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Higher Braid Groups and Regular Semigroups from Polyadic-Binary Correspondence

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Abstract: In this note, we first consider a ternary matrix group related to the von Neumann regular semigroups and to the Artin braid group (in an algebraic way). The product of a special kind of ternary matrices (idempotent and of finite order) reproduces the regular semigroups and braid groups with their binary multiplication of components. We then generalize the construction to the higher arity case, which allows us to obtain some higher degree versions (in our sense) of the regular semigroups and braid groups. The latter are connected with the generalized polyadic braid equation and R-matrix introduced by the author, which differ from any version of the well-known tetrahedron equation and higher-dimensional analogs of the Yang-Baxter equation, n-simplex equations. The higher degree (in our sense) Coxeter group and symmetry groups are then defined, and it is shown that these are connected only in the non-higher case.

Keywords: regular semigroup; braid group; generator; relation; presentation; Coxeter group; symmetric group; polyadic matrix group; querelement; idempotence; finite order element

MSC: 16T25; 17A42; 20B30; 20F36; 20M17; 20N15

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1. Introduction

We begin by observing that the defining relations of the von Neumann regular semigroups (e.g., References [1–3]) and the Artin braid group [4,5] correspond to such properties of ternary matrices (over the same set) as idempotence and the orders of elements (period). We then generalize the correspondence thus introduced to the polyadic case and thereby obtain higher degree (in our definition) analogs of the former. The higher (degree) regular semigroups obtained in this way have appeared previously in semisupermanifold theory [6] and higher regular categories in Topological Quantum Field Theory [7]. The representations of the higher braid relations in vector spaces coincide with the higher braid equation and corresponding generalized R-matrix obtained in Reference [8], as do the ordinary braid group and the Yang-Baxter equation [9,10]. The proposed constructions use polyadic group methods and differ from the tetrahedron equation [11] and n-simplex equations [12] connected with the braid group representations [13,14], as well as from higher braid groups of Reference [15]. Finally, we define higher degree (in our sense) versions of the Coxeter group and the symmetric group and show that they are connected in the classical (i.e., non-higher) case only.

2. Preliminaries

There is a general observation [16] that a block-matrix, forming a semisimple $(2,k)$-ring (Artinian ring with binary addition and k-ary multiplication) has the shape:

$$\mathsf{M}(k-1) \equiv \mathsf{M}^{((k-1)\times(k-1))}$$
$$= \begin{pmatrix} 0 & \mathsf{m}^{(i_1\times i_2)} & 0 & \dots & 0 \\ 0 & 0 & \mathsf{m}^{(i_2\times i_3)} & \dots & 0 \\ 0 & 0 & \vdots & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & \mathsf{m}^{(i_{k-2}\times i_{k-1})} \\ \mathsf{m}^{(i_{k-1}\times i_1)} & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{1}$$

In other words, it is given by the cyclic shift $(k-1)\times(k-1)$ matrix, in which identities are replaced by blocks of suitable sizes and with arbitrary entries.

The set $\{\mathsf{M}(k-1)\}$ is closed with respect to the product of k matrices, while the product of two (or less than k of such matrices) has not the same form as (1). In this sense, the k-ary multiplication is not reducible (or derived) to the binary multiplication; therefore, we will call them k-ary matrices. This "non-reducibility" is the key property of k-ary matrices, which will be used in the below constructions. The matrices of the shape (1) form a k-ary semigroup (which cannot be reduced to binary semigroups), and, when the blocks are over an associative binary ring, then the total k-ary associativity follows from the ordinary associativity of the binary matrix multiplication of the blocks.

Our proposal is to use single arbitrary elements (from rings with associative multiplication) in place of the blocks $\mathsf{m}^{(i\times j)}$, supposing that the elements of the multiplicative part G of the rings form binary (semi)groups having some special properties. Then, we investigate the similar correspondence between the (multiplicative) properties of the matrices $\mathsf{M}^{(k-1)}$, related to idempotence and order, and the appearance of the relations in G leading to regular semigroups and braid groups, respectively. We call this connection a polyadic matrix-binary (semi)group correspondence (or in short the *polyadic-binary correspondence*).

In the lowest arity case $k=3$, the ternary case, the 2×2 matrices $\mathsf{M}^{(2)}$ are anti-triangle. From $(\mathsf{M}(2))^3 = \mathsf{M}(2)$ and $(\mathsf{M}(2))^3 \sim \mathsf{E}(2)$ (where $\mathsf{E}(2)$ is the ternary identity; see below), we obtain the correspondences of the above conditions on $\mathsf{M}(2)$ with the ordinary regular semigroups and braid groups, respectively. In this way, we extend the polyadic-binary correspondence on -arities $k\geq 4$ to get the higher relations

$$(\mathsf{M}(k-1))^k = \begin{cases} = \mathsf{M}(k-1) & \text{corresponds to higher } k\text{-degree regular semigroups,} \\ = q\mathsf{E}(k-1) & \text{corresponds to higher } k\text{-degree braid groups,} \end{cases} \tag{2}$$

where $\mathsf{E}(k-1)$ is the k-ary identity (see below), and q is a fixed element of the braid group.

3. Ternary Matrix Group Corresponding to the Regular Semigroup

Let $G_{free} = \left\{ \mathrm{G} \mid \mu_2^g \right\}$ be a free semigroup with the underlying set $\mathrm{G} = \left\{ g^{(i)} \right\}$ and the binary multiplication. The anti-diagonal matrices over G_{free}

$$M^g(2) = M^{(2\times 2)}\left(g^{(1)}, g^{(2)}\right) = \begin{pmatrix} 0 & g^{(1)} \\ g^{(2)} & 0 \end{pmatrix}, \quad g^{(1)}, g^{(2)} \in G_{free} \tag{3}$$

form a ternary semigroup $\mathcal{M}_3^g \equiv \mathcal{M}_{k=3}^g = \left\{ \mathrm{M}^g(2) \mid \mu_3^g \right\}$, where $\mathrm{M}^g(2) = \{M^g(2)\}$ is the set of ternary matrices (3) closed under the ternary multiplication

$$\mu_3^g\left[M_1^g(2), M_2^g(2), M_3^g(2)\right] = M_1^g(2) M_2^g(2) M_3^g(2), \quad \forall M_1^g(2), M_2^g(2), M_3^g(2) \in \mathcal{M}_3^g, \tag{4}$$

being the ordinary matrix product. Recall that an element $M^g(2) \in \mathcal{M}_3^g$ is idempotent, if

$$\mu_3^g[M^g(2), M^g(2), M^g(2)] = M^g(2), \tag{5}$$

which, in the matrix form (4), leads to

$$\left(M^{g}(2)\right)^{3}=M^{g}(2). \tag{6}$$

We denote the set of idempotent ternary matrices by $\mathrm{M}^{g}_{\mathrm{id}}(2)=\left\{M^{g}_{\mathrm{id}}(2)\right\}$.

Definition 1. *A ternary matrix semigroup in which every element is idempotent (6) is called an idempotent ternary semigroup.*

Using (3) and (6), the idempotence expressed in components gives the regularity conditions

$$g^{(1)}g^{(2)}g^{(1)}=g^{(1)}, \tag{7}$$

$$g^{(2)}g^{(1)}g^{(2)}=g^{(2)},\quad \forall g^{(1)},g^{(2)}\in G_{free}. \tag{8}$$

Definition 2. *A binary semigroup G_{free} in which any two elements are mutually regular (7) and (8) is called a regular semigroup G_{reg}.*

Proposition 1. *The set of idempotent ternary matrices (6) form a ternary semigroup $\mathcal{M}^{g}_{3,\mathrm{id}}=\{\mathrm{M}_{\mathrm{id}}(2)\mid\mu_{3}\}$, if G_{reg} is abelian.*

Proof. It follows from (7) and (8) that idempotence (and following from it regularity) is preserved with respect to the ternary multiplication (4), only when any $g^{(1)},g^{(2)}\in G_{free}$ commute. □

Definition 3. *We say that the set of idempotent ternary matrices $\mathrm{M}^{g}_{\mathrm{id}}(2)$ (6) is in ternary-binary correspondence with the regular (binary) semigroup G_{reg} and write this as*

$$\mathrm{M}^{g}_{\mathrm{id}}(2)\backsimeq G_{reg}. \tag{9}$$

This means that such property of the ternary matrices as their idempotence (6) leads to the regularity conditions (7) and (8) in the correspondent binary group G_{free}.

