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Polyadic Rings of p-Adic Integers

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Abstract: In this note, we first recall that the sets of all representatives of some special ordinary residue classes become (m,n)-rings. Second, we introduce a possible p-adic analog of the residue class modulo a p-adic integer. Then, we find the relations which determine when the representatives form a (m,n)-ring. At very short spacetime scales, such rings could lead to new symmetries of modern particle models.

Keywords: polyadic semigroup; polyadic ring; arity; querelement; residue class; representative; p-adic integer

MSC: 17A42; 20N15; 11A07; 11S31

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1. Introduction

The fundamental conception of p-adic numbers is based on a special extension of rational numbers that is an alternative to real and complex numbers. The main idea is the completion of the rational numbers with respect to the p-adic norm, which is non-Archimedean. Nowadays, p-adic methods are widely used in number theory [1,2], arithmetic geometry [3,4] and algorithmic computations [5]. In mathematical physics, a non-Archimedean approach to spacetime and string dynamics at the Planck scale leads to new symmetries of particle models (see, e.g., [6,7] and the references therein). For some special applications, see, e.g., [8,9]. General reviews are given in [10–12].

Previously, we have studied the algebraic structure of the representative set in a fixed ordinary residue class [13]. We found that the set of representatives becomes a polyadic or (m,n)-ring, if the parameters of a class satisfy special "quantization" conditions. We have found that similar polyadic structures appear naturally for p-adic integers, if we introduce informally a p-adic analog of the residue classes, and we investigate here the set of its representatives along the lines of [13–15].

2. (m,n)-Rings of Integer Numbers from Residue Classes

Here we recall that representatives of special residue (congruence) classes can form polyadic rings, as was found in [13,14] (see also notation from [15]).

Let us denote the residue (congruence) class of an integer a modulo b by

$$[a]_b = \{\{r_k(a,b)\} \mid k \in \mathbb{Z}, a \in \mathbb{Z}_+, b \in \mathbb{N}, 0 \leq a \leq b-1\}, \tag{1}$$

where $r_k(a,b) = a + bk$ is a generic representative element of the class $[a]_b$. The canonical representative is the smallest non-negative number of these. Informally, a is the remainder of $r_k(a,b)$ when divided by b. The corresponding equivalence relation (congruence modulo b) is denoted by

$$r = a(\operatorname{mod} b). \tag{2}$$

Introducing the binary operations between classes $(+_{cl}, \times_{cl})$, the addition $[a_1]_b +_{cl} [a_2]_b = [a_1 + a_2]_b$, and the multiplication $[a_1]_b \times_{cl} [a_2]_b = [a_1 a_2]_b$, the residue class (binary) finite commutative ring $\mathbb{Z}/b\mathbb{Z}$ (with identity) is defined in the standard method (which was named "external" [13]). If $a \neq 0$ and b is prime, then $\mathbb{Z}/b\mathbb{Z}$ becomes a finite field.

The set of representatives $\{r_k(a,b)\}$ in a given class $[a]_b$ does not form a binary ring, because there are no binary operations (addition and multiplication) which are simultaneously closed for arbitrary a and b. Nevertheless, the following polyadic operations on representatives $r_k = r_k(a,b)$, m-ary addition ν_m

$$\nu_m\left[r_{k_1}, r_{k_2}, \dots, r_{k_m}\right] = r_{k_1} + r_{k_2} + \dots + r_{k_m}, \tag{3}$$

and n-ary multiplication μ_n

$$\mu_n\left[r_{k_1}, r_{k_2}, \dots, r_{k_n}\right] = r_{k_1} r_{k_2} \dots r_{k_n}, \quad r_{k_i} \in [a]_b,\ k_i \in \mathbb{Z}, \tag{4}$$

can be closed but only for special values of $a = a_q$ and $b = b_q$, which defines the nonderived (m,n)-ary ring

$$\mathbb{Z}_{(m,n)}\left(a_q, b_q\right) = \left\langle \left[a_q\right]_{b_q} \mid \nu_m, \mu_n \right\rangle \tag{5}$$

of polyadic integers (that was called the "internal" way [13]). The conditions of closure for the operations between representatives can be formulated in terms of the (arity shape [14]) invariants (which may be seen as some form of "quantization")

