. For details, see [36,37]. In the graded case, the k square blocks A in (23) are full supermatrix rings of the size $\left(q_{even}^{(j)} \mid q_{odd}^{(j)}\right) \times \left(q_{even}^{(j)} \mid q_{odd}^{(j)}\right)$, while the square Bs (20) are block-diagonal supermatrices, and the block-shift n-ary supermatrices have a nonstandard form (19).

We assume that in super case a polyadic analog of the Wedderburn–Artin theorem for semisimple Artinian superrings (of the first kind) is also valid, with the forms of the double decomposition (19) and (20) being the same; however, now the As and B are corresponding supermatrices.

4. Polyadization Concept

Here we propose a general procedure for how to construct new polyadic algebraic structures from binary (or lower arity) ones, using the "inverse" (informally) to the block-shift matrix decomposition (8). It can be considered as a polyadic analog of the inverse problem of the determination of an algebraic structure from the knowledge of its Wedderburn decomposition [38].

4.1. Polyadization of Binary Algebraic Structures

Let a binary algebraic structure $\mathcal{X}$ be represented by $p \times p$ matrices $\mathrm{B}_{\mathbf{y}} \equiv \mathrm{B}_{\mathbf{y}}(p \times p)$ over a ring $\mathcal{R}$ (a linear representation), where $\mathbf{y}$ is the set of N_y parameters corresponding to an element x of $\mathcal{X}$. As the binary addition in $\mathcal{R}$ transfers to the matrix addition without restrictions (as opposed to the polyadic case, see below), we will consider only the multiplicative part of the resulting polyadic matrix ring. In this way, we propose a special block-shift matrix method to obtain n-ary semigroups (n-ary groups) from the binary ones, but the former are not derived from the latter [11,25]. In general, this can lead to new algebraic structures that were not known before.

Definition 2. *A (block-matrix) polyadization* $\mathbf{\Phi}_{pol}$ *of a binary semigroup (or group)* $\mathcal{X}$ *represented by square* $p \times p$ *matrices* $\mathrm{B}_{\mathbf{y}}$ *is an* n*-ary semigroup (or an* n*-ary group) represented by the* $d \times d$ *block-shift matrices (over a ring* $\mathcal{R}$*) of the form (8) as follows:*

$$\mathrm{Q}_{\mathbf{y}_1,\dots,\mathbf{y}_{n-1}} \equiv \mathrm{Q}_{\mathbf{y}_1,\dots,\mathbf{y}_{n-1}}^{Bshift(n)}(d \times d) = \begin{pmatrix} 0 & \mathrm{B}_{\mathbf{y}_1} & \dots & 0 & 0 \\ 0 & 0 & \mathrm{B}_{\mathbf{y}_2} & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & \mathrm{B}_{\mathbf{y}_{n-2}} \\ \mathrm{B}_{\mathbf{y}_{n-1}} & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{38}$$

where $d = (n-1)p$, and the n-ary multiplication $\mu^{[[n]]}$ is given by the product of n matrices (38).

In terms of the block-matrices B, the multiplication

$$\mu^{[[n]]}\left[\overbrace{\mathrm{Q}_{\mathbf{y}'_1,\dots,\mathbf{y}'_{n-1}},\mathrm{Q}_{\mathbf{y}''_1,\dots,\mathbf{y}''_{n-1}},\dots,\mathrm{Q}_{\mathbf{y}'''_1,\dots,\mathbf{y}'''_{n-1}}\mathrm{Q}_{\mathbf{y}''''_1,\dots,\mathbf{y}''''_{n-1}}}^{n}\right]=\mathrm{Q}_{\mathbf{y}_1,\dots,\mathbf{y}_{n-1}} \tag{39}$$

has the cyclic product form (see [39])

$$\overbrace{\mathrm{B}_{\mathbf{y}'_1}\mathrm{B}_{\mathbf{y}''_2}\dots\mathrm{B}_{\mathbf{y}'''_{n-1}}\mathrm{B}_{\mathbf{y}''''_1}}^{n}=\mathrm{B}_{\mathbf{y}_1}, \tag{40}$$

$$\mathrm{B}_{\mathbf{y}'_2}\mathrm{B}_{\mathbf{y}''_3}\dots\mathrm{B}_{\mathbf{y}'''_1}\mathrm{B}_{\mathbf{y}''''_2}=\mathrm{B}_{\mathbf{y}_2}, \tag{41}$$

$$\vdots$$

$$\mathrm{B}_{\mathbf{y}'_{n-1}}\mathrm{B}_{\mathbf{y}''_1}\dots\mathrm{B}_{\mathbf{y}'''_{n-2}}\mathrm{B}_{\mathbf{y}''''_{n-1}}=\mathrm{B}_{\mathbf{y}_{n-1}}. \tag{42}$$

Remark 4. *The number of parameters N_y describing an element $x \in \mathcal{X}$ increases to $(n-1)N_y$, and the corresponding algebraic structure $\left\langle \left\{\mathrm{Q}_{\mathbf{y}_1,\dots,\mathbf{y}_{n-1}}\right\} \mid \mu^{[[n]]}\right\rangle$ becomes n-ary, and so (38) can be treated as a new algebraic structure, which we denote by the same letter with the arities in double square brackets $\mathcal{X}^{[[n]]}$.*

We now analyze some of the most general properties of the polyadization map $\mathbf{\Phi}_{pol}$, which are independent of the concrete form of the block-matrices B and over which algebraic structure (ring, field, etc.) they are defined. We then present some concrete examples.

Definition 3. *A unique polyadization $\mathbf{\Phi}_{Upol}$ is a polyadization where all sets of parameters coincide*

$$\mathbf{y}=\mathbf{y}_1=\mathbf{y}_2\dots=\mathbf{y}_{n-1}. \tag{43}$$

Proposition 2. *The unique polyadization is an n-ary-binary homomorphism.*

Proof. In the case of (43), all $(n-1)$ relations (40)–(42) coincide

$$\overbrace{\mathrm{B}_{\mathbf{y}'}\mathrm{B}_{\mathbf{y}''}\dots\mathrm{B}_{\mathbf{y}'''}\mathrm{B}_{\mathbf{y}''''}}^{n}=\mathrm{B}_{\mathbf{y}}, \tag{44}$$

which means that the ordinary (binary) product of n matrices $\mathrm{B}_{\mathbf{y}}$ is mapped to the n-ary product of matrices $\mathrm{Q}_{\mathbf{y}}$ (39)

$$\mu^{[[n]]}\left[\overbrace{\mathrm{Q}_{\mathbf{y}'},\mathrm{Q}_{\mathbf{y}''},\dots,\mathrm{Q}_{\mathbf{y}'''}\mathrm{Q}_{\mathbf{y}''''}}^{n}\right]=\mathrm{Q}_{\mathbf{y}}, \tag{45}$$

as it should be for an n-ary-binary homomorphism, but not for a homomorphism. □

Assertion 3. *If matrices $\mathrm{B}_{\mathbf{y}} \equiv \mathrm{B}_{\mathbf{y}}(p\times p)$ contain the identity matrix E_p, then the n-ary identity $\mathrm{E}_d^{(n)}$ in $\left\langle \left\{\mathrm{Q}_{\mathbf{y}}(d\times d)\right\} \mid \mu^{[[n]]}\right\rangle$, $d=(n-1)p$ has the form*

$$\mathrm{E}_d^{(n)}=\begin{pmatrix} 0 & \mathrm{E}_p & \dots & 0 & 0 \\ 0 & 0 & \mathrm{E}_p & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & \mathrm{E}_p \\ \mathrm{E}_p & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{46}$$

Proof. It follows from (38), (39), and (44). □

In this case, the unique polyadization maps the identity matrix to the n-ary identity $\mathbf{\Phi}_{Upol} : \mathrm{E}_p \to \mathrm{E}_d^{(n)}$.