Remark 1. *The correspondence (9) is not a homomorphism and not a bi-element mapping [17], and also not a heteromorphism in the sense of Reference [18], because we do not demand that the set of idempotent matrices $M^{g}_{\mathrm{id}}(2)$ form a ternary semigroup (which is possible in commutative case of G_{free} only; see Proposition 1).*

4. Polyadic Matrix Semigroup Corresponding to the Higher Regular Semigroup

Next we extend the ternary-binary correspondence (9) to the k-ary matrix case (1) and thereby obtain higher k-regular binary semigroups. We use the following notation:

- Round brackets: (k) is size of matrix $k\times k$, as well as the sequential number of a matrix element.
- Square brackets: $[k]$ is number of multipliers in the regularity and braid conditions.
- Angle brackets: $\langle\ell\rangle_{k}$ is the polyadic power (number of k-ary multiplications).

Let us introduce the $(k-1)\times(k-1)$ matrix over a binary group G_{free} of the form (1)

$$M^{g}(k-1)\equiv M^{((k-1)\times(k-1))}\left(g^{(1)},g^{(2)},\ldots,g^{(k-1)}\right)$$
$$=\begin{pmatrix} 0 & g^{(1)} & 0 & \ldots & 0\\ 0 & 0 & g^{(2)} & \ldots & 0\\ 0 & 0 & \vdots & \ddots & \vdots\\ \vdots & \vdots & 0 & \ddots & g^{(k-2)}\\ g^{(k-1)} & 0 & 0 & \ldots & 0 \end{pmatrix}, \tag{10}$$

where $g^{(i)} \in G_{free}$.

Definition 4. *The set of k-ary matrices* $\mathrm{M}^{g}(k-1)$ *(10) over* G_{free} *is a k-ary matrix semigroup* $\mathcal{M}^{g}_{k} = \left\{ \mathrm{M}^{g}(k-1) \mid \mu^{g}_{k} \right\}$*, where the multiplication*

$$\mu^{g}_{k}\left[M^{g}_{1}(k-1), M^{g}_{2}(k-1), \dots, M^{g}_{k}(k-1) \right]$$
$$= M^{g}_{1}(k-1) M^{g}_{2}(k-1) \dots M^{g}_{k}(k-1), \;\; M^{g}_{i}(k-1) \in \mathcal{M}^{g}_{k} \tag{11}$$

is the ordinary product of k matrices $M^{g}_{i}(k-1) \equiv M^{((k-1)\times(k-1))}\left(g^{(1)}_{i}, g^{(2)}_{i}, \dots, g^{(k-1)}_{i} \right)$*; see (10).*

Recall that the polyadic power ℓ of an element M from a k-ary semigroup $\mathcal{M}_k$ is defined by (e.g., Reference [19])

$$M^{\langle \ell \rangle_k} = (\mu_k)^{\ell} \left[\overbrace{M, \dots, M}^{\ell(k-1)+1} \right], \tag{12}$$

such that ℓ coincides with the *number of k-ary multiplications*. In the binary case $k = 2$ the polyadic power is connected with the ordinary power p (*number of elements* in the product) as $p = \ell + 1$, i.e., $M^{\langle \ell \rangle_2} = M^{\ell+1} = M^{p}$. In the ternary case $k = 3$, we have $\langle \ell \rangle_3 = 2\ell + 1$, and so the l.h.s. of (6) is of polyadic power $\ell = 1$.

Definition 5. *An element of a k-ary semigroup* $M \in \mathcal{M}_3$ *is called idempotent, if its first polyadic power coincides with itself*

$$M^{\langle 1 \rangle_k} = M, \tag{13}$$

and $\langle \ell \rangle$*-idempotent, if*

$$M^{\langle \ell \rangle_k} = M, \;\; M^{\langle \ell - 1 \rangle_k} \neq M. \tag{14}$$

Definition 6. *A k-ary semigroup* $\mathcal{M}_k$ *is called idempotent (*ℓ*-idempotent), if each of its elements* $M \in \mathcal{M}_k$ *is idempotent (*$\langle \ell \rangle$*-idempotent).*

Assertion 1. *From* $M^{\langle 1 \rangle_k} = M$ *it follows that* $M^{\langle \ell \rangle_k} = M$*, but not vice-versa; therefore, all* $\langle 1 \rangle$*-idempotent elements are* $\langle \ell \rangle$*-idempotent, but an* $\langle \ell \rangle$*-idempotent element need not be* $\langle 1 \rangle$*-idempotent.*

Therefore, the definition given in(14) makes sense.

Proposition 2. *If a k-ary matrix* $M^{g}(k-1) \in \mathcal{M}^{g}_{k}$ *is idempotent (13), then its elements satisfy the* $(k-1)$ *relations*

$$g^{(1)} g^{(2)}, \dots g^{(k-2)} g^{(k-1)} g^{(1)} = g^{(1)}, \tag{15}$$
$$g^{(2)} g^{(3)}, \dots g^{(k-1)} g^{(1)} g^{(2)} = g^{(2)}, \tag{16}$$
$$\vdots$$
$$g^{(k-1)} g^{(1)} g^{(2)}, \dots g^{(k-2)} g^{(k-1)} = g^{(k-1)}, \quad \forall g^{(1)}, \dots, g^{(k-1)} \in G_{free}. \tag{17}$$

Proof. This follows from (10), (11) and (13). □

Definition 7. *The relations (15)–(17) are called (higher)* $[k]$*-regularity (or higher k-degree regularity). The case* $k = 3$ *is the standard regularity (*$[3]$*-regularity in our notation) (7) and (8).*

Proposition 3. *If a k-ary matrix $M^g(k-1) \in \mathcal{M}^g_k$ is $\langle\ell\rangle$-idempotent (14), then its elements satisfy the following $(k-1)$ relations*

$$\overbrace{\left(g^{(1)}g^{(2)}\dots g^{(k-2)}g^{(k-1)}\right)\dots\left(g^{(1)}g^{(2)},\dots g^{(k-2)}g^{(k-1)}\right)}^{\ell}g^{(1)} = g^{(1)}, \tag{18}$$

$$\overbrace{\left(g^{(2)}g^{(3)},\dots g^{(k-2)}g^{(k-1)}g^{(1)}\right)\dots\left(g^{(2)}g^{(3)}\dots g^{(k-2)}g^{(k-1)}g^{(1)}\right)}^{\ell}g^{(2)} = g^{(2)}, \tag{19}$$

$$\vdots$$

$$\overbrace{\left(g^{(k-1)}g^{(1)}g^{(2)}\dots g^{(k-3)}g^{(k-2)}\right)\dots\left(g^{(k-1)}g^{(1)}g^{(2)}\dots g^{(k-3)}g^{(k-2)}\right)}^{\ell}g^{(k-1)} = g^{(k-1)}, \tag{20}$$

$$\forall g^{(1)},\dots,g^{(k-1)} \in G_{free}.$$

Proof. This also follows from (10), (11), and (14). □

Definition 8. *The relations (15)–(17) are called (higher) $[k]$-$\langle\ell\rangle$-regularity. The case $k=3$ (7) and (8) is the standard regularity ($[3]$-$\langle 1\rangle$-regularity in this notation).*

Definition 9. *A binary semigroup G_{free}, in which any $k-1$ elements are $[k]$-regular ($[k]$-$\langle\ell\rangle$-regular), is called a higher $[k]$-regular ($[k]$-$\langle\ell\rangle$-regular) semigroup $G_{reg}[k]$ ($G_{\langle\ell\rangle\text{-}reg}[k]$).*