$$(m-1)\frac{a_q}{b_q} = I_m\left(a_q, b_q\right) \in \mathbb{N}, \tag{6}$$

$$a_q^{n-1}\frac{a_q - 1}{b_q} = J_n\left(a_q, b_q\right) \in \mathbb{N}, \tag{7}$$

or, equivalently, using the congruence relations [13]

$$m a_q \equiv a_q \left(\bmod\, b_q\right), \tag{8}$$

$$a_q^n \equiv a_q \left(\bmod\, b_q\right), \tag{9}$$

where we have denoted by a_q and b_q the concrete solutions of the "quantization" Equations (6)–(9). To understand the nature of the "quantization", we consider in detail the concrete example of nonderived m-ary addition and n-ary multiplication appearance for representatives in a fixed residue class.

Example 1. *Let us consider the following residue class*

$$[[3]]_4 = \{\dots -45, -33, -29, -25, -21, -17, -13, -9, -5, -1, 3, 7, 11, 15, 19, 23, 27, 31 \dots\}, \tag{10}$$

where the representatives are

$$r_k = r_k(3,4) = 3 + 4k, \quad k \in \mathbb{Z}. \tag{11}$$

We first obtain the condition for when the sum of m representatives belongs to the class (10). So, we compute step by step

$$m = 2, \quad r_{k_1} + r_{k_2} = r_k + 3 \notin [[3]]_4, \quad k = k_1 + k_2, \tag{12}$$

$$m = 3, \quad r_{k_1} + r_{k_2} + r_{k_3} = r_k + 6 \notin [[3]]_4, \quad k = k_1 + k_2 + k_3, \tag{13}$$

$$m = 4, \quad r_{k_1} + r_{k_2} + r_{k_3} + r_{k_4} = r_k + 9 \notin [[3]]_4, \quad k = k_1 + k_2 + k_3 + k_4, \tag{14}$$

$$m = 5, \quad r_{k_1} + r_{k_2} + r_{k_3} + r_{k_4} + r_{k_5} = r_k \in [[3]]_4, \quad k = k_1 + k_2 + k_3 + k_4 + k_5 + 3. \tag{15}$$

$$\vdots$$

Thus, the binary, ternary, and 4-ary additions are not closed, while 5-ary addition is. In general, the closure of m-ary addition holds valid when $4 \mid (m-1)$, that is for $m = 5, 9, 13, 17, \dots$, such that

$$\sum_{i=1}^{m} r_{k_i} = r_k \in [[3]]_4, \quad k = \sum_{i=1}^{m} k_i + 3\ell_\nu, \tag{16}$$

where $\ell_\nu = \frac{m-1}{4} \in \mathbb{N}$ is a natural number. The "quantization" rule for the arity of addition (6), (8) becomes $m = 4\ell_\nu + 1$.

If we consider the minimal arity $m = 5$, we arrive at the conclusion that $\langle \{r_k\} \mid \nu_5 \rangle$, is a 5-ary commutative semigroup, where ν_5 is the nonderived (i.e., not composed from lower arity operations) 5-ary addition

$$\nu_5\left[r_{k_1}, r_{k_2}, r_{k_3}, r_{k_4}, r_{k_5}\right] = r_{k_1} + r_{k_2} + r_{k_3} + r_{k_4} + r_{k_5}, \quad r_{k_i} \in [3]_4, \tag{17}$$

given by (15), and total 5-ary associativity follows from that of the binary addition in (3) and (17). In this case, ℓ_ν is the "number" of composed 5-ary additions (the polyadic power). There is no neutral element z for 5-ary addition ν_5 (17) defined by $\nu_5[z, z, z, z, r_k] = r_k$, and so the semigroup $\langle \{r_k\} \mid \nu_5 \rangle$ is zeroless. Nevertheless, $\langle \{r_k\} \mid \nu_5 \rangle$ is a 5-ary group (which is impossible in the binary case, where all groups contain a neutral element, the identity), because each element r_k has a unique querelement $\widetilde{r_k}$ defined by (see, e.g., [15])

$$\nu_5[r_k, r_k, r_k, r_k, \widetilde{r_k}] = r_k, \quad r_k, \widetilde{r_k} \in [3_4], \tag{18}$$

Therefore, from (15) and (17) it follows that $\tilde{r}_k = -3r_k = r_{-9-12k}$. For instance, for the first several elements of the residue class $[3]_4$ (10), we have the following querelements

$$\widetilde{7} = -21, \ \ \widetilde{11} = -33, \ \ \widetilde{15} = -45, \tag{19}$$

$$\widetilde{-1} = 3, \ \ \widetilde{-5} = 15, \ \ \widetilde{-9} = 27. \tag{20}$$

Note that in the 5-ary group $\left\langle \{r_k\} \mid \nu_5, \widetilde{(\cdot)} \right\rangle$, the additive quermapping defined by $r_k \mapsto \widetilde{r_k}$ is not a reflection (of any order) for $m \geq 3$, i.e., $\widetilde{\widetilde{r_k}} \neq r_k$ (as opposed to the inverse in the binary case) [15].

