

# POLYADIC SUPERSYMMETRY

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**ABSTRACT.** We introduce a polyadic analog of supersymmetry by considering the polyadization procedure (proposed by the author) applied to the toy model of one-dimensional supersymmetric quantum mechanics. The supercharges are generalized to polyadic ones using the  $n$ -ary sigma matrices defined in earlier work. In this way, polyadic analogs of supercharges and Hamiltonians take the cyclic shift block matrix form, and they can describe multidegenerated quantum states in a way that is different from the  $N$ -extended and multigraded SQM. While constructing the corresponding supersymmetry as an  $n$ -ary Lie superalgebra ( $n$  is the arity of the initial associative multiplication), we have found new brackets with a reduced arity of  $2 \leq m < n$  and a related series of  $m$ -ary superalgebras (which is impossible for binary superalgebras). In the case of even reduced arity  $m$  we obtain a tower of higher order (as differential operators) even Hamiltonians, while for  $m$  odd we get a tower of higher order odd supercharges, and the corresponding algebra consists of the odd sector only.

## CONTENTS

1. INTRODUCTION	2
2. POLYADIC SIGMA MATRICES	2
3. GENERAL SCHEME	4
3.1. Standard binary SQM	4
3.2. Polyadic superalgebra with reduced arity brackets	5
3.3. Algebras of polyadic supercharges	6
3.4. SQM with reduced arity binary bracket	8
4. SQM FROM TERNARY SUPERALGEBRA	11
5. SQM FROM 4-ARY SUPERALGEBRA	12
REFERENCES	13

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## 1. INTRODUCTION

The most natural way to introduce new symmetries and to investigate their initial properties is by considering toy physical models. In the case of supersymmetry, such a model is supersymmetric quantum mechanics (SQM), for a review, see [COOPER ET AL. \[2001\]](#), [JUNKER \[1996\]](#), [GANGOPADHYAYA ET AL. \[2017\]](#). Its simplest one-dimensional  $N = 2$  version was considered in [WITTEN \[1981\]](#), and then numerous generalizations were proposed, such as, e.g., higher  $N$  SQM [AKULOV AND KUDINOV \[1999\]](#), [PASHNEV \[1986\]](#), higher order SQM [KHARE \[1992\]](#), [ROBERT AND SMILGA \[2008\]](#), [FERNÁNDEZ C. AND FERNÁNDEZ-GARCÍA \[2004\]](#), parasupersymmetric quantum mechanics [BECKERS AND DEBERGH \[1991\]](#), [RUBAKOV AND SPIRIDONOV \[1988\]](#), and multigraded SQM [BRUCE AND DUPLIJ \[2020\]](#), [TOPPAN \[2021\]](#), [AIZAWA ET AL. \[2021\]](#).

From the other side, polyadic (or higher arity) algebraic structures (for a mathematical review, see [DUPLIJ \[2022\]](#) and refs. therein) appeared in some physical applications, e.g., [KERNER \[2000\]](#), [CASTRO \[2010\]](#), including supersymmetry [BARS AND GÜNAYDIN \[1979\]](#), [BAGGER AND LAMBERT \[2008\]](#). In addition, independently of any models, the polyadic analog of sigma matrices ( $\sigma$ -matrices, or Pauli matrices) of cyclic shift shape was proposed in [DUPLIJ \[2024\]](#).

In this paper we use the polyadic (nonderived  $n$ -ary) sigma matrices to construct the corresponding polyadic analogs of supercharges and Hamiltonians of cyclic shift block matrix form which can describe multi-degenerated quantum states in a special way that is different from  $N$ -extended and multigraded SQM. While it is common to endow the  $n$ -ary associative algebra of generators with an  $n$ -ary Lie superbracket to obtain an  $n$ -ary Lie superalgebra, we have constructed (using polyadic units) a series of additional superbrackets having reduced arities  $2 \leq m < n$  and their related  $m$ -ary superalgebras. The cases of even and odd reduced arities  $m$  are significantly different: in the former we have the tower of higher order (as differential operators) even Hamiltonians, while in the latter case we get higher order odd supercharges and no even elements at all. This shows that polyadic supersymmetry has a non-trivially rich and complicated structure even in the simplest example of the SQM model.

## 2. POLYADIC SIGMA MATRICES

Polyadic sigma matrices were introduced in [DUPLIJ \[2024\]](#) by using the polyadization procedure proposed in [DUPLIJ \[2022\]](#). In explicit form the full polyadic  $\Sigma$ -matrices over  $\mathbb{C}$  of size  $2(n-1) \times 2(n-1)$  are

$$\Sigma_j = \Sigma_j^{[n]} = \begin{pmatrix} 0 & \boxed{\sigma_j^{(1)}} & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \boxed{\sigma_j^{(2)}} & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \boxed{\sigma_j^{(k)}} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \dots & \boxed{\sigma_j^{(n-2)}} \\ \boxed{\sigma_j^{(n-1)}} & 0 & 0 & \dots & 0 & \dots & 0 \end{pmatrix}, \quad j = 0, 1, 2, 3, \quad (2.1)$$

where  $\sigma_j$  are sigma matrices (Pauli matrices with  $\sigma_0 = I_2$ ) and 0 is the  $2 \times 2$  zero matrix, such that  $\Sigma_j^{[n]} \in GL(2(n-1), \mathbb{C})$  and  $\det \Sigma_{1,2,3}^{[n]} = (-1)^n$ ,  $\det \Sigma_0^{[n]} = 1$ . The set  $\{\mathbf{M}_{shift}\}$  of matrices of the

general shape (2.1), cyclic shift block matrices, with invertible  $GL(2, \mathbb{C})$  blocks, form a nonderived  $n$ -ary group  $\mathcal{G}_{shift}^{[n]}$ , because the multiplication of  $n$  matrices (only, and not fewer) is closed. We define

$$\boldsymbol{\mu}^{[n]} [M_1, M_2, \dots, M_n] = M_1 \cdot M_2 \cdot \dots \cdot M_n, \quad M_j \in \mathcal{G}_{shift}^{[n]} = \langle \{M_{shift}\} \mid \boldsymbol{\mu}^{[n]}, \overline{(\cdot)} \rangle, \quad (2.2)$$

where  $(\cdot)$  is the ordinary matrix product, and  $\overline{(\cdot)}$  is the querelement. Moreover, the  $n$ -ary product (2.2) is defined only for the product of  $r(n-1) + 1$  matrices, where  $n \geq 3$  and  $r \in \mathbb{N}$  is the polyadic power. In this notation  $\boldsymbol{\mu}^{[2]}$  coincides with the ordinary binary multiplication, which allows an arbitrary number of multipliers (see DUPLIJ [2022] and DUPLIJ [2024] for details).