Similarly to Assertion 1, it is seen that $[k]$-$\langle\ell\rangle$-regularity (18)–(20) follows from $[k]$-regularity (15)–(17), but not the other way around; therefore, we have:

Assertion 2. *If a binary semigroup $G_{reg}[k]$ is $[k]$-regular, then it is $[k]$-$\langle\ell\rangle$-regular as well, but not vice-versa.*

Proposition 4. *The set of idempotent ($\langle\ell\rangle$-idempotent) k-ary matrices $\mathrm{M}^g_{\mathrm{id}}(k-1)$ form a k-ary semigroup $\mathcal{M}^g_{3,\mathrm{id}} = \left\{\mathrm{M}^g_{\mathrm{id}}(k-1) \mid \mu^g_k\right\}$, if and only if $G_{reg}[k]$ ($G_{\ell\text{-}reg}[k]$) is abelian.*

Proof. It follows from (15)–(20) that the idempotence ($\langle\ell\rangle$-idempotence) and the following $[k]$-regularity ($[k]$-$\langle\ell\rangle$-regularity) are preserved with respect the k-ary multiplication (11) only in the case, when all $g^{(1)},\dots,g^{(k-1)} \in G_{free}$ mutually commute. □

By analogy with (9), we have:

Definition 10. *We will say that the set of k-ary $(k-1)\times(k-1)$ matrices $\mathrm{M}^g_{\mathrm{id}}(k-1)$ (10) over the underlying set G is in polyadic-binary correspondence with the binary $[k]$-regular semigroup $G_{reg}[k]$ and write this as*

$$\mathrm{M}^g_{\mathrm{id}}(k-1) \mathrel{\substack{\frown\\\smile}} G_{reg}[k]. \tag{21}$$

Thus, using the idempotence condition for k-ary matrices in components (being simultaneously elements of a binary semigroup G_{free}) and the polyadic-binary correspondence (21) we obtain the higher regularity conditions (15)–(20) generalizing the ordinary regularity (7) and (8), which allows us to define the higher $[k]$-regular binary semigroups $G_{reg}[k]$ ($G_{\langle\ell\rangle\text{-}reg}[k]$).

Example 1. *The lowest nontrivial ($k\geq 3$) case is $k=4$, where the 3×3 matrices over G_{free} are of the shape*

$$M(3) = M^{(3\times 3)} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix}, \quad a,b,c \in G_{free}, \tag{22}$$

and they form the 4*-ary matrix semigroup* $\mathcal{M}_4^g$*. The idempotence* $(M(3))^{\langle 1\rangle_4} = (M(3))^3 = M(3)$ *gives three* $[4]$*-regularity conditions*

$$abca = a, \tag{23}$$
$$bcab = b, \tag{24}$$
$$cabc = c. \tag{25}$$

According to the polyadic-binary correspondence (21), the conditions (23)–(25) are $[4]$*-regularity relations for the binary semigroup* G_{free}*, which defines to the higher* $[4]$*-regular binary semigroup* $G_{reg}[4]$*.*

In the case $\ell = 2$*, we have* $(M(3))^{\langle 2\rangle_4} = (M(3))^7 = M(3)$*, which gives three* $[4]$*-*$\langle 2\rangle$*-regularity conditions (they are different from* $[7]$*-regularity)*

$$abcabca = a, \tag{26}$$
$$bcabcab = b, \tag{27}$$
$$cabcabc = c, \tag{28}$$

and these define the higher $[4]$*-*$\langle 2\rangle$*-regular binary semigroup* $G_{\langle 2\rangle\text{-}reg}[4]$*. Obviously, (26)–(28) follow from (23)–(25), but not vice-versa.*

The higher regularity conditions (23)–(25) obtained above from the idempotence of polyadic matrices using the polyadic-binary correspondence, appeared first in Reference [20] and were then used for transition functions in the investigation of semisupermanifolds [6] and higher regular categories in TQFT [7,21].

Now, we turn to the second line of (2), and in the same way as above introduce higher degree braid groups.

5. Ternary Matrix Group Corresponding to the Braid Group

Recall the definition of the Artin braid group [22] in terms of generators and relations [4] (we follow the algebraic approach; see, e.g., Reference [23]).

The *Artin braid group* B_n (with n strands and the identity $e \in B_n$) has the presentation by $n-1$ generators $\sigma_1, \dots, \sigma_{n-1}$ satisfying $n(n-1)/2$ relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad 1 \le i \le n-2, \tag{29}$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i-j| \ge 2, \tag{30}$$

where (29) are called the *braid relations*, and (30) are called *far commutativity*. A general element of B_n is a word of the form

$$w = \sigma_{i_1}^{p_1} \dots \sigma_{i_r}^{p_r} \dots \sigma_{i_m}^{p_m}, \quad i_m = 1, \dots, n, \tag{31}$$

where $p_r \in \mathbb{Z}$ are (positive or negative) powers of the generators σ_{i_r}, $r = 1, \dots, m$ and $m \in \mathbb{N}$.

For instance, B_3 is generated by σ_1 and σ_2 satisfying one relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, and is isomorphic to the trefoil knot group. The group B_4 has 3 generators $\sigma_1, \sigma_2, \sigma_3$ satisfying

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \tag{32}$$
$$\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3, \tag{33}$$
$$\sigma_1\sigma_3 = \sigma_3\sigma_1. \tag{34}$$

The representation theory of B_n is well known and well established [4,5]. The connections with the Yang-Baxter equation were investigated, e.g., in Reference [9].

Now, we build a ternary group of matrices over B_n having generators satisfying relations which are connected with the braid relations (29) and (30). We then generalize our construction to a k-ary matrix group, which gives us the possibility to "go back" and define some special higher analogs of the Artin braid group.

Let us consider the set of anti-diagonal 2×2 matrices over B_n

$$M(2) = M^{(2\times 2)}\left(b^{(1)}, b^{(2)}\right) = \begin{pmatrix} 0 & b^{(1)} \\ b^{(2)} & 0 \end{pmatrix}, \quad b^{(1)}, b^{(2)} \in B_n. \tag{35}$$

Definition 11. *The set of matrices* $\mathrm{M}(2) = \{M(2)\}$ *(35) over* B_n *form a ternary matrix semigroup* $\mathcal{M}_{k=3} = \mathcal{M}_3 = \{\mathrm{M}(2) \mid \mu_3\}$, *where* $k = 3$ *is the arity of the following multiplication*

$$\mu_3[M_1(2), M_2(2), M_3(2)] \equiv M_1(2), M_2(2), M_3(2) = M(2), \tag{36}$$

$$b_1^{(1)} b_2^{(2)} b_3^{(1)} = b^{(1)}, \tag{37}$$

$$b_1^{(2)} b_2^{(1)} b_3^{(2)} = b^{(2)}, \quad b_i^{(1)}, b_i^{(2)} \in B_n, \quad M_i(2) = \begin{pmatrix} 0 & b_i^{(1)} \\ b_i^{(2)} & 0 \end{pmatrix} \tag{38}$$

and the associativity is governed by the associativity of both the ordinary matrix product in the r.h.s. of (36) and B_n.