Now, we turn to the multiplication of n representatives (11) of the residue class $[3]_4$ (10). By analogy with (12)–(15) we obtain, step by step

$$\begin{aligned}
&n = 2 \quad r_{k_1} r_{k_2} = r_k + 6 \notin [[3]]_4, \quad k = 3k_1 + 3k_2 + 4k_1k_2, \\
&n = 3 \quad \begin{cases} r_{k_1} r_{k_2} r_{k_3} = r_k \in [[3]]_4, \\ k = 9k_1 + 9k_2 + 9k_3 + 12k_1k_2 + 12k_1k_3 + 12k_2k_3 + 16k_1k_2k_3 + 6, \end{cases} \\
&n = 4 \quad \begin{cases} r_{k_1} r_{k_2} r_{k_3} r_{k_4} = r_k + 2 \notin [[3]]_4, \\ k = 27k_1 + 27k_2 + 27k_3 + 27k_4 + 36k_1k_2 + 36k_1k_3 + 36k_1k_4 \\ +36k_2k_3 + 36k_2k_4 + 36k_3k_4 + 48k_1k_2k_3 + 48k_1k_2k_4 + 48k_1k_3k_4 \\ +48k_2k_3k_4 + 64k_1k_2k_3k_4 + 19, \end{cases} \\
&n = 5 \quad \begin{cases} r_{k_1} r_{k_2} r_{k_3} r_{k_4} r_{k_5} = r_k \in [[3]]_4, \\ k = 81k_1 + 81k_2 + 81k_3 + 81k_4 + 81k_5 + 108k_1k_2 + 108k_1k_3 + 108k_1k_4 \\ +108k_2k_3 + 108k_1k_5 + 108k_2k_4 + 108k_2k_5 + 108k_3k_4 + 108k_3k_5 + 108k_4k_5 \\ +144k_1k_2k_3 + 144k_1k_2k_4 + 144k_1k_2k_5 + 144k_1k_3k_4 + 144k_1k_3k_5 + 144k_2k_3k_4 \\ +144k_1k_4k_5 + 144k_2k_3k_5 + 144k_2k_4k_5 + 144k_3k_4k_5 + 192k_1k_2k_3k_4 + 192k_1k_2k_3k_5 \\ +192k_1k_2k_4k_5 + 192k_1k_3k_4k_5 + 192k_2k_3k_4k_5 + 256k_1k_2k_3k_4k_5 + 60. \end{cases}
\end{aligned} \tag{21}$$

By direct computation, we observe that the binary and 4-ary multiplications are not closed, but the ternary and 5-ary ones are closed. In general, the product of $n = 2\ell_\mu + 1$ ($\ell_\mu \in \mathbb{N}$) elements of the residue class $[3]_4$ is in the class, which is the "quantization" rule for multiplication (7) and (9). Again, we consider the minimal arity $n = 3$ of multiplication and observe that $\langle \{r_k\} \mid \mu_3 \rangle$ is a commutative ternary semigroup, where

$$\mu_3\left[r_{k_1}, r_{k_2}, r_{k_3}\right] = r_{k_1} r_{k_2} r_{k_3}, \quad r_{k_i} \in [3]_4 \tag{22}$$

is a nonderived ternary multiplication, and the total ternary associativity of μ_3 is governed by associativity of the binary product in (4) and (22). As opposed to the 5-ary addition, $\langle\{r_k\} \mid \mu_3\rangle$ is not a group, because not all elements have a unique querelement. However, a polyadic identity e defined by (i.e., as a neutral element of the ternary multiplication μ_3)

$$\mu_3[e, e, r_k] = r_k, \quad e, r_k \in [3]_4, \tag{23}$$

exists and is equal to $e = -1$.

