Journal of Mathematical Physics, Analysis, Geometry
2019, Vol. 15, No. 1, pp. 3–56
doi: https://doi.org/10.15407/mag15.01.003

Arity Shape of Polyadic Algebraic Structures

Steven Duplij

Concrete two-set (module-like and algebra-like) algebraic structures are investigated from the viewpoint that the initial arities of all operations are arbitrary. Relations between operations arising from the structure definitions, however, lead to the restrictions which determine their possible arity shapes and lead us to formulate a partial arity freedom principle. Polyadic vector spaces and algebras, dual vector spaces, direct sums, tensor products and inner pairing spaces are reconsidered.

Elements of polyadic operator theory are outlined: multistars and polyadic analogs of adjoints, operator norms, isometries and projections are introduced, as well as polyadic C^*-algebras, Toeplitz algebras and Cuntz algebras represented by polyadic operators.

It is shown that congruence classes are polyadic rings of a special kind. Polyadic numbers are introduced (see Definition 7.17), and Diophantine equations over these polyadic rings are then considered. Polyadic analogs of the Lander–Parkin–Selfridge conjecture and Fermat's Last Theorem are formulated. For polyadic numbers neither of these statements holds. Polyadic versions of Frolov's theorem and the Tarry–Escott problem are presented.

Key words: polyadic ring, polyadic vector space, multiaction, multistar, Diophantine equation, Fermat's Last Theorem, Lander–Parkin–Selfridge conjecture.

Mathematical Subject Classification 2010: 11D41, 11R04, 11R06, 17A42, 20N15, 47A05, 47L30, 47L70, 47L80.

1. Introduction

The study of polyadic (higher arity) algebraic structures has a two-century long history, commencing with works by Cayley, Sylvester, Kasner, Prüfer, Dörnte, Lehmer, Post, etc. They took a single set, closed under one (main) binary operation having special properties (the so-called group-like structure), and "generalized" it by increasing the arity of that operation, which can then be called a *polyadic operation* and the corresponding algebraic structure *polyadic* as well. We use the term "polyadic" in this sense only, while there are other uses extant in the literature (see, e.g., [28]). An "abstract way" to study polyadic algebraic structures is via the use of universal algebras defined as sets with different axioms (equational laws) for polyadic operations [3,9,27]. However, in this language some important algebraic structures cannot be described, e.g., ordered groups, fields, etc. [14]. Therefore, it is worthwhile also to pursue a "concrete approach" which is to study examples of binary algebraic structures and then to "polyadize" them properly. This has initiated the development of a corresponding theory of n-ary quasigroups [2], n-ary semigroups [38, 45] and n-ary groups [25, 41] (for a more recent review, see, e.g., [16] and a comprehensive list of references therein). The binary algebraic structures with two operations (addition and multiplication) on one set (the so-called ring-like structures) were later on generalized to (m, n)-rings [8, 10, 31] and (m, n)-fields [29] (for recent study, see [18]), while these were studied mostly in a more restrictive manner by considering particular cases: ternary rings (or $(2, 3)$-rings) [33], $(m, 2)$-rings [4, 40], as well as $(3, 2)$-fields [21].

In the case of one set, speaking informally, the "polyadization" of two operations' "interaction" is straightforward, giving only polyadic distributivity which does not connect or restrict their arities. However, when the number of sets becomes greater than one, the "polyadization" turns out to be non-trivial, leading to special relations between the operation arities, and also introduces additional (to the arities) parameters, allowing us to classify them. We call a selection of such relations an *arity shape* and formulate the *arity partial freedom principle* that not all arities of the operations that arise during "polyadization" of binary operations are possible.

In this paper, we consider two-set algebraic structures in the "concrete way" and provide the consequent "polyadization" of binary operations on them for the so-called module-like structures (vector spaces) and algebra-like structures (algebras and inner product spaces). The "polyadization" of binary scalar multiplication is defined in terms of the multiactions introduced in [16], having special arity shapes parametrized by the number of intact elements (ℓ_{id}) in the corresponding multiactions. We then "polyadize" related constructions, such as dual vector spaces, direct sums and tensor products, and show that, as opposed to the binary case, they can be implemented in spaces of different arity signatures. The "polyadization" of inner product spaces and related norms gives us additional arity shapes and restrictions. In the resulting Table 5.1 we present the arity signatures and shapes of the polyadic algebraic structures under consideration.

As applications we note some starting points for polyadic operator theory by introducing multistars and polyadic analogs of adjoints, operator norms, isometries and projections. It is proved (Theorem 6.7) that if the polyadic inner pairing (the analog of the inner product) is symmetric, then all multistars coincide and all polyadic operators are self-adjoint (in contrast to the binary case). The polyadic analogs of C^*-algebras, Toeplitz algebras and Cuntz algebras are presented in terms of the polyadic operators introduced here, and a ternary example is given.

Another application is connected with number theory: we show that the internal structure of congruence classes is described by a polyadic ring having a special arity signature (Table 7.1), and these we will call the polyadic integers (or numbers) $\mathbb{Z}_{(m,n)}$ (Definition 7.17). They are classified by polyadic shape invariants, and the relations between them which give the same arity signature are established. Also, the limiting cases are analyzed, and it is shown that in one such case the polyadic rings can be embedded into polyadic fields with binary multiplication, which leads to the so-called polyadic rational numbers [11].

We then consider Diophantine equations over these polyadic rings in a straightforward manner: we change only the arities of the operations ("additions" and "multiplications"), but save their mutual "interaction". In this way we try to "polyadize" the equal sums of like powers equation and formulate polyadic analogs of the Lander–Parkin–Selfridge conjecture and of Fermat's Last Theorem [30]. It is shown that in the simplest case, when the polyadic "addition" and "multiplication" are nonderived (e.g., for polyadic numbers), neither conjecture is valid, and counterexamples are presented. Finally, we apply Frolov's theorem to the Tarry–Escott problem [15, 39] over polyadic rings to obtain new solutions to the equal sums of like powers equation for fixed congruence classes.

2. One set polyadic "linear" structures

We use concise notations from our previous work on polyadic structures [16, 17]. Take a non-empty set A, then an *n-tuple* (or *polyad*) consisting of the elements $(a_1, \ldots, a_n)$, $a_i \in A$, is denoted by a bold letter $(\mathbf{a})$ taking its values in the Cartesian product $A^{\times n}$. If the number of elements in the n-tuple is important, we denote it as $\left(\mathbf{a}^{(n)}\right)$, and an n-tuple with equal elements is denoted by (a^n). On the Cartesian product $A^{\times n}$ one can define a polyadic operation $\mu_n : A^{\times n} \to A$, and use the notation $\mu_n[\mathbf{a}]$.

A *polyadic structure* $\mathcal{A}$ is a set A which is closed under polyadic operations, and a *polyadic signature* is the selection of their arities. For formal definitions, see, e.g., [9].

2.1. Polyadic distributivity.

Let us consider a polyadic structure with two operations on the same set A: the "chief" (*multiplication*) n-ary operation $\mu_n : A^n \to A$ and the additional m-ary operation $\nu_m : A^m \to A$, that is $\langle A \mid \mu_n, \nu_m \rangle$. If there are no relations between μ_n and ν_m, then nothing new, as compared with the polyadic structures having a single operation $\langle A \mid \mu_n \rangle$ or $\langle A \mid \nu_m \rangle$, can be said. Informally, the "interaction" between operations can be described using the important relation of distributivity (an analog of $a \cdot (b + c) = a \cdot b + a \cdot c$, $a, b, c \in A$ in the binary case).

Definition 2.1. The *polyadic distributivity* for the operations μ_n and ν_m (no additional properties are implied for now) consists of n relations:

$$\mu_n[\nu_m[a_1, \ldots, a_m], b_2, b_3, \ldots, b_n]$$
$$= \nu_m[\mu_n[a_1, b_2, b_3, \ldots, b_n], \mu_n[a_2, b_2, b_3, \ldots, b_n], \ldots, \mu_n[a_m, b_2, b_3, \ldots, b_n]], \quad (2.1)$$
$$\mu_n[b_1, \nu_m[a_1, \ldots a_m], b_3, \ldots, b_n]$$
$$= \nu_m[\mu_n[b_1, a_1, b_3, \ldots, b_n], \mu_n[b_1, a_2, b_3, \ldots, b_n], \ldots, \mu_n[b_1, a_m, b_3, \ldots, b_n]], \quad (2.2)$$
$$\ldots$$
$$\mu_n[b_1, b_2, \ldots, b_{n-1}, \nu_m[a_1, \ldots, a_m]]$$

$$= \nu_m[\mu_n[b_1, b_2, \ldots, b_{n-1}, a_1], \mu_n[b_1, b_2, \ldots, b_{n-1}, a_2], \ldots, \mu_n[b_1, b_2, \ldots, b_{n-1}, a_m]], \tag{2.3}$$

where $a_i, b_j \in A$.

It is seen that the operations μ_n and ν_m enter into (2.1)–(2.3) in a non-symmetric way, which allows us to distinguish them: one of them (μ_n, the n-ary multiplication) "distributes" over the other one ν_m, and therefore ν_m is called the *addition*. If only some of the relations (2.1)–(2.3) hold, then such distributivity is *partial* (an analog of the left and right distributivity in the binary case). Obviously, the operations μ_n and ν_m need have nothing to do with ordinary multiplication (in the binary case denoted by $\mu_2 \Longrightarrow (\cdot)$) and addition (in the binary case denoted by $\nu_2 \Longrightarrow (+)$) as in the example below.

Example 2.2. Let $A = \mathbb{R}$, $n = 2$, $m = 3$, and $\mu_2 [b_1, b_2] = b_1^{b_2}$, $\nu_3 [a_1, a_2, a_3] = a_1 a_2 a_3$ (product in $\mathbb{R}$). The partial distributivity now is $(a_1 a_2 a_3)^{b_2} = a_1^{b_2} a_2^{b_2} a_3^{b_2}$ (only the first relation (2.1) holds).

2.2. Polyadic rings and fields. Here we briefly remind the reader of one-set (ring-like) polyadic structures (informally). Let both operations μ_n and ν_m be (totally) *associative*, which (in our definition [16]) means independence of the composition of two operations under placement of the internal operations (there are n and m such placements and therefore $(n+m)$ corresponding relations):

$$\mu_n [\mathbf{a}, \mu_n [\mathbf{b}], \mathbf{c}] = \text{invariant}, \tag{2.4}$$

$$\nu_m [\mathbf{d}, \nu_m [\mathbf{e}], \mathbf{f}] = \text{invariant}, \tag{2.5}$$

where the polyads $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ have a corresponding length, and then both $\langle A \mid \mu_n \mid \text{assoc}\rangle$ and $\langle A \mid \nu_m \mid \textit{assoc}\rangle$ are *polyadic semigroups* $\mathcal{S}_n$ and $\mathcal{S}_m$. A *commutative semigroup* $\langle A \mid \nu_m \mid \text{assoc}, \text{comm}\rangle$ is defined by $\nu_m [\mathbf{a}] = \nu_m [\sigma \circ \mathbf{a}]$, for all $\sigma \in S_n$, where S_n is the symmetry group. If the equation $\nu_m [\mathbf{a}, x, \mathbf{b}] = c$ is solvable for any place of x, then $\langle A \mid \nu_m \mid \text{assoc}, \text{solv}\rangle$ is a *polyadic group* $\mathcal{G}_m$, and such $x = \tilde{c}$ is called a (additive) *querelement* for c, which defines the (additive) *unary queroperation* $\tilde{\nu}_1$ by $\tilde{\nu}_1 [c] = \tilde{c}$.

Definition 2.3. A *polyadic* (m, n)*-ring* $\mathcal{R}_{m,n}$ is a set A with two operations $\mu_n : A^n \to A$ and $\nu_m : A^m \to A$, such that:

1) they are distributive (2.1)–(2.3);

2) $\langle A \mid \mu_n \mid \text{assoc}\rangle$ is a polyadic semigroup;

3) $\langle A \mid \nu_m \mid \text{assoc}, \text{comm}, \text{solv}\rangle$ is a commutative polyadic group.

It is obvious that a $(2, 2)$-ring $\mathcal{R}_{2,2}$ is an ordinary (binary) ring. Polyadic rings have much richer structure and can have unusual properties [8,10,13,31]. If the multiplicative semigroup $\langle A \mid \mu_n \mid \text{assoc}\rangle$ is commutative, $\mu_n [\mathbf{a}] = \mu_n [\sigma \circ \mathbf{a}]$, for all $\sigma \in S_n$, then $\mathcal{R}_{m,n}$ is called a *commutative polyadic ring*, and if it contains the identity, then $\mathcal{R}_{m,n}$ is a (polyadic) (m,n)-*semiring*. If the distributivity is only partial, then $\mathcal{R}_{m,n}$ is called a *polyadic near-ring*.

Introduce in $\mathcal{R}_{m,n}$ the additive and multiplicative idempotent elements by $\nu_m[a^m] = a$ and $\mu_n[b^n] = b$, respectively. A *zero* z of $\mathcal{R}_{m,n}$ is defined by $\mu_n[z,\mathbf{a}] = z$ for any $\mathbf{a} \in A^{n-1}$, where z can be in any place. Evidently, a zero (if it exists) is a multiplicative idempotent and is unique, and if a polyadic ring has an additive idempotent, it is a zero [31]. Due to the distributivity (2.1)–(2.3), there can be at most one zero in a polyadic ring. If a zero z exists, denote $A^* = A\setminus\{z\}$, and observe that (in distinction to binary rings) $\langle A^* \mid \mu_n \mid \text{assoc}\rangle$ is not a polyadic group, in general. In the case where $\langle A^* \mid \mu_n \mid \text{assoc}\rangle$ is a commutative n-ary group, such a polyadic ring is called a (*polyadic*) (m,n)-*field* and $\mathbb{K}_{m,n}$ ("polyadic scalars") (see [29,31]).

A multiplicative *identity* e in $\mathcal{R}_{m,n}$ is a distinguished element e such that

$$\mu_n\left[a,\left(e^{n-1}\right)\right] = a \tag{2.6}$$

for any $a \in A$ and where a can be in any place. In binary rings the identity is the only neutral element, while in polyadic rings there can exist many *neutral* $(n-1)$-*polyads* $\mathbf{e}$ satisfying

$$\mu_n[a,\mathbf{e}] = a, \tag{2.7}$$

for any $a \in A$ which can also be in any place. The neutral polyads $\mathbf{e}$ are not determined uniquely. Obviously, the polyad $\left(e^{n-1}\right)$ is neutral. There exist exotic polyadic rings which have no zero, no identity, and no additive idempotents at all (see, e.g., [10]), but if $m = 2$, then a zero always exists [31].