In this notation  $\Sigma_0 = \Sigma_0^{[n]} = \mathbf{E}$  is the polyadic unit of the group  $\mathcal{G}_{shift}^{[n]}$  satisfying

$$\boldsymbol{\mu}^{[n]} [\mathbf{E}, \mathbf{E}, \dots, M] = M, \quad (2.3)$$

where  $M$  in the l.h.s. can be on any place, and

$$\Sigma_0^{[n]} = \mathbf{E} = \begin{pmatrix} 0 & I_2 & \dots & 0 & 0 \\ 0 & 0 & I_2 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & I_2 \\ I_2 & 0 & \dots & 0 & 0 \end{pmatrix} \in \mathcal{G}_{shift}^{[n]} \neq \mathbf{I} = \begin{pmatrix} I_2 & 0 & \dots & 0 & 0 \\ 0 & I_2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & I_2 & 0 \\ 0 & 0 & \dots & 0 & I_2 \end{pmatrix} \notin \mathcal{G}_{shift}^{[n]}. \quad (2.4)$$

The corresponding associative (because the ordinary matrix product (2.2) is such)  $n$ -ary algebra  $\mathcal{A}_{shift}^{[n]}$  consists of the cyclic shift matrices of the general form (2.1) and the  $n$ -ary multiplication (2.2) together with the (binary) matrix addition and multiplication by scalars (as for ordinary matrices)

$$\mathcal{A}_{shift}^{[n]} = \langle \{M_{shift}\} \mid \boldsymbol{\mu}^{[n]}, (+), \overline{(\cdot)} \rangle. \quad (2.5)$$

In general, only the product of  $r(n-1) + 1$  matrices, for  $r \in \mathbb{N}$ , is defined. For instance, the Cayley table of the full ternary  $\Sigma$ -matrices is (recall that multiplication of two  $\Sigma_j = \Sigma_j^{[3]}$  is not closed)

$$\boldsymbol{\mu}^{[3]} [\Sigma_k, \Sigma_l, \Sigma_m] = \delta_{kl}\Sigma_m - \delta_{km}\Sigma_l + \delta_{lm}\Sigma_k + i\epsilon_{klm}\Sigma_0, \quad (2.6)$$

$$\boldsymbol{\mu}^{[3]} [\Sigma_k, \Sigma_l, \Sigma_0] = \boldsymbol{\mu}^{[3]} [\Sigma_k, \Sigma_0, \Sigma_l] = \boldsymbol{\mu}^{[3]} [\Sigma_0, \Sigma_k, \Sigma_l] = \delta_{kl}\Sigma_0 + i\epsilon_{klm}\Sigma_m, \quad (2.7)$$

$$\boldsymbol{\mu}^{[3]} [\Sigma_k, \Sigma_0, \Sigma_0] = \boldsymbol{\mu}^{[3]} [\Sigma_0, \Sigma_k, \Sigma_0] = \boldsymbol{\mu}^{[3]} [\Sigma_0, \Sigma_0, \Sigma_k] = \Sigma_k, \quad k, l, m = 1, 2, 3, \quad (2.8)$$

where we have to include products with  $\Sigma_0$ , because of (2.4).

We get, for full ternary  $\Sigma$ -matrices, the ternary commutators

$$\begin{aligned} [M_1, M_2, M_3]^{[3]} &= \boldsymbol{\mu}^{[3]} [M_1, M_2, M_3] + \boldsymbol{\mu}^{[3]} [M_3, M_1, M_2] + \boldsymbol{\mu}^{[3]} [M_2, M_3, M_1] \\ &\quad - \boldsymbol{\mu}^{[3]} [M_1, M_3, M_2] - \boldsymbol{\mu}^{[3]} [M_3, M_2, M_1] - \boldsymbol{\mu}^{[3]} [M_2, M_1, M_3], \end{aligned} \quad (2.9)$$

as

$$[\boldsymbol{\mu}^{[3]} [\Sigma_k, \Sigma_l, \Sigma_m]]^{[3]} = 6i\epsilon_{klm}\Sigma_0, \quad k, l, m = 1, 2, 3, \quad (2.10)$$

and standard ternary anticommutators  $\{-\}^{[3]}$  (all  $+$ 's in the r.h.s. of (2.9))

$$\{\boldsymbol{\mu}^{[3]} [\Sigma_k, \Sigma_l, \Sigma_m]\}^{[3]} = 2\delta_{kl}\Sigma_m + 2\delta_{km}\Sigma_l + 2\delta_{lm}\Sigma_k. \quad (2.11)$$

Below we will omit  $\boldsymbol{\mu}^{[n]}$ , if it will be clear from the context. Further properties of general  $\Sigma$ -matrices can be found in DUPLIJ [2024].

## 3. GENERAL SCHEME

One-dimensional supersymmetric quantum mechanics (SQM) is the simplest model which has dynamical supersymmetry [WITTEN](#) [1981]. This means that there exist transformations converting ‘‘bosons’’ and ‘‘fermions’’, the Hamiltonian is among the generators, and the symmetry algebra, in addition to commutators (Lie algebra), contains anticommutators, thus becoming its graded version: a Lie superalgebra (for a review, see [KAC](#) [1977]).

**3.1. Standard binary SQM.** Recall, just to introduce notation and to show the relations to be generalized polyadically, that in the binary case (where the arity of multiplication is  $n = 2$ ), informally, the transition from the associative superalgebra  $\mathcal{A}^{[2]} = \langle A \mid \mu^{[2]} = (\cdot), s\text{-assoc} \rangle$  to the Lie superalgebra  $\mathcal{A}_{sLie}^{[2]} = \langle A \mid \mathcal{L}^{[2]}, s\text{-Jacobi} \rangle$  can be done by replacing the binary multiplication  $\mu^{[2]}$  by the binary Lie superbracket ( $\mathbb{Z}_2$ -graded commutator)

$$\mathcal{L}^{[2]} [a_1, a_2] = \mu^{[2]} [a_1, a_2] - (-1)^{\pi(a_1)\pi(a_2)} \mu^{[2]} [a_2, a_1], \quad a_j \in A, \quad (3.1)$$

and the graded associativity by the super Jacobi identity (over the same underlying set, a graded linear vector space  $A = A_0 \oplus A_1$ ), where the parity (even, odd) is  $\pi(a) = 0, 1 \in \mathbb{Z}_2$ , with  $a \in A_{0,1}$ . The superbracket (3.1) satisfies the Lie-anticommutation relation

$$\mathcal{L}^{[2]} [a_1, a_2] = -(-1)^{\pi(a_1)\pi(a_2)} \mathcal{L}^{[2]} [a_2, a_1]. \quad (3.2)$$