Assertion 4. *If the matrices* $\mathrm{B}_{\mathbf{y}}$ *are invertible* $\mathrm{B}_{\mathbf{y}}\mathrm{B}_{\mathbf{y}}^{-1} = \mathrm{B}_{\mathbf{y}}^{-1}\mathrm{B}_{\mathbf{y}} = \mathrm{E}_p$*, then each* $\mathrm{Q}_{\mathbf{y}_1,\dots,\mathbf{y}_{n-1}}$ *has a querelement*

$$\overline{\mathrm{Q}}_{\mathbf{y}_1,\dots,\mathbf{y}_{n-1}} = \begin{pmatrix} 0 & \overline{\mathrm{B}}_{\mathbf{y}_1} & \dots & 0 & 0 \\ 0 & 0 & \overline{\mathrm{B}}_{\mathbf{y}_2} & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & \overline{\mathrm{B}}_{\mathbf{y}_{n-2}} \\ \overline{\mathrm{B}}_{\mathbf{y}_{n-1}} & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{47}$$

satisfying

$$\mu^{[[n]]}\left[\overbrace{\mathrm{Q}_{\mathbf{y}_1,\dots,\mathbf{y}_{n-1}}, \mathrm{Q}_{\mathbf{y}_1,\dots,\mathbf{y}_{n-1}}, \dots, \mathrm{Q}_{\mathbf{y}_1,\dots,\mathbf{y}_{n-1}} \overline{\mathrm{Q}}_{\mathbf{y}_1,\dots,\mathbf{y}_{n-1}}}^{n}\right] = \mathrm{Q}_{\mathbf{y}_1,\dots,\mathbf{y}_{n-1}} \tag{48}$$

where $\overline{\mathrm{Q}}_{\mathbf{y}_1,\dots,\mathbf{y}_{n-1}}$ *can be on any places and*

$$\overline{\mathrm{B}}_{\mathbf{y}_i} = \mathrm{B}_{\mathbf{y}_{i-1}}^{-1}\mathrm{B}_{\mathbf{y}_{i-2}}^{-1}\dots\mathrm{B}_{\mathbf{y}_2}^{-1}\mathrm{B}_{\mathbf{y}_1}^{-1}\mathrm{B}_{\mathbf{y}_{n-1}}^{-1}\mathrm{B}_{\mathbf{y}_{n-2}}^{-1}\dots\mathrm{B}_{\mathbf{y}_{i+2}}^{-1}\mathrm{B}_{\mathbf{y}_{i+1}}^{-1}. \tag{49}$$

Proof. This follows from (47), (48), and (40), (42); consequently, by applying $\mathrm{B}_{\mathbf{y}_i}^{-1}$ (with suitable indices) on both sides, we obtain (49). □

Let us suppose that on the set of matrices $\{\mathrm{B}_{\mathbf{y}}\}$ over a binary ring $\mathcal{R}$, one can consider some analog of a multiplicative character $\chi : \{\mathrm{B}_{\mathbf{y}}\} \to \mathcal{R}$, being a (binary) homomorphism, such that

$$\chi(\mathrm{B}_{\mathbf{y}_1})\chi(\mathrm{B}_{\mathbf{y}_2}) = \chi(\mathrm{B}_{\mathbf{y}_1}\mathrm{B}_{\mathbf{y}_2}). \tag{50}$$

For instance, in case $\mathrm{B} \in GL(p, \mathbb{C})$, the determinant can be considered to have a (binary) multiplicative character. Similarly, we can introduce

Definition 4. *A polyadized multiplicative character* $\boldsymbol{\varnothing} : \{\mathrm{Q}_{\mathbf{y}_1,\dots,\mathbf{y}_{n-1}}\} \to \mathcal{R}$ *is proportional to a product of the binary multiplicative characters of the blocks* $\chi(\mathrm{B}_{\mathbf{y}_i})$

$$\boldsymbol{\varnothing}(\mathrm{Q}_{\mathbf{y}_1,\dots,\mathbf{y}_{n-1}}) = (-1)^n \chi(\mathrm{B}_{\mathbf{y}_1})\chi(\mathrm{B}_{\mathbf{y}_2})\dots\chi(\mathrm{B}_{\mathbf{y}_{n-1}}). \tag{51}$$

The normalization factor $(-1)^n$ in (51) is needed to be consistent with the case when $\mathcal{R}$ is commutative, and the multiplicative characters are determinants. It can also be consistent in other cases.

Proposition 3. *If the ring* $\mathcal{R}$ *is commutative, then the polyadized multiplicative character* $\boldsymbol{\varnothing}$ *is an n-ary-binary homomorphism.*

Proof. It follows from (44), (45), (51), and the commutativity of $\mathcal{R}$. □

Proposition 4. *If the ring* $\mathcal{R}$ *is commutative and unital with the unit* E_p*, then the algebraic structure* $\left\langle \{\mathrm{Q}_{\mathbf{y}_1,\dots,\mathbf{y}_{n-1}}\} \mid \mu^{[[n]]} \right\rangle$ *contains polyadic (n-ary) idempotents satisfying*

$$\mathrm{B}_{\mathbf{y}_1}\mathrm{B}_{\mathbf{y}_2}\dots\mathrm{B}_{\mathbf{y}_{n-1}} = E_p. \tag{52}$$