The polyadic distributivity between ν_5 and μ_3 [15] follows from the binary distributivity in $\mathbb{Z}$ and (17), (22), and therefore, the residue class $[3]_4$ has the algebraic structure of the polyadic ring

$$\mathbb{Z}_{(5,3)} = \mathbb{Z}_{(5,3)}(3,4) = \left\langle \{r_k\} \mid \nu_5, \widetilde{(\cdot)}, \mu_3, e \right\rangle, \quad e, r_k \in [3]_4, \tag{24}$$

which is a commutative zeroless $(5,3)$-ring with the additive quermapping $\widetilde{(\cdot)}$ (18) and the multiplicative neutral element e (23).

The arity shape of the ring of polyadic integers $\mathbb{Z}_{(m,n)}(a_q, b_q)$ (5) is the (surjective) mapping

$$(a_q, b_q) \Longrightarrow (m, n). \tag{25}$$

The mapping (25) for the lowest values of a_q, b_q is given in Table 1 ($I = I_m(a_q, b_q)$, $J = J_n(a_q, b_q)$).

Table 1. The arity shape mapping (25) for the polyadic ring $\mathbb{Z}_{(m,n)}(a_q, b_q)$ (5). Empty cells indicate that no such structure exists.

$a_q \backslash b_q$	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	$m = \mathbf{3}$ $n = \mathbf{2}$ $I = 1$ $J = 0$	$m = \mathbf{4}$ $n = \mathbf{2}$ $I = 1$ $J = 0$	$m = \mathbf{5}$ $n = \mathbf{2}$ $I = 1$ $J = 0$	$m = \mathbf{6}$ $n = \mathbf{2}$ $I = 1$ $J = 0$	$m = \mathbf{7}$ $n = \mathbf{2}$ $I = 1$ $J = 0$	$m = \mathbf{8}$ $n = \mathbf{2}$ $I = 1$ $J = 0$	$m = \mathbf{9}$ $n = \mathbf{2}$ $I = 1$ $J = 0$	$m = \mathbf{10}$ $n = \mathbf{2}$ $I = 1$ $J = 0$	$m = \mathbf{11}$ $n = \mathbf{2}$ $I = 1$ $J = 0$
2		$m = \mathbf{4}$ $n = \mathbf{3}$ $I = 2$ $J = 2$		$m = \mathbf{6}$ $n = \mathbf{5}$ $I = 2$ $J = 6$	$m = \mathbf{4}$ $n = \mathbf{3}$ $I = 1$ $J = 1$	$m = \mathbf{8}$ $n = \mathbf{4}$ $I = 2$ $J = 2$		$m = \mathbf{10}$ $n = \mathbf{7}$ $I = 2$ $J = 14$	$m = \mathbf{6}$ $n = \mathbf{5}$ $I = 1$ $J = 3$
3			$m = \mathbf{5}$ $n = \mathbf{3}$ $I = 3$ $J = 6$	$m = \mathbf{6}$ $n = \mathbf{5}$ $I = 3$ $J = 48$	$m = \mathbf{3}$ $n = \mathbf{2}$ $I = 1$ $J = 1$	$m = \mathbf{8}$ $n = \mathbf{7}$ $I = 3$ $J = 312$	$m = \mathbf{9}$ $n = \mathbf{3}$ $I = 3$ $J = 3$		$m = \mathbf{11}$ $n = \mathbf{5}$ $I = 3$ $J = 24$
4				$m = \mathbf{6}$ $n = \mathbf{3}$ $I = 4$ $J = 12$	$m = \mathbf{4}$ $n = \mathbf{2}$ $I = 2$ $J = 2$	$m = \mathbf{8}$ $n = \mathbf{4}$ $I = 4$ $J = 36$		$m = \mathbf{10}$ $n = \mathbf{4}$ $I = 4$ $J = 28$	$m = \mathbf{6}$ $n = \mathbf{3}$ $I = 2$ $J = 6$
5					$m = \mathbf{7}$ $n = \mathbf{3}$ $I = 5$ $J = 20$	$m = \mathbf{8}$ $n = \mathbf{7}$ $I = 11$ $J = 11{,}160$	$m = \mathbf{9}$ $n = \mathbf{3}$ $I = 5$ $J = 15$	$m = \mathbf{10}$ $n = \mathbf{7}$ $I = 5$ $J = 8680$	$m = \mathbf{3}$ $n = \mathbf{2}$ $I = 1$ $J = 2$
6						$m = \mathbf{8}$ $n = \mathbf{3}$ $I = 6$ $J = 30$			$m = \mathbf{6}$ $n = \mathbf{2}$ $I = 3$ $J = 3$
7							$m = \mathbf{9}$ $n = \mathbf{3}$ $I = 7$ $J = 42$	$m = \mathbf{10}$ $n = \mathbf{4}$ $I = 7$ $J = 266$	$m = \mathbf{11}$ $n = \mathbf{5}$ $I = 7$ $J = 1680$