Proposition 5. $\mathcal{M}^{(3)}$ *is a ternary matrix group.*

Proof. Each element of the ternary matrix semigroup $M(2) \in \mathcal{M}_3$ is invertible (in the ternary sense) and has a *querelement* $\bar{M}(2)$ (a polyadic analog of the group inverse [24]) defined by

$$\mu_3[M(2), M(2), \bar{M}(2)] = \mu_3[M(2), \bar{M}(2), M(2)] = \mu_3[\bar{M}(2), M(2), M(2)] = M(2). \tag{39}$$

It follows from (36)–(38) that

$$\bar{M}(2) = (M(2))^{-1} = \begin{pmatrix} 0 & \left(b^{(1)}\right)^{-1} \\ \left(b^{(2)}\right)^{-1} & 0 \end{pmatrix}, \quad b^{(1)}, b^{(2)} \in B_n, \tag{40}$$

where $\left(M^{(2)}\right)^{-1}$ denotes the ordinary matrix inverse (but not the binary group inverse which does not exist in the k-ary case, $k \geq 3$). Non-commutativity of μ_3 is provided by (37) and (38). □

The ternary matrix group $\mathcal{M}_3$ has the *ternary identity*

$$E(2) = \begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix}, \quad e \in B_n, \tag{41}$$

where e is the identity of the binary group B_n, and

$$\mu_3[M(2), E(2), E(2)] = \mu_3[E(2), M(2), E(2)] = \mu_3[E(2), E(2), M(2)] = M(2). \tag{42}$$

We observe that the ternary product μ_3 in components is "naturally braided" (37) and (38). This allows us to ask the question: which generators of the ternary group $\mathcal{M}_3$ can be constructed using the Artin braid group generators $\sigma_i \in B_n$ and the relations (29) and (30)?

6. Ternary Matrix Generators

Let us introduce $(n-1)^2$ ternary 2×2 *matrix generators*

$$\Sigma_{ij}(2) = \Sigma_{ij}^{(2\times 2)}(\sigma_i, \sigma_j) = \begin{pmatrix} 0 & \sigma_i \\ \sigma_j & 0 \end{pmatrix}, \tag{43}$$

where $\sigma_i \in B_n$, $i = 1\ldots, n-1$ are generators of the Artin braid group. The querelement of $\Sigma_{ij}(2)$ is defined by analogy with (40) as

$$\bar{\Sigma}_{ij}(2) = \left(\Sigma_{ij}(2)\right)^{-1} = \begin{pmatrix} 0 & \sigma_j^{-1} \\ \sigma_i^{-1} & 0 \end{pmatrix}. \quad (44)$$

Now, we are in a position to present a ternary matrix group with multiplication μ_3 in terms of generators and relations in such a way that the braid group relations (29) and (30) will be reproduced.

Proposition 6. *The relations for the matrix generators $\Sigma_{ij}(2)$ corresponding to the braid group relations for σ_i (29) and (30) have the form*

$$\mu_3[\Sigma_{i,i+1}(2), \Sigma_{i,i+1}(2), \Sigma_{i,i+1}(2)] = \mu_3[\Sigma_{i+1,i}(2), \Sigma_{i+1,i}(2), \Sigma_{i+1,i}(2)]$$
$$= q_i^{[3]} E(2), \quad 1 \leq i \leq n-2, \quad (45)$$
$$\mu_3\left[\Sigma_{ij}(2), \Sigma_{ij}(2), E(2)\right] = \mu_3\left[\Sigma_{ji}(2), \Sigma_{ji}(2), E(2)\right], \quad |i-j| \geq 2, \quad (46)$$

where $q_i^{[3]} = \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, and $E(2)$ is the ternary identity (41).

Proof. Use μ_3 as the triple matrix product (36)–(38) and the braid relations (29) and (30). □

Definition 12. *We say that the ternary matrix group $\mathcal{M}_3^{gen\text{-}\Sigma}$ generated by the matrix generators $\Sigma_{ij}(2)$ satisfying the relations (45) and (46) is in ternary-binary correspondence with the braid (binary) group B_n, which is denoted as (cf. (9))*

$$\mathcal{M}_3^{gen\text{-}\Sigma} \,\hat{\approx}\, B_n. \quad (47)$$

Indeed, in components the relations (45) give (29), and (46) leads to (30).

Remark 2. *Note that the above construction is totally different from the bi-element representations of ternary groups considered in Reference [17] (for k-ary groups see [18]).*

Definition 13. *An element $M(2) \in \mathcal{M}_3$ is of finite polyadic (ternary) order, if there exists a finite ℓ such that*

$$M(2)^{\langle \ell \rangle_3} = M(2)^{2\ell+1} = E(2), \quad (48)$$

where $E(2)$ is the ternary matrix identity (41).

Definition 14. *An element $M(2) \in \mathcal{M}_3$ is of finite q-polyadic (q-ternary) order, if there exists a finite ℓ such that*

$$M(2)^{\langle \ell \rangle_3} = M(2)^{2\ell+1} = qE(2), \quad q \in B_n. \quad (49)$$

The relations (45), therefore, say that the ternary matrix generators $\Sigma_{i,j+1}(2)$ are of finite q-ternary order. Each element of $\mathcal{M}_3^{gen\text{-}\Sigma}$ is a ternary matrix word (analogous to the binary word (31)), being the ternary product of the polyadic powers (12) of the 2×2 matrix generators $\Sigma_{ij}^{(2)}$ and their querelements $\bar{\Sigma}_{ij}^{(2)}$ (on choosing the first or second row)

$$W = \begin{pmatrix} \Sigma_{i_1 j_1}(2) \\ \bar{\Sigma}_{i_1 j_1}(2) \end{pmatrix}^{\langle \ell_1 \rangle_3}, \ldots, \begin{pmatrix} \Sigma_{i_r j_r}(2) \\ \bar{\Sigma}_{i_r j_r}(2) \end{pmatrix}^{\langle \ell_r \rangle_3}, \ldots, \begin{pmatrix} \Sigma_{i_m j_m}(2) \\ \bar{\Sigma}_{i_m j_m}(2) \end{pmatrix}^{\langle \ell_m \rangle_3}$$
$$= \begin{pmatrix} \Sigma_{i_1 j_1}(2) \\ \bar{\Sigma}_{i_1 j_1}(2) \end{pmatrix}^{2\ell_1+1}, \ldots, \begin{pmatrix} \Sigma_{i_r j_r}(2) \\ \bar{\Sigma}_{i_r j_r}(2) \end{pmatrix}^{2\ell_r+1}, \ldots, \begin{pmatrix} \Sigma_{i_m j_m}(2) \\ \bar{\Sigma}_{i_m j_m}(2) \end{pmatrix}^{2\ell_m+1}, \quad (50)$$

where $r = 1, \ldots, m$, $i_r, j_r = 1, \ldots, n$ (from B_n), $\ell_r, m \in \mathbb{N}$. In the ternary case, the total number of multipliers in (50) should be compatible with (12), i.e., $(2\ell_1 + 1) + \ldots + (2\ell_r + 1) + \ldots + (2\ell_m + 1) = 2\ell_W + 1$, $\ell_W \in \mathbb{N}$, and m is, therefore, odd. Thus, we have:

Remark 3. *The ternary words (50) in components give only a subset of the binary words (31), and so $\mathcal{M}_3^{gen\text{-}\Sigma}$ corresponds to B_n, but does not present it.*

Example 2. *For B_3, we have only two ternary 2×2 matrix generators*

$$\Sigma_{12}(2) = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_2 & 0 \end{pmatrix}, \quad \Sigma_{21}(2) = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_1 & 0 \end{pmatrix}, \tag{51}$$

satisfying

$$(\Sigma_{12}(2))^{\langle 1 \rangle_3} = (\Sigma_{12}(2))^3 = q_1^{[3]} E(2), \tag{52}$$

$$(\Sigma_{21}(2))^{\langle 1 \rangle_3} = (\Sigma_{21}(2))^3 = q_1^{[3]} E(2), \tag{53}$$

where $q_1^{[3]} = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, and both matrix relations (52) and (53) coincide in components.

Example 3. *For B_4, the ternary matrix group $\mathcal{M}_3^{gen\text{-}\Sigma}$ is generated by more generators satisfying the relations*

$$(\Sigma_{12}(2))^3 = q_1^{(3)} E(2), \tag{54}$$

$$(\Sigma_{21}(2))^3 = q_1^{(3)} E(2), \tag{55}$$

$$(\Sigma_{23}(2))^3 = q_2^{(3)} E(2), \tag{56}$$

$$(\Sigma_{32}(2))^3 = q_2^{(3)} E(2), \tag{57}$$

$$\Sigma_{13}(2)\Sigma_{13}(2)E(2) = \Sigma_{31}(2)\Sigma_{31}(2)E(2), \tag{58}$$

where $q_1^{[3]} = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ and $q_2^{[3]} = \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3$. The first two relations give the braid relations (32) and (33), while the last relation corresponds to far commutativity (34).