Proof. It follows from (45) and (46). □

4.2. Concrete Examples of the Polyadization Procedure

4.2.1. Polyadization of $GL(2,\mathbb{C})$

Consider the polyadization procedure for the general linear group $GL(2,\mathbb{C})$. We have for the 4-parameter block matrices $\mathrm{B}_{\mathbf{y}_i} = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in GL(2,\mathbb{C})$, $\mathbf{y}_i = (a_i, b_i, c_i, d_i) \in \mathbb{C}\times\mathbb{C}\times\mathbb{C}\times\mathbb{C}$, $i = 1,2,3$. Thus, the 12-parameter 4-ary group $GL^{[[4]]}(2,\mathbb{C}) = \left\langle \{\mathrm{Q}_{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3}\} \mid \mu^{[[4]]} \right\rangle$ is represented by the following 6×6 Q-matrices:

$$\mathrm{Q}_{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3} = \begin{pmatrix} 0 & \mathrm{B}_{\mathbf{y}_1} & 0 \\ 0 & 0 & \mathrm{B}_{\mathbf{y}_2} \\ \mathrm{B}_{\mathbf{y}_3} & 0 & 0 \end{pmatrix} \in GL^{[[4]]}(2,\mathbb{C}), \quad \mathrm{B}_{\mathbf{y}_i} \in GL(2,\mathbb{C}), \quad i = 1,2,3, \tag{53}$$

obeying the 4-ary multiplication:

$$\begin{aligned} &\mu^{[[4]]}\left[\mathrm{Q}_{\mathbf{y}'_1,\mathbf{y}'_2,\mathbf{y}'_3}, \mathrm{Q}_{\mathbf{y}''_1,\mathbf{y}''_2,\mathbf{y}''_3}, \mathrm{Q}_{\mathbf{y}'''_1,\mathbf{y}'''_2,\mathbf{y}'''_3}, \mathrm{Q}_{\mathbf{y}''''_1,\mathbf{y}''''_2,\mathbf{y}''''_3}\right] \\ &= \mathrm{Q}_{\mathbf{y}'_1,\mathbf{y}'_2,\mathbf{y}'_3}\mathrm{Q}_{\mathbf{y}''_1,\mathbf{y}''_2,\mathbf{y}''_3}\mathrm{Q}_{\mathbf{y}'''_1,\mathbf{y}'''_2,\mathbf{y}'''_3}\mathrm{Q}_{\mathbf{y}''''_1,\mathbf{y}''''_2,\mathbf{y}''''_3} = \mathrm{Q}_{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3}. \end{aligned} \tag{54}$$

In terms of the block matrices $\mathrm{B}_{\mathbf{y}_i}$, the multiplication (54) becomes (see (39)–(42))

$$\mathrm{B}_{\mathbf{y}'_1}\mathrm{B}_{\mathbf{y}''_2}\mathrm{B}_{\mathbf{y}'''_3}\mathrm{B}_{\mathbf{y}''''_1} = \mathrm{B}_{\mathbf{y}_1}, \tag{55}$$

$$\mathrm{B}_{\mathbf{y}'_2}\mathrm{B}_{\mathbf{y}''_3}\mathrm{B}_{\mathbf{y}'''_1}\mathrm{B}_{\mathbf{y}''''_2} = \mathrm{B}_{\mathbf{y}_2}, \tag{56}$$

$$\mathrm{B}_{\mathbf{y}'_3}\mathrm{B}_{\mathbf{y}''_1}\mathrm{B}_{\mathbf{y}'''_2}\mathrm{B}_{\mathbf{y}''''_3} = \mathrm{B}_{\mathbf{y}_3}, \tag{57}$$

which can be further expressed in the B-matrix entries (its manifest form is too cumbersome to give here).

For $\{\mathrm{Q}_{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3}\}$ to be a 4-ary group, each Q-matrix should have the unique querelement determined by the equation (see (48)):

$$\mathrm{Q}_{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3}\mathrm{Q}_{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3}\mathrm{Q}_{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3}\overline{\mathrm{Q}}_{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3} = \mathrm{Q}_{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3}, \tag{58}$$

which has the solution

$$\overline{\mathrm{Q}}_{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3} = \begin{pmatrix} 0 & \overline{\mathrm{B}}_{\mathbf{y}_1} & 0 \\ 0 & 0 & \overline{\mathrm{B}}_{\mathbf{y}_2} \\ \overline{\mathrm{B}}_{\mathbf{y}_3} & 0 & 0 \end{pmatrix}, \tag{59}$$

where (see (49))

$$\overline{\mathrm{B}}_{\mathbf{y}_1} = \mathrm{B}^{-1}_{\mathbf{y}_3}\mathrm{B}^{-1}_{\mathbf{y}_2}, \quad \overline{\mathrm{B}}_{\mathbf{y}_2} = \mathrm{B}^{-1}_{\mathbf{y}_1}\mathrm{B}^{-1}_{\mathbf{y}_3}, \quad \overline{\mathrm{B}}_{\mathbf{y}_3} = \mathrm{B}^{-1}_{\mathbf{y}_2}\mathrm{B}^{-1}_{\mathbf{y}_1}. \tag{60}$$

In the manifest form, the querelements of $GL^{[[4]]}(2,\mathbb{C})$ are (59), where

$$\overline{\mathrm{B}}_{\mathbf{y}_1} = \frac{1}{\Delta_3\Delta_2}\begin{pmatrix} b_3c_2 + d_3d_2 & -b_3a_2 - d_3b_2 \\ -a_3c_2 - c_3d_2 & a_3a_2 + c_3b_2 \end{pmatrix} \tag{61}$$

$$\overline{\mathrm{B}}_{\mathbf{y}_2} = \frac{1}{\Delta_2\Delta_3}\begin{pmatrix} b_1c_3 + d_1d_3 & -b_1a_3 - d_1b_3 \\ -a_1c_3 - c_1d_3 & a_1a_3 + c_1b_3 \end{pmatrix} \tag{62}$$

$$\overline{\mathrm{B}}_{\mathbf{y}_3} = \frac{1}{\Delta_2\Delta_1}\begin{pmatrix} b_2c_1 + d_2d_1 & -b_2a_1 - d_2b_1 \\ -a_2c_1 - c_2d_1 & a_2a_1 + c_2b_1 \end{pmatrix}, \tag{63}$$

where $\Delta_i = a_id_i - b_ic_i \neq 0$ are the (nonvanishing) determinants of $\mathrm{B}_{\mathbf{y}_i}$.

Definition 5. *We call $GL^{[[4]]}(2,\mathbb{C})$ a polyadic (4-ary) general linear group.*

If we take the binary multiplicative characters to be determinants $\chi\left(\mathrm{B}_{\mathbf{y}_i}\right)=\Delta_i\neq 0$, then the polyadized multiplicative character in $GL^{[[4]]}(2,\mathbb{C})$ becomes

$$\text{\o}\left(\mathrm{Q}_{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3}\right)=\Delta_1\Delta_2\Delta_3, \tag{64}$$

which is a 4-ary-binary homomorphism, because (see (55)–(57))

$$\begin{aligned}
&\text{\o}\left(\mathrm{Q}_{\mathbf{y}'_1,\mathbf{y}'_2,\mathbf{y}'_3}\right)\text{\o}\left(\mathrm{Q}_{\mathbf{y}''_1,\mathbf{y}''_2,\mathbf{y}''_3}\right)\text{\o}\left(\mathrm{Q}_{\mathbf{y}'''_1,\mathbf{y}'''_2,\mathbf{y}'''_3}\right)\text{\o}\left(\mathrm{Q}_{\mathbf{y}''''_1,\mathbf{y}''''_2,\mathbf{y}''''_3}\right)\\
&=\left(\Delta'_1\Delta'_2\Delta'_3\right)\left(\Delta''_1\Delta''_2\Delta''_3\right)\left(\Delta'''_1\Delta'''_2\Delta'''_3\right)\left(\Delta''''_1\Delta''''_2\Delta''''_3\right)\\
&=\left(\Delta'_1\Delta''_2\Delta'''_3\Delta''''_1\right)\left(\Delta'_2\Delta''_3\Delta'''_1\Delta''''_2\right)\left(\Delta'_3\Delta''_1\Delta'''_2\Delta''''_3\right)\\
&=\text{\o}\left(\mathrm{Q}_{\mathbf{y}'_1,\mathbf{y}'_2,\mathbf{y}'_3}\mathrm{Q}_{\mathbf{y}''_1,\mathbf{y}''_2,\mathbf{y}''_3}\mathrm{Q}_{\mathbf{y}'''_1,\mathbf{y}'''_2,\mathbf{y}'''_3}\mathrm{Q}_{\mathbf{y}''''_1,\mathbf{y}''''_2,\mathbf{y}''''_3}\right).
\end{aligned} \tag{65}$$