Table 1. *Cont.*

$a_q \backslash b_q$	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
8								$m = \mathbf{10}$ $n = \mathbf{3}$ $I = 8$ $J = 56$	$m = \mathbf{6}$ $n = \mathbf{5}$ $I = 4$ $J = 3276$
9									$m = \mathbf{11}$ $n = \mathbf{3}$ $I = 9$ $J = 72$

The binary ring of ordinary integers $\mathbb{Z}$ corresponds to $\left(a_q = 0, b_q = 1\right) \Longrightarrow (2,2)$ or $\mathbb{Z} = \mathbb{Z}_{(2,2)}(0,1)$, $I = J = 0$.

3. Representations of p-Adic Integers

Let us explore briefly some well-known definitions regarding p-adic integers to establish notations (for reviews, see [10,11,16]).

A p-adic integer is an infinite formal sum of the form

$$\boldsymbol{x} = \boldsymbol{x}(p) = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \ldots + \alpha_{i-1} p^{i-1} + \alpha_i p^i + \alpha_{i+1} p^{i+1} + \ldots, \quad \alpha_i \in \mathbb{Z}, \tag{26}$$

where the digits (denoted by Greek letters from the beginning of alphabet) $0 \leq \alpha_i \leq p-1$, and $p \geq 2$ is a fixed prime number. The expansion (26) is called standard (or canonical), and α_i are the p-adic digits which are usually written from the right to the left (positional notation) $\boldsymbol{x} = \boldsymbol{.} \ldots \alpha_{i+1}\alpha_i\alpha_{i-1} \ldots \alpha_2\alpha_1\alpha_0$ or sometimes $\boldsymbol{x} = \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_i, \alpha_{i+1} \ldots\}$. The set of p-adic integers is a commutative ring (of p-adic integers) denoted by $\mathbb{Z}_p = \{\boldsymbol{x}\}$, and the ring of ordinary integers (sometimes called "rational" integers) $\mathbb{Z}$ is its (binary) subring.

The so-called coherent representation of $\mathbb{Z}_p$ is based on the (inverse) projective limit of finite fields $\mathbb{Z} \diagup p^l \mathbb{Z}$, because the surjective map $\mathbb{Z}_p \longrightarrow \mathbb{Z} \diagup p^l \mathbb{Z}$ defined by

$$\alpha_0 + \alpha_1 p + \alpha_2 p^2 + \ldots + \alpha_i p^i + \ldots \mapsto \left(\alpha_0 + \alpha_1 p + \alpha_2 p^2 + \ldots + \alpha_{l-1} p^{l-1}\right) \bmod p^l \tag{27}$$

is a ring homomorphism. In this case, a p-adic integer is the infinite Cauchy sequence that converges to

$$\boldsymbol{x} = \boldsymbol{x}(p) = \{x_i(p)\}_{i=1}^{\infty} = \{x_1(p), x_2(p), \ldots, x_i(p) \ldots\}, \tag{28}$$

where

$$x_i(p) = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \ldots + \alpha_{l-1} p^{l-1} \tag{29}$$

with the coherency condition

$$x_{i+1}(p) \equiv x_i(p) \bmod p^i, \quad \forall i \geq 1, \tag{30}$$

and the p-adic digits are $0 \leq \alpha_i \leq p-1$.

If $0 \leq x_i(p) \leq p^i - 1$ for all $i \geq 1$, then the coherent representation (28) is said to be reduced. The ordinary integers $x \in \mathbb{Z}$ embed into p-adic integers as constant infinite sequences by $x \mapsto \{x, x, \ldots, x, \ldots\}$.