In this way, the 1D and  $N = 2$  SQM (with 2 odd supercharges  $Q_{1,2}$  and the even Hamiltonian  $H$ ) has a symmetry which is defined by the following Lie superalgebra  $\mathfrak{osp}(1 \mid 2)$  relations

$$\mathcal{L}^{[2]} [Q_j, Q_k] = \{Q_j, Q_k\}^{[2]} = \{Q_j, Q_k\} = 2\delta_{jk}H, \quad (3.3)$$

$$\mathcal{L}^{[2]} [H, Q_k] = [H, Q_k]^{[2]} = [H, Q_k] = 0, \quad j, k = 1, 2, \quad (3.4)$$

where  $[-, -]^{[2]}$  and  $\{-, -\}^{[2]}$  are the binary commutator and anticommutator, respectively. In a matrix representation (quantization) the supercharges and the Hamiltonian (as operators in the two-dimensional Hilbert superspace) have the manifest form

$$\hat{Q}_1 = \frac{1}{\sqrt{2}} (\sigma_1 \hat{p} + \sigma_2 W(x)), \quad (3.5)$$

$$\hat{Q}_2 = \frac{1}{\sqrt{2}} (\sigma_2 \hat{p} - \sigma_1 W(x)), \quad (3.6)$$

$$\hat{H} = \hat{H}_{Witten} = \frac{\sigma_0}{2} (\hat{p}^2 + W^2(x)) + \frac{\sigma_3}{2} W'(x), \quad \hat{p} = -i \frac{d}{dx}, \quad (3.7)$$

where  $W(x)$  is a superpotential (an arbitrary even complex analytic function of  $x$ ), and the parities are

$$\pi(\hat{Q}_{1,2}) = 1, \pi(\hat{H}) = 0, \pi(\sigma_{1,2}) = 1, \pi(\sigma_{0,3}) = 0, \pi(\hat{p}) = 0. \quad (3.8)$$

The fermionic charge  $\hat{F}$  (an even operator) satisfies (in this presentation)

$$[\hat{F}, \hat{Q}_1] = i\hat{Q}_2, \quad [\hat{F}, \hat{Q}_2] = -i\hat{Q}_1, \quad [\hat{F}, \hat{H}] = 0, \quad \hat{F} = \frac{1}{2}\sigma_3. \quad (3.9)$$

For further details, see [COOPER ET AL.](#) [2001], [JUNKER](#) [1996].

**3.2. Polyadic superalgebra with reduced arity brackets.** We now propose a generalization of the supercharges (3.5)–(3.6) by formal substitution of sigma matrices  $\sigma_j$  by the corresponding polyadic sigma matrices  $\Sigma_j^{[n]}$  (2.1). This means that the Hilbert superspace becomes  $2(n-1)$ -dimensional. In this way, two consequences can occur:

- 1) Mathematical: a special polyadic analog of Lie superalgebra (with reduced arity brackets) can be defined.
- 2) Physical: a polyadic analog of even Hamiltonians with higher order derivatives and higher order odd supercharges can appear, closing the algebra, which means that the polyadic dynamics can be richer and more prosperous.

Because polyadic sigma matrices obey a nonderived  $n$ -ary multiplication, any variables constructed from them (linearly) form a nonderived  $n$ -ary algebra (using polyadic distributivity). After endowing them parities similar to  $\sigma$ -matrices (3.8), this algebra becomes the totally associative  $n$ -ary superalgebra  $\mathcal{A}^{[n]} = \langle \mathbf{A} \mid \mu^{[n]}, n\text{-}s\text{-}assoc \rangle$  satisfying  $n$ -ary  $\mathbb{Z}_2$ -graded commutativity

$$\mu^{[n]} [a_1, a_2, \dots, a_{j-1}, a_j, \dots, a_n] = (-1)^{\pi(a_{j-1})\pi(a_j)} \mu^{[n]} [a_1, a_2, \dots, a_j, a_{j-1}, \dots, a_n], \quad (3.10)$$

where the parity is  $\pi(a_j) = 0, 1$ ,  $a_j \in \mathbf{A}_{0,1}$ , and has total  $n$ -ary associativity (for further definitions and review of polyadic structures, see DUPLIJ [2022]).

To obtain the  $n$ -ary Lie superalgebra we exchange (as in the binary case) the  $n$ -ary multiplication  $\mu^{[n]}$  for the  $n$ -ary Lie superbracket ( $n$ -ary  $\mathbb{Z}_2$ -graded commutator)  $\mathcal{L}^{[n]} [a_1, a_2, \dots, a_n]$  (of the same arity  $n$  as the initial multiplication), and graded associativity for the  $n$ -ary super Jacobi identity (over the same underlying set, a graded linear vector space  $\mathbf{A} = \mathbf{A}_0 \oplus \mathbf{A}_1$ ) to get

$$\mathcal{A}_{sLie}^{[n]} = \langle \mathbf{A} \mid \mathcal{L}^{[n]}, n\text{-}s\text{-}Jacobi \rangle. \quad (3.11)$$

The  $n$ -ary Lie superbracket  $\mathcal{L}^{[n]}$  satisfies (on homogeneous elements) the following  $n$ -ary anticommutation relation

$$\mathcal{L}^{[n]} [a_1, a_2, \dots, a_{j-1}, a_j, \dots, a_n] = -(-1)^{\pi(a_{j-1})\pi(a_j)} \mathcal{L}^{[n]} [a_1, a_2, \dots, a_j, a_{j-1}, \dots, a_n], \quad (3.12)$$

and the  $n$ -ary super Jacobi identity (see, e.g., POJIDAEV [2003], BARREIRO ET AL. [2019]). In the limiting cases, when all arguments  $a_j$  are in one subspace  $\mathbf{A}_0$  or  $\mathbf{A}_1$ , the  $n$ -ary Lie superbracket becomes an  $n$ -ary anticommutator or the ordinary  $n$ -ary commutator respectively

$$\mathcal{L}^{[n]} [a_1, a_2, \dots, a_n] = \begin{cases} \sum_{\sigma \in S_n} \mu^{[n]} [a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)}] = \{a_1, a_2, \dots, a_n\}^{[n]}, & \text{if } a_j \in \mathbf{A}_1, \\ \sum_{\sigma \in S_n} (-1)^{\pi(\sigma)} \mu^{[n]} [a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)}] = [a_1, a_2, \dots, a_n]^{[n]}, & \text{if } a_j \in \mathbf{A}_0. \end{cases} \quad (3.13)$$

The general case was given in BARREIRO ET AL. [2019] and is too cumbersome to be presented here, we therefore restrict consideration of its manifest form to some lower arity examples below.