The 4-ary identity $\mathrm{E}_6^{(4)}$ of $GL^{[[4]]}(2,\mathbb{C})$ is unique and has the form (see (46))

$$\mathrm{E}_6^{(4)}=\begin{pmatrix} 0 & \mathrm{E}_2 & 0\\ 0 & 0 & \mathrm{E}_2\\ \mathrm{E}_2 & 0 & 0\end{pmatrix}, \tag{66}$$

where E_2 is the identity of $GL(2,\mathbb{C})$. The 4-ary identity $\mathrm{E}_6^{(4)}$ satisfies the 4-ary idempotence relation

$$\mathrm{E}_6^{(4)}\mathrm{E}_6^{(4)}\mathrm{E}_6^{(4)}\mathrm{E}_6^{(4)}=\mathrm{E}_6^{(4)}. \tag{67}$$

In general, the 4-ary group $GL^{[[4]]}(2,\mathbb{C})$ contains an infinite number of 4-ary idempotents $\mathrm{Q}^{idemp}_{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3}$ defined by the system of equations

$$\mathrm{Q}^{idemp}_{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3}\mathrm{Q}^{idemp}_{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3}\mathrm{Q}^{idemp}_{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3}\mathrm{Q}^{idemp}_{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3}=\mathrm{Q}^{idemp}_{\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3}, \tag{68}$$

which gives

$$\mathrm{B}^{idemp}_{\mathbf{y}_1}\mathrm{B}^{idemp}_{\mathbf{y}_2}\mathrm{B}^{idemp}_{\mathbf{y}_3}=\mathrm{E}_2, \tag{69}$$

or manifestly

$$a_1a_2a_3+a_1b_2c_3+a_3b_1c_2+b_1c_3d_2=1, \tag{70}$$

$$a_2b_3c_1+b_2c_1d_3+b_3c_2d_1+d_1d_2d_3=1, \tag{71}$$

$$a_1a_2b_3+a_1b_2d_3+b_1b_3c_2+b_1d_2d_3=0, \tag{72}$$

$$a_2a_3c_1+a_3c_2d_1+b_2c_1c_3+c_3d_1d_2=0. \tag{73}$$

The infinite set of idempotents in $GL^{[[4]]}(2,\mathbb{C})$ is determined by $12-4=8$ complex parameters, because one block matrix (with 4 complex parameters) can always be excluded using the Equation (69).

Remark 5. *The above example shows how "far" polyadic groups can be formed from ordinary (binary) groups: the former can contain an infinite number of 4-ary idempotents determined by (70)–(73), in addition to the standard idempotent in any group, the 4-ary identity (66).*

4.2.2. Polyadization of $SO(2,\mathbb{R})$

Here we provide a polyadization for the simplest subgroup of $GL(2,\mathbb{C})$, the special orthogonal group $SO(2,\mathbb{R})$. In the matrix form, $SO(2,\mathbb{R})$ is represented by the one-parameter rotation matrix

$$B(\alpha)=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\in SO(2,\mathbb{R}),\qquad \alpha\in\mathbb{R}/2\pi\mathbb{Z}, \tag{74}$$

satisfying the commutative multiplication

$$B(\alpha)B(\beta) = B(\alpha+\beta), \tag{75}$$

and the (binary) identity E_2 is $B(0)$. Therefore, the inverse element for $B(\alpha)$ is $B(-\alpha)$.

The 4-ary polyadization of $SO(2,\mathbb{R})$ is given by the 3-parameter 4-ary group of Q-matrices $SO^{[[4]]}(2,\mathbb{R}) = \left\langle \{Q(\alpha,\beta,\gamma)\} \mid \mu^{[[4]]} \right\rangle$, where (cf. (53))

$$Q(\alpha,\beta,\gamma) = \begin{pmatrix} 0 & B(\alpha) & 0 \\ 0 & 0 & B(\beta) \\ B(\gamma) & 0 & 0 \end{pmatrix} \tag{76}$$

$$= \begin{pmatrix} 0 & 0 & \cos\alpha & -\sin\alpha & 0 & 0 \\ 0 & 0 & \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & 0 & 0 & \sin\beta & \cos\beta \\ \cos\gamma & -\sin\gamma & 0 & 0 & 0 & 0 \\ \sin\gamma & \cos\gamma & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha,\beta,\gamma \in \mathbb{R}/2\pi\mathbb{Z}, \tag{77}$$

and the 4-ary multiplication is

$$\begin{aligned} &\mu^{[[4]]}[Q(\alpha_1,\beta_1,\gamma_1), Q(\alpha_2,\beta_2,\gamma_2), Q(\alpha_3,\beta_3,\gamma_3), Q(\alpha_4,\beta_4,\gamma_4)] \\ &= Q(\alpha_1,\beta_1,\gamma_1)Q(\alpha_2,\beta_2,\gamma_2)Q(\alpha_3,\beta_3,\gamma_3)Q(\alpha_4,\beta_4,\gamma_4) \\ &= Q(\alpha_1+\beta_2+\gamma_3+\alpha_4, \beta_1+\gamma_2+\alpha_3+\beta_4, \gamma_1+\alpha_2+\beta_3+\gamma_4) = Q(\alpha,\beta,\gamma), \end{aligned} \tag{78}$$

which is noncommutative, as opposed to the binary product of B-matrices (75).

The querelement $\overline{Q}(\alpha,\beta,\gamma)$ for a given $Q(\alpha,\beta,\gamma)$ is defined by the equation (see (58))

$$Q(\alpha,\beta,\gamma)Q(\alpha,\beta,\gamma)Q(\alpha,\beta,\gamma)\overline{Q}(\alpha,\beta,\gamma) = Q(\alpha,\beta,\gamma), \tag{79}$$

which has the solution

$$\overline{Q}(\alpha,\beta,\gamma) = Q(-\beta-\gamma, -\alpha-\gamma, -\alpha-\beta). \tag{80}$$

Definition 6. *We call $SO^{[[4]]}(2,\mathbb{R})$ a polyadic (4-ary) special orthogonal group, and $Q(\alpha,\beta,\gamma)$ is called a polyadic (4-ary) rotation matrix.*

Informally, the matrix $Q(\alpha,\beta,\gamma)$ represents the polyadic (4-ary) rotation. There is an infinite number of polyadic (4-ary) identities (neutral elements) $E(\alpha,\beta,\gamma)$ which are defined by

$$E(\alpha,\beta,\gamma)E(\alpha,\beta,\gamma)E(\alpha,\beta,\gamma)Q(\alpha,\beta,\gamma) = Q(\alpha,\beta,\gamma), \tag{81}$$

and the solution is

$$E(\alpha,\beta,\gamma) = Q(\alpha,\beta,\gamma), \quad \alpha+\beta+\gamma = 0. \tag{82}$$

It follows from (81) that $E(\alpha,\beta,\gamma)$ are 4-ary idempotents (see (67) and Remark 5).