Using the fact that the process of reducing modulo p^i is equivalent to vanishing the last i digits, the coherency condition (30) leads to a sequence of partial sums [16]

$$\boldsymbol{x} = \boldsymbol{x}(p) = \{y_i(p)\}_{i=1}^{\infty} = \{y_1(p), y_2(p), \ldots, y_i(p) \ldots\}, \tag{31}$$

where

$$y_1(p) = \alpha_0,\ y_2(p) = \alpha_0 + \alpha_1 p,$$
$$y_3(p) = \alpha_0 + \alpha_1 p + \alpha_2 p^2,\ y_4(p) = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \alpha_3 p^3,\ \dots\ . \tag{32}$$

Sometimes, the partial sum representation (31) is simpler for p-adic integer computations.

4. (m,n)-Rings of p-Adic Integers

As may be seen from Section 2 and [13,14], the construction of the nonderived (m,n)-rings of ordinary ("rational") integers $\mathbb{Z}_{(m,n)}(a_q, b_q)$ (5) can be performed in terms of residue class representatives (1). To introduce a p-adic analog of the residue class (1), one needs some ordering concept, which does not exist for p-adic integers [16]. Nevertheless, one could informally define the following analog of ordering.

Definition 1. *A "componentwise strict order" $<_{comp}$ is a multicomponent binary relation between p-adic numbers $\boldsymbol{a} = \{\alpha_i\}_{i=0}^{\infty}$, $0 \leq \alpha_i \leq p-1$ and $\boldsymbol{b} = \{\beta_i\}_{i=0}^{\infty}$, $0 \leq \beta_i \leq p-1$, such that*

$$\boldsymbol{a} <_{comp} \boldsymbol{b} \Longleftrightarrow \alpha_i < \beta_i, \quad \text{for all } i = 0,\dots,\infty, \quad \boldsymbol{a},\boldsymbol{b} \in \mathbb{Z}_p, \quad \alpha_i, \beta_i \in \mathbb{Z}. \tag{33}$$

A "componentwise nonstrict order" $\leq_{comp}$ is defined in the same way, but using the nonstrict order $\leq$ for component integers from $\mathbb{Z}$ (digits).

Using this definition, we can define a p-adic analog of the residue class informally by changing $\mathbb{Z}$ to $\mathbb{Z}_p$ in (1).

Definition 2. *A p-adic analog of the residue class of $\boldsymbol{a}$ modulo $\boldsymbol{b}$ is*

$$[\boldsymbol{a}]_{\boldsymbol{b}} = \left\{ \{ \boldsymbol{r}_{\boldsymbol{k}}(\boldsymbol{a},\boldsymbol{b}) \} \mid \boldsymbol{a},\boldsymbol{b},\boldsymbol{k} \in \mathbb{Z}_p, 0 \leq \boldsymbol{a} < \boldsymbol{b} \right\}, \tag{34}$$

and the generic representative of the class is

$$\boldsymbol{r}_{\boldsymbol{k}}(\boldsymbol{a},\boldsymbol{b}) = \boldsymbol{a} +_p \boldsymbol{b} \bullet_p \boldsymbol{k}, \tag{35}$$

where $+_p$ and $\bullet_p$ are the binary sum and the binary product of p-adic integers (we treat them componentwise in the partial sum representation (32)), and the ith component of (35)'s right hand side is computed by mod p^i.

As with the ordinary ("rational") integers (1), the p-adic integer $\boldsymbol{a}$ can be treated as a type of remainder for the representative $\boldsymbol{r}_{\boldsymbol{k}}(\boldsymbol{a},\boldsymbol{b})$ when divided by the p-adic integer $\boldsymbol{b}$. We denote the corresponding p-adic analog of (2) (informally, a p-adic analog of the congruence modulo $\boldsymbol{b}$) as

$$\boldsymbol{r} = \boldsymbol{a}(\mathrm{Mod}_p\, \boldsymbol{b}). \tag{36}$$

Remark 1. *In general, to build a nonderived (m,n)-ring along the lines of Section 2, we do not need any analog of the residue class at all, but only the concrete form of the representative (35). Then, demanding the closure of m-ary addition (3) and n-ary multiplication (4), we obtain conditions on the parameters (now digits of p-adic integers) similar to (6) and (7).*

In the partial sum representation (31), the case of ordinary ("rational") integers corresponds to the first component (first digit α_0) of the p-adic integer (32), and higher components can be computed using the explicit formulas for sum and product of p-adic integers [17]. Because they are too cumbersome, we present here the "block-schemes" of the computations, while concrete examples can be obtained componentwise using (32).