Example 2.4. Let us consider a polyadic ring $\mathcal{R}_{3,4}$, generated by 2 elements a, b and the relations

$$\mu_4\left[a^4\right] = a,\quad \mu_4\left[a^3,b\right] = b,\quad \mu_4\left[a^2,b^2\right] = a,\quad \mu_4\left[a,b^3\right] = b,\quad \mu_4\left[b^4\right] = a, \tag{2.8}$$

$$\nu_3\left[a^3\right] = b,\quad \nu_3\left[a^2,b\right] = a,\quad \nu_3\left[a,b^2\right] = b,\quad \nu_3\left[b^3\right] = a, \tag{2.9}$$

which has a multiplicative idempotent a only, but has no zero and no identity.

Proposition 2.5. *In the case of polyadic structures with two operations on one set there are no conditions between the arities of operations which could follow from distributivity* (2.1)–(2.3) *or the other relations above, and therefore they have no arity shape.*

Such conditions will appear below, when we consider more complicated universal algebraic structures with two or more sets with operations and relations.

3. Two set polyadic structures

3.1. Polyadic vector spaces. Let us consider a polyadic field $\mathbb{K}_{m_K,n_K} = \langle K \mid \sigma_{m_K}, \kappa_{n_K}\rangle$ ("polyadic scalars"), having the m_K-ary addition $\sigma_{m_K} : K^{m_K} \to K$ and n_K-ary multiplication $\kappa_{n_K} : K^{n_K} \to K$, and the identity $e_K \in K$, a neutral element with respect to multiplication $\kappa_{n_K}\left[e_K^{n_K-1},\lambda\right] = \lambda$ for all $\lambda \in K$.

In polyadic structures, one can introduce a neutral $(n_K - 1)$-polyad (*identity polyad* for "scalars") $\mathbf{e}_K \in K^{n_K - 1}$ by

$$\kappa_{n_K}[\mathbf{e}_K, \lambda] = \lambda, \tag{3.1}$$

where $\lambda \in K$ can be in any place.

Next, take an m_V-ary commutative (abelian) group $\langle \mathsf{V} \mid \nu_{m_V} \rangle$, which can be treated as "polyadic vectors" with m_V-ary addition $\nu_{m_V} : \mathsf{V}^{m_V} \to \mathsf{V}$. Define in $\langle \mathsf{V} \mid \nu_{m_V} \rangle$ an additive neutral element (zero) $\mathsf{z}_V \in \mathsf{V}$ by

$$\nu_{m_V}\left[\mathsf{z}_V^{m_V - 1}, \mathsf{v}\right] = \mathsf{v} \tag{3.2}$$

for any $\mathsf{v} \in \mathsf{V}$, and a "negative vector" $\bar{\mathsf{v}} \in \mathsf{V}$ as its querelement

$$\nu_{m_V}[\mathbf{a}_V, \bar{\mathsf{v}}, \mathbf{b}_V] = \mathsf{v}, \tag{3.3}$$

where $\bar{\mathsf{v}}$ can be in any place in the l.h.s., and $\mathbf{a}_V$, $\mathbf{b}_V$ are polyads in V. Here, instead of one neutral element we can also introduce the $(m_V - 1)$-polyad z_V (which may not be unique), and so, for a *zero polyad* (for "vectors"), we have

$$\nu_{m_V}[\mathsf{z}_V, \mathsf{v}] = \mathsf{v}, \quad \forall \mathsf{v} \in \mathsf{V}, \tag{3.4}$$

where $\mathsf{v} \in \mathsf{V}$ can be in any place. The "interaction" between "polyadic scalars" and "polyadic vectors" (the analog of binary multiplication by a scalar $\lambda \mathsf{v}$) can be defined as a multiaction (k_ρ-place action) introduced in [16],

$$\rho_{k_\rho} : K^{k_\rho} \times \mathsf{V} \longrightarrow \mathsf{V}. \tag{3.5}$$

The set of all multiactions forms an n_ρ-ary semigroup $\mathcal{S}_\rho$ under composition. We can "normalize" the multiactions in a similar way, as multiplace representations [16], by (an analog of $1\mathsf{v} = \mathsf{v}$, $\mathsf{v} \in \mathsf{V}$, $1 \in K$)

$$\rho_{k_\rho}\left\{\begin{array}{c} e_K \\ \vdots \\ e_K \end{array}\middle| \mathsf{v}\right\} = \mathsf{v}, \tag{3.6}$$

for all $\mathsf{v} \in \mathsf{V}$, where e_K is the identity of $\mathbb{K}_{m_K, n_K}$. In the case of an (ordinary) 1-place (left) action (as an external binary operation) $\rho_1 : K \times \mathsf{V} \to \mathsf{V}$, its consistency with the polyadic field multiplication κ_{n_K} under composition of the binary operations $\rho_1\{\lambda | a\}$ gives a product of the same arity

$$n_\rho = n_K,$$

that is (a polyadic analog of $\lambda(\mu \mathsf{v}) = (\lambda\mu)\mathsf{v}$, $\mathsf{v} \in \mathsf{V}$, $\lambda, \mu \in K$)

$$\rho_1\{\lambda_1 | \rho_1\{\lambda_2 | \ldots | \rho_1\{\lambda_{n_K} | \mathsf{v}\}\} \ldots\} = \rho_1\{\kappa_{n_K}[\lambda_1, \lambda_2, \ldots \lambda_{n_K}] | \mathsf{v}\}, \\ \lambda_1, \ldots, \lambda_n \in K, \ \mathsf{v} \in \mathsf{V}. \tag{3.7}$$

In the general case of k_ρ-place actions, the multiplication in the n_ρ-ary semigroup $\mathcal{S}_\rho$ can be defined by the *arity changing formula* [16] (schematically)

$$\overbrace{\rho_{k_\rho}\left\{\begin{array}{c}\lambda_1\\ \vdots\\ \lambda_{k_\rho}\end{array}\right| \ldots \left|\rho_{k\rho}\left\{\begin{array}{c}\lambda_{k_\rho(n_\rho-1)+1}\\ \vdots\\ \lambda_{k_\rho n_\rho}\end{array}\right| \mathsf{v}\right\} \ldots\right\}}^{n_\rho}$$
$$= \rho_{k_\rho}\left\{\begin{array}{c}\left.\begin{array}{c}\kappa_{n_K}\left[\lambda_1,\ldots\lambda_{n_K}\right],\\ \vdots\\ \kappa_{n_K}\left[\lambda_{n_K(\ell_\mu-1)},\ldots\lambda_{n_K\ell_\mu}\right]\end{array}\right\}\ell_\mu\\ \left.\begin{array}{c}\lambda_{n_K\ell_\mu+1},\\ \vdots\\ \lambda_{n_K\ell_\mu+\ell_{\mathrm{id}}}\end{array}\right\}\ell_{\mathrm{id}}\end{array}\right| \mathsf{v}\right\}, \tag{3.8}$$

where ℓ_μ and ℓ_{id} are both integers. The associativity of (3.8) in each concrete case can be achieved by applying the *associativity quiver* concept from [16].

Definition 3.1. The *ℓ-shape* is a pair $(\ell_\mu, \ell_{\mathrm{id}})$, where ℓ_μ is the number of multiplications and ℓ_{id} is the number of *intact elements* in the composition of operations.

It follows from (3.8),

Proposition 3.2. *The arities of the polyadic field $\mathbb{K}_{m_K,n_K}$, the arity n_ρ of the multiaction semigroup $\mathfrak{S}_\rho$ and the ℓ-shape of the composition satisfy*

$$k_\rho n_\rho = n_K \ell_\mu + \ell_{\mathrm{id}}, \tag{3.9}$$
$$k_\rho = \ell_\mu + \ell_{\mathrm{id}}. \tag{3.10}$$

We can exclude ℓ_μ or ℓ_{id} and obtain

$$n_\rho = n_K - \frac{n_K - 1}{k_\rho}\ell_{\mathrm{id}}, \quad n_\rho = \frac{n_K - 1}{k_\rho}\ell_\mu + 1, \tag{3.11}$$

respectively, where $\frac{n_K-1}{k_\rho}\ell_{\mathrm{id}} \geq 1$ and $\frac{n_K-1}{k_\rho}\ell_\mu \geq 1$ are integers. The following inequalities hold:

$$1 \leq \ell_\mu \leq k_\rho, \quad 0 \leq \ell_{\mathrm{id}} \leq k_\rho - 1, \quad \ell_\mu \leq k_\rho \leq (n_K - 1)\,\ell_\mu, \quad 2 \leq n_\rho \leq n_K. \tag{3.12}$$

Remark 3.3. The formulas (3.11) coincide with the *arity changing formulas* for heteromorphisms [16] applied to (3.8).

It follows from (3.9) that the ℓ-shape is determined by the arities and the number of places k_ρ by

$$\ell_\mu = \frac{k_\rho\,(n_\rho - 1)}{n_K - 1}, \quad \ell_{\mathrm{id}} = \frac{k_\rho\,(n_K - n_\rho)}{n_K - 1}. \tag{3.13}$$

Because we have two polyadic "additions" ν_{m_V} and σ_{m_K}, we need to consider how the multiaction ρ_{k_ρ} "distributes" between each of them. First, consider the distributivity of the multiaction ρ_{k_ρ} with respect to "vector addition" ν_{m_V} (a polyadic analog of the binary $\lambda(\mathsf{v}+\mathsf{u}) = \lambda\mathsf{v}+\lambda\mathsf{u}$, $\mathsf{v},\mathsf{u}\in\mathsf{V}$, $\lambda,\mu\in K$),

$$\rho_{k_\rho}\left\{\begin{array}{c}\lambda_1\\ \vdots\\ \lambda_{k_\rho}\end{array}\middle|\,\nu_{m_V}[\mathsf{v}_1,\ldots,\mathsf{v}_{m_V}]\right\} = \nu_{m_V}\left[\rho_{k_\rho}\left\{\begin{array}{c}\lambda_1\\ \vdots\\ \lambda_{k_\rho}\end{array}\middle|\,\mathsf{v}_1\right\},\ldots,\rho_{k_\rho}\left\{\begin{array}{c}\lambda_1\\ \vdots\\ \lambda_{k_\rho}\end{array}\middle|\,\mathsf{v}_{m_V}\right\}\right]. \tag{3.14}$$

Observe that here, in distinction to (3.8), there is no connection between the arities m_V and k_ρ.

Secondly, the distributivity of the multiaction ρ_{k_ρ} ("multiplication by scalar") with respect to the "field addition" (a polyadic analog of $\lambda\mathsf{v}+\mu\mathsf{v}=(\lambda+\mu)\,\mathsf{v}$, $\mathsf{v}\in A$, $\lambda,\mu\in K$) has a form similar to (3.8) (which can be obtained from the arity changing formula [16]),

$$\nu_{m_V}\left[\rho_{k_\rho}\left\{\begin{array}{c}\lambda_1\\ \vdots\\ \lambda_{k_\rho}\end{array}\middle|\,\mathsf{v}\right\},\ldots,\rho_{k_\rho}\left\{\begin{array}{c}\lambda_{k_\rho(m_V-1)+1}\\ \vdots\\ \lambda_{k_\rho m_V}\end{array}\middle|\,\mathsf{v}\right\}\right]$$
$$= \rho_{k_\rho}\left\{\begin{array}{c}\left.\begin{array}{c}\sigma_{m_K}[\lambda_1,\ldots\lambda_{m_K}],\\ \vdots\\ \sigma_{m_K}\left[\lambda_{m_K(\ell'_\mu-1)},\ldots\lambda_{m_K\ell'_\mu}\right]\end{array}\right\}\ell'_\mu\\ \left.\begin{array}{c}\lambda_{m_K\ell'_\mu+1},\\ \vdots\\ \lambda_{m_K\ell'_\mu+\ell'_{\mathrm{id}}}\end{array}\right\}\ell'_{\mathrm{id}}\end{array}\middle|\,\mathsf{v}\right\}, \tag{3.15}$$

where ℓ'_ρ and ℓ'_{id} are the numbers of multiplications and *intact elements* in the resulting multiaction, respectively. Here the arities are not independent as in (3.14), and so we have

Proposition 3.4. *The arities of the polyadic field $\mathbb{K}_{m_K,n_K}$, the arity n_ρ of the multiaction semigroup $\mathcal{S}_\rho$ and the ℓ-shape of the distributivity satisfy*

$$k_\rho m_V = m_K\ell'_\mu+\ell'_{\mathrm{id}}, \tag{3.16}$$
$$k_\rho = \ell'_\mu+\ell'_{\mathrm{id}}. \tag{3.17}$$

It follows from (3.16), (3.17),

$$m_V = m_K - \frac{m_K-1}{k_\rho}\ell'_{\mathrm{id}},\quad m_V = \frac{m_K-1}{k_\rho}\ell'_\mu+1.$$

Here $\frac{m_K-1}{k_\rho}\ell'_{\mathrm{id}}\geq 1$ and $\frac{m_K-1}{k_\rho}\ell'_\mu\geq 1$ are integers, and we have the inequalities

$$1\leq\ell'_\mu\leq k_\rho,\quad 0\leq\ell'_{\mathrm{id}}\leq k_\rho-1,\quad \ell'_\mu\leq k_\rho\leq(m_K-1)\,\ell'_\mu,\quad 2\leq m_V\leq m_K. \tag{3.18}$$

Now, the ℓ-shape of the distributivity is fully determined from the arities and the number of places k_ρ by the arity shape formulas

$$\ell'_\rho = \frac{k_\rho (m_V - 1)}{m_K - 1}, \quad \ell'_{\mathrm{id}} = \frac{k_\rho (m_K - m_V)}{m_K - 1}. \tag{3.19}$$

It follows from (3.18) that:

Corollary 3.5. *The arity m_V of the vector addition is less than or equal to the arity m_K of the field addition.*

Definition 3.6. A polyadic ($\mathbb{K}$)-vector ("linear") space over a polyadic field is the 2-set 4-operation algebraic structure

$$\mathcal{V}_{m_K,n_K,m_V,k_\rho} = \left\langle K; \mathsf{V} \mid \sigma_{m_K}, \kappa_{n_K}; \nu_{m_V} \mid \rho_{k_\rho} \right\rangle \tag{3.20}$$

such that the following axioms hold:

1) $\langle K \mid \sigma_{m_K}, \kappa_{n_K} \rangle$ is a polyadic (m_K, n_K)-field $\mathbb{K}_{m_K,n_K}$;
2) $\langle \mathsf{V} \mid \nu_{m_V} \rangle$ is a commutative m_V-ary group;
3) $\left\langle \rho_{k_\rho} \mid \text{composition} \right\rangle$ is an n_ρ-ary semigroup $\mathfrak{S}_\rho$;
4) Distributivity of the multiaction ρ_{k_ρ} with respect to the "vector addition" ν_{m_V} (3.14);
5) Distributivity of ρ_{k_ρ} with respect to the "scalar addition" σ_{m_K} (3.15);
6) Compatibility of ρ_{k_ρ} with the "scalar multiplication" κ_{n_K} (3.8);
7) Normalization of the multiaction ρ_{k_ρ} (3.6).