7. Generated k-Ary Matrix Group Corresponding the Higher Braid Group

The above construction of the ternary matrix group $\mathcal{M}_3^{gen\text{-}\Sigma}$ corresponding to the braid group B_n can be naturally extended to the k-ary case, which will allow us to "go in the opposite way" and build so called higher degree analogs of B_n (in our sense: the number of factors in braid relations more than 3). We denote such a braid-like group with n generators by $\mathcal{B}_n[k]$, where k is the number of generator multipliers in the braid relations (as in the regularity relations (15)–(17)). Simultaneously, k is the -arity of the matrices (35); therefore, we call $\mathcal{B}_n[k]$ a higher *k-degree analog* of the braid group B_n. In this notation, the Artin braid group B_n is $\mathcal{B}_n[3]$. Now, we build $\mathcal{B}_n[k]$ for any degree k exploiting the "reverse" procedure, as for $k = 3$ and B_n in Section 5. For that, we need a k-ary generalization of the matrices over B_n, which, in the ternary case, are the anti-diagonal matrices $M(2)$ (35), and the generator matrices $\Sigma_{ij}(2)$ (43). Then, using the k-ary analog of multiplication (37) and (38) we will obtain the higher degree (than (29)) braid relations which generate the so called *higher k-degree braid group*. In distinction to the higher degree regular semigroup construction from Section 4, where the k-ary matrices form a semigroup for the Abelian group G_{free}, using the generator matrices, we construct a k-ary matrix semigroup (presented by generators and relations) for any (even non-commutative) matrix entries. In this way, the polyadic-binary correspondence will connect k-ary matrix groups of finite order with higher binary braid groups (cf. idempotent k-ary matrices and higher regular semigroups (21)).

Let us consider a free binary group $\mathcal{B}_{free}$ and construct over it a k-ary matrix group along the lines of Reference [16], similarly to the ternary matrix group $\mathcal{M}_3$ in (35)–(38).

Definition 15. *A set* $\mathrm{M}(k-1) = \{M(k-1)\}$ *of k-ary* $(k-1)\times(k-1)$ *matrices*

$$M(k-1) = M^{((k-1)\times(k-1))}\left(\mathsf{b}^{(1)}, \mathsf{b}^{(2)}, \ldots, \mathsf{b}^{(k-1)}\right) = \begin{pmatrix} 0 & \mathsf{b}^{(1)} & 0 & \ldots & 0 \\ 0 & 0 & \mathsf{b}^{(2)} & \ldots & 0 \\ 0 & 0 & \vdots & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & \mathsf{b}^{(k-2)} \\ \mathsf{b}^{(k-1)} & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad (59)$$

$$\mathsf{b}^{(j)} \in \mathcal{B}_{free}, \quad j = 1, \ldots, k-1,$$

form a k-ary matrix semigroup $\mathcal{M}_k = \{\mathrm{M}(k-1) \mid \mu_k\}$, *where* μ_k *is the k-ary multiplication*

$$\mu_k[M_1(k-1), M_2(k-1), \ldots, M_k(k-1)] = M_1(k-1)M_2(k-1)\ldots M_k(k-1) = M(k-1), \quad (60)$$

$$\mathsf{b}_1^{(1)}\mathsf{b}_2^{(2)}, \ldots \mathsf{b}_{k-1}^{(k-1)}\mathsf{b}_k^{(1)} = \mathsf{b}^{(1)}, \quad (61)$$

$$\mathsf{b}_1^{(2)}\mathsf{b}_2^{(3)}, \ldots \mathsf{b}_{k-1}^{(1)}\mathsf{b}_k^{(2)} = \mathsf{b}^{(2)}, \quad (62)$$

$$\vdots$$

$$\mathsf{b}_1^{(k-1)}\mathsf{b}_2^{(1)}, \ldots \mathsf{b}_{k-1}^{(k-2)}\mathsf{b}_k^{(k-1)} = \mathsf{b}^{(k-1)}, \quad (63)$$

where the r.h.s. of (60) is the ordinary matrix multiplication of k-ary matrices (59) $M_i(k-1) = M^{((k-1)\times(k-1))}\left(\mathsf{b}_i^{(1)}, \mathsf{b}_i^{(2)}, \ldots, \mathsf{b}_i^{(k-1)}\right)$, $i = 1, \ldots, k$.

Proposition 7. $\mathcal{M}_k$ *is a k-ary matrix group.*

Proof. Because $\mathcal{B}_{free}$ is a (binary) group with the identity $e \in \mathcal{B}_{free}$, each element of the k-ary matrix semigroup $M(k-1) \in \mathcal{M}_k$ is invertible (in the k-ary sense) and has a *querelement* $\bar{M}(k-1)$ (see Reference [24]) defined by (cf. (42))

$$\mu_k\left[\overbrace{M(k-1), \ldots, M(k-1)}^{k-1}, \bar{M}(k-1)\right] = \ldots = M(k-1), \quad (64)$$

where $\bar{M}(k-1)$ can be on any place, and so we have k conditions (cf. (39) for $k = 3$). □

The k-ary matrix group has the polyadic identity

$$E(k-1) = E^{((k-1)\times(k-1))} = \begin{pmatrix} 0 & e & 0 & \ldots & 0 \\ 0 & 0 & e & \ldots & 0 \\ 0 & 0 & \vdots & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & e \\ e & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad e \in \mathcal{B}_{free}, \quad (65)$$

satisfying

$$\mu_k[M(k-1), E(k-1), \ldots, E(k-1)] = \ldots = M(k-1), \quad (66)$$

where $M(k-1)$ can be on any place, and so we have k conditions (cf. (42)).

Definition 16. *An element of a k-ary group* $M(k-1) \in \mathcal{M}_k$ *has the polyadic order* ℓ, *if*

$$(M(k-1))^{\langle \ell \rangle_k} \equiv (M(k-1))^{\ell(k-1)+1} = E(k-1), \quad (67)$$

where $E(k-1) \in \mathcal{M}_k$ *is the polyadic identity (65), for* $k=3$*; see (41).*

Definition 17. *An element of the* $(k-1)\times(k-1)$*-matrix over* $\mathcal{B}_{free}$ *that is* $M(k-1) \in \mathcal{M}^{\{k\}}$ *is of finite q-polyadic order, if there exists a finite* ℓ *such that*

$$(M(k-1))^{\langle \ell \rangle_k} \equiv (M(k-1))^{\ell(k-1)+1} = \mathsf{q}E(k-1), \quad \mathsf{q} \in \mathcal{B}_{free}. \tag{68}$$

Let us assume that the binary group $\mathcal{B}_{free}$ is presented by generators and relations (cf. the Artin braid group (29) and (30)), i.e., it is generated by $n-1$ generators $\boldsymbol{\sigma}_i$, $i=1,\dots,n-1$. An element of $\mathcal{B}_n^{gen\text{-}\sigma} \equiv \mathcal{B}_{free}(\boldsymbol{e},\boldsymbol{\sigma}_i)$ is the word of the form (31). To find the relations between $\boldsymbol{\sigma}_i$ we construct the corresponding k-ary matrix generators analogous to the ternary ones (43). Then, using a k-ary version of the relations (45) and (46) for the matrix generators, as the finite order conditions (68), we will obtain the corresponding higher degree braid relations for the binary generators $\boldsymbol{\sigma}_i$ and can, therefore, present a higher degree braid group $\mathcal{B}_n[k]$ in the form of generators and relations.