The determinants of $B(\alpha)$ and $Q(\alpha,\beta,\gamma)$ are 1, and therefore the corresponding multiplicative characters and polyadized multiplicative characters (51) are also equal to one.

Compared with the successive products of four B-matrices (74)

$$B(\alpha_1)B(\alpha_2)B(\alpha_3)B(\alpha_4) = B(\alpha_1+\alpha_2+\alpha_3+\alpha_4), \tag{83}$$

we observe that 4-ary multiplication (78) gives a shifted sum of four angles.

More exactly, for the triple (α,β,γ) we introduce the circle (left) shift operator by

$$\mathsf{s}\alpha = \beta, \quad \mathsf{s}\beta = \gamma, \quad \mathsf{s}\gamma = \alpha \tag{84}$$

with the property $\mathsf{s}^3 = \mathsf{id}$. Then, the 4-ary multiplication (78) becomes

$$\begin{aligned}&\mu^{[[4]]}[Q(\alpha_1,\beta_1,\gamma_1),Q(\alpha_2,\beta_2,\gamma_2),Q(\alpha_3,\beta_3,\gamma_3),Q(\alpha_4,\beta_4,\gamma_4)]\\&=Q\big(\alpha_1+\mathsf{s}\alpha_2+\mathsf{s}^2\alpha_3+\alpha_4,\beta_1+\mathsf{s}\beta_2+\mathsf{s}^2\beta_3+\beta_4,\gamma_1+\mathsf{s}\gamma_2+\mathsf{s}^2\gamma_3+\gamma_4\big).\end{aligned} \tag{85}$$

The querelement has the form

$$\overline{Q}(\alpha,\beta,\gamma)=Q\Big(-\mathsf{s}\alpha-\mathsf{s}^2\alpha,-\mathsf{s}\beta-\mathsf{s}^2\beta,-\mathsf{s}\gamma-\mathsf{s}^2\gamma\Big). \tag{86}$$

The multiplication (85) can be (informally) expressed in terms of a new operation, the 4-ary "cyclic shift addition" defined on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ by (see (78))

$$\begin{aligned}&\circ_{\mathsf{s}}^{[4]}[(\alpha_1,\beta_1,\gamma_1),(\alpha_2,\beta_2,\gamma_2),(\alpha_3,\beta_3,\gamma_3),(\alpha_4,\beta_4,\gamma_4)]\\&=(\alpha_1+\beta_2+\gamma_3+\alpha_4,\beta_1+\gamma_2+\alpha_3+\beta_4,\gamma_1+\alpha_2+\beta_3+\gamma_4)\\&=\Big(\nu_{\mathsf{s}}^{[4]}[\alpha_1,\alpha_2,\alpha_3,\alpha_4],\nu_{\mathsf{s}}^{[4]}[\beta_1,\beta_2,\beta_3,\beta_4],\nu_{\mathsf{s}}^{[4]}[\gamma_1,\gamma_2,\gamma_3,\gamma_4]\Big),\end{aligned} \tag{87}$$

where $\nu_{\mathsf{s}}^{[4]}$ is (informally)

$$\nu_{\mathsf{s}}^{[4]}[\alpha_1,\alpha_2,\alpha_3,\alpha_4]=\mathsf{s}^0\alpha_1+\mathsf{s}^1\alpha_2+\mathsf{s}^2\alpha_3+\mathsf{s}^3\alpha_4=\alpha_1+\mathsf{s}\alpha_2+\mathsf{s}^2\alpha_3+\alpha_4, \tag{88}$$

and $\mathsf{s}^0 = \mathsf{id}$. This can also be treated as some "deformation" of the repeated binary additions by shifts. It is seen that the 4-ary operation $\circ_{\mathsf{s}}^{[4]}$ (87) is not derived and cannot be obtained by consequent binary operations on the triples (α,β,γ) as (83).

In terms of the 4-ary cyclic shift addition, the 4-ary multiplication (85) becomes

$$\begin{aligned}&\mu^{[[4]]}[Q(\alpha_1,\beta_1,\gamma_1),Q(\alpha_2,\beta_2,\gamma_2),Q(\alpha_3,\beta_3,\gamma_3),Q(\alpha_4,\beta_4,\gamma_4)]\\&=Q\Big(\circ_{\mathsf{s}}^{[4]}[(\alpha_1,\beta_1,\gamma_1),(\alpha_2,\beta_2,\gamma_2),(\alpha_3,\beta_3,\gamma_3),(\alpha_4,\beta_4,\gamma_4)]\Big).\end{aligned} \tag{89}$$

The binary case corresponds to $\mathsf{s} = \mathsf{id}$, because in (74) we have only one angle α, as opposed to three angles in (84).

Thus, we conclude that just as the binary product of B-matrices corresponds to the ordinary angle addition (75), the 4-ary multiplication of polyadic rotation Q-matrices (76) corresponds to the 4-ary cyclic shift addition (88) through (89).

4.3. "Deformation" of Binary Operations by Shifts

The concrete example from the previous subsection shows the strong connection (89) between the polyadization procedure and the shifted operations (88). Here we generalize it to an n-ary case for any semigroup.

Let $\mathcal{A} = \langle A \mid (+)\rangle$ be a binary semigroup, where A is its underlying set and $(+)$ is the binary operation (which can be noncommutative). The simplest way to construct an n-ary operation $\nu^{[n]} : A^n \to A$ is the consequent repetition of the binary operation (see (83))

$$\nu^{[n]}[\alpha_1,\alpha_2,\ldots,\alpha_n]=\alpha_1+\alpha_2+\ldots+\alpha_n, \tag{90}$$

where the n-ary multiplication $\nu^{[n]}$ (90) is called derived [26,40].

To construct a nonderived operation, we now consider the (external) mth direct power $\mathcal{A}^m$ of the semigroup $\mathcal{A}$ by introducing m-tuples

$$\mathbf{a}\equiv\mathbf{a}^{(m)}=\Big(\overbrace{\alpha,\beta,\ldots,\gamma}^{m}\Big),\quad \alpha,\beta,\ldots,\gamma\in A,\quad \mathbf{a}\in A^m. \tag{91}$$

The mth direct power becomes a binary semigroup by endowing m-tuples with the componentwise binary operation $(\hat{+})$ as

$$\mathbf{a}_1 \hat{+} \mathbf{a}_2 = \left(\overbrace{\alpha_1, \beta_1, \ldots, \gamma_1}^{m}\right) \hat{+} \left(\overbrace{\alpha_2, \beta_2, \ldots, \gamma_2}^{m}\right) = \left(\overbrace{\alpha_1 + \alpha_2, \beta_1 + \beta_2, \ldots, \gamma_1 + \gamma_2}^{m}\right). \tag{92}$$

The derived n-ary operation for m-tuples (on the mth direct power) is then defined componentwise by analogy with (90)

$$\circ^{[n]}[\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n] = \mathbf{a}_1 \hat{+} \mathbf{a}_2 \hat{+} \ldots \hat{+} \mathbf{a}_n. \tag{93}$$

Now using shifts, instead of (93) we construct a nonderived n-ary operation on the direct power.