All of the arities in (3.20) are independent and can be chosen arbitrarily, but they fix the ℓ-shape of the multiaction composition (3.8) and the distributivity (3.15) by (3.13) and (3.19), respectively. Note that the main distinction from the binary case is a possibility for the arity n_ρ of the multiaction semigroup $\mathcal{S}_\rho$ to be arbitrary.

Definition 3.7. A polyadic $\mathbb{K}$-vector subspace is

$$\mathcal{V}^{\mathrm{sub}}_{m_K,n_K,m_V,k_\rho} = \left\langle K; \mathsf{V}^{\mathrm{sub}} \mid \sigma_{m_K}, \kappa_{n_K}; \nu_{m_V} \mid \rho_{k_\rho} \right\rangle, \tag{3.21}$$

where the subset $\mathsf{V}^{\mathrm{sub}} \subset \mathsf{V}$ is closed under all operations $\sigma_{m_K}, \kappa_{n_K}, \nu_{m_V}, \rho_{k_\rho}$ and the axioms 1)–7).

Let us consider a subset $\mathsf{S} = \{\mathsf{v}_1, \ldots, \mathsf{v}_{d_V}\} \subseteq \mathsf{V}$ (of d_V "vectors"), then a *polyadic span* of S is (a "linear combination")

$$\mathrm{Span}^{\lambda}_{\mathrm{pol}} (\mathsf{v}_1, \ldots, \mathsf{v}_{d_V}) = \{w\}, \tag{3.22}$$

$$w=\nu_{m_V}^{\ell_\nu}\left[\rho_{k_\rho}\left\{\begin{array}{c}\lambda_1\\ \vdots\\ \lambda_{k_\rho}\end{array}\middle|\,\mathsf{v}_1\right\},\ldots,\rho_{k_\rho}\left\{\begin{array}{c}\lambda_{(d_V-1)k_\rho}\\ \vdots\\ \lambda_{d_V k_\rho}\end{array}\middle|\,\mathsf{v}_s\right\}\right], \tag{3.23}$$

where $(d_V\cdot k_\rho)$ "scalars" play the role of coefficients (or coordinates as columns consisting of k_ρ elements from the polyadic field $\mathbb{K}_{m_K,n_K}$), and the number of "vectors" s is connected with the "number of m_V-ary additions" ℓ_ν by

$$d_V=\ell_\nu\left(m_V-1\right)+1,$$

while $\mathrm{Span}^{\lambda}_{\mathrm{pol}}\,\mathsf{S}$ is the set of all "vectors" of this form (3.22) (we consider here only finite "sums").

Definition 3.8. A polyadic span $\mathsf{S}=\{\mathsf{v}_1,\ldots,\mathsf{v}_{d_V}\}\subseteq\mathsf{V}$ is *nontrivial* if at least one multiaction ρ_{k_ρ} in (3.22) is nonzero.

Since polyadic fields and groups do not contain zeroes, we need to redefine the basic notions of equivalences. Let us take two different spans of the set S.

Definition 3.9. A set $\{\mathsf{v}_1,\ldots,\mathsf{v}_{d_V}\}$ is called "linear" *polyadic independent* if from the equality of nontrivial spans, as $\mathrm{Span}^{\lambda}_{\mathrm{pol}}\left(\mathsf{v}_1,\ldots,\mathsf{v}_{d_V}\right)=\mathrm{Span}^{\lambda'}_{\mathrm{pol}}\left(\mathsf{v}_1,\ldots,\mathsf{v}_{d_V}\right)$, it follows that all $\lambda_i=\lambda'_i$, $i=1,\ldots,d_V k_\rho$.

Definition 3.10. A set $\{\mathsf{v}_1,\ldots,\mathsf{v}_{d_V}\}$ is called a *polyadic basis* of a polyadic vector space $\mathcal{V}_{m_K n_K m_V k_\rho}$ if it spans the whole space $\mathrm{Span}^{\lambda}_{\mathrm{pol}}\left(\mathsf{v}_1,\ldots,\mathsf{v}_{d_V}\right)=\mathsf{V}$.

In other words, any element of V can be uniquely presented in the form of the polyadic "linear combination"(3.22). If a polyadic vector space $\mathcal{V}_{m_K n_K m_V k_\rho}$ has a finite basis $\{\mathsf{v}_1,\ldots,\mathsf{v}_{d_V}\}$, then any other basis $\left\{\mathsf{v}'_1,\ldots,\mathsf{v}'_{d_V}\right\}$ has the same number of elements.

Definition 3.11. The number of elements in the polyadic basis $\{\mathsf{v}_1,\ldots,\mathsf{v}_{d_V}\}$ is called the *polyadic dimension* of $\mathcal{V}_{m_K,n_K,m_V,k_\rho}$.

Remark 3.12. The so-called 3-vector space, introduced and studied in [21], corresponds to $\mathcal{V}_{m_K=3,n_K=2,m_V=3,k_\rho=1}$.

3.2. One-set polyadic vector space. A particular polyadic vector space is important: consider $\mathsf{V}=K$, $\nu_{m_V}=\sigma_{m_K}$ and $m_V=m_K$, which gives the following one-set "linear" algebraic structure (we call it a *one-set polyadic vector space*):

$$\mathcal{K}_{m_K,n_K,k_\rho}=\left\langle K\mid\sigma_{m_K},\kappa_{n_K}\mid\rho^{\lambda}_{k_\rho}\right\rangle,$$

where now the multiaction

$$\rho^{\lambda}_{k_\rho}\left\{\begin{array}{c}\lambda_1\\ \vdots\\ \lambda_{k_\rho}\end{array}\middle|\,\lambda\right\},\lambda,\lambda_i\in K,$$

acts on K itself (in some special way), and therefore can be called a *regular multiaction*. In the binary case $n_K = m_K = 2$, the only possibility for the regular action is the multiplication (by "scalars") in the field $\rho_1^\lambda \{\lambda_1 | \lambda\} = \kappa_2 [\lambda_1 \lambda] (\equiv \lambda_1 \lambda)$, which obviously satisfies the axioms 4)–7) of a vector space in Definition 3.6. In this way we arrive at the definition of the binary field $\mathbb{K} \equiv \mathbb{K}_{2,2} = \langle K \mid \sigma_2, \kappa_2 \rangle$, and so a one-set *binary vector space* coincides with the underlying field $\mathcal{K}_{m_K=2, n_K=2, k_\rho=1} = \mathbb{K}$, or as it is said "a field is a (one-dimensional) vector space over itself".

Remark 3.13. In the polyadic case, the regular multiaction $\rho_{k_\rho}^\lambda$ can be chosen, as any (additional to σ_{m_K}, κ_{n_K}) function satisfying axioms 4)–7) of a polyadic vector space and the number of places k_ρ and the arity of the semigroup of multiactions $\mathcal{S}_\rho$ can be *arbitrary*, in general. Also, $\rho_{k_\rho}^\lambda$ can be taken as some nontrivial combination of σ_{m_K}, κ_{n_K} satisfying axioms 4)–7) (which admits a nontrivial "multiplication by scalars").

In the simplest regular (similar to the binary) case,

$$\rho_{k_\rho}^\lambda \left\{ \begin{array}{c|c} \lambda_1 & \\ \vdots & \lambda \\ \lambda_{k_\rho} & \end{array} \right\} = \kappa_{n_K}^{\ell_\kappa} \left[\lambda_1, \ldots, \lambda_{k_\rho}, \lambda \right], \tag{3.24}$$

where ℓ_κ is the number of multiplications κ_{n_K}, and the number of places k_ρ is now fixed by

$$k_\rho = \ell_\kappa (n_K - 1), \tag{3.25}$$

while λ in (3.24) can be in any place due to the commutativity of the field multiplication κ_{n_K}.

Remark 3.14. In general, the one-set polyadic vector space need not to coincide with the underlying polyadic field, $\mathcal{K}_{m_K, n_K, k_\rho} \neq \mathbb{K}_{n_K m_K}$ (as *opposed* to the binary case), but can have a more complicated structure which is determined by an additional multiplace function, the multiaction $\rho_{k_\rho}^\lambda$.

3.3. Polyadic algebras. By analogy with the binary case, introducing an additional operation on vectors, a multiplication which is distributive and "linear" with respect to "scalars", leads to a polyadic generalization of the (associative) algebra notion [7]. Here, we denote the second (except for the 'scalars' K) set by A (instead of V as above), on which we define *two* operations: the m_A-ary "addition" $\nu_{m_A} : \mathsf{A}^{\times m_A} \to \mathsf{A}$ and the n_A-ary "multiplication" $\mu_{n_A} : \mathsf{A}^{\times n_A} \to \mathsf{A}$. To interpret the n_A-ary operation as a true multiplication, the operations μ_{n_A} and ν_{m_A} should satisfy polyadic distributivity (2.1)–(2.3) (an analog of $(\mathsf{a} + \mathsf{b}) \cdot c = \mathsf{a} \cdot \mathsf{c} + \mathsf{b} \cdot \mathsf{c}$, with $\mathsf{a}, \mathsf{b}, \mathsf{c} \in \mathsf{A}$). Then we should consider the "interaction" of this new operation μ_{n_A} with the multiaction ρ_{k_ρ} (an analog of the "compatibility with scalars" $(\lambda \mathsf{a}) \cdot (\mu \mathsf{b}) = (\lambda \mu)\, \mathsf{a} \cdot \mathsf{b}$, $\mathsf{a}, \mathsf{b} \in \mathsf{A}$, $\lambda, \mu \in K$). In the most general case, when all arities are arbitrary, we have the *polyadic compatibility* of μ_{n_A} with the field multiplication κ_{n_K} as follows:

$$\mu_{n_A}\left[\rho_{k_\rho}\left\{\begin{array}{c}\lambda_1\\ \vdots\\ \lambda_{k_\rho}\end{array}\middle|\,\mathsf{a}_1\right\},\ldots,\rho_{k_\rho}\left\{\begin{array}{c}\lambda_{k_\rho(n_A-1)}\\ \vdots\\ \lambda_{k_\rho n_A}\end{array}\middle|\,\mathsf{a}_{n_A}\right\}\right]$$
$$=\rho_{k_\rho}\left\{\begin{array}{c}\left.\begin{array}{c}\kappa_{n_K}\left[\lambda_1,\ldots,\lambda_{n_K}\right],\\ \vdots\\ \kappa_{n_K}\left[\lambda_{n_K(\ell''_\mu-1)},\ldots,\lambda_{n_K\ell''_\mu}\right]\end{array}\right\}\ell''_\mu\\ \left.\begin{array}{c}\lambda_{n_K\ell''_\mu+1},\\ \vdots\\ \lambda_{n_K\ell''_\mu+\ell''_{\mathrm{id}}}\end{array}\right\}\ell''_{\mathrm{id}}\end{array}\middle|\,\mu_{n_A}\left[\mathsf{a}_1\ldots\mathsf{a}_{n_A}\right]\right\}, \quad (3.26)$$

where ℓ''_μ and ℓ''_{id} are the numbers of multiplications and intact elements in the resulting multiaction, respectively.

Proposition 3.15. *The arities of the polyadic field $\mathbb{K}_{m_K,n_K}$, the arity n_ρ of the multiaction semigroup $\mathfrak{S}_\rho$ and the ℓ-shape of the polyadic compatibility (3.26) satisfy*

$$k_\rho n_A = n_K\ell''_\mu + \ell''_{\mathrm{id}}, \quad k_\rho = \ell''_\mu + \ell''_{\mathrm{id}}. \quad (3.27)$$

We can exclude from (3.27) ℓ''_ρ or ℓ''_{id} and obtain

$$n_A = n_K - \frac{n_K-1}{k_\rho}\ell''_{\mathrm{id}}, \quad n_A = \frac{n_K-1}{k_\rho}\ell''_\mu + 1,$$

where $\frac{n_K-1}{k_\rho}\ell''_{\mathrm{id}} \geq 1$ and $\frac{n_K-1}{k_\rho}\ell''_\mu \geq 1$ are integers, and the inequalities hold

$$1 \leq \ell''_\mu \leq k_\rho, \quad 0 \leq \ell''_{\mathrm{id}} \leq k_\rho - 1, \quad \ell''_\mu \leq k_\rho \leq (n_K-1)\,\ell''_\mu, \quad 2 \leq n_A \leq n_K. \quad (3.28)$$

It follows from (3.27), that the ℓ-shape is determined by the arities and the number of places k_ρ as

$$\ell''_\mu = \frac{k_\rho(n_A-1)}{n_K-1}, \quad \ell''_{\mathrm{id}} = \frac{k_\rho(n_K-n_A)}{n_K-1}. \quad (3.29)$$

Definition 3.16. A polyadic ("linear") algebra over a polyadic field is the 2-set 5-operation algebraic structure,

$$\mathcal{A}_{m_K,n_K,m_A,n_A,k_\rho} = \left\langle K; \mathsf{A} \mid \sigma_{m_K}, \kappa_{n_K}; \nu_{m_A}, \mu_{n_A} \mid \rho_{k_\rho}\right\rangle, \quad (3.30)$$

such that the following axioms hold:

1) $\left\langle K; \mathsf{A} \mid \sigma_{m_K}, \kappa_{n_K}; \nu_{m_A} \mid \rho_{k_\rho}\right\rangle$ is a polyadic vector space over a polyadic field $\mathbb{K}_{m_K,n_K}$;

2) The algebra multiplication μ_{n_A} and the algebra addition ν_{m_A} satisfy the polyadic distributivity (2.1)–(2.3);

3) The multiplications in the algebra μ_{n_A} and in the field κ_{n_K} are compatible by (3.26).