Let us look more closely at the definitions of the binary Lie superbracket (3.1) and  $n$ -ary Lie superbracket (3.13). If the initial associative  $n$ -ary superalgebra  $\mathcal{A}^{[n]}$  has a polyadic multiplicative unit  $e$  (obeying (2.3)–(2.4)), then we can define a superbracket with a lower arity,  $m$ , than the initial  $n$ -ary multiplication  $\mu^{[n]}$ . So instead of (3.13), we propose using the reduced  $m$ -ary superbracket (which is,

obviously, non-Lie for  $m \neq n$ )

$$\begin{aligned} & \mathcal{R}_{(n)}^{[m]} [a_1, a_2, \dots, a_m] \\ &= \begin{cases} \sum_{\sigma \in S_n} \mu^{[n]} \left[ a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(m)}, \overbrace{e, \dots, e}^{n-m} \right] = \{a_1, a_2, \dots, a_n\}_{(n)}^{[m]}, & \text{if } a_j \in A_1, \\ \sum_{\sigma \in S_n} (-1)^{\pi(\sigma)} \mu^{[n]} \left[ a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(m)}, \overbrace{e, \dots, e}^{n-m} \right] = [a_1, a_2, \dots, a_n]_{(n)}^{[m]}, & \text{if } a_j \in A_0. \end{cases} \end{aligned} \quad (3.14)$$

where  $2 \leq m \leq n$ , and  $2 \leq j \leq m$ . The reduced  $m$ -ary superbracket satisfies the  $m$ -ary anticommutation

$$\mathcal{R}_{(n)}^{[m]} [a_1, a_2, \dots, a_{j-1}, a_j, \dots, a_m] = -(-1)^{\pi(a_{j-1})\pi(a_j)} \mathcal{R}_{(n)}^{[m]} [a_1, a_2, \dots, a_j, a_{j-1}, \dots, a_m], \quad (3.15)$$

and, possibly, the reduced  $m$ -ary super Jacobi identity. In the limiting case of equal arities  $m = n$  the reduced superbracket coincides with the  $n$ -ary Lie superbracket

$$\mathcal{R}_{(n)}^{[n]} = \mathcal{L}^{[n]}. \quad (3.16)$$

In the simplest case  $m = 2$ , the reduced binary superbracket becomes (cf. (3.1))

$$\mathcal{R}_{(n)}^{[2]} [a_1, a_2] = \begin{cases} \mu^{[n]} \left[ a_1, a_2, \overbrace{e, \dots, e}^{n-2} \right] + \mu^{[n]} \left[ a_2, a_1, \overbrace{e, \dots, e}^{n-2} \right] = \{a_1, a_2\}_{(n)}^{[2]}, & \text{if } a_1, a_2 \in A_1, \\ \mu^{[n]} \left[ a_1, a_2, \overbrace{e, \dots, e}^{n-2} \right] - \mu^{[n]} \left[ a_2, a_1, \overbrace{e, \dots, e}^{n-2} \right] = [a_1, a_2]_{(n)}^{[2]}, & \text{in other cases,} \end{cases} \quad (3.17)$$

which may be called the reduced binary anticommutator and the reduced binary commutator, respectively.

In this way, the polyadic superalgebra with reduced arity bracket (having  $m$ -ary multiplication, while constructed from an  $n$ -ary associative superalgebra with the polyadic unit  $e$ ) becomes

$$\mathcal{A}_{sRed(n)}^{[m]} = \left\langle A \mid \mathcal{R}_{(n)}^{[m]}, e, m\text{-}s\text{-Jacobi} \right\rangle. \quad (3.18)$$

If the reduced  $m$ -ary superbracket  $\mathcal{R}_{(n)}^{[m]}$  satisfies the standard  $m$ -ary super Jacobi identity [POJIDAEV \[2003\]](#), [BARREIRO ET AL. \[2019\]](#), then we can call  $\mathcal{A}_{sRed(n)}^{[m]}$  (3.18) a  $m$ -ary polyadic analog of the  $n$ -ary Lie superalgebra.

Thus, from each associative  $n$ -ary superalgebra  $\mathcal{A}^{[n]}$  (by substituting  $\mu^{[n]} \rightarrow \mathcal{R}_{(n)}^{[m]}$  and  $n$ -ary associativity condition of  $\mathcal{A}^{[n]}$  with the lower arity analog of  $m$ -ary super Jacobi identity, if it exists) we can build not only a superalgebra  $\mathcal{A}_{sRed(n)}^{[2]}$  with the reduced arity binary bracket (which does not coincide with the ordinary binary Lie superalgebra), but also a series of  $n - 2$  superalgebras  $\mathcal{A}_{sRed(n)}^{[m]}$  having the  $m$ -ary reduced superbrackets  $\mathcal{R}_{(n)}^{[m]}$  with arities  $2 \leq m \leq n$  (with  $\mathcal{R}_{(n)}^{[m]} = \mathcal{L}^{[n]}$  if  $m = n$ , see (3.16)).

**3.3. Algebras of polyadic supercharges.** Let us introduce polyadic ( $n$ -ary) supercharges  $\hat{Q}_{1,2}$  in the most general form (by analogy with their binary versions (3.5)–(3.6))

$$\hat{Q}_1 = \frac{1}{\sqrt{2}} (\Sigma_1 \hat{p} + \Sigma_2 W(x)), \quad (3.19)$$

$$\hat{Q}_2 = \frac{1}{\sqrt{2}} (\Sigma_2 \hat{p} - \Sigma_1 W(x)), \quad (3.20)$$

where  $W(x)$ , are even complex analytic functions. Regarding parities we agree that (cf. (3.8))

$$\pi(\hat{\mathbf{Q}}_{1,2}) = 1, \quad \pi(\Sigma_{1,2}) = 1, \quad \pi(\Sigma_{0,3}) = 0, \quad \pi(\mathbf{E}) = 0. \quad (3.21)$$

In these parity prescriptions the set of matrix operators  $\{\hat{\mathbf{M}}_{shift}\}$  of the cycled shift form (2.1) generated by the odd polyadic supercharges  $\hat{\mathbf{Q}}_{1,2}$  (3.19)–(3.20) with respect to the matrix operator multiplication  $\hat{\mu}^{[n]}$  (2.5) becomes the noncommutative, nonassociative  $n$ -ary graded algebra of operators  $\hat{\mathcal{A}}_{shift}^{[n]} = \langle \{\hat{\mathbf{M}}_{shift}\} | \hat{\mu}^{[n]}, \mathbf{E} \rangle$  (cf. (2.5)), where  $\mathbf{E}$  is the polyadic unit (2.4). We endow the  $n$ -ary superalgebra  $\hat{\mathcal{A}}_{shift}^{[n]}$  with the reduced arity  $m$ -ary superbracket  $\hat{\mu}^{[n]} \longrightarrow \hat{\mathcal{R}}_{(n)}^{[m]}$  (3.14),  $2 \leq m \leq n$ , which is polyadic anticommutative