Using $n-1$ generators $\boldsymbol{\sigma}_i$ of $\mathcal{B}_n^{gen\text{-}\sigma}$, we build $(n-1)^k$ polyadic (or k-ary) $(k-1)\times(k-1)$-matrix generators having $k-1$ indices $i_1,\dots,i_{k-1}=1,\dots,n-1$, as follows

$$\Sigma_{i_1,\dots,i_{k-1}}(k-1) \equiv \Sigma^{((k-1)\times(k-1))}_{i_1,\dots,i_{k-1}}\left(\boldsymbol{\sigma}_{i_1},\dots,\boldsymbol{\sigma}_{i_{k-1}}\right) = \begin{pmatrix} 0 & \boldsymbol{\sigma}_{i_1} & 0 & \dots & 0 \\ 0 & 0 & \boldsymbol{\sigma}_{i_2} & \dots & 0 \\ 0 & 0 & \vdots & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & \boldsymbol{\sigma}_{i_{k-2}} \\ \boldsymbol{\sigma}_{i_{k-1}} & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{69}$$

For the matrix generator $\Sigma_{i_1,\dots,i_{k-1}}(k-1)$ (69), its querelement $\bar{\Sigma}_{i_1,\dots,i_{k-1}}(k-1)$ is defined by (64).

We now build a k-ary matrix analog of the braid relations (29), (45) and of far commutativity (30), (46). Using (69), we obtain $(k-1)$ conditions that the matrix generators are of finite polyadic order (analog of (45))

$$\mu_k[\Sigma_{i,i+1,\dots,i+k-2}(k-1),\Sigma_{i,i+1,\dots,i+k-2}(k-1),\dots,\Sigma_{i,i+1,\dots,i+k-2}(k-1)] \tag{70}$$
$$= \mu_k[\Sigma_{i+1,i+2,\dots,i+k-2,i}(k-1),\Sigma_{i+1,i+2,\dots,i+k-2,i}(k-1),\dots,\Sigma_{i+1,i+2,\dots,i+k-2,i}(k-1)] \tag{71}$$
$$\vdots$$
$$\mu_k[\Sigma_{i+k-2,i,i+1,\dots,i+k-3}(k-1),\Sigma_{i+k-2,i,i+1,\dots,i+k-3}(k-1),\dots,\Sigma_{i+k-2,i,i+1,\dots,i+k-3}(k-1)] \tag{72}$$
$$= q_i^{[k]}E(k-1), \;\; 1\le i\le n-k+1, \tag{73}$$

where $E(k-1)$ are polyadic identities (65) and $q_i^{[k]} \in \mathcal{B}_n^{gen\text{-}\sigma}$.

We propose a k-ary version of the far commutativity relation (46) in the following form:

$$\mu_k\Bigg[\overbrace{\Sigma_{i_1,\dots,i_{k-1}}(k-1),\dots,\Sigma_{i_1,\dots,i_{k-1}}(k-1)}^{k-1},E(k-1)\Bigg] = \dots \tag{74}$$

$$= \mu_k\Bigg[\overbrace{\Sigma_{\tau(i_1),\tau(i_2),\dots,\tau(i_{k-1})}(k-1),\dots,\Sigma_{\tau(i_1),\tau(i_2),\dots,\tau(i_{k-1})}(k-1)}^{k-1},E(k-1)\Bigg], \tag{75}$$

$$\text{if all } \left|i_p - i_s\right| \ge k-1, \quad p,s=1,\dots,k-1, \tag{76}$$

where τ is an element the permutation symmetry group $\tau \in S_{k-1}$.

In matrix form, we can define

Definition 18. *A k-ary (generated) matrix group $\mathcal{M}_k^{gen\text{-}\Sigma}$ is presented by the $(k-1)\times(k-1)$ matrix generators $\Sigma_{i_1,\dots,i_{k-1}}(k-1)$ (69) and the relations (we use (60))*

$$\left(\Sigma_{i,i+1,\dots,i+k-2}(k-1)\right)^k, \tag{77}$$
$$= \left(\Sigma_{i+1,i+2,\dots,i+k-2,i}(k-1)\right)^k, \tag{78}$$
$$\vdots$$
$$\left(\Sigma_{i+k-2,i,i+1,\dots,i+k-3}(k-1)\right)^k, \tag{79}$$
$$= q_i^{[k]} E(k-1), \;\; 1 \le i \le n-k+1,$$

and

$$\left(\Sigma^{(k-1)}_{i_1,i_2,\dots,i_{k-1}}\right)^{k-1} E(k-1) = \left(\Sigma^{(k-1)}_{\tau(i_1),\tau(i_2),\dots,\tau(i_{k-1})}\right)^{k-1} E(k-1), \tag{80}$$
$$\text{if all } \left|i_p - i_s\right| \ge k-1, \quad p,s = 1,\dots,k-1,$$

where $\tau \in S_{k-1}$ and $q_i^{[k]} \in \mathcal{B}_n^{gen\text{-}\sigma}$.

Each element of $\mathcal{M}_k^{gen\text{-}\Sigma}$ is a k-ary matrix word (analogous to the binary word (31)) being the k-ary product of the polyadic powers (12) of the matrix generators $\Sigma_{i_1,\dots,i_{k-1}}(k-1)$ and their querelements $\bar{\Sigma}_{i_1,\dots,i_{k-1}}(k-1)$ as in (50).

Similarly to the ternary case $k=3$ (Section 5), we now develop the k-ary "reverse" procedure and build from $\mathcal{B}_n^{gen\text{-}\sigma}$ the higher k-degree braid group $\mathcal{B}_n[k]$ using (69). Because the presentation of $\mathcal{M}_k^{gen\text{-}\Sigma}$ by generators and relations has already been given in (77) and (80), we need to expand them into components and postulate that these new relations between the (binary) generators σ_i present a new higher degree analog of the braid group. This gives:

Definition 19. *A higher k-degree braid (binary) group $\mathcal{B}_n[k]$ is presented by $(n-1)$ generators $\sigma_i \equiv \sigma_i^{[k]}$ (and the identity $\mathbf{e}$) satisfying the following relations:*

- *$(k-1)$ higher braid relations*

$$\overbrace{\sigma_i\sigma_{i+1}\dots\sigma_{i+k-3}\sigma_{i+k-2}\sigma_i}^{k} \tag{81}$$
$$= \sigma_{i+1}\sigma_{i+2}\dots\sigma_{i+k-2}\sigma_i\sigma_{i+1} \tag{82}$$
$$\vdots$$
$$= \sigma_{i+k-2}\sigma_i\sigma_{i+1}\sigma_{i+2}\dots\sigma_i\sigma_{i+1}\sigma_{i+k-2} \equiv q_i^{[k]} \quad q_i^{[k]} \in \mathcal{B}_n[k], \tag{83}$$
$$i \in \mathbf{I}_{braid} = \{1,\dots,n-k+1\}, \tag{84}$$

- *$(k-1)$-ary far commutativity*

$$\overbrace{\sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_{k-3}}\sigma_{i_{k-2}}\sigma_{i_{k-1}}}^{k-1} \tag{85}$$
$$\vdots$$
$$= \sigma_{\tau(i_1)}\sigma_{\tau(i_2)}\dots\sigma_{\tau(i_{k-3})}\sigma_{\tau(i_{k-2})}\sigma_{\tau(i_{k-1})}, \tag{86}$$
$$\text{if all } \left|i_p - i_s\right| \ge k-1, \quad p,s = 1,\dots,k-1, \tag{87}$$
$$\mathbf{I}_{far} = \{n-k,\dots,n-1\}, \tag{88}$$

where τ is an element of the permutation symmetry group $\tau \in S_{k-1}$.

A general element of the higher k-degree braid group $\mathcal{B}_n[k]$ is a word of the form

$$w = \sigma_{i_1}^{p_1} \dots \sigma_{i_r}^{p_r} \dots \sigma_{i_m}^{p_m}, \quad i_m = 1, \dots, n, \tag{89}$$

where $p_r \in \mathbb{Z}$ are (positive or negative) powers of the generators σ_{i_r}, $r = 1, \dots, m$ and $m \in \mathbb{N}$.