If the algebra multiplication μ_{n_A} is associative (2.4), then $\mathcal{A}_{m_K,n_K,m_A,n_A,k_\rho}$ is an *associative polyadic algebra* (for $k_\rho = 1$ see [7]). If μ_{n_A} is commutative, $\mu_{n_A}[\mathbf{a}_A] = \mu_{n_A}[\sigma \circ \mathbf{a}_A]$, for any polyad in algebra $\mathbf{a}_A \in \mathsf{A}^{\times n_A}$ for all permutations $\sigma \in S_n$, where S_n is the symmetry group, then $\mathcal{A}_{m_K,n_K,m_A,n_A,k_\rho}$ is called a *commutative polyadic algebra*. As in the n-ary (semi)group theory, for polyadic algebras one can introduce special kinds of associativity and partial commutativity. If the multiplication μ_{n_A} contains the identity e_A (2.6) or a neutral polyad for any element, then a polyadic algebra is called *unital* or *neutral-unital*, respectively. It follows from (3.28) that:

Corollary 3.17. *In a polyadic ("linear") algebra the arity of the algebra multiplication n_A is less than or equal to the arity of the field multiplication n_K.*

Proposition 3.18. *If all the operation ℓ-shapes in* (3.8)*,* (3.15)*, and* (3.26) *coincide*

$$\ell''_\mu = \ell'_\mu = \ell_\mu, \quad \ell''_{\mathrm{id}} = \ell'_{\mathrm{id}} = \ell_{\mathrm{id}},$$

then we obtain the conditions for the arities

$$n_K = m_K, \quad n_\rho = n_A, \tag{3.31}$$

while m_A and k_ρ are not connected.

Proof. Use (3.13) and (3.29). □

Proposition 3.19. *In the case of equal ℓ-shapes the multiplication and addition of the polyadic ground field ("scalars") should coincide, while the arity n_ρ of the multiaction semigroup $\mathcal{S}_\rho$ should be the same as of the algebra multiplication n_A, while the arity of the algebra addition m_A and the number of places k_ρ remain arbitrary.*

Remark 3.20. The above ℓ-shapes (3.13), (3.19), and (3.29) are defined by a pair of *integers*, and therefore the arities in them are not arbitrary, but should be "quantized" in the same manner as the arities of heteromorphisms in [16].

Therefore, numerically the "quantization" rules for the ℓ-shapes (3.13), (3.19), and (3.29) coincide and are given in Table 3.1.

Thus, we arrive at the following

Theorem 3.21 (The arity partial freedom principle)**.** *The basic two-set polyadic algebraic structures have non-free underlying operation arities which are "quantized" in such a way that their ℓ-shape is given by integers.*

The above definitions can be generalized, as in the binary case, by considering a polyadic ring $\mathcal{R}_{m_K,n_K}$ instead of a polyadic field $\mathbb{K}_{m_K,n_K}$. In this way, a polyadic vector space becomes a *polyadic module over a ring* or *polyadic $\mathcal{R}$-module*, while a polyadic algebra over a polyadic field becomes a *polyadic algebra over a ring* or a *polyadic $\mathcal{R}$-algebra*. All the axioms and relations between arities in Definition 3.6 and Definition 3.16 remain the same. However, one should take into account that

Table 3.1: "Quantization" of arity ℓ-shapes

.

k_ρ	$\ell_\mu \mid \ell'_\mu \mid \ell''_\mu$	$\ell_{\mathrm{id}} \mid \ell'_{\mathrm{id}} \mid \ell''_{\mathrm{id}}$	$\frac{n_K}{n_\rho} \mid \frac{m_K}{n_\rho} \mid \frac{n_K}{n_A}$
2	1	1	3, 5, 7, ... 2, 3, 4, ...
3	1	2	4, 7, 10, ... 2, 3, 4, ...
3	2	1	4, 7, 10, ... 3, 5, 7, ...
4	1	3	5, 9, 13, ... 2, 3, 4, ...
4	2	2	3, 5, 7, ... 2, 3, 4, ...
4	3	1	5, 9, 13, ... 4, 7, 10, ...

the ring multiplication κ_{n_K} can be noncommutative, and therefore for *polyadic* $\mathcal{R}$-modules and $\mathcal{R}$-algebras it is necessary to consider many different kinds of multiactions ρ_{k_ρ} (all of them are described in (3.8)). For instance, in the ternary case this corresponds to trimodules [6] or ternary module structure [35].

4. Mappings between polyadic algebraic structures

Let us consider D_V different polyadic vector spaces over the same polyadic field $\mathbb{K}_{m_K,n_K}$, as

$$\mathcal{V}^{(i)}_{m_K,n_K,m_V^{(i)},k_\rho^{(i)}} = \left\langle K; \mathsf{V}^{(i)} \mid \sigma_{m_K}, \kappa_{n_K}; \nu^{(i)}_{m_V^{(i)}} \mid \rho^{(i)}_{k_\rho^{(i)}} \right\rangle, \quad i = 1, \ldots, D_V < \infty.$$

Here we define a polyadic analog of a "linear" mapping for polyadic vector spaces which "commutes" with the "vector addition" and the "multiplication by scalar" (an analog of the additivity $\mathbf{F}(\mathsf{v}+\mathsf{u}) = \mathbf{F}(\mathsf{v}) + \mathbf{F}(\mathsf{u})$, and the homogeneity of degree one $\mathbf{F}(\lambda\mathsf{v}) = \lambda\mathbf{F}(\mathsf{v})$, $\mathsf{v},\mathsf{u} \in \mathsf{V}$, $\lambda \in K$).

Definition 4.1. A *1-place ("$\mathbb{K}$-linear") mapping* between the polyadic vector spaces $\mathcal{V}_{m_K,n_K,m_V,k_\rho} = \left\langle K; \mathsf{V} \mid \sigma_{m_K}, \kappa_{n_K}; \nu_{m_V} \mid \rho_{k_\rho} \right\rangle$ and $\mathcal{V}_{m_K,n_K,m_V,k_\rho} = \left\langle K; \mathsf{V}' \mid \sigma_{m_K}, \kappa_{n_K}; \nu'_{m_V} \mid \rho'_{k_\rho} \right\rangle$ over the same polyadic field $\mathbb{K}_{m_K,n_K} = \left\langle K \mid \sigma_{m_K}, \kappa_{n_K} \right\rangle$ is $\mathbf{F}_1 : \mathsf{V} \to \mathsf{V}'$ such that

$$\mathbf{F}_1\left(\nu_{m_V}\left[\mathsf{v}_1, \ldots, \mathsf{v}_{m_V}\right]\right) = \nu'_{m_V}\left[\mathbf{F}_1(\mathsf{v}_1), \ldots, \mathbf{F}_1(\mathsf{v}_{m_V})\right], \tag{4.1}$$

$$\mathbf{F}_1\left(\rho_{k_\rho}\left\{\begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_{k_\rho} \end{array}\middle| \mathsf{v}\right\}\right) = \rho'_{k_\rho}\left\{\begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_{k_\rho} \end{array}\middle| \mathbf{F}_1(\mathsf{v})\right\}, \tag{4.2}$$

where $\mathsf{v}_1,\ldots,\mathsf{v}_{m_V},\mathsf{v}\in\mathsf{V}$, $\lambda_1,\ldots,\lambda_{k_\rho}\in K$.

If z_V is a "zero vector" in V and $\mathsf{z}_{V'}$ is a "zero vector" in V' (see (3.2)), then it follows from (4.1), (4.2) that $\mathbf{F}_1(\mathsf{z}_V)=\mathsf{z}_{V'}$.

The initial and final arities of ν_{m_V} ("vector addition") and the multiaction ρ_{k_ρ} ("multiplication by scalar") coincide because $\mathbf{F}_1$ is a 1-place mapping (a linear homomorphism). In [16] *multiplace mappings* and corresponding *heteromorphisms* were introduced. The latter allows us to change the arrities $m_V\to m'_V$, $k_\rho\to k'_\rho$), which is the main difference between the binary and polyadic mappings.

Definition 4.2. A *k_F-place ("$\mathbb{K}$-linear") mapping* between two polyadic vector spaces $\mathcal{V}_{m_K,n_K,m_V,k_\rho}=\left\langle K;\mathsf{V}\mid\sigma_{m_K},\kappa_{n_K};\nu_{m_V}\mid\rho_{k_\rho}\right\rangle$ and $\mathcal{V}_{m_K,n_K,m_V,k_\rho}=\left\langle K;\mathsf{V}'\mid\sigma_{m_K},\kappa_{n_K};\nu'_{m'_V}\mid\rho'_{k'_\rho}\right\rangle$ over the same polyadic field $\mathbb{K}_{m_K,n_K}=\left\langle K\mid\sigma_{m_K},\kappa_{n_K}\right\rangle$ is defined if there exists $\mathbf{F}_{k_F}:\mathsf{V}^{\times k_F}\to\mathsf{V}'$ such that

$$\mathbf{F}_{k_F}\left(\begin{array}{c}\left.\begin{array}{c}\nu_{m_V}\left[\mathsf{v}_1,\ldots,\mathsf{v}_{m_V}\right]\\ \vdots\\ \nu_{m_V}\left[\mathsf{v}_{m_V\left(\ell^k_\mu-1\right)},\ldots\mathsf{v}_{m_V\ell^k_\mu}\right]\end{array}\right\}\ell^k_\mu\\ \left.\begin{array}{c}\mathsf{v}_{m_V\ell^k_\mu+1},\\ \vdots\\ \mathsf{v}_{m_V\ell^k_\mu+\ell^k_{\mathrm{id}}}\end{array}\right\}\ell^k_{\mathrm{id}}\end{array}\right)=\nu'_{m'_V}\left[\mathbf{F}_{k_F}\begin{pmatrix}\mathsf{v}_1\\ \vdots\\ \mathsf{v}_{k_F}\end{pmatrix},\ldots,\mathbf{F}_{k_F}\begin{pmatrix}\mathsf{v}_{k_F\left(m'_V-1\right)}\\ \vdots\\ \mathsf{v}_{k_Fm'_V}\end{pmatrix}\right],\tag{4.3}$$

$$\mathbf{F}_{k_F}\left(\begin{array}{c}\left.\begin{array}{c}\rho_{k_\rho}\left\{\begin{array}{c}\lambda_1\\ \vdots\\ \lambda_{k_\rho}\end{array}\middle|\,\mathsf{v}_1\right\}\\ \vdots\\ \rho_{k_\rho}\left\{\begin{array}{c}\lambda_{k_\rho\left(\ell^f_\mu-1\right)}\\ \vdots\\ \lambda_{k_\rho\ell^f_\mu}\end{array}\middle|\,\mathsf{v}_{\ell^f_\mu}\right\}\end{array}\right\}\ell^f_\mu\\ \left.\begin{array}{c}\mathsf{v}_{\ell^f_\mu+1}\\ \vdots\\ \mathsf{v}_{k_F}\end{array}\right\}\ell^f_{\mathrm{id}}\end{array}\right)=\rho'_{k'_\rho}\left\{\begin{array}{c}\lambda_1\\ \vdots\\ \lambda_{k'_\rho}\end{array}\middle|\,\mathbf{F}_{k_F}\begin{pmatrix}\mathsf{v}_1\\ \vdots\\ \mathsf{v}_{k_F}\end{pmatrix}\right\},\tag{4.4}$$

where $\mathsf{v}_1,\ldots,\mathsf{v}_{m_V},\mathsf{v}\in\mathsf{V}$, $\lambda_1,\ldots,\lambda_{k_\rho}\in K$, and the four integers ℓ^k_ρ, ℓ^k_{id}, ℓ^f_ρ, ℓ^f_{id} define the ℓ-shape of the mapping.

It follows from (4.3), (4.4) that the arities satisfy

$$k_Fm'_V=m_V\ell^k_\mu+\ell^k_{\mathrm{id}},\quad k_F=\ell^k_\mu+\ell^k_{\mathrm{id}},\quad k_F=\ell^f_\mu+\ell^f_{\mathrm{id}},\quad k'_\mu=k_\rho\ell^f_\mu.$$

The following inequalities hold:

$$1 \leq \ell_{\mu}^{k} \leq k_F, \quad 0 \leq \ell_{\mathrm{id}}^{k} \leq k_F - 1,$$
$$\ell_{\mu}^{k} \leq k_F \leq (m_V - 1)\, \ell_{\mu}^{k}, \quad 2 \leq m_V' \leq m_V, \quad 2 \leq k_\rho \leq k_\rho'.$$

Thus, the ℓ-shape of the k_F-place mapping between polyadic vector spaces is determined by

$$\ell_{\mu}^{k} = \frac{k_F\,(m_V - 1)}{m_V - 1}, \quad \ell_{\mathrm{id}}^{k} = \frac{k_F\,(m_V - m_V')}{m_V - 1}, \quad \ell_{\mu}^{f} = \frac{k_\rho}{k_\rho'}, \quad \ell_{\mathrm{id}}^{f} = k_F - \frac{k_\rho}{k_\rho'}.$$

4.1. Polyadic functionals and dual polyadic vector spaces. An important particular case of the k_F-place mapping can be considered, where the final polyadic vector space coincides with the underlying field (an analog of a "linear functional").