$$\begin{aligned} & \hat{\mathcal{R}}_{(n)}^{[m]} [\hat{\mathbf{M}}_1, \hat{\mathbf{M}}_2, \dots, \hat{\mathbf{M}}_{j-1}, \hat{\mathbf{M}}_j, \dots, \hat{\mathbf{M}}_m] \\ &= -(-1)^{\pi(\hat{\mathbf{M}}_{j-1})\pi(\hat{\mathbf{M}}_j)} \hat{\mathcal{R}}_{(n)}^{[m]} [\hat{\mathbf{M}}_1, \hat{\mathbf{M}}_2, \dots, \hat{\mathbf{M}}_j, \hat{\mathbf{M}}_{j-1}, \dots, \hat{\mathbf{M}}_m], \end{aligned} \quad (3.22)$$

and can satisfy the  $m$ -ary super Jacobi identity [POJIDAEV \[2003\]](#), [BARREIRO ET AL. \[2019\]](#). Thus, by analogy with (3.18), we obtain the  $m$ -ary superalgebra of operators with the reduced bracket  $\hat{\mathcal{R}}_{(n)}^{[m]}$  (3.22) as

$$\hat{\mathcal{A}}_{sRed(n)}^{[m]} = \langle \{\hat{\mathbf{M}}_{shift}\} | \hat{\mathcal{R}}_{(n)}^{[m]}, \mathbf{E}, m\text{-s-Jacobi} \rangle. \quad (3.23)$$

Because the  $n$ -ary supercharges  $\hat{\mathbf{Q}}_{\ell_j}$  (3.19)–(3.20) are odd, the polyadic superalgebra  $\hat{\mathcal{A}}_{sRed(n)}^{[m]}$  has different properties for different parities of the reduced arity  $m$ . Indeed:

- 1) Even reduced arity  $m = 2m' \leq n$ ,  $m' \in \mathbb{N}$ . We construct the even elements of  $\hat{\mathcal{A}}_{sRed(n)}^{[m]}$  which can be treated as higher polyadic analogs of the Hamiltonian which describe the dynamics of  $m$ -ary supersymmetric quantum mechanics. The reduced arity superbracket (3.22) of  $n$ -ary supercharges gives the higher Hamiltonian (tower)

$$\hat{\mathbf{H}}_{(n)}^{[2m']} = \frac{1}{(2m')!} \hat{\mathcal{R}}_{(n)}^{[2m']} \left[ \overbrace{\hat{\mathbf{Q}}_{\ell_1}, \hat{\mathbf{Q}}_{\ell_2}, \dots, \hat{\mathbf{Q}}_{\ell_m}}^{2m'} \right], \quad \hat{\mathbf{Q}}_{\ell_j} \in \hat{\mathcal{A}}_{sRed(n)}^{[2m']}, \quad \ell_j = 1, 2. \quad (3.24)$$

In this way, we obtain polyadic supersymmetry, because informally we have “odd”<sup>2m'</sup> = “even”, as the polyadic analog of ordinary binary supersymmetry “odd”<sup>2</sup> = “even”. We can use the  $n$ -ary anticommutator (3.13), because all supercharges are odd, and add a reduced arity bracket of the higher order Hamiltonian with polyadic supercharges as “even” • “odd”<sup>2m'-1</sup> = 0, and also the polyadic analog of orthogonality of supercharges (3.3) with  $j \neq k$ . In this way we obtain an example of polyadic supersymmetry, as  $2m'$ -ary supersymmetric quantum mechanics described by the algebra (3.23) with the  $2m'$ -ary reduced bracket (cf. the standard binary SQM (3.3)–(3.4))

$$\hat{\mathbf{H}}_{(n)}^{[2m']} = \frac{1}{(2m')!} \left\{ \overbrace{\hat{\mathbf{Q}}_{\ell_1}, \hat{\mathbf{Q}}_{\ell_2}, \dots, \hat{\mathbf{Q}}_{\ell_{2m'}}}^{2m'} \right\}_{(n)}^{[2m']}, \quad \hat{\mathbf{Q}}_{\ell_j} \in \hat{\mathcal{A}}_{sRed(n)}^{[2m']}, \quad \ell_j = 1, 2. \quad (3.25)$$

$$\hat{\mathcal{R}}_{(n)}^{[2m']} \left[ \hat{\mathbf{H}}_{(n)}^{[2m']}, \overbrace{\hat{\mathbf{Q}}_{\ell_1}, \hat{\mathbf{Q}}_{\ell_2}, \dots, \hat{\mathbf{Q}}_{\ell_{2m'-1}}}^{2m'-1} \right] = 0, \quad m' \in \mathbb{N}, \quad (3.26)$$

$$\hat{\mathcal{R}}_{(n)}^{[2m']} \left[ \overbrace{\hat{\mathbf{Q}}_1, \hat{\mathbf{Q}}_2, \hat{\mathbf{Q}}_2, \dots, \hat{\mathbf{Q}}_2}^{2m'-1} \right] = \left\{ \overbrace{\hat{\mathbf{Q}}_1, \hat{\mathbf{Q}}_2, \hat{\mathbf{Q}}_2, \dots, \hat{\mathbf{Q}}_2}^{2m'-1} \right\}_{(n)}^{[2m']} = 0, \quad (3.27)$$

$$\hat{\mathcal{R}}_{(n)}^{[2m']} \left[ \overbrace{\hat{\mathbf{Q}}_2, \hat{\mathbf{Q}}_1, \hat{\mathbf{Q}}_1, \dots, \hat{\mathbf{Q}}_1}^{2m'-1} \right] = \left\{ \overbrace{\hat{\mathbf{Q}}_2, \hat{\mathbf{Q}}_1, \hat{\mathbf{Q}}_1, \dots, \hat{\mathbf{Q}}_1}^{2m'-1} \right\}_{(n)}^{[2m']} = 0, \quad (3.28)$$

where  $\{ \}_{(n)}^{[m]}$  is the  $m$ -ary anticommutator (3.14). Note that the polyadic Hamiltonians (3.25) are invariant with respect to the interchange  $\hat{\mathbf{Q}}_1 \leftrightarrow \hat{\mathbf{Q}}_2$ . It is seen from (3.19)–(3.20), that the reduced arity  $m$  coincides with the order of the polyadic Hamiltonians (3.24) as differential operators (higher order SQM was considered, e.g. in FERNÁNDEZ C. AND FERNÁNDEZ-GARCÍA [2004]).