Remark 4. *The ternary case $k = 3$ coincides with the Artin braid group $\mathcal{B}_n^{[3]} = B_n$ (29) and (30).*

Remark 5. *The representation of the higher k-degree braid relations in $\mathcal{B}_n[k]$ in the tensor product of vector spaces (similarly to B_n and the Yang-Baxter equation [9]) can be obtained using the n'-ary braid equation introduced in Reference [8] (Proposition 7.2 and next there).*

Definition 20. *We say that the k-ary matrix group $\mathcal{M}_k^{gen\text{-}\Sigma}$ generated by the matrix generators $\Sigma_{i_1,i_2,\dots,i_{k-1}}(k-1)$ satisfying the relations (77)–(80) is in polyadic-binary correspondence with the higher k-degree braid group $\mathcal{B}_n[k]$, which is denoted as (cf. (47))*

$$\mathcal{M}_k^{gen\text{-}\Sigma} \frown\!\!\!\!\smile \mathcal{B}_n[k]. \tag{90}$$

Example 4. *Let $k = 4$; then, the 4-ary matrix group $\mathcal{M}_4^{gen\text{-}\Sigma}$ is generated by the matrix generators $\Sigma_{i_1,i_2,i_3}(3)$ satisfying (77)–(80)*

- *4-ary relations of q-polyadic order (48)*

$$\left(\Sigma_{i,i+1,i+2}(3)\right)^4 = \left(\Sigma_{i+2,i,i+1}(3)\right)^4 = \left(\Sigma_{i+1,i+2,i}(3)\right)^4 = q_i^{[3]} E(3), \; 1 \le i \le n-3, \tag{91}$$

- *far commutativity*

$$\begin{gathered}\left(\Sigma_{i_1,i_2,i_3}(3)\right)^3 E(3) = \left(\Sigma_{i_3,i_1,i_2}(3)\right)^3 E(3) = \left(\Sigma_{i_2,i_3,i_1}(3)\right)^3 E(3) \\ = \left(\Sigma_{i_1,i_3,i_2}(3)\right)^3 E(3) = \left(\Sigma_{i_3,i_2,i_1}(3)\right)^3 E(3) = \left(\Sigma_{i_2,i_1,i_3}(3)\right)^3 E(3), \\ |i_1 - i_2| \ge 3, \quad |i_1 - i_3| \ge 3, \quad |i_2 - i_3| \ge 3.\end{gathered} \tag{92}$$

Let $\sigma_i \equiv \sigma_i^{[4]} \in \mathcal{B}_n[4]$, $i = 1, \dots, n-1$; then, we use the 4-ary 3×3 matrix presentation for the generators (cf. Example 1):

$$\Sigma_{i_1,i_2,i_3}(3) \equiv \Sigma^{(3\times3)}(\sigma_{i_1}, \sigma_{i_2}, \sigma_{i_3}) = \begin{pmatrix} 0 & \sigma_{i_1} & 0 \\ 0 & 0 & \sigma_{i_2} \\ \sigma_{i_3} & 0 & 0 \end{pmatrix}, \quad i_1, i_2, i_3 = 1, \dots, n-1. \tag{93}$$

The querelement $\bar{\Sigma}_{i_1,i_2,i_3}(3)$ satisfying

$$\left(\Sigma_{i_1,i_2,i_3}(3)\right)^3 \bar{\Sigma}_{i_1,i_2,i_3}(3) = \Sigma_{i_1,i_2,i_3}(3) \tag{94}$$

has the form

$$\bar{\Sigma}_{i_1,i_2,i_3}(3) = \begin{pmatrix} 0 & \sigma_{i_3}^{-1}\sigma_{i_2}^{-1} & 0 \\ 0 & 0 & \sigma_{i_1}^{-1}\sigma_{i_3}^{-1} \\ \sigma_{i_2}^{-1}\sigma_{i_1}^{-1} & 0 & 0 \end{pmatrix}. \tag{95}$$

Expanding (91)–(92) in components, we obtain the relations for the higher 4-degree braid group $\mathcal{B}_n[4]$ as follows.

- *higher 4-degree braid relations*

$$\sigma_i \sigma_{i+1} \sigma_{i+2} \sigma_i = \sigma_{i+1} \sigma_{i+2} \sigma_i \sigma_{i+1} = \sigma_{i+2} \sigma_i \sigma_{i+1} \sigma_{i+2} \equiv q_i^{[4]}, \; 1 \le i \le n-3, \tag{96}$$

- *ternary far (total) commutativity*

$$\sigma_{i_1}\sigma_{i_2}\sigma_{i_3} = \sigma_{i_2}\sigma_{i_3}\sigma_{i_1} = \sigma_{i_3}\sigma_{i_1}\sigma_{i_2} = \sigma_{i_1}\sigma_{i_3}\sigma_{i_2} = \sigma_{i_2}\sigma_{i_1}\sigma_{i_3} = \sigma_{i_3}\sigma_{i_2}\sigma_{i_1}, \quad (97)$$
$$|i_1 - i_2| \geq 3, \ \ |i_1 - i_3| \geq 3, \ \ |i_2 - i_3| \geq 3. \quad (98)$$

In the higher 4-degree braid group, the minimum number of generators is 4, which follows from (96). In this case, we have a braid relation for $i = 1$ only and no far commutativity relations because of (98). Then:

Example 5. *The higher 4-degree braid group $\mathcal{B}_4[4]$ is generated by 3 generators σ_1, σ_2, σ_3, which satisfy only the braid relation*

$$\sigma_1\sigma_2\sigma_3\sigma_1 = \sigma_2\sigma_3\sigma_1\sigma_2 = \sigma_3\sigma_1\sigma_2\sigma_3. \quad (99)$$

If $n \leq 7$, then there will be no far commutativity relations at all, which follows from (98), and so the first higher 4-degree braid group containing far commutativity should have $n = 8$ elements.

Example 6. *The higher 4-degree braid group $\mathcal{B}_8[4]$ is generated by 7 generators $\sigma_1, \dots, \sigma_7$, which satisfy the braid relations with $i = 1, \dots, 5$*

$$\sigma_1\sigma_2\sigma_3\sigma_1 = \sigma_2\sigma_3\sigma_1\sigma_2 = \sigma_3\sigma_1\sigma_2\sigma_3, \quad (100)$$
$$\sigma_2\sigma_3\sigma_4\sigma_2 = \sigma_3\sigma_4\sigma_2\sigma_3 = \sigma_4\sigma_2\sigma_3\sigma_4, \quad (101)$$
$$\sigma_3\sigma_4\sigma_5\sigma_3 = \sigma_4\sigma_5\sigma_3\sigma_4 = \sigma_5\sigma_3\sigma_4\sigma_5, \quad (102)$$
$$\sigma_4\sigma_5\sigma_6\sigma_4 = \sigma_5\sigma_6\sigma_4\sigma_5 = \sigma_6\sigma_4\sigma_5\sigma_6, \quad (103)$$
$$\sigma_5\sigma_6\sigma_7\sigma_5 = \sigma_6\sigma_7\sigma_5\sigma_6 = \sigma_7\sigma_5\sigma_6\sigma_7, \quad (104)$$

together with the ternary far commutativity relation

$$\sigma_1\sigma_4\sigma_7 = \sigma_4\sigma_7\sigma_1 = \sigma_7\sigma_1\sigma_4 = \sigma_1\sigma_7\sigma_4 = \sigma_4\sigma_1\sigma_7 = \sigma_7\sigma_4\sigma_1. \quad (105)$$

Remark 6. *In polyadic group theory, there are several possible modifications of the commutativity property; but, nevertheless, we assume here the total commutativity relations in the k-ary matrix generators and the corresponding far commutativity relations in the higher degree braid groups.*

If $\mathcal{B}_n[k] \to \mathbb{Z}$ is the abelianization defined by $\sigma_i^{\pm} \to \pm 1$, then $\sigma_i^p = e$, if and only if $p = 0$, and σ_i are of infinite order. Moreover, we can prove (as in the ordinary case $k = 3$ [25]).