Definition 4.3. A "linear" *polyadic functional* (or *polyadic dual vector*, *polyadic covector*) is a k_L-place mapping of a polyadic vector space $\mathcal{V}_{m_K,n_K,m_V,k_\rho} = \left\langle K; \mathsf{V} \mid \sigma_{m_K}, \kappa_{n_K}; \nu_{m_V} \mid \rho_{k_\rho} \right\rangle$ into its polyadic field $\mathbb{K}_{m_K,n_K} = \langle K \mid \sigma_{m_K}, \kappa_{n_K} \rangle$ such that there exists $L_{k_L} : \mathsf{V}^{\times k_L} \to K$, and

$$\mathbf{L}_{k_L} \begin{pmatrix} \left.\begin{matrix} \nu_{m_V}\left[\mathsf{v}_1, \ldots, \mathsf{v}_{m_V}\right] \\ \vdots \\ \nu_{m_V}\left[\mathsf{v}_{m_V(\ell_{\nu}^{k}-1)}, \ldots \mathsf{v}_{m_V \ell_{\nu}^{k}}\right] \end{matrix}\right\} \ell_{\nu}^{k} \\ \left.\begin{matrix} \mathsf{v}_{m_V \ell_{\nu}^{k}+1}, \\ \vdots \\ \mathsf{v}_{n_K \ell_{\nu}^{k} + \ell_{\mathrm{id}}^{\nu}} \end{matrix}\right\} \ell_{\mathrm{id}}^{\nu} \end{pmatrix} = \sigma_{m_K}\left[\mathbf{L}_{k_L}\begin{pmatrix} \mathsf{v}_1 \\ \vdots \\ \mathsf{v}_{k_L} \end{pmatrix}, \ldots, \mathbf{L}_{k_L}\begin{pmatrix} \mathsf{v}_{k_L(m_K-1)} \\ \vdots \\ \mathsf{v}_{k_L m_K} \end{pmatrix}\right], \quad (4.5)$$

$$\mathbf{L}_{k_L} \begin{pmatrix} \left.\begin{matrix} \rho_{k_\rho}\left\{\begin{matrix} \lambda_1 \\ \vdots \\ \lambda_{k_\rho} \end{matrix} \middle| \mathsf{v}_1 \right\} \\ \vdots \\ \rho_{k_\rho}\left\{\begin{matrix} \lambda_{k_\rho(\ell_{\mu}^{L}-1)} \\ \vdots \\ \lambda_{k_\rho \ell_{\mu}^{L}} \end{matrix} \middle| \mathsf{v}_{\ell_{\mu}^{L}} \right\} \end{matrix}\right\} \ell_{\mu}^{L} \\ \left.\begin{matrix} \mathsf{v}_{\ell_{\mu}^{L}+1} \\ \vdots \\ \mathsf{v}_{k_L} \end{matrix}\right\} \ell_{\mathrm{id}}^{L} \end{pmatrix} = \kappa_{n_K}\left[\lambda_1, \ldots, \lambda_{n_K-1}, \mathbf{L}_{k_L}\begin{pmatrix} \mathsf{v}_1 \\ \vdots \\ \mathsf{v}_{k_L} \end{pmatrix}\right], \quad (4.6)$$

where $\mathsf{v}_1, \ldots, \mathsf{v}_{m_V}, \mathsf{v} \in V$, $\lambda_1, \ldots, \lambda_{n_K} \in K$, and the integers ℓ_{ν}^{k}, ℓ_{id}^{ν}, ℓ_{μ}^{L}, ℓ_{id}^{L} define the ℓ-shape of L_{k_L}.

It follows from (4.3), (4.4) that the arities satisfy

$$k_L m_K = m_V \ell_\nu^k + \ell_{\mathrm{id}}^\nu, \quad k_L = \ell_\nu^k + \ell_{\mathrm{id}}^\nu, \quad k_L = \ell_\mu^h + \ell_{\mathrm{id}}^h, \quad n_K - 1 = k_\rho \ell_\mu^h,$$

and for them

$$\begin{array}{lll} 1 \leq \ell_\nu^k \leq k_L, & 0 \leq \ell_{\mathrm{id}}^\nu \leq k_L - 1, & \ell_\nu^k \leq k_L \leq (m_V - 1)\,\ell_\nu^k, \\ 2 \leq m_K \leq m_V, & 2 \leq k_\rho \leq n_K - 1. & \end{array}$$

Thus, the ℓ-shape of the polyadic functional is determined by

$$\ell_\nu^k = \frac{k_L (m_K - 1)}{m_V - 1}, \quad \ell_{\mathrm{id}}^\nu = \frac{k_L (m_V - m_K)}{m_V - 1}, \quad \ell_\mu^h = \frac{k_\rho}{n_K - 1}, \quad \ell_{\mathrm{id}}^h = k_L - \frac{k_\rho}{n_K - 1}.$$

In the binary case, because the dual vectors (linear functionals) take their values in the underlying field, new operations between them, such that the dual vector "addition" ($+^*$) and the "multiplication by a scalar" ($\bullet^*$) can be naturally introduced by $\left(L^{(1)} +^* L^{(2)}\right)(\mathsf{v}) = L^{(1)}(\mathsf{v}) + L^{(2)}(\mathsf{v})$, $(\lambda \bullet^* L)(\mathsf{v}) = \lambda \bullet L(\mathsf{v})$, which leads to another vector space structure, called a dual vector space. Note that the operations $+^*$ and $+$, $\bullet^*$ and $\bullet$ are different, because $+$ and $\bullet$ are the operations in the underlying field $\mathbb{K}$. In the polyadic case, we have more complicated arity changing formulas, and here we consider only finite-dimensional spaces. The arities of operations between dual vectors can differ from those in the underlying polyadic field $\mathbb{K}_{m_K n_K}$, in general. In this way, we arrive at the following

Definition 4.4. A *polyadic dual vector space* over a polyadic field $\mathbb{K}_{m_K,n_K}$ is

$$\mathcal{V}^*_{m_K,n_K,m_V^*,k_\rho^*} = \left\langle K; \left\{ \mathbf{L}_{k_L}^{(i)} \right\} \mid \sigma_{m_K}, \kappa_{n_K}; \nu_{m_L}^* \mid \rho_{k_L}^* \right\rangle,$$

and the axioms are:

1) $\langle K \mid \sigma_{m_K}, \kappa_{n_K} \rangle$ is a polyadic (m_K, n_K)-field $\mathbb{K}_{m_K,n_K}$;

2) $\left\langle \left\{ \mathbf{L}_{k_L}^{(i)} \right\} \mid \nu_{m_L}^*, i = 1, \ldots, D_L \right\rangle$ is a commutative m_L-ary group (which is finite if $D_L < \infty$);

3) The "dual vector addition" $\nu_{m_L}^*$ is compatible with the polyadic field addition σ_{m_K} by

$$\nu_{m_L}^* \left[\mathbf{L}_{k_L}^{(1)}, \ldots, \mathbf{L}_{k_L}^{(m_L)} \right] \left(\mathbf{a}^{(k_L)} \right) = \sigma_{m_K} \left[\mathbf{L}_{k_L}^{(1)} \left(\mathbf{a}^{(k_L)} \right), \ldots, \mathbf{L}_{k_L}^{(m_K)} \left(\mathbf{v}^{(k_L)} \right) \right],$$

where $\mathbf{v}^{(k_L)} = \begin{pmatrix} \mathsf{v}_1 \\ \vdots \\ \mathsf{v}_{k_L} \end{pmatrix}$, $\mathsf{v}_1, \ldots, \mathsf{v}_{k_L} \in \mathsf{V}$, and it follows that

$$m_L = m_K;$$

4) The compatibility of $\rho^*_{k_L}$ with the "multiplication by a scalar" in the underlying polyadic field

$$\rho^*_{k_L}\left\{\begin{array}{c}\lambda_1\\ \vdots\\ \lambda_{k_L}\end{array}\middle|\mathbf{L}_{k_L}\right\}\left(\mathbf{v}^{(k_L)}\right)=\kappa_{n_K}\left[\lambda_1,\ldots,\lambda_{n_K-1},\mathbf{L}_{k_L}\left(\mathbf{v}^{(k_L)}\right)\right], \tag{4.7}$$

and then

$$k_L=n_K-1; \tag{4.8}$$

5) $\left\langle\left\{\rho^*_{k_L}\right\}\mid composition\right\rangle$ is an n_L-ary semigroup $\mathcal{S}_L$ (similar to (3.8))

$$\overbrace{\rho^*_{k_L}\left\{\begin{array}{c}\lambda_1\\ \vdots\\ \lambda_{k_L}\end{array}\middle|\ldots\middle|\rho^*_{k_L}\left\{\begin{array}{c}\lambda_{k_L(n_L-1)}\\ \vdots\\ \lambda_{k_Ln_L}\end{array}\middle|\mathbf{L}_{k_L}\right\}\ldots\right\}}^{n_L}\left(\mathbf{v}^{(k_L)}\right)$$

$$=\rho^*_{k_L}\left\{\begin{array}{c}\left.\begin{array}{c}\kappa_{n_K}\left[\lambda_1,\ldots\lambda_{n_K}\right],\\ \vdots\\ \kappa_{n_K}\left[\lambda_{n_K(\ell^L_\mu-1)},\ldots\lambda_{n_K\ell^L_\mu}\right]\end{array}\right\}\ell^L_\mu\\ \left.\begin{array}{c}\lambda_{n_K\ell^L_\mu+1},\\ \vdots\\ \lambda_{n_K\ell^L_\mu+\ell^L_{\mathrm{id}}}\end{array}\right\}\ell^L_{\mathrm{id}}\end{array}\middle|\mathbf{L}_{k_L}\right\}\left(\mathbf{v}^{(k_L)}\right),$$

where the ℓ-shape is determined by the system

$$k_Ln_L=n_K\ell^L_\mu+\ell^L_{\mathrm{id}},\quad k_L=\ell^L_\mu+\ell^L_{\mathrm{id}}. \tag{4.9}$$

Using (4.8) and (4.9), we obtain the ℓ-shape as

$$\ell^L_\mu=n_L-1,\quad \ell^L_{\mathrm{id}}=n_K-n_L. \tag{4.10}$$

Corollary 4.5. *The arity n_L of the semigroup $\mathcal{S}_L$ is less than or equal to the arity n_K of the underlying polyadic field $n_L\leq n_K$.*

4.2. Polyadic direct sum and tensor product. The Cartesian product of D_V polyadic vector spaces $\times\Pi^{m_V}_{i=1}\mathcal{V}^{(i)}_{m_Kn_Km^{(i)}_Vk^{(i)}_\rho}$ (sometimes we use the concise notation $\times\Pi\mathcal{V}^{(i)}$), $i=1,\ldots,D_V$ is given by the D_V-ples (an analog of the Cartesian pair (v,u), $\mathsf{v}\in\mathsf{V}^{(1)}$, $\mathsf{u}\in\mathsf{V}^{(2)}$)

$$\begin{pmatrix}\mathsf{v}^{(1)}\\ \vdots\\ \mathsf{v}^{(D_V)}\end{pmatrix}\equiv\left(\mathsf{v}^{(D_V)}\right)\in\mathsf{V}^{\times D_V}. \tag{4.11}$$

We introduce polyadic generalizations of the direct sum and tensor product of vector spaces by considering "linear" operations on the D_V-ples (4.11).

In the first case, to endow $\times\Pi\mathcal{V}^{(i)}$ with the structure of a vector space we need to define a new operation between the D_V-ples (4.11) (this is similar to the vector addition, but between the elements from *different* spaces) and a rule, specifying how they are "multiplied by scalars" (analogs of $(\mathsf{v}_1, \mathsf{v}_2) + (\mathsf{u}_1, \mathsf{u}_2) = (\mathsf{v}_1 + \mathsf{u}_1, \mathsf{v}_2 + \mathsf{u}_2)$ and $\lambda(\mathsf{v}_1, \mathsf{v}_2) = (\lambda\mathsf{v}_1, \lambda\mathsf{v}_2)$). In the binary case, a formal summation is used, but it can differ from the addition in the initial vector spaces. Therefore, we can define on the set of the D_V-ples (4.11) new operations χ_{m_V} ("addition of vectors from *different spaces*") and "multiplication by a scalar" τ_{k_ρ}, which *does not* need to coincide with the corresponding operations $\nu^{(i)}_{m_V^{(i)}}$ and $\rho^{(i)}_{k_\rho^{(i)}}$ of the initial polyadic vector spaces $\mathcal{V}^{(i)}_{m_K, n_K, m_V^{(i)}, k_\rho^{(i)}}$.

If all D_V-ples (4.11) are of fixed length, then we can define their "addition" χ_{m_V} in the standard way when all the arities $m_V^{(i)}$ coincide and equal the arity of the resulting vector space

$$m_V = m_V^{(1)} = \ldots = m_V^{(D_V)}, \tag{4.12}$$

while the operations ("additions") themselves $\nu^{(i)}_{m_V}$ between vectors in different spaces can be still different. Thus, a new commutative m_V-ary operation ("addition") χ_{m_V} of the D_V-ples of the same length is defined by

$$\chi_{m_V}\left[\begin{pmatrix}\mathsf{v}_1^{(1)}\\ \vdots \\ \mathsf{v}_1^{(D_V)}\end{pmatrix}, \ldots, \begin{pmatrix}\mathsf{v}_{m_V}^{(1)}\\ \vdots \\ \mathsf{v}_{m_V}^{(D_V)}\end{pmatrix}\right] = \begin{pmatrix}\nu^{(1)}_{m_V}\left[\mathsf{v}_1^{(1)}, \ldots, \mathsf{v}_{m_V}^{(1)}\right]\\ \vdots \\ \nu^{(D_V)}_{m_V}\left[\mathsf{v}_1^{(D_V)}, \ldots, \mathsf{v}_{m_V}^{(D_V)}\right]\end{pmatrix}, \tag{4.13}$$

where $D_V \neq m_V$, in general. However, it is also possible to add D_V-ples of *different length* such that the operation (4.13) is compatible with all arities $m_V^{(i)}$, $i = 1, \ldots, m_V$. For instance, if $m_V = 3$, $m_V^{(1)} = m_V^{(2)} = 3$, $m_V^{(3)} = 2$, then

$$\chi_3\left[\begin{pmatrix}\mathsf{v}_1^{(1)}\\ \mathsf{v}_1^{(2)}\\ \mathsf{v}_1^{(3)}\end{pmatrix}, \begin{pmatrix}\mathsf{v}_2^{(1)}\\ \mathsf{v}_2^{(2)}\\ \mathsf{v}_2^{(3)}\end{pmatrix}, \begin{pmatrix}\mathsf{v}_3^{(1)}\\ \mathsf{v}_3^{(2)}\\ \end{pmatrix}\right] = \begin{pmatrix}\nu_3^{(1)}\left[\mathsf{v}_1^{(1)}, \mathsf{v}_2^{(1)}, \mathsf{v}_3^{(1)}\right]\\ \nu_3^{(2)}\left[\mathsf{v}_1^{(2)}, \mathsf{v}_2^{(2)}, \mathsf{v}_3^{(2)}\right]\\ \nu_2^{(3)}\left[\mathsf{v}_1^{(3)}, \mathsf{v}_2^{(3)}\right]\end{pmatrix}. \tag{4.14}$$

Assertion 4.6. *In the polyadic case, a direct sum of polyadic vector spaces having different arities of "vector addition" $m_V^{(i)}$ can be defined.*