- 2) Odd reduced arity  $m = 2m' + 1 \geq 3$ ,  $m' \in \mathbb{N}$ . In this case we have, informally, “odd”<sup>2m'+1</sup> = “odd”, and so the algebra  $\hat{\mathcal{A}}_{sRed(n)}^{[2m'+1]}$  contains no even elements at all, and therefore – no supersymmetry (in its “odd”<sup>2</sup> = “even” definition). Using the reduced arity superbracket (3.22), we obtain only the higher order supercharges which are of  $(2m' + 1)$  order as differential operators

$$\hat{\mathbf{Q}}_{(n)}^{[2m'+1]} = \frac{1}{(2m' + 1)!} \left\{ \overbrace{\hat{\mathbf{Q}}_{\ell_1}, \hat{\mathbf{Q}}_{\ell_2}, \dots, \hat{\mathbf{Q}}_{\ell_{2m'+1}}}^{2m'+1} \right\}_{(n)}^{[2m'+1]}, \quad \hat{\mathbf{Q}}_{\ell_j} \in \hat{\mathcal{A}}_{sRed(n)}^{[2m'+1]}, \quad \ell_j = 1, 2. \quad (3.29)$$

Note that, despite  $\hat{\mathcal{A}}_{sRed(n)}^{[2m'+1]}$  containing only odd elements (because only odd number of multipliers are allowed to close multiplication), it is actually a  $(2m' + 1)$ -ary superalgebra (consisting of the odd part only) with respect to the reduced arity superbracket which is  $(2m' + 1)$ -anticommutative (3.15). This contrasts with the ordinary (binary) superalgebras, where the odd part by itself is not an algebra at all, since the multiplication is not closed.

Note that the (classical) odd Hamiltonians were obtained by changing the even Poisson bracket to another one, the odd Poisson bracket VOLKOV ET AL. [1986], while we have the reduced arity (quantum) superbracket  $\hat{\mathcal{R}}_{(n)}^{[m]}$  in both ( $m$  is even or odd) cases.

The fermionic charge  $\hat{\mathbf{F}}$ , as an even matrix operator, becomes (cf. (3.9))

$$\hat{\mathbf{F}} = \frac{1}{2} \Sigma_3. \quad (3.30)$$

The normalization in (3.24) is chosen such that all the polyadic Hamiltonians of the binary reduced arity  $m = 2$  would reproduce the standard (binary) Hamiltonian (3.7) and the main SQM relation (3.3). Also, it is important to note that algebras with different reduced arity  $m$  of the superbrackets  $\hat{\mathcal{R}}_{(n)}^{[m]}$  do not intersect.

**3.4. SQM with reduced arity binary bracket.** The standard binary SQM corresponds to the case  $m = n = 2$ , and  $\Sigma_j = \sigma_j$  (see Subsection 3.1). If  $n \geq 3$  and  $m = 2$ , we have more interesting dynamics.



Indeed, the (still) binary polyadic algebra of Hamiltonian and supercharges  $\hat{\mathcal{A}}_{sRed(n)}^{[2]}$  with the reduced arity bracket becomes

$$\hat{\mathbf{H}}_{(n)}^{[m=2]} = \frac{1}{2} \left\{ \hat{\mathbf{Q}}_{1,2}, \hat{\mathbf{Q}}_{1,2} \right\}_{(n)}^{[2]} = \frac{1}{2} \Sigma_0 (\hat{p}^2 + W^2(x)) + \frac{1}{2} \Sigma_3 W'(x), \quad (3.31)$$

$$\left\{ \hat{\mathbf{Q}}_1, \hat{\mathbf{Q}}_2 \right\}_{(n)}^{[2]} = 0, \quad (3.32)$$

$$\left[ \hat{\mathbf{H}}_{(2)}^{[2]}, \hat{\mathbf{Q}}_1 \right]_{(n)}^{[2]} = \left[ \hat{\mathbf{H}}_{(2)}^{[2]}, \hat{\mathbf{Q}}_2 \right]_{(n)}^{[2]} = 0. \quad (3.33)$$

Thus, using (2.1), (3.19)–(3.20), (3.24) and (3.7), for the case  $= 1$ , we obtain the polyadic SQM Hamiltonian of the reduced arity  $m = 2$  in the matrix form

$$\hat{\mathbf{H}}_{(n)}^{[m=2]} = \begin{pmatrix} 0 & \boxed{\hat{\mathbf{H}}_{Witten}^{(1)}} & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \boxed{\hat{\mathbf{H}}_{Witten}^{(2)}} & \dots & 0 & \dots & 0 \\ 0 & 0 & 0 & \ddots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \boxed{\hat{\mathbf{H}}_{Witten}^{(k)}} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \dots & \boxed{\hat{\mathbf{H}}_{Witten}^{(n-2)}} \\ \boxed{\hat{\mathbf{H}}_{Witten}^{(n-1)}} & 0 & 0 & \dots & 0 & \dots & 0 \end{pmatrix}. \quad (3.34)$$

In comparison with binary SQM (3.7), the polyadic SQM Hamiltonians with higher reduced arity of bracket  $m \geq 3$  would have new features and properties, giving novel genuine characterizations of polyadic dynamical systems.

The general structure of the polyadic higher order even Hamiltonians and the  $m$ -ary higher order odd supercharges is as follows

$$\hat{\mathbf{H}}_{(n)}^{[2m']} = C_0^{(2m')}(x) \Sigma_0^{[n]} + C_3^{(2m')}(x) \Sigma_3^{[n]}, \quad (3.35)$$

$$\hat{\mathbf{Q}}_{(n),1}^{[2m'+1]} = C_1^{(2m'+1)}(x) \Sigma_1^{[n]} + C_2^{(2m'+1)}(x) \Sigma_2^{[n]}, \quad (3.36)$$

$$\hat{\mathbf{Q}}_{(n),2}^{[2m'+1]} = C_1^{(2m'+1)}(x) \Sigma_2^{[n]} - C_2^{(2m'+1)}(x) \Sigma_1^{[n]}, \quad m' \in \mathbb{N}, \quad (3.37)$$

where  $C_j^{(m)}(x)$ , depending on the potential  $W(x)$ , being complex analytic functions of  $x$  and derivatives of order up to  $m$ . The matrix block form of (3.35)–(3.37) coincides with (3.34), but with the equal  $2 \times 2$  blocks  $C_0^{(2m')}(x) \sigma_0 + C_3^{(2m')}(x) \sigma_3$  and  $C_1^{(2m'+1)}(x) \sigma_1 + C_2^{(2m'+1)}(x) \sigma_2$ ,  $C_1^{(2m'+1)}(x) \sigma_2 - C_2^{(2m'+1)}(x) \sigma_1$ ,  $m' \in \mathbb{N}$ . The initial values coincide with the standard SQM (3.7) and (3.19)–(3.20), that is  $\hat{\mathbf{H}}_{(2)}^{[2]} = \hat{\mathbf{H}}_{Witten}$  and  $\hat{\mathbf{Q}}_{(2),1}^{[1]} = Q_1$ ,  $\hat{\mathbf{Q}}_{(2),2}^{[1]} = Q_2$  (since  $\Sigma_j^{[2]} = \sigma_j$ ).