Theorem 1. *The higher k-degree braid group $\mathcal{B}_n[k]$ is torsion-free.*

Recall (see, e.g., Reference [4]) that there exists a surjective homomorphism of the braid group onto the finite symmetry group $B_n \to S_n$ by $\sigma_i \to s_i = (i, i+1) \in S_n$. The generators s_i satisfy (29) and (30), together with the finite order demand

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad 1 \leq i \leq n-2, \quad (106)$$
$$s_i s_j = s_j s_i, \quad |i - j| \geq 2, \quad (107)$$
$$s_i^2 = e, \quad i = 1, \dots, n-1, \quad (108)$$

which is called the *Coxeter presentation* of the symmetry group S_n. Indeed, multiplying both sides of (106) from the right successively by s_{i+1}, s_i, and s_{i+1}, using (108), we obtain

$\left(s_i s_{i+1}\right)^3 = 1$, and (107) on s_i and s_j, we get $\left(s_i s_j\right)^2 = 1$. Therefore, a Coxeter group [26] corresponding to (106)–(108) is presented by the same generators s_i and the relations

$$\left(s_i s_{i+1}\right)^3 = 1, \quad 1 \le i \le n-2, \tag{109}$$

$$\left(s_i s_j\right)^2 = 1, \quad |i-j| \ge 2, \tag{110}$$

$$s_i^2 = e, \quad i = 1, \ldots, n-1. \tag{111}$$

A general Coxeter group $W_n = W_n(e, r_i)$ is presented by n generators r_i and the relations [27]

$$\left(r_i r_j\right)^{m_{ij}} = e, \quad m_{ij} = \begin{cases} 1, & i = j, \\ \ge 2, & i \ne j. \end{cases} \tag{112}$$

By analogy with (106)–(108), we make the following.

Definition 21. *A higher analog of S_n, the k-degree symmetry group $\mathcal{S}_n[k] = \mathcal{S}_n^{[k]}(\boldsymbol{e}, \boldsymbol{s}_i)$, is presented by generators $\boldsymbol{s}_i$, $i = 1, \ldots, n-1$ satisfying (81)–(86) together with the additional condition of finite $(k-1)$-order $\boldsymbol{s}_i^{(k-1)} = \boldsymbol{e}$, $i = 1, \ldots, n$.*

Example 7. *The lowest higher degree case is $\mathcal{S}_4[4]$ which is presented by three generators $\boldsymbol{s}_1$, $\boldsymbol{s}_2$, $\boldsymbol{s}_3$ satisfying (see (99))*

$$\boldsymbol{s}_1 \boldsymbol{s}_2 \boldsymbol{s}_3 \boldsymbol{s}_1 = \boldsymbol{s}_2 \boldsymbol{s}_3 \boldsymbol{s}_1 \boldsymbol{s}_2 = \boldsymbol{s}_3 \boldsymbol{s}_1 \boldsymbol{s}_2 \boldsymbol{s}_3, \tag{113}$$

$$\boldsymbol{s}_1^3 = \boldsymbol{s}_2^3 = \boldsymbol{s}_3^3 = \boldsymbol{e}. \tag{114}$$

In a similar way, we define a higher degree analog of the Coxeter group (112).

Definition 22. *A higher k-degree Coxeter group $\mathcal{W}_n[k] = \mathcal{W}_n^{[k]}(\boldsymbol{e}, \boldsymbol{r}_i)$ is presented by n generators $\boldsymbol{r}_i$ obeying the relations*

$$\left(\boldsymbol{r}_{i_1} \boldsymbol{r}_{i_2} \ldots \boldsymbol{r}_{i_{k-1}}\right)^{\boldsymbol{m}_{i_1 i_2, \ldots, i_{k-1}}} = \boldsymbol{e}, \tag{115}$$

$$\boldsymbol{m}_{i_1 i_2, \ldots, i_{k-1}} = \begin{cases} 1, & i_1 = i_2 = \ldots = i_{k-1}, \\ \ge k-1, & |i_p - i_s| \ge k-1, \quad p, s = 1, \ldots, k-1. \end{cases} \tag{116}$$

It follows from (116) that all generators are of $(k-1)$ order $\boldsymbol{r}_i^{k-1} = \boldsymbol{e}$. A *higher k-degree Coxeter matrix* is a hypermatrix $M_{n,Cox}^{[k-1]}\left(\overbrace{n \times n \times \ldots \times n}^{k-1}\right)$ having 1 on the main diagonal and other entries $\boldsymbol{m}_{i_1 i_2, \ldots, i_{k-1}}$.

Example 8. *In the lowest higher degree case, $k = 4$ and all $\boldsymbol{m}_{i_1 i_2, \ldots, i_{k-1}} = 3$, we have (instead of commutativity in the ordinary case $k = 3$)*

$$\left(\boldsymbol{r}_i \boldsymbol{r}_j\right)^2 = \boldsymbol{r}_j^2 \boldsymbol{r}_i^2, \tag{117}$$

$$\boldsymbol{r}_i \boldsymbol{r}_j \boldsymbol{r}_i = \boldsymbol{r}_j^2 \boldsymbol{r}_i^2 \boldsymbol{r}_j^2. \tag{118}$$

Example 9. *A higher 4-degree analog of (109)–(111) is given by*

$$\left(\boldsymbol{r}_i \boldsymbol{r}_{i+1} \boldsymbol{r}_{i+2}\right)^4 = 1, \quad 1 \le i \le n-3, \tag{119}$$

$$\left(\boldsymbol{r}_{i_1} \boldsymbol{r}_{i_2} \boldsymbol{r}_{i_3}\right)^3 = 1, \quad |i_1 - i_2| \ge 3, \quad |i_1 - i_3| \ge 3, \quad |i_2 - i_3| \ge 3, \tag{120}$$

$$\boldsymbol{r}_i^3 = \boldsymbol{e}, \quad i = 1, \ldots, n-1. \tag{121}$$

It follows from (120) that

$$\left(r_{i_1} r_{i_2} r_{i_3}\right)^2 = r_{i_3}^2 r_{i_2}^2 r_{i_1}^2, \tag{122}$$

which cannot be reduced to total commutativity (97). From the first relation (119), we obtain

$$r_i r_{i+1} r_{i+2} r_i = r_{i+2}^2 r_{i+1}^2, \tag{123}$$

which differs from the higher 4-degree braid relations (96).

Example 10. *In the simplest case, the higher 4-degree Coxeter group $\mathcal{W}_4[4]$ has 3 generator r_1, r_2, r_3 satisfying*

$$(r_1 r_2 r_3)^4 = r_1^3 = r_2^3 = r_3^3 = e. \tag{124}$$

Example 11. *The minimal case, when the conditions (120) appear is $\mathcal{W}_8[4]$*

$$r_1 r_2 r_3 r_1 = r_3^2 r_2^2, \tag{125}$$
$$r_2 r_3 r_4 r_2 = r_4^2 r_3^2, \tag{126}$$
$$r_3 r_4 r_5 r_3 = r_5^2 r_4^2, \tag{127}$$
$$r_4 r_5 r_6 r_4 = r_6^2 r_5^2, \tag{128}$$
$$r_5 r_6 r_7 r_5 = r_7^2 r_6^2, \tag{129}$$

and an analog of commutativity

$$(r_1 r_4 r_7)^2 = r_7^2 r_4^2 r_1^2. \tag{130}$$

Thus, we arrive at:

Theorem 2. *The higher k-degree Coxeter group can present the k-degree symmetry group in the lowest case only, if and only if $k = 3$.*

As a further development, it would be interesting to consider the higher degree (in our sense) groups constructed here from a geometric viewpoint (e.g., Reference [5,28]).

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