Lemma 1. *The p-adic analog of the residue class (34) is a commutative m-ary group $\langle [\boldsymbol{a}]_{\boldsymbol{b}} \mid \boldsymbol{\nu}_m \rangle$, if*

$$(m-1)\boldsymbol{a} = \boldsymbol{b} \bullet_p \boldsymbol{I}, \tag{37}$$

where $\boldsymbol{I}$ is a p-adic integer (addition shape invariant), and the nonderived m-ary addition $\boldsymbol{\nu}_m$ is the repeated binary sum of m representatives $\boldsymbol{r_k} = \boldsymbol{r_k}(\boldsymbol{a}, \boldsymbol{b})$

$$\boldsymbol{\nu}_m\left[\boldsymbol{r}_{\boldsymbol{k}_1}, \boldsymbol{r}_{\boldsymbol{k}_2}, \ldots, \boldsymbol{r}_{\boldsymbol{k}_m}\right] = \boldsymbol{r}_{\boldsymbol{k}_1} +_p \boldsymbol{r}_{\boldsymbol{k}_2} +_p \ldots +_p \boldsymbol{r}_{\boldsymbol{k}_m}. \tag{38}$$

Proof. The condition of closure for the m-ary addition $\boldsymbol{\nu}_m$ is $\boldsymbol{r}_{\boldsymbol{k}_1} +_p \boldsymbol{r}_{\boldsymbol{k}_2} +_p \ldots +_p \boldsymbol{r}_{\boldsymbol{k}_m} = \boldsymbol{r}_{\boldsymbol{k}_0}$ in the notation of (34). Using (35), it provides $m\boldsymbol{a} + \boldsymbol{b} \bullet_p (\boldsymbol{k}_1 +_p \boldsymbol{k}_2 +_p \ldots +_p \boldsymbol{k}_m) = \boldsymbol{a} +_p \boldsymbol{b} \bullet_p \boldsymbol{k}_0$, which is equivalent to (37), where $\boldsymbol{I} = \boldsymbol{k}_0 -_p (\boldsymbol{k}_1 +_p \boldsymbol{k}_2 +_p \ldots +_p \boldsymbol{k}_m)$. The querelement $\boldsymbol{r}_{\bar{\boldsymbol{k}}}$ [18] satisfies

$$\boldsymbol{\nu}_m[\boldsymbol{r}_{\boldsymbol{k}}, \boldsymbol{r}_{\boldsymbol{k}}, \ldots, \boldsymbol{r}_{\boldsymbol{k}}, \boldsymbol{r}_{\bar{\boldsymbol{k}}}] = \boldsymbol{r}_{\boldsymbol{k}}, \tag{39}$$

which has a unique solution $\bar{\boldsymbol{k}} = (2-m)\boldsymbol{k} - \boldsymbol{I}$. Therefore, each element of $[\boldsymbol{a}]_{\boldsymbol{b}}$ is invertible with respect to $\boldsymbol{\nu}_m$, and $\langle [\boldsymbol{a}]_{\boldsymbol{b}} \mid \boldsymbol{\nu}_m \rangle$ is a commutative m-ary group. □

Lemma 2. *The p-adic analog of the residue class (34) is a commutative n-ary semigroup $\langle [\boldsymbol{a}]_{\boldsymbol{b}} \mid \boldsymbol{\mu}_n \rangle$, if*

$$\boldsymbol{a}^n - \boldsymbol{a} = \boldsymbol{b} \bullet_p \boldsymbol{J}, \tag{40}$$

where $\boldsymbol{J}$ is a p-adic integer (multiplication shape invariant), and the nonderived m-ary multiplication $\boldsymbol{\nu}_m$ is the repeated binary product of n representatives

$$\boldsymbol{\mu}_n\left[\boldsymbol{r}_{\boldsymbol{k}_1}, \boldsymbol{r}_{\boldsymbol{k}_2}, \ldots, \boldsymbol{r}_{\boldsymbol{k}_n}\right] = \boldsymbol{r}_{\boldsymbol{k}_1}\boldsymbol{r}_{\boldsymbol{k}_2}\ldots\boldsymbol{r}_{\boldsymbol{k}_n}. \tag{41}$$