In this way we obtain the component form for the even polyadic Hamiltonian

$$\hat{\mathbf{H}}_{(n)}^{[2m']} = \begin{pmatrix} 0 & \begin{pmatrix} \hat{H}_+ & 0 \\ 0 & \hat{H}_- \end{pmatrix} & 0 & 0 & \dots & 0 \\ 0 & 0 & \begin{pmatrix} \hat{H}_+ & 0 \\ 0 & \hat{H}_- \end{pmatrix} & 0 & \dots & 0 \\ 0 & 0 & 0 & \begin{pmatrix} \hat{H}_+ & 0 \\ 0 & \hat{H}_- \end{pmatrix} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \begin{pmatrix} \hat{H}_+ & 0 \\ 0 & \hat{H}_- \end{pmatrix} \\ \begin{pmatrix} \hat{H}_+ & 0 \\ 0 & \hat{H}_- \end{pmatrix} & 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad (3.38)$$

where (from (3.7))

$$\hat{H}_\pm = C_0^{(2m')}(x) \pm C_3^{(2m')}(x), \quad (3.39)$$

$$C_0^{(2)}(x) = \frac{1}{2}(\hat{p}^2 + W^2(x)), \quad C_3^{(2)}(x) = \frac{1}{2}W'(x), \quad (3.40)$$

and the higher order  $(2m' + 1)$ -ary odd supercharges

$$\hat{\mathbf{Q}}_{(n,\ell)}^{[2m'+1]} = \begin{pmatrix} 0 & \begin{pmatrix} 0 & \hat{Q}_{\ell+} \\ \hat{Q}_{\ell-} & 0 \end{pmatrix} & 0 & 0 & \dots & 0 \\ 0 & 0 & \begin{pmatrix} 0 & \hat{Q}_{\ell+} \\ \hat{Q}_{\ell-} & 0 \end{pmatrix} & 0 & \dots & 0 \\ 0 & 0 & 0 & \begin{pmatrix} 0 & \hat{Q}_{\ell+} \\ \hat{Q}_{\ell-} & 0 \end{pmatrix} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \begin{pmatrix} 0 & \hat{Q}_{\ell+} \\ \hat{Q}_{\ell-} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \hat{Q}_{\ell+} \\ \hat{Q}_{\ell-} & 0 \end{pmatrix} & 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad (3.41)$$

where (from (3.5)–(3.6))

$$\hat{Q}_{1\pm} = C_1^{(2m'+1)}(x) \mp iC_2^{(2m'+1)}(x), \quad (3.42)$$

$$\hat{Q}_{2\pm} = \mp C_1^{(2m'+1)}(x) - iC_2^{(2m'+1)}(x), \quad (3.43)$$

$$C_1^{(1)}(x) = \frac{1}{\sqrt{2}}\hat{p}, \quad C_2^{(1)}(x) = \frac{1}{\sqrt{2}}W(x). \quad (3.44)$$

In the examples below we present the concrete expressions for  $\hat{\mathbf{H}}_{(n)}^{[m]}$  with  $n = 3, 4$ .

## 4. SQM FROM TERNARY SUPERALGEBRA

If  $n = 3$ , we have an even binary Hamiltonian presented by the general formulas (3.31)–(3.33) and (3.34), such that

$$\hat{\mathbf{H}}_{(3)}^{[2]} = \frac{1}{2} \hat{\mathcal{R}}_{(3)}^{[2]} \left[ \hat{\mathcal{Q}}_{1,2}, \hat{\mathcal{Q}}_{1,2} \right]_{(3)}^{[2]} = \frac{1}{2} \left\{ \hat{\mathcal{Q}}_{1,2}, \hat{\mathcal{Q}}_{1,2} \right\}_{(3)}^{[2]} = \begin{pmatrix} 0 & \hat{H}_{Witten} \\ \hat{H}_{Witten} & 0 \end{pmatrix}, \quad (4.1)$$

and other binary SQM relations are

$$\left\{ \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_2 \right\}_{(3)}^{[2]} = 0, \quad (4.2)$$

$$\left[ \hat{\mathbf{H}}_{(3)}^{[2]}, \hat{\mathcal{Q}}_1 \right]_{(3)}^{[2]} = \left[ \hat{\mathbf{H}}_{(3)}^{[2]}, \hat{\mathcal{Q}}_2 \right]_{(3)}^{[2]} = 0, \quad (4.3)$$

where  $\hat{H}_{Witten}$  is given in (3.7). In components

$$\hat{\mathbf{H}}_{(3)}^{[2]} = \begin{pmatrix} 0 & \begin{pmatrix} \hat{H}_+ & 0 \\ 0 & \hat{H}_- \end{pmatrix} \\ \begin{pmatrix} \hat{H}_+ & 0 \\ 0 & \hat{H}_- \end{pmatrix} & 0 \end{pmatrix}, \quad (4.4)$$

where  $\hat{H}_{\pm} = \frac{1}{2} (\hat{p}^2 + W^2(x)) \pm \frac{1}{2} W'(x)$ .

Next we consider the odd reduced arity  $m = 3$ . The higher order odd supercharges are

$$\hat{\mathcal{Q}}_{(n=3),1}^{[m=3]} = \frac{1}{6} \left\{ \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_1 \right\}_{(3)}^{[3]} = 3 \hat{\mathcal{Q}}_{(n=3),21}^{[m=3]} = 3 \left( \frac{1}{6} \left\{ \hat{\mathcal{Q}}_2, \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_2 \right\}_{(3)}^{[3]} \right), \quad (4.5)$$

$$\hat{\mathcal{Q}}_{(n=3),2}^{[m=3]} = \frac{1}{6} \left\{ \hat{\mathcal{Q}}_2, \hat{\mathcal{Q}}_2, \hat{\mathcal{Q}}_2 \right\}_{(3)}^{[3]} = 3 \hat{\mathcal{Q}}_{(n=3),12}^{[m=3]} = 3 \left( \frac{1}{6} \left\{ \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_2, \hat{\mathcal{Q}}_1 \right\}_{(3)}^{[3]} \right). \quad (4.6)$$

In manifest matrix form they are

$$\hat{\mathcal{Q}}_{(3),\ell}^{[3]} = \begin{pmatrix} 0 & \hat{\mathcal{Q}}_{(3),\ell}^{[3]} \\ \hat{\mathcal{Q}}_{(3),\ell}^{[3]} & 0 \end{pmatrix}, \quad (4.7)$$

where (cf. (3.19)–(3.20))

$$\hat{\mathcal{Q}}_{(3),1}^{[3]} = \frac{1}{2\sqrt{2}} \left( \sigma_1 (\hat{p}^3 + W^2(x) \hat{p} - iWW'(x)) + \sigma_2 (W(x) \hat{p}^2 - iW'(x) \hat{p} - W''(x) + W^3(x)) \right), \quad (4.8)$$

$$\hat{\mathcal{Q}}_{(3),2}^{[3]} = \frac{1}{2\sqrt{2}} \left( \sigma_2 (\hat{p}^3 + W^2(x) \hat{p} - iWW'(x)) - \sigma_1 (W(x) \hat{p}^2 - iW'(x) \hat{p} - W''(x) + W^3(x)) \right). \quad (4.9)$$