Definition 7. *A cyclic m-shift operator s is defined for the m-tuple (91) by*

$$\overbrace{\mathsf{s}\alpha = \beta, \mathsf{s}\beta = \gamma, \ \ldots \ , \mathsf{s}\gamma = \alpha}^{m}, \tag{94}$$

and $\mathsf{s}^m = \mathsf{id}$.

For instance, in this notation, if $m = 3$ and $\mathbf{a} = (\alpha, \beta, \gamma)$, then $\mathsf{s}\mathbf{a} = (\gamma, \alpha, \beta)$, $\mathsf{s}^2\mathbf{a} = (\beta, \gamma, \alpha)$, $\mathsf{s}^3\mathbf{a} = \mathbf{a}$ (as in the previous subsection).

To obtain a nonderived n-ary operation, by analogy with (87), we deform by shifts the derived n-ary operation (93).

Definition 8. *The shift deformation by (94) of the derived operation $\circ^{[n]}$ on the direct power $\mathcal{A}^m$ is defined noncomponentwise by*

$$\circ_{\mathsf{s}}^{[n]}[\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n] = \sum_{i=1}^{n} \mathsf{s}^{i-1}\mathbf{a}_i = \mathbf{a}_1 \hat{+} \mathsf{s}\mathbf{a}_2 \hat{+} \ldots \hat{+} \mathsf{s}^{n-1}\mathbf{a}_n, \tag{95}$$

where $\mathbf{a} \in A^m$ *(91) and* $\mathsf{s}^0 = \mathsf{id}$.

Note that till now there exist no relations between n and m.

Proposition 5. *The shift deformed operation $\nu_{\mathsf{s}}^{[n]}$ is totally associative, if*

$$\mathsf{s}^{n-1} = \mathsf{id}, \tag{96}$$

$$m = n - 1. \tag{97}$$

Proof. We compute

$$\begin{aligned} &\circ_{\mathsf{s}}^{[n]}\left[\circ_{\mathsf{s}}^{[n]}[\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n], \mathbf{a}_{n+1}, \mathbf{a}_{n+2}, \ldots, \mathbf{a}_{2n-1}\right] \\ &= \left(\mathbf{a}_1 \hat{+} \mathsf{s}\mathbf{a}_2 \hat{+} \ldots \hat{+} \mathsf{s}^{n-1}\mathbf{a}_n\right) \hat{+} \mathsf{s}\mathbf{a}_{n+1} \hat{+} \mathsf{s}^2\mathbf{a}_{n+2} \hat{+} \ldots \hat{+} \mathsf{s}^{n-1}\mathbf{a}_{2n-1} \\ &= \mathbf{a}_1 \hat{+} \mathsf{s}\left(\mathbf{a}_2 \hat{+} \mathsf{s}\mathbf{a}_3 \hat{+} \ldots \hat{+} \mathsf{s}^{n-1}\mathbf{a}_{n+1}\right) \hat{+} \mathsf{s}^2\mathbf{a}_{n+2} \hat{+} \mathsf{s}^3\mathbf{a}_{n+3} \hat{+} \ldots \hat{+} \mathsf{s}^{n-1}\mathbf{a}_{2n-1} \\ &\vdots \\ &\mathbf{a}_1 \hat{+} \mathsf{s}\mathbf{a}_2 \hat{+} \ldots \hat{+} \mathsf{s}^{n-2}\mathbf{a}_n \hat{+} \mathsf{s}^{n-1}\left(\mathbf{a}_{n+1} \hat{+} \mathsf{s}\mathbf{a}_{n+2} \hat{+} \mathsf{s}^2\mathbf{a}_{n+3} \hat{+} \ldots \hat{+} \mathsf{s}^{n-1}\mathbf{a}_{2n-1}\right) \\ &\circ_{\mathsf{s}}^{[n]}\left[\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-1}, \circ_{\mathsf{s}}^{[n]}[\mathbf{a}_n, \mathbf{a}_{n+1}, \mathbf{a}_{n+2}, \ldots, \mathbf{a}_{2n-1}]\right], \end{aligned} \tag{98}$$

which are satisfied in all lines, if $\mathsf{s}^{n-1} = \mathsf{id}$ (96). □

Corollary 2. *The set of $(n-1)$-tuples (91) with the shift deformed associative operation (95) is a nonderived n-ary semigroup $\mathcal{S}^{[n]}_{shift} = \left\langle \{\mathbf{a}\} \mid \overset{\circ}{\mathsf{s}}{}^{[n]} \right\rangle$ constructed from the binary semigroup $\mathcal{A}$.*

Proposition 6. *If the binary semigroup $\mathcal{A}$ is commutative, then $\mathcal{S}^{[n]}_{shift}$ becomes a nonderived n-ary group $\mathcal{G}^{[n]}_{shift} = \left\langle \{\mathbf{a}\} \mid \overset{\circ}{\mathsf{s}}{}^{[n]}, \overset{\bar{\circ}}{\mathsf{s}}{}^{[1]} \right\rangle$, such that each element $\mathbf{a} \in \mathcal{A}^{n-1}$ has a unique querelement $\bar{\mathbf{a}}$ (an analog of inverse) by*

$$\bar{\mathbf{a}} = \overset{\bar{\circ}}{\mathsf{s}}{}^{[1]}[\mathbf{a}] = -\left(\mathsf{s}\mathbf{a} \hat{+} \mathsf{s}^2\mathbf{a} \hat{+} \ldots \hat{+} \mathsf{s}^{n-2}\mathbf{a}\right), \tag{99}$$

where $\overset{\bar{\circ}}{\mathsf{s}}{}^{[1]} : \mathcal{A}^{n-1} \to \mathcal{A}^{n-1}$ is an unary queroperation.