Let us introduce the multiaction τ_{k_ρ} ("multiplication by a scalar") acting on

D_V-ple $\left(\mathsf{v}^{(m_V)}\right)$. Then

$$\tau_{k_\rho}\left\{\begin{array}{c|}\lambda_1\\ \vdots\\ \lambda_{k_\rho}\end{array}\begin{pmatrix}\mathsf{v}^{(1)}\\ \vdots\\ \mathsf{v}^{(D_V)}\end{pmatrix}\right\}=\begin{pmatrix}\rho^{(1)}_{k_\rho^{(1)}}\left\{\begin{array}{c|}\lambda_1\\ \vdots\\ \lambda_{k_\rho^{(1)}}\end{array}\mathsf{v}^{(1)}\right\}\\ \vdots\\ \rho^{(m_V)}_{k_\rho^{(D_V)}}\left\{\begin{array}{c|}\lambda_{k_\rho^{(1)}+\ldots+k_\rho^{(D_V-1)}+1}\\ \vdots\\ \lambda_{k_\rho^{(1)}+\ldots+k_\rho^{(D_V)}}\end{array}\mathsf{v}^{(D_V)}\right\}\end{pmatrix}, \tag{4.15}$$

where

$$k_\rho^{(1)}+\ldots+k_\rho^{(D_V)}=k_\rho. \tag{4.16}$$

Definition 4.7. A *polyadic direct sum* of m_V polyadic vector spaces is their Cartesian product equipped with the m_V-ary addition χ_{m_V} and the k_ρ-place multiaction τ_{k_ρ}, satisfying (4.13) and (4.15) respectively

$$\oplus\Pi_{i=1}^{D_V}\mathcal{V}^{(i)}_{m_K,n_K,m_V^{(i)},k_\rho^{(i)}}=\left\{\times\Pi_{i=1}^{D_V}\mathcal{V}^{(i)}_{m_K,n_K,m_V^{(i)},k_\rho^{(i)}}\mid\chi_{m_V},\tau_{k_\rho}\right\}.$$

Let us consider another way to define a vector space structure on the D_V-ples from the Cartesian product $\times\Pi\mathcal{V}^{(i)}$. Remember that in the binary case, the concept of bilinearity is used, which means "distributivity" and "multiplicativity by scalars" on *each place* separately in the Cartesian pair $(\mathsf{v}_1,\mathsf{v}_2)\in\mathsf{V}^{(1)}\times\mathsf{V}^{(2)}$ (*as opposed to* the direct sum, where these relations hold on all places simultaneously, see (4.13) and (4.15)) such that

$$(\mathsf{v}_1+\mathsf{u}_1,\mathsf{v}_2)=(\mathsf{v}_1,\mathsf{v}_2)+(\mathsf{u}_1,\mathsf{v}_2)\,,\qquad(\mathsf{v}_1,\mathsf{v}_2+\mathsf{u}_2)=(\mathsf{v}_1,\mathsf{v}_2)+(\mathsf{v}_1,\mathsf{u}_2)\,, \tag{4.17}$$

$$\lambda(\mathsf{v}_1,\mathsf{v}_2)=(\lambda\mathsf{v}_1,\mathsf{v}_2)=(\mathsf{v}_1,\lambda\mathsf{v}_2)\,, \tag{4.18}$$

respectively. If we denote the ideal corresponding to the relations (4.17), (4.18) by $\mathfrak{B}_2$, then the binary tensor product of the vector spaces can be defined from their Cartesian product by factoring out this ideal as $\mathcal{V}^{(1)}\otimes\mathcal{V}^{(2)}\overset{def}{=}\mathcal{V}^{(1)}\times\mathcal{V}^{(2)}/\mathfrak{B}_2$. Note first that the additions and multiplications by a scalar on both sides of (4.17), (4.18) "work" in different spaces, which sometimes can be concealed by adding the word "formal" to them. Second, all these operations have the *same* arity (binary ones), which need not to be the case when considering polyadic structures.

As in the case of the polyadic direct sum, we first define a new operation $\tilde{\chi}_{m_V}$ ("addition") of the D_V-ples of fixed length (different from χ_{m_V} in (4.13)), when all the arities $m_V^{(i)}$ coincide and are equal to m_V (4.12). Then, a straightforward generalization of (4.17) can be defined for m_V-ples similar to polyadic distributivity (2.1)–(2.3), as in the following m_V relations:

$$\begin{pmatrix} \nu_{m_V}^{(1)}\left[\mathsf{v}_1^{(1)},\ldots,\mathsf{v}_{m_V}^{(1)}\right] \\ \mathsf{u}_2 \\ \vdots \\ \mathsf{u}_{D_V} \end{pmatrix} = \tilde{\chi}_{m_V}\left[\begin{pmatrix} \mathsf{v}_1^{(1)} \\ \mathsf{u}_2 \\ \vdots \\ \mathsf{u}_{D_V} \end{pmatrix},\ldots,\begin{pmatrix} \mathsf{v}_{m_V}^{(1)} \\ \mathsf{u}_2 \\ \vdots \\ \mathsf{u}_{D_V} \end{pmatrix}\right], \tag{4.19}$$

$$\begin{pmatrix} \mathsf{u}_1 \\ \nu_{m_V}^{(2)}\left[\mathsf{v}_1^{(2)},\ldots,\mathsf{v}_{m_V}^{(2)}\right] \\ \vdots \\ \mathsf{u}_{m_V} \end{pmatrix} = \tilde{\chi}_{m_V}\left[\begin{pmatrix} \mathsf{u}_1 \\ \mathsf{v}_1^{(2)} \\ \vdots \\ \mathsf{u}_{m_V} \end{pmatrix},\ldots,\begin{pmatrix} \mathsf{u}_1 \\ \mathsf{v}_{m_V}^{(2)} \\ \vdots \\ \mathsf{u}_{m_V} \end{pmatrix}\right], \tag{4.20}$$

$$\ldots$$

$$\begin{pmatrix} \mathsf{u}_1 \\ \mathsf{u}_2 \\ \vdots \\ \nu_{m_V}^{(m_V)}\left[\mathsf{v}_1^{(D_V)},\ldots,\mathsf{v}_{m_V}^{(D_V)}\right] \end{pmatrix} = \tilde{\chi}_{m_V}\left[\begin{pmatrix} \mathsf{u}_1 \\ \mathsf{u}_2 \\ \vdots \\ \mathsf{v}_1^{(D_V)} \end{pmatrix},\ldots,\begin{pmatrix} \mathsf{u}_1 \\ \mathsf{u}_2 \\ \vdots \\ \mathsf{v}_{m_V}^{(D_V)} \end{pmatrix}\right]. \tag{4.21}$$

By analogy, if all $k_\rho^{(i)}$ are equal, we can define a new multiaction $\tilde{\tau}_{k_\rho}$ (different from τ_{k_ρ} (4.15)) but with the same number of places

$$k_\rho = k_\rho^{(1)} = \ldots = k_\rho^{(D_V)} \tag{4.22}$$

as the D_V relations (an analog of (4.18))

$$\tilde{\tau}_{k_\rho}\left\{\begin{matrix} \lambda_1 \\ \vdots \\ \lambda_{k_\rho} \end{matrix}\middle| \begin{pmatrix} \mathsf{v}^{(1)} \\ \vdots \\ \mathsf{v}^{(D_V)} \end{pmatrix}\right\} = \begin{pmatrix} \rho_{k_\rho}^{(1)}\left\{\begin{matrix} \lambda_1 \\ \vdots \\ \lambda_{k_\rho} \end{matrix}\middle| \mathsf{v}^{(1)}\right\} \\ \mathsf{v}^{(2)} \\ \vdots \\ \mathsf{v}^{(D_V)} \end{pmatrix} = \begin{pmatrix} \mathsf{v}^{(1)} \\ \rho_{k_\rho}^{(2)}\left\{\begin{matrix} \lambda_1 \\ \vdots \\ \lambda_{k_\rho} \end{matrix}\middle| \mathsf{v}^{(2)}\right\} \\ \vdots \\ \mathsf{v}^{(D_V)} \end{pmatrix}$$

$$\cdots = \begin{pmatrix} \mathsf{v}^{(1)} \\ \mathsf{v}^{(2)} \\ \vdots \\ \rho_{k_\rho}^{(D_V)}\left\{\begin{matrix} \lambda_1 \\ \vdots \\ \lambda_{k_\rho} \end{matrix}\middle| \mathsf{v}^{(D_V)}\right\} \end{pmatrix}. \tag{4.23}$$

Let us denote the ideal corresponding to the relations (4.19)–(4.21), (4.23) by $\mathfrak{B}_{D_V}$.

Definition 4.8. A *polyadic tensor product* of D_V polyadic vector spaces $\mathcal{V}^{(i)}_{m_K,n_K,m_V^{(i)},k_\rho^{(i)}}$ is obtained from their Cartesian product equipped with the m_V-

ary addition $\tilde{\chi}_{m_V}$ (of D_V-ples) and the k_ρ-place multiaction $\tilde{\tau}_{k_\rho}$, satisfying (4.19)–(4.21) and (4.23), respectively, by factoring out the ideal $\mathfrak{B}_{D_V}$

$$\otimes\Pi_{i=1}^{m_V}\mathcal{V}^{(i)}_{m_K,n_K,m_V^{(i)},k_\rho^{(i)}}=\left\{\times\Pi_{i=1}^{m_V}\mathcal{V}^{(i)}_{m_K,n_K,m_V^{(i)},k_\rho^{(i)}}\mid\tilde{\chi}_{m_V},\tilde{\tau}_{k_\rho}\right\}\diagup\mathfrak{B}_{D_V}.$$

As in the case of the polyadic direct sum, we can consider distributivity for D_V-ples of different length. In a similar example (4.14), if $m_V=3$, $m_V^{(1)}=m_V^{(2)}=3$, $m_V^{(3)}=2$, we have

$$\begin{pmatrix}\nu_3^{(1)}\left[\mathsf{v}_1^{(1)},\mathsf{v}_2^{(1)},\mathsf{v}_3^{(1)}\right]\\ \mathsf{u}_2\\ \mathsf{u}_3\end{pmatrix}=\tilde{\chi}_3\left[\begin{pmatrix}\mathsf{v}_1^{(1)}\\ \mathsf{u}_2\\ \mathsf{u}_3\end{pmatrix},\begin{pmatrix}\mathsf{v}_2^{(1)}\\ \mathsf{u}_2\\ \mathsf{u}_3\end{pmatrix},\begin{pmatrix}\mathsf{v}_3^{(1)}\\ \mathsf{u}_2\end{pmatrix}\right],$$

$$\begin{pmatrix}\mathsf{u}_1\\ \nu_3^{(2)}\left[\mathsf{v}_1^{(2)},\mathsf{v}_2^{(2)},\mathsf{v}_3^{(2)}\right]\\ \mathsf{u}_3\end{pmatrix}=\tilde{\chi}_3\left[\begin{pmatrix}\mathsf{u}_1\\ \mathsf{v}_1^{(2)}\\ \mathsf{u}_3\end{pmatrix},\begin{pmatrix}\mathsf{u}_1\\ \mathsf{v}_2^{(2)}\\ \mathsf{u}_3\end{pmatrix},\begin{pmatrix}\mathsf{u}_1\\ \mathsf{v}_3^{(2)}\end{pmatrix}\right],$$

$$\begin{pmatrix}\mathsf{u}_1\\ \mathsf{u}_2\\ \nu_2^{(3)}\left[\mathsf{v}_1^{(3)},\mathsf{v}_2^{(3)}\right]\end{pmatrix}=\tilde{\chi}_3\left[\begin{pmatrix}\mathsf{u}_1\\ \mathsf{u}_2\\ \mathsf{v}_1^{(3)}\end{pmatrix},\begin{pmatrix}\mathsf{u}_1\\ \mathsf{u}_2\\ \mathsf{v}_2^{(3)}\end{pmatrix},\begin{pmatrix}\mathsf{u}_1\\ \mathsf{u}_2\end{pmatrix}\right].$$

Assertion 4.9. *A tensor product of polyadic vector spaces having different arities of the "vector addition" $m_V^{(i)}$ can be defined.*

In the case of modules over a polyadic ring, the formulas connecting arities and ℓ-shapes similar to those above hold, while their concrete properties (noncommutativity, mediality, etc.) should be taken into account.

5. Polyadic inner pairing spaces and norms

Here we introduce the next important operation: a polyadic analog of the inner product for polyadic vector spaces - a polyadic inner pairing. However, this concept differs from the n-inner product spaces considered, e.g., in [37].

Let $\mathcal{V}_{m_K,n_K,m_V,k_\rho}=\left\langle K;\mathsf{V}\mid\sigma_{m_K},\kappa_{n_K};\nu_{m_V}\mid\rho_{k_\rho}\right\rangle$ be a polyadic vector space over the polyadic field $\mathbb{K}_{m_K,n_K}$ (3.20). By analogy with the binary inner product, we next introduce its polyadic counterpart and study its arity shape.

Definition 5.1. *A polyadic N-place* inner pairing *(an analog of the inner product) is a mapping*

$$\overbrace{\langle\!\langle\bullet|\bullet|\ldots|\bullet\rangle\!\rangle}^{N}:\mathsf{V}^{\times N}\to K,\tag{5.1}$$

satisfying the following conditions:

1) Polyadic "linearity" (3.8) (for first argument):

$$\left\langle\!\!\left\langle\rho_{k_\rho}\left\{\begin{array}{c|}\lambda_1\\ \vdots\\ \lambda_{k_\rho}\end{array}\,\mathsf{v}_1\right\}|\mathsf{v}_2|\ldots|\mathsf{v}_N\right\rangle\!\!\right\rangle=\kappa_{n_K}\left[\lambda_1,\ldots,\lambda_{k_\rho},\langle\!\langle\mathsf{v}_1|\mathsf{v}_2|\ldots|\mathsf{v}_N\rangle\!\rangle\right].\tag{5.2}$$

2) Polyadic "distributivity" (2.1)–(2.3) (on each place):

$$\langle\langle \nu_{m_V}[\mathsf{v}_1, \mathsf{u}_1, \dots \mathsf{u}_{m_V-1}] | \mathsf{v}_2 | \dots | \mathsf{v}_N \rangle\rangle = \sigma_{m_K}[\langle\langle \mathsf{v}_1 | \mathsf{v}_2 | \dots | \mathsf{v}_N \rangle\rangle, \langle\langle \mathsf{u}_1 | \mathsf{v}_2 | \dots | \mathsf{v}_N \rangle\rangle \dots \langle\langle \mathsf{u}_{m_V-1} | \mathsf{v}_2 | \dots | \mathsf{v}_N \rangle\rangle]. \quad (5.3)$$

If the polyadic field $\mathbb{K}_{m_K,n_K}$ contains the zero z_K and $\langle \mathsf{V} \mid m_V \rangle$ has the zero "vector" z_V (which is not always true in the polyadic case), we have the additional axiom:

3) The polyadic inner pairing vanishes $\langle\langle \mathsf{v}_1 | \mathsf{v}_2 | \dots | \mathsf{v}_N \rangle\rangle = z_K$ iff any of the "vectors" vanishes, $\exists i \in 1, \dots, N$, such that $\mathsf{v}_i = \mathsf{z}_V$.