In components, see (3.41), we obtain

$$\hat{\mathcal{Q}}_{(n=3)}^{[m=3]} = \begin{pmatrix} 0 & \begin{pmatrix} 0 & \hat{\mathcal{Q}}_+ \\ \hat{\mathcal{Q}}_- & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \hat{\mathcal{Q}}_+ \\ \hat{\mathcal{Q}}_- & 0 \end{pmatrix} & 0 \end{pmatrix}, \quad (4.10)$$

where

$$\hat{\mathcal{Q}}_{\pm} = \frac{1}{2\sqrt{2}} \left( \hat{p}^3 + W^2(x) \hat{p} - iWW'(x) \pm (-iW(x) \hat{p}^2 - W'(x) \hat{p} + iW''(x) - iW^3(x)) \right). \quad (4.11)$$

## 5. SQM FROM 4-ARY SUPERALGEBRA

In the case  $n = 4$ , the polyadic SQM is described by the binary Hamiltonian (of reduced arity  $m = 2$ ) and higher order 4-ary Hamiltonians. For the former we have

$$\hat{\mathbf{H}}_{(4)}^{[2]} = \hat{\mathbf{H}}_{(4),1}^{[2]} = \frac{1}{2} \hat{\mathcal{R}}_{(4)}^{[2]} [\hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_1]_{(4)}^{[2]} = \frac{1}{2} \left\{ \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_1 \right\}_{(4)}^{[2]} = \hat{\mathbf{H}}_{(4),2}^{[2]} = \frac{1}{2} \left\{ \hat{\mathcal{Q}}_2, \hat{\mathcal{Q}}_2 \right\}_{(4)}^{[2]}.$$

The second order orthogonality of supercharges is (cf. (5.8))

$$\left\{ \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_2, \right\}_{(4)}^{[2]} = 0. \quad (5.1)$$

In matrix form the binary Hamiltonian is (see (3.7) and (3.34))

$$\hat{\mathbf{H}}_{(4)}^{[2]} = \begin{pmatrix} 0 & \hat{\mathbf{H}}_{Witten} & 0 \\ 0 & 0 & \hat{\mathbf{H}}_{Witten} \\ \hat{\mathbf{H}}_{Witten} & 0 & 0 \end{pmatrix}. \quad (5.2)$$

The 4-ary Hamiltonian of 4th order is

$$\hat{\mathbf{H}}_{(4)}^{[4]} = \hat{\mathbf{H}}_{(4),4}^{[4]} = \frac{1}{24} \left\{ \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_1 \right\}_{(4)}^{[4]} = \hat{\mathbf{H}}_{(4),1}^{[4]} = \frac{1}{24} \left\{ \hat{\mathcal{Q}}_2, \hat{\mathcal{Q}}_2, \hat{\mathcal{Q}}_2, \hat{\mathcal{Q}}_2 \right\}_{(4)}^{[4]} \quad (5.3)$$

$$= 3\hat{\mathbf{H}}_{(4),12}^{[4]} = 3 \left( \frac{1}{24} \left\{ \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_2, \hat{\mathcal{Q}}_2 \right\}_{(4)}^{[4]} \right). \quad (5.4)$$

In matrix form

$$\hat{\mathbf{H}}_{(4)}^{[4]} = \begin{pmatrix} 0 & \hat{\mathbf{H}}_{(4)}^{[4]} & 0 \\ 0 & 0 & \hat{\mathbf{H}}_{(4)}^{[4]} \\ \hat{\mathbf{H}}_{(4)}^{[4]} & 0 & 0 \end{pmatrix}, \quad (5.5)$$

where

$$\hat{\mathbf{H}}_{(4)}^{[4]} = \frac{\sigma_0}{4} (\hat{p}^4 + 2W^2(x) \hat{p}^2 - 4W(x) W'(x) \hat{p} + W^4(x) - 2W(x) W''(x) - W'^2(x)) \quad (5.6)$$

$$+ \frac{\sigma_3}{4} (2W'(x) \hat{p}^2 - W''(x) \hat{p} + 2W^2(x) W'(x) - W'''(x)). \quad (5.7)$$

The condition of fourth order orthogonality for the supercharges becomes

$$\left\{ \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_2, \hat{\mathcal{Q}}_2 \right\}_{(4)}^{[4]} = \left\{ \hat{\mathcal{Q}}_2, \hat{\mathcal{Q}}_2, \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_1 \right\}_{(4)}^{[4]} = 0. \quad (5.8)$$

The the higher order odd supercharges of third reduced arity  $m = 3$  are

$$\hat{\mathcal{Q}}_{(4),1}^{[3]} = \frac{1}{6} \left\{ \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_1 \right\}_{(4)}^{[3]} = 3\hat{\mathcal{Q}}_{(4),21}^{[3]} = 3 \left( \frac{1}{6} \left\{ \hat{\mathcal{Q}}_2, \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_2 \right\}_{(4)}^{[3]} \right), \quad (5.9)$$

$$\hat{\mathcal{Q}}_{(4),2}^{[3]} = \frac{1}{6} \left\{ \hat{\mathcal{Q}}_2, \hat{\mathcal{Q}}_2, \hat{\mathcal{Q}}_2 \right\}_{(4)}^{[3]} = 3\hat{\mathcal{Q}}_{(4),12}^{[3]} = 3 \left( \frac{1}{6} \left\{ \hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_2, \hat{\mathcal{Q}}_1 \right\}_{(4)}^{[3]} \right). \quad (5.10)$$

In matrix form we have (cf. (5.2))

$$\hat{\mathcal{Q}}_{(4)}^{[3]} = \begin{pmatrix} 0 & \hat{\mathcal{Q}}_{(4)}^{[3]} & 0 \\ 0 & 0 & \hat{\mathcal{Q}}_{(4)}^{[3]} \\ \hat{\mathcal{Q}}_{(4)}^{[3]} & 0 & 0 \end{pmatrix}. \quad (5.11)$$

The manifest form of third order supercharges  $\hat{Q}_{(4)}^{[3]}$  from (5.11) are

$$\hat{Q}_{(4),1}^{[3]} = \frac{\sqrt{2}}{4} (\sigma_1 (\hat{p}^3 + W^2(x) \hat{p} - iW(x) W'(x)) + \sigma_2 (W(x) \hat{p}^2 - W'(x) \hat{p} - W''(x) + W^3(x))), \quad (5.12)$$

$$\hat{Q}_{(4),2}^{[3]} = \frac{\sqrt{2}}{4} (\sigma_2 (\hat{p}^3 + W'^2(x) \hat{p} - iW(x) W'(x)) - \sigma_1 (W(x) \hat{p}^2 - W'(x) \hat{p} - W''(x) + W^3(x))), \quad (5.13)$$

which can be compared with (3.19)–(3.20).

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