Proof. The condition of closure for the n-ary multiplication $\boldsymbol{\mu}_n$ is $\boldsymbol{r}_{\boldsymbol{k}_1} \bullet_p \boldsymbol{r}_{\boldsymbol{k}_2} \bullet_p \ldots \bullet_p \boldsymbol{r}_{\boldsymbol{k}_m} = \boldsymbol{r}_{\boldsymbol{k}_0}$. Using (35) and opening brackets, we obtain $n\boldsymbol{a} + \boldsymbol{b} \bullet_p \boldsymbol{J}_1 = \boldsymbol{a} +_p \boldsymbol{b} \bullet_p \boldsymbol{k}_0$, where $\boldsymbol{J}_1$ is some p-adic integer, which gives (40) with $\boldsymbol{J} = \boldsymbol{k}_0 -_p \boldsymbol{J}_1$. □

Combining the conditions (37) and (40), we arrive at

Theorem 1. *The p-adic analog of the residue class (34) becomes a (m,n)-ring with m-ary addition (38) and n-ary multiplication (41)*

$$\mathbb{Z}_{(m,n)}(\boldsymbol{a}_q, \boldsymbol{b}_q) = \left\langle [\boldsymbol{a}_q]_{\boldsymbol{b}_q} \mid \boldsymbol{\nu}_m, \boldsymbol{\mu}_n \right\rangle, \tag{42}$$

when the p-adic integers $\boldsymbol{a}_q, \boldsymbol{b}_q \in \mathbb{Z}_p$ are solutions of the equations

$$m\boldsymbol{a}_q = \boldsymbol{a}_q(\mathrm{Mod}_p\, \boldsymbol{b}_q), \tag{43}$$
$$\boldsymbol{a}_q^n = \boldsymbol{a}_q(\mathrm{Mod}_p\, \boldsymbol{b}_q). \tag{44}$$

Proof. The conditions (43)–(44) are equivalent to (37) and (40), respectively, which shows that $[\boldsymbol{a}_q]_{\boldsymbol{b}_q}$ (considered as a set of representatives (35)) is simultaneously an m-ary group with respect to $\boldsymbol{\nu}_m$, and an n-ary semigroup with respect to $\boldsymbol{\mu}_n$, and is therefore a (m,n)-ring. □

If we work in the partial sum representation (32), the procedure of finding the digits of p-adic integers $\boldsymbol{a}_q, \boldsymbol{b}_q \in \mathbb{Z}_p$ such that $[\boldsymbol{a}_q]_{\boldsymbol{b}_q}$ becomes a (m,n)-ring with initially fixed arities is recursive. To find the first digits α_0 and β_0 that are ordinary integers, we use the Equations (6)–(9), and for their arity shape, we use Table 1. Next, we consider the second components of (32) to find the digits α_1 and β_1 of $\boldsymbol{a}_q$ and $\boldsymbol{b}_q$ by solving Equations (37) and (40) (these having initially given arities m and n from the first step) by application of the exact formulas from [17]. In this way, we can find as many digits $(\alpha_0, , \alpha_{i_{\max}})$, $(\beta_0, , \beta_{i_{\max}})$ of $\boldsymbol{a}_q$, and $\boldsymbol{b}_q$ as needed for our accuracy preferences in building the polyadic ring of p-adic integers $\mathbb{Z}_{(m,n)}(\boldsymbol{a}_q, \boldsymbol{b}_q)$ (42).

Further development and examples will appear elsewhere.

5. Conclusions

The study of "external" residue class properties is a foundational subject in standard number theory. We have investigated their "internal" properties to understand the algebraic structure of the representative set of a fixed residue class. We found that if the parameters of a class satisfy some special "quantization" conditions, the set of representatives becomes a polyadic ring. We introduced the arity shape, a surjective-like mapping of the residue class parameters to the arity of addition m, and the arity of multiplication n, which result in commutative (m, n)-rings (see Table 1).

We then generalized the approach thus introduced to p-adic integers by defining an analog of a residue class for them. Using the coherent representation for p-adic integers as partial sums we defined the p-adic analog of the "quantization" conditions in a componentwise manner for when the set of p-adic representatives form a polyadic ring. Finally, we proposed a recursive procedure to find any desired digits of the p-adic residue class parameters.

The proposed polyadic algebraic structure of p-adic numbers may lead to new symmetries and features in p-adic mathematical physics and the corresponding particle models.

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