If the standard binary ordering on $\mathbb{K}_{m_K,n_K}$ can be defined, then the polyadic inner pairing satisfies:

4) The positivity condition

$$\overbrace{\langle\langle \mathsf{v} | \mathsf{v} | \dots | \mathsf{v} \rangle\rangle}^{N} \geq z_K,$$

5) The polyadic *Cauchy–Schwarz inequality ("triangle" inequality)*

$$\kappa_{n_K}\Big[\overbrace{\overbrace{\langle\langle \mathsf{v}_1 | \mathsf{v}_1 | \dots | \mathsf{v}_1 \rangle\rangle}^{N}, \overbrace{\langle\langle \mathsf{v}_2 | \mathsf{v}_2 | \dots | \mathsf{v}_2 \rangle\rangle}^{N} \dots \overbrace{\langle\langle \mathsf{v}_{n_K} | \mathsf{v}_{n_K} | \dots | \mathsf{v}_{n_K} \rangle\rangle}^{N}}^{n_K}\Big] \geq \kappa_{n_K}\Big[\overbrace{\langle\langle \mathsf{v}_1 | \mathsf{v}_2 | \dots | \mathsf{v}_N \rangle\rangle, \langle\langle \mathsf{v}_1 | \mathsf{v}_2 | \dots | \mathsf{v}_N \rangle\rangle \dots, \langle\langle \mathsf{v}_1 | \mathsf{v}_2 | \dots | \mathsf{v}_N \rangle\rangle}^{n_K}\Big]. \quad (5.4)$$

To make the above relations consistent, the arity shapes should be fixed.

Definition 5.2. If the inner pairing is fully symmetric under permutations it is called a *polyadic inner product*.

Proposition 5.3. *The number of places in the multiaction ρ_{k_ρ} differs by 1 from the multiplication arity of the polyadic field*

$$n_K - k_\rho = 1. \quad (5.5)$$

Proof. It follows from the polyadic "linearity"(5.2). □

Proposition 5.4. *The arities of "vector addition" and "field addition" coincide*

$$m_V = m_K. \quad (5.6)$$

Proof. Implied by the polyadic "distributivity"(5.3). □

Proposition 5.5. *The arity of the "field multiplication" is equal to the arity of the polyadic inner pairing space*

$$n_K = N. \quad (5.7)$$

Proof. The proof follows from the polyadic Cauchy-Schwarz inequality (5.4). □

Definition 5.6. The polyadic vector space $\mathcal{V}_{m_K,n_K,m_V,k_\rho}$ equipped with the polyadic inner pairing $\overbrace{\langle\langle \bullet|\bullet|\ldots|\bullet\rangle\rangle}^{N} : \mathsf{V}^{\times N} \to K$ is called a *polyadic inner pairing space* $\mathcal{H}_{m_K,n_K,m_V,k_\rho,N}$.

A polyadic analog of the binary norm $\|\bullet\| : \mathsf{V} \to K$ can be induced by the inner pairing similarly to the binary case for the inner product (we use the form $\|\mathsf{v}\|^2 = \langle\langle \mathsf{v}|\mathsf{v}\rangle\rangle$).

Definition 5.7. A polyadic norm of a "vector" v in the polyadic inner pairing space $\mathcal{H}_{m_K,n_K,m_V,k_\rho,N}$ is a mapping $\|\bullet\|_N : \mathsf{V} \to K$, such that

$$\kappa_{n_K}\Big[\overbrace{\|\mathsf{v}\|_N, \|\mathsf{v}\|_N, \ldots, \|\mathsf{v}\|_N}^{n_K}\Big] = \overbrace{\langle\langle \mathsf{v}|\mathsf{v}|\ldots|\mathsf{v}\rangle\rangle}^{N}, \qquad n_K = N, \tag{5.8}$$

and the following axioms apply:

1) The polyadic "linearity"

$$\left\| \rho_{k_\rho} \left\{ \begin{array}{c|} \lambda_1 \\ \vdots \\ \lambda_{k_\rho} \end{array} \mathsf{v} \right\} \right\|_N = \kappa_{n_K}\left[\lambda_1, \ldots, \lambda_{k_\rho}, \|\mathsf{v}\|_N\right], \tag{5.9}$$

$$n_K - k_\rho = 1. \tag{5.10}$$

If the polyadic field $\mathbb{K}_{m_K,n_K}$ contains the zero z_K and $\langle \mathsf{V} \mid m_V\rangle$ has a zero "vector" z_V, then:

2) The polyadic norm vanishes $\|\mathsf{v}\|\, N = z_K$ iff $\mathsf{v} = \mathsf{z}_V$.

If the binary ordering on $\langle \mathsf{V} \mid m_V\rangle$ can be defined, then:

3) The polyadic norm is positive $\|\mathsf{v}\|_N \geq z_K$.

4) The polyadic "triangle" inequality holds

$$\sigma_{m_K}\Big[\overbrace{\|\mathsf{v}_1\|_N, \|\mathsf{v}_2\|_N, \ldots, \|\mathsf{v}_N\|_N}^{m_K}\Big] \geq \Big\|\nu_{m_V}\Big[\overbrace{\|\mathsf{v}_1\|_N, \|\mathsf{v}_2\|_N, \ldots, \|\mathsf{v}_N\|_N}^{m_V}\Big]\Big\|,$$
$$m_K = m_V = N.$$

Definition 5.8. The polyadic inner pairing space $\mathcal{H}_{m_K,n_K,m_V,k_\rho,N}$ equipped with the polyadic norm $\|\mathsf{v}\|_N$ is called a *polyadic normed space*.

Recall that in the binary vector space V over the field $\mathbb{K}$ equipped with the inner product $\langle\langle \bullet|\bullet\rangle\rangle$ and the norm $\|\bullet\|$, one can introduce the angle between the vectors $\|\mathsf{v}_1\|\,\|\mathsf{v}_2\|\cos\theta = \langle\langle \mathsf{v}_1|\mathsf{v}_2\rangle\rangle$, where in the l.h.s. there are *two* binary multiplications $(\cdot)$.

Definition 5.9. A *polyadic angle* between N vectors $\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_{n_K}$ of the polyadic inner pairing space $\mathcal{H}_{m_K, n_K, m_V, k_\rho, N}$ is defined as a *set* of angles $\vartheta = \{\{\theta_i\} \mid i = 1, 2, \ldots, n_K - 1\}$ satisfying

$$\kappa_{n_K}^{(2)} \left[\|\mathsf{v}_1\|_N , \|\mathsf{v}_2\|_N , \ldots , \|\mathsf{v}_{n_K}\|_N , \cos\theta_1, \cos\theta_2, \ldots, \cos\theta_{n_K-1} \right]$$
$$= \langle\!\langle \mathsf{v}_1 | \mathsf{v}_2 | \ldots | \mathsf{v}_{n_K} \rangle\!\rangle ,$$

where $\kappa_{n_K}^{(2)}$ is a long product of two n_K-ary multiplications, which consists of $2(n_K - 1) + 1$ terms.

We will not consider the completion with respect to the above norm (to obtain a polyadic analog of the Hilbert space) and corresponding limits and boundedness questions, because this will not give us additional arity shapes in which we are mostly interested here. Instead, we turn below to some applications and new general constructions which appear from the above polyadic structures.

Table 5.1: The arity signature and arity shape of polyadic algebraic structures.

Structures	**Sets**		**Operations and arities**						**Arity shape**
	N	Name	N	Multiplications		Additions		Multiactions	
Group-like polyadic algebraic structures									
n-ary magma (or groupoid)	**1**	M	**1**	$\mu_n : M^n \to M$					
n-ary semigroup (and monoid)	**1**	S	**1**	$\mu_n : S^n \to S$					
n-ary quasigroup (and loop)	**1**	Q	**1**	$\mu_n : Q^n \to Q$					
n-ary group	**1**	G	**1**	$\mu_n : G^n \to G$					
Ring-like polyadic algebraic structures									
(m, n)-ary ring	**1**	R	**2**	$\mu_n : R^n \to R$		$\nu_m : R^m \to R$			
(m, n)-ary field	**1**	K	**2**	$\mu_n : K^n \to K$		$\nu_m : K^m \to K$			
Module-like polyadic algebraic structures									
Module over (m, n)-ring	**2**	R, M	**4**	$\sigma_n : R^n \to R$		$\kappa_m : R^m \to R$	$\nu_{m_M} : \mathsf{M}^{m_M} \to \mathsf{M}$	$\rho_{k_\rho} : R^{k_\rho} \times \mathsf{M} \to \mathsf{M}$	
Vector space over (m_K, n_K)-field	**2**	K, V	**4**	$\sigma_{n_K} : K^{n_K} \to K$		$\kappa_{m_K} : K^{m_K} \to K$	$\nu_{m_V} : \mathsf{V}^{m_V} \to \mathsf{V}$	$\rho_{k_\rho} : K^{k_\rho} \times \mathsf{V} \to \mathsf{V}$	(3.13) (3.19)
Algebra-like polyadic algebraic structures									
Inner pairing space over (m_K, n_K)-field	**2**	K, V	**5**	$\sigma_{n_K} : K^{n_K} \to K$	N-Form $\langle\!\langle \bullet .. \bullet \rangle\!\rangle : \mathsf{V}^N \to K$	$\kappa_{m_K} : K^{m_K} \to K$	$\nu_{m_V} : \mathsf{V}^{m_V} \to \mathsf{V}$	$\rho_{k_\rho} : K^{k_\rho} \times \mathsf{V} \to \mathsf{V}$	(5.5) (5.6) (5.7)
(m_A, n_A)-algebra over (m_K, n_K)-field	**2**	K, A	**5**	$\sigma_{n_K} : K^{n_K} \to K$	$\mu_{n_A} : \mathsf{A}^n \to \mathsf{A}$	$\kappa_{m_K} : K^{m_K} \to K$	$\nu_{m_A} : \mathsf{A}^{m_M} \to \mathsf{A}$	$\rho_{k_\rho} : K^{k_\rho} \times \mathsf{A} \to \mathsf{A}$	(3.29)

To conclude, we present the resulting Table 5.1 in which the polyadic algebraic structures are listed together with their arity shapes.

Applications

6. Elements of polyadic operator theory

Here we consider the 1-place polyadic operators $\mathbf{T}=\mathbf{F}_{k_F=1}$ (the case $k_F=1$ of the mapping $\mathbf{F}_{k_F}$ in Definition 4.2) on polyadic inner pairing spaces and structurally generalize the concepts of adjointness and involution.

Remark 6.1. A polyadic operator is a complicated mapping between polyadic vector spaces having nontrivial arity shapes (4.3) which is actually an action on a set of "vectors". However, only for $k_F=1$ it can be written in a formal way multiplicatively, as is always done in the binary case.

Recall (to fix notations and observe analogies) the informal standard introduction of the operator algebra and the adjoint operator on a binary pre-Hilbert space $\mathcal{H}$ $(\equiv \mathcal{H}_{m_K=2,n_K=2,m_V=2,k_\rho=1,N=2})$ over a binary field $\mathbb{K}$ $(\equiv \mathbb{K}_{m_K=2,n_K=2})$ (having the underlying set $\{K;\mathsf{V}\}$). For the operator norm $\|\bullet\|_T:\{\mathbf{T}\}\to K$, we use (among many others) the definition

$$\|\mathbf{T}\|_T=\inf\{M\in K \mid \forall \mathsf{v}\in\mathsf{V}\ \ \|\mathbf{T}\mathsf{v}\|\leq M\|\mathsf{v}\|\}, \tag{6.1}$$

which is convenient for further polyadic generalization. *Bounded* operators have $M<\infty$. If on the set of operators $\{\mathbf{T}\}$ (as 1-place mappings $\mathsf{V}\to\mathsf{V}$) one defines the addition $(+_T)$, product $(\circ_T)$ and scalar multiplication $(\cdot_T)$ in the standard way:

$$(\mathbf{T}_1+_T\mathbf{T}_2)(\mathsf{v})=\mathbf{T}_1\mathsf{v}+\mathbf{T}_2\mathsf{v},$$
$$(\mathbf{T}_1\circ_T\mathbf{T}_2)(\mathsf{v})=\mathbf{T}_1(\mathbf{T}_2\mathsf{v}),$$
$$(\lambda\cdot_T\mathbf{T})(\mathsf{v})=\lambda(\mathbf{T}\mathsf{v}),\quad \lambda\in K,\quad \mathsf{v}\in\mathsf{V},$$

then $\langle\{\mathbf{T}\}\mid +_T,\circ_T|\cdot_T\rangle$ becomes an operator algebra $\mathcal{A}_T$ (associativity and distributivity are obvious). The unity $\mathbf{I}$ and zero $\mathbf{Z}$ of $\mathcal{A}_T$ (if they exist) satisfy

$$\mathbf{I}\mathsf{v}=\mathsf{v}, \tag{6.2}$$
$$\mathbf{Z}\mathsf{v}=\mathsf{z}_V,\quad \mathsf{v}\in\mathsf{V}, \tag{6.3}$$

respectively, where $\mathsf{z}_V\in\mathsf{V}$ is the polyadic "zero-vector".