Proof. We have the definition of the querelement

$$\overset{\circ}{\mathsf{s}}{}^{[n]}[\bar{\mathbf{a}}, \mathbf{a}, \ldots, \mathbf{a}] = \mathbf{a}, \tag{100}$$

where $\bar{\mathbf{a}}$ can be on any place. Thus, (95) gives the equation

$$\bar{\mathbf{a}} \hat{+} \mathsf{s}\mathbf{a} \hat{+} \mathsf{s}^2\mathbf{a} \hat{+} \ldots \hat{+} \mathsf{s}^{n-2}\mathbf{a} \hat{+} \mathbf{a} = \mathbf{a}, \tag{101}$$

which can be resolved for the commutative and cancellative semigroup $\mathcal{A}$ only, and the solution is (99). If $\bar{\mathbf{a}}$ is on the ith place in (100), then it has the coefficient s^{i-1}, and we multiply both sides by s^{n-i} to get $\bar{\mathbf{a}}$ without any shift operator coefficient using (96), which gives the same solution (99). □

For $n = 4$ and $\mathbf{a} = (\alpha, \beta, \gamma)$, the equation (100) is

$$\bar{\mathbf{a}} \hat{+} \mathsf{s}\mathbf{a} \hat{+} \mathsf{s}^2\mathbf{a} \hat{+} \mathbf{a} = \mathbf{a} \tag{102}$$

and (see (86))

$$\bar{\mathbf{a}} = -\left(\mathsf{s}\mathbf{a} \hat{+} \mathsf{s}^2\mathbf{a}\right) \tag{103}$$

so (cf. (80))

$$\bar{\mathbf{a}} = \overline{(\alpha, \beta, \gamma)} = -(\gamma + \beta, \alpha + \gamma, \beta + \alpha). \tag{104}$$

It is known that the existence of an identity (as a neutral element) is not necessary for polyadic groups, and only a querelement is important [26,27]. Nevertheless, we have

Proposition 7. *If the commutative and cancellative semigroup $\mathcal{A}$ has zero $0 \in A$, then the n-ary group $\mathcal{G}^{[n]}_{shift}$ has a set of polyadic identities (idempotents) satisfying*

$$\mathbf{e} \hat{+} \mathsf{s}\mathbf{e} \hat{+} \ldots \hat{+} \mathsf{s}^{n-2}\mathbf{e} = \mathbf{0}, \tag{105}$$

where $\mathbf{0} = \left(\overbrace{0, 0, \ldots, 0}^{n-1}\right)$ is the zero $(n-1)$-tuple.

Proof. The definition of polyadic identity in terms of the deformed n-ary product in the direct power is

$$\overset{\circ}{\mathsf{s}}{}^{[n]}\left[\overbrace{\mathbf{e}, \mathbf{e}, \ldots, \mathbf{e}}^{n-1}, \mathbf{a}\right] = \mathbf{a}, \quad \forall \mathbf{a} \in \mathcal{A}^{n-1}. \tag{106}$$

Using (95) we get the equation

$$\mathbf{e} \hat{+} \mathsf{s}\mathbf{e} \hat{+} \mathsf{s}^2\mathbf{e} \hat{+} \ldots \hat{+} \mathsf{s}^{n-2}\mathbf{e} \hat{+} \mathbf{a} = \mathbf{a}. \tag{107}$$

After cancellation by $\mathbf{a}$, we obtain (105). □

For $n = 4$ and $\mathbf{e} = (\alpha_0, \beta_0, \gamma_0)$, we obtain an infinite set of identities satisfying

$$\mathbf{e} = (\alpha_0, \beta_0, \gamma_0), \quad \alpha_0 + \beta_0 + \gamma_0 = 0. \tag{108}$$

To see that they are 4-ary idempotents, insert $\mathbf{a} = \mathbf{e}$ into (106).

Thus, starting from a binary semigroup $\mathcal{A}$, using our polyadization procedure, we have obtained a nonderived n-ary group on $(n-1)$th direct power $\mathcal{A}^{n-1}$ with the shift deformed multiplication. This construction draws on the post-like associative quiver from [11,22], and allows us to construct a nonderived n-ary group from any semigroup in the unified way presented here.

4.4. Polyadization of Binary Supergroups

Here we consider a more exotic possibility, when the B-matrices are defined over the Grassmann algebra, and therefore can represent supergroups (see (37) and below). In this case, Bs can be supermatrices of two kinds, even and odd, which have different properties [36,37]. The general polyadization procedure remains the same as for the ordinary matrices considered before (see Definition 2), and therefore we confine ourselves to examples only.

Indeed, to obtain an n-ary matrix (semi)group represented now by the Q-supermatrices (38) of the nonstandard form, we should take $(n-1)$ initial B-supermatrices which present a simple ($k = 1$ in (20)) binary (semi)supergroup, which now have different parameters $\mathrm{B}_{\mathbf{y}_i} \equiv \mathrm{B}_{\mathbf{y}_i}((p_{even} \mid p_{odd}) \times (p_{even} \mid p_{odd}))$, $i = 1, \ldots, n-1$, where p_{even} and p_{odd} are even and odd dimensions of the B-supermatrix. The closure of the Q-supermatrix multiplication is governed by the closure of B-supermatrix multiplication (40)–(42) in the initial binary (semi)supergroup.

Polyadization of $GL(1 \mid 1, \Lambda)$

Let $\Lambda = \Lambda_{even} \oplus \Lambda_{odd}$ be a Grassmann algebra over $\mathbb{C}$, where Λ_{even} and Λ_{odd} are its even and odd parts (it can be also any commutative superalgebra). We provide (in brief) the polyadization procedure of the general linear supergroup $GL(1 \mid 1, \Lambda)$ for $n = 3$. The 4-parameter block (invertible) supermatrices become $\mathrm{B}_{\mathbf{y}_i} = \begin{pmatrix} a_i & \alpha_i \\ \beta_i & b_i \end{pmatrix} \in GL(1 \mid 1, \Lambda)$, where the parameters are $\mathbf{y}_i = (a_i, b_i, \alpha_i, \beta_i) \in \Lambda_{even} \times \Lambda_{even} \times \Lambda_{odd} \times \Lambda_{odd}$, $i = 1, 2$. Thus, the 8-parameter ternary supergroup $GL^{[[3]]}(1 \mid 1, \Lambda) = \left\langle \{\mathrm{Q}_{\mathbf{y}_1,\mathbf{y}_2}\} \mid \mu^{[[3]]} \right\rangle$ is represented by the following 4×4 Q-supermatrices:

$$\mathrm{Q}_{\mathbf{y}_1,\mathbf{y}_2} = \begin{pmatrix} 0 & \mathrm{B}_{\mathbf{y}_1} \\ \mathrm{B}_{\mathbf{y}_2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_1 & \alpha_1 \\ 0 & 0 & \beta_1 & b_1 \\ a_2 & \alpha_2 & 0 & 0 \\ \beta_2 & b_2 & 0 & 0 \end{pmatrix} \in GL^{[[3]]}(1 \mid 1, \Lambda), \tag{109}$$

which satisfy the ternary (nonderived) multiplication

$$\mu^{[[3]]}\left[\mathrm{Q}_{\mathbf{y}'_1,\mathbf{y}'_2}, \mathrm{Q}_{\mathbf{y}''_1,\mathbf{y}''_2}, \mathrm{Q}_{\mathbf{y}'''_1,\mathbf{y}'''_2}\right] = \mathrm{Q}_{\mathbf{y}'_1,\mathbf{y}'_2}\mathrm{Q}_{\mathbf{y}''_1,\mathbf{y}''_2}\mathrm{Q}_{\mathbf{y}'''_1,\mathbf{y}'''_2} = \mathrm{Q}_{\mathbf{y}_1,\mathbf{y}_2}. \tag{110}$$

In terms of the block matrices $\mathrm{B}_{\mathbf{y}_i}$, the multiplication (54) becomes (see (39)–(42))

$$\mathrm{B}_{\mathbf{y}'_1}\mathrm{B}_{\mathbf{y}''_2}\mathrm{B}_{\mathbf{y}'''_1} = \mathrm{B}_{\mathbf{y}_1}, \tag{111}$$

$$\mathrm{B}_{\mathbf{y}'_2}\mathrm{B}_{\mathbf{y}'''_1}\mathrm{B}_{\mathbf{y}'''_2} = \mathrm{B}_{\mathbf{y}_2}. \tag{112}$$

In terms of the B-supermatrix parameters, the supergroup $GL^{[[3]]}(1 \mid 1, \Lambda)$ is defined by the following component ternary multiplication

$$
\begin{aligned}
\alpha_1'\beta_2''a_1''' + a_1'\alpha_2''\beta_1''' + \alpha_1'b_2''\beta_1''' + a_1'a_2''a_1''' &= a_1, \;\; \beta_1'a_2''\alpha_1''' + \beta_1'\alpha_2''b_1''' + b_1'\beta_2''\alpha_1''' + b_1'b_2''b_1''' = b_1, \\
\alpha_1'\beta_2''\alpha_1''' + a_1'a_2''\alpha_1''' + a_1'\alpha_2''b_1''' + \alpha_1'b_2''b_1''' &= \alpha_1, \;\; \beta_1'\alpha_2''\beta_1''' + \beta_1'a_2''a_1''' + b_1'\beta_2''a_1''' + b_1'b_2''\beta_1''' = \beta_1, \\
\alpha_2'\beta_1''a_2''' + a_2'\alpha_1''\beta_2''' + \alpha_2'b_1''\beta_2''' + a_2'a_1''a_2''' &= a_2, \;\; \beta_2'a_1''\alpha_2''' + \beta_2'\alpha_1''b_2''' + b_2'\beta_1''\alpha_2''' + b_2'b_1''b_2''' = b_2, \\
\alpha_2'\beta_1''\alpha_2''' + a_2'a_1''\alpha_2''' + a_2'\alpha_1''b_2''' + \alpha_2'b_1''b_2''' &= \alpha_2, \;\; \beta_2'\alpha_1''\beta_2''' + \beta_2'a_1''a_2''' + b_2'\beta_1''a_2''' + b_2'b_1''\beta_2''' = \beta_2.
\end{aligned}
\tag{113}
$$

The unique querelement in $GL^{[[3]]}(1 \mid 1, \Lambda)$ can be found from the equation (see (48)):

$$\mathrm{Q}_{\mathbf{y}_1,\mathbf{y}_2}\mathrm{Q}_{\mathbf{y}_1,\mathbf{y}_2}\overline{\mathrm{Q}}_{\mathbf{y}_1,\mathbf{y}_2} = \mathrm{Q}_{\mathbf{y}_1,\mathbf{y}_2}, \tag{114}$$

where the solution is

$$\overline{\mathrm{Q}}_{\mathbf{y}_1,\mathbf{y}_2} = \begin{pmatrix} 0 & \overline{\mathrm{B}}_{\mathbf{y}_1} \\ \overline{\mathrm{B}}_{\mathbf{y}_2} & 0 \end{pmatrix}, \tag{115}$$

with (see (49))

$$\overline{\mathrm{B}}_{\mathbf{y}_1} = \mathrm{B}_{\mathbf{y}_2}^{-1}, \quad \overline{\mathrm{B}}_{\mathbf{y}_2} = \mathrm{B}_{\mathbf{y}_1}^{-1}, \tag{116}$$

and $\mathrm{B}_{\mathbf{y}_1}^{-1}, \mathrm{B}_{\mathbf{y}_2}^{-1} \in GL(1 \mid 1, \Lambda)$.

Definition 9. *We call $GL^{[[3]]}(1 \mid 1, \Lambda)$ a polyadic (ternary) general linear supergroup obtained by the polyadization procedure from the binary linear supergroup $GL(1 \mid 1, \Lambda)$.*

The ternary identity $\mathrm{E}_4^{(3)}$ of $GL^{[[3]]}(1 \mid 1, \Lambda)$ has the form (see (46))

$$\mathrm{E}_4^{(3)} = \begin{pmatrix} 0 & \mathrm{E}_2 \\ \mathrm{E}_2 & 0 \end{pmatrix}, \tag{117}$$

where E_2 is the identity of $GL(1 \mid 1, \Lambda)$, and it is ternary idempotent:

$$\mathrm{E}_4^{(3)}\mathrm{E}_4^{(3)}\mathrm{E}_4^{(3)} = \mathrm{E}_4^{(3)}. \tag{118}$$

The ternary supergroup $GL^{[[3]]}(1 \mid 1, \Lambda)$ contains the infinite number of ternary idempotents $\mathrm{Q}_{\mathbf{y}_1,\mathbf{y}_2}^{idemp}$ defined by the system of equations

$$\mathrm{Q}_{\mathbf{y}_1,\mathbf{y}_2}^{idemp}\mathrm{Q}_{\mathbf{y}_1,\mathbf{y}_2}^{idemp}\mathrm{Q}_{\mathbf{y}_1,\mathbf{y}_2}^{idemp} = \mathrm{Q}_{\mathbf{y}_1,\mathbf{y}_2}^{idemp}, \tag{119}$$

which gives

$$\mathrm{B}_{\mathbf{y}_1}^{idemp}\mathrm{B}_{\mathbf{y}_2}^{idemp} = \mathrm{E}_2. \tag{120}$$

Therefore, the idempotents are determined by $8 - 4 = 4$ Grassmann parameters. One of the ways to realize this is to exclude from (120) the 2×2 B-supermatrix. In this case, the idempotents in the supergroup $GL^{[[3]]}(1 \mid 1, \Lambda)$ become

$$\mathrm{Q}_{\mathbf{y}_1,\mathbf{y}_2}^{idemp} = \begin{pmatrix} 0 & \mathrm{B}_{\mathbf{y}_1} \\ \left(\mathrm{B}_{\mathbf{y}_1}\right)^{-1} & 0 \end{pmatrix}, \tag{121}$$

where $\mathrm{B}_{\mathbf{y}_i} \in GL(1 \mid 1, \Lambda)$ is an invertible 2×2 supermatrix of the standard form (see Remark 5).

In the same way one, can polyadize any supergroup that can be presented by supermatrices.

5. Conclusions

In this paper we have given answers to the following important questions: how can one obtain nonderived polyadic structures from binary ones, and what would be a matrix form of their semisimple versions? First, we introduced a general matrix form for polyadic structures in terms of block-shift matrices. If the blocks correspond to a binary structure (a ring, semigroup, group, or supergroup), this can be treated as a polyadization procedure for them. Second, the semisimple blocks which further have a block-diagonal form give rise to semisimple nonderived polyadic structures. For a deeper and expanded understanding of the new constructions introduced, we have given clarifying examples. The polyadic structures presented can be used, e.g., for the further development of differential geometry and operad theory, as well as in other directions which use higher arity and nontrivial properties of the constituent universal objects.

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