The connection between operators, linear functionals and inner products is given by the Riesz representation theorem. Informally, it states that in a binary pre-Hilbert space $\mathcal{H}=\{K;\mathsf{V}\}$ a (bounded) linear functional (sesquilinear form) $\mathbf{L}:\mathsf{V}\times\mathsf{V}\to K$ can be *uniquely* represented as

$$\mathbf{L}(\mathsf{v}_1,\mathsf{v}_2)=\langle\!\langle\mathbf{T}\mathsf{v}_1|\mathsf{v}_2\rangle\!\rangle_{\mathrm{sym}},\quad \mathsf{v}_1,\mathsf{v}_2\in\mathsf{V}, \tag{6.4}$$

where $\langle\!\langle\bullet|\bullet\rangle\!\rangle_{\mathrm{sym}}:\mathsf{V}\times\mathsf{V}\to K$ is a (binary) inner product with standard properties and $\mathbf{T}:\mathsf{V}\to\mathsf{V}$ is a bounded linear operator such that the norms of $\mathbf{L}$ and $\mathbf{T}$ coincide. Because the linear functionals form a dual space (see Subsection 4.1), the relation (6.4) fixes the shape of its elements. The main consequence of the Riesz representation theorem is the existence of the adjoint: for any (bounded) linear operator $\mathbf{T} : \mathsf{V} \to \mathsf{V}$ there exists a (unique bounded) *adjoint operator* $\mathbf{T}^* : \mathsf{V} \to \mathsf{V}$ satisfying

$$\mathbf{L}\left(\mathsf{v}_1, \mathsf{v}_2\right) = \langle\!\langle \mathbf{T}\mathsf{v}_1 | \mathsf{v}_2 \rangle\!\rangle_{\text{sym}} = \langle\!\langle \mathsf{v}_1 | \mathbf{T}^* \mathsf{v}_2 \rangle\!\rangle_{\text{sym}}, \quad \mathsf{v}_1, \mathsf{v}_2 \in \mathsf{V}, \tag{6.5}$$

and the norms of $\mathbf{T}$ and $\mathbf{T}^*$ are equal. It follows from the conjugation symmetry of the standard binary inner product that (6.5) coincides with

$$\langle\!\langle \mathsf{v}_1 | \mathbf{T}\mathsf{v}_2 \rangle\!\rangle_{\text{sym}} = \langle\!\langle \mathbf{T}^* \mathsf{v}_1 | \mathsf{v}_2 \rangle\!\rangle_{\text{sym}}, \quad \mathsf{v}_1, \mathsf{v}_2 \in \mathsf{V}. \tag{6.6}$$

However, when $\langle\!\langle \bullet | \bullet \rangle\!\rangle$ has no symmetry (permutation, conjugation, etc., see, e.g., [36]), it becomes the binary ($N = 2$) inner pairing (5.1), the binary adjoint consists of 2 operators $(\mathbf{T}^{\star_{12}}) \neq (\mathbf{T}^{\star_{21}})$, $\mathbf{T}^{\star_{ij}} : \mathsf{V} \to \mathsf{V}$, which should be defined by 2 equations

$$\langle\!\langle \mathbf{T}\mathsf{v}_1 | \mathsf{v}_2 \rangle\!\rangle = \langle\!\langle \mathsf{v}_1 | \mathbf{T}^{\star_{12}} \mathsf{v}_2 \rangle\!\rangle,$$
$$\langle\!\langle \mathsf{v}_1 | \mathbf{T}\mathsf{v}_2 \rangle\!\rangle = \langle\!\langle \mathbf{T}^{\star_{21}} \mathsf{v}_1 | \mathsf{v}_2 \rangle\!\rangle,$$

where $(\star_{12}) \neq (\star_{21})$ are 2 different star operations satisfying 2 relations

$$\mathbf{T}^{\star_{12}\star_{21}} = \mathbf{T}, \tag{6.7}$$
$$\mathbf{T}^{\star_{21}\star_{12}} = \mathbf{T}. \tag{6.8}$$

If $\langle\!\langle \bullet | \bullet \rangle\!\rangle = \langle\!\langle \bullet | \bullet \rangle\!\rangle_{\text{sym}}$ is symmetric, it becomes the inner product in the pre-Hilbert space $\mathcal{H}$ and equations (6.7), (6.8) coincide, while the operation $(*) = (\star_{12}) = (\star_{21})$ stands for the standard involution

$$\mathbf{T}^{**} = \mathbf{T}. \tag{6.9}$$

6.1. Multistars and polyadic adjoints. Consider now a special case of the polyadic inner pairing space (see Definition 5.6)

$$\mathcal{H}_{m_K, n_K, m_V, k_\rho = 1, N} = \left\langle K; \mathsf{V} \mid \sigma_{m_K}, \kappa_{n_K}; \nu_{m_V} \mid \rho_{k_\rho=1} \mid \overbrace{\langle\!\langle \bullet | \ldots | \bullet \rangle\!\rangle}^{N} \right\rangle$$

with a 1-place multiaction $\rho_{k_\rho=1}$.

Definition 6.2. The set of 1-place operators $\mathbf{T} : \mathsf{V} \to \mathsf{V}$ together with the set of "scalars" K becomes a polyadic operator algebra $\mathcal{A}_T = \langle K; \{\mathbf{T}\} \mid \sigma_{m_K}, \kappa_{n_K}; \eta_{m_T}, \omega_{n_T} \mid \theta_{k_F=1} \rangle$ if the operations $\eta_{m_T}, \omega_{n_T}, \theta_{k_F=1}$ are defined by

$$\eta_{m_T}\left[\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_{m_T}\right](\mathsf{v}) = \nu_{m_V}\left[\mathbf{T}_1\mathsf{v}, \mathbf{T}_2\mathsf{v}, \ldots, \mathbf{T}_{m_T}\mathsf{v}\right], \tag{6.10}$$
$$\omega_{n_T}\left[\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_{n_T}\right](\mathsf{v}) = \mathbf{T}_1\left(\mathbf{T}_2 \ldots \left(\mathbf{T}_{n_T}\mathsf{v}\right)\right), \tag{6.11}$$
$$\theta_{k_F=1}\left\{\lambda \mid \mathbf{T}\right\}(\mathsf{v}) = \rho_{k_\rho=1}\left\{\lambda \mid \mathbf{T}\mathsf{v}\right\}, \quad \lambda \in K, \ \mathsf{v} \in \mathsf{V}. \tag{6.12}$$

The arity shape is fixed by

Proposition 6.3. *In the polyadic algebra $\mathcal{A}_T$ the arity of the operator addition m_T coincides with the "vector" addition of the inner pairing space m_V, i.e.,*

$$m_T = m_V. \tag{6.13}$$

Proof. The proof follows from (6.10). □

To get relations between operators we assume (as in the binary case) the uniqueness: for any $\mathbf{T}_1, \mathbf{T}_2 : \mathsf{V} \to \mathsf{V}$ it follows from

$$\langle\!\langle \mathsf{v}_1 | \mathsf{v}_2 | \ldots | \mathbf{T}_1 \mathsf{v}_i | \ldots \mathsf{v}_{N-1} | \mathsf{v}_N \rangle\!\rangle = \langle\!\langle \mathsf{v}_1 | \mathsf{v}_2 | \ldots | \mathbf{T}_2 \mathsf{v}_i | \ldots \mathsf{v}_{N-1} | \mathsf{v}_N \rangle\!\rangle \tag{6.14}$$

that $\mathbf{T}_1 = \mathbf{T}_2$ in any place $i = 1, \ldots, N$.

First, by analogy with the binary adjoint (6.5), we define N different adjoints for each operator $\mathbf{T}$.

Definition 6.4. Given a polyadic operator $\mathbf{T} : \mathsf{V} \to \mathsf{V}$ on the polyadic inner pairing space $\mathcal{H}_{m_K, n_K, m_V, k_\rho = 1, N}$, we define a *polyadic adjoint* as the set $\{\mathbf{T}^{\star ij}\}$ of N operators $\mathbf{T}^{\star ij}$ satisfying the following N equations:

$$\begin{aligned}
\langle\!\langle \mathbf{T}\mathsf{v}_1 | \mathsf{v}_2 | \mathsf{v}_3 | \ldots | \mathsf{v}_N \rangle\!\rangle &= \langle\!\langle \mathsf{v}_1 | \mathbf{T}^{\star 12} \mathsf{v}_2 | \mathsf{v}_3 | \ldots | \mathsf{v}_N \rangle\!\rangle , \\
\langle\!\langle \mathsf{v}_1 | \mathbf{T}\mathsf{v}_2 | \mathsf{v}_3 | \ldots | \mathsf{v}_N \rangle\!\rangle &= \langle\!\langle \mathsf{v}_1 | \mathsf{v}_2 | \mathbf{T}^{\star 23} \mathsf{v}_3 | \ldots | \mathsf{v}_N \rangle\!\rangle , \\
&\ldots \\
\langle\!\langle \mathsf{v}_1 | \mathsf{v}_2 | \mathsf{v}_3 | \ldots \mathbf{T}\mathsf{v}_{N-1} | \mathsf{v}_N \rangle\!\rangle &= \langle\!\langle \mathsf{v}_1 | \mathsf{v}_2 | \mathsf{v}_3 | \ldots | \mathbf{T}^{\star N-1,N} \mathsf{v}_N \rangle\!\rangle , \\
\langle\!\langle \mathsf{v}_1 | \mathsf{v}_2 | \mathsf{v}_3 | \ldots \mathsf{v}_{N-1} | \mathbf{T}\mathsf{v}_N \rangle\!\rangle &= \langle\!\langle \mathbf{T}^{\star N,1} \mathsf{v}_1 | \mathsf{v}_2 | \mathsf{v}_3 | \ldots | \mathsf{v}_N \rangle\!\rangle , \qquad \mathsf{v}_i \in \mathsf{V}.
\end{aligned} \tag{6.15}$$

In what follows, for the composition we will use the notation $(\mathbf{T}^{\star ij})^{\star kl \cdots} \equiv \mathbf{T}^{\star ij \star kl \cdots}$. From (6.15), we have the N relations:

$$\begin{aligned}
\mathbf{T}^{\star 12 \star 23 \star 34 \cdots \star N-1,N \star N,1} &= \mathbf{T}, \\
\mathbf{T}^{\star 23 \star 34 \cdots \star N-1,N \star N,1 \star 12} &= \mathbf{T}, \\
&\ldots \\
\mathbf{T}^{\star N,1 \star 12 \star 23 \star 34 \cdots \star N-1,N} &= \mathbf{T},
\end{aligned} \tag{6.16}$$

which are called *multistar cycles*.

Definition 6.5. We call the set of adjoint mappings $(\bullet^{\star ij}) : \mathbf{T} \to \mathbf{T}^{\star ij}$ a *polyadic involution* if they satisfy the multistar cycles (6.16).

If the inner pairing $\langle\!\langle \bullet | \ldots | \bullet \rangle\!\rangle$ has more than two places $N \geq 3$, we have some additional structural issues which do not exist in the binary case.

First, we observe that the set of the adjointness relations (6.15) can be described in the framework of the associativity quiver approach introduced in [16] for polyadic representations. That is, for general $N \geq 3$ in addition to (6.15) which corresponds to the so called Post-like associativity quiver (they will be called the *Post-like adjointness relations*), there also exist other sets. It is cumbersome to write additional general formulas like (6.15) for other non-Post-like cases, so instead we give a clear example for $N = 4$.

Example 6.6. The polyadic adjointness relations for $N = 4$ consist of the sets corresponding to different associativity quivers:

1) *Post-like adjointness relations*

$$\langle\!\langle \mathbf{T}\mathsf{v}_1|\mathsf{v}_2|\mathsf{v}_3|\mathsf{v}_4\rangle\!\rangle = \langle\!\langle \mathsf{v}_1|\mathbf{T}^{\star 12}\mathsf{v}_2|\mathsf{v}_3|\mathsf{v}_4\rangle\!\rangle,$$
$$\langle\!\langle \mathsf{v}_1|\mathbf{T}\mathsf{v}_2|\mathsf{v}_3|\mathsf{v}_4\rangle\!\rangle = \langle\!\langle \mathsf{v}_1|\mathsf{v}_2|\mathbf{T}^{\star 23}\mathsf{v}_3|\mathsf{v}_4\rangle\!\rangle,$$
$$\langle\!\langle \mathsf{v}_1|\mathsf{v}_2|\mathbf{T}\mathsf{v}_3|\mathsf{v}_4\rangle\!\rangle = \langle\!\langle \mathsf{v}_1|\mathsf{v}_2|\mathsf{v}_3|\mathbf{T}^{\star 34}\mathsf{v}_4\rangle\!\rangle,$$
$$\langle\!\langle \mathsf{v}_1|\mathsf{v}_2|\mathsf{v}_3|\mathbf{T}\mathsf{v}_4\rangle\!\rangle = \langle\!\langle \mathbf{T}^{\star 41}\mathsf{v}_1|\mathsf{v}_2|\mathsf{v}_3|\mathsf{v}_4\rangle\!\rangle,$$

2) *Non-Post-like adjointness relations*

$$\langle\!\langle \mathbf{T}\mathsf{v}_1|\mathsf{v}_2|\mathsf{v}_3|\mathsf{v}_4\rangle\!\rangle = \langle\!\langle \mathsf{v}_1|\mathsf{v}_2|\mathsf{v}_3|\mathbf{T}^{\star 14}\mathsf{v}_4\rangle\!\rangle,$$
$$\langle\!\langle \mathsf{v}_1|\mathsf{v}_2|\mathsf{v}_3|\mathbf{T}\mathsf{v}_4\rangle\!\rangle = \langle\!\langle \mathsf{v}_1|\mathsf{v}_2|\mathbf{T}^{\star 43}\mathsf{v}_3|\mathsf{v}_4\rangle\!\rangle,$$
$$\langle\!\langle \mathsf{v}_1|\mathsf{v}_2|\mathbf{T}\mathsf{v}_3|\mathsf{v}_4\rangle\!\rangle = \langle\!\langle \mathsf{v}_1|\mathbf{T}^{\star 32}\mathsf{v}_2|\mathsf{v}_3|\mathsf{v}_4\rangle\!\rangle,$$
$$\langle\!\langle \mathsf{v}_1|\mathbf{T}\mathsf{v}_2|\mathsf{v}_3|\mathsf{v}_4\rangle\!\rangle = \langle\!\langle \mathbf{T}^{\star 21}\mathsf{v}_1|\mathsf{v}_2|\mathsf{v}_3|\mathsf{v}_4\rangle\!\rangle$$

and the corresponding multistar cycles:

1) *Post-like multistar cycles*

$$\mathbf{T}^{\star 12\star 23\star 34\star 41} = \mathbf{T},$$
$$\mathbf{T}^{\star 23\star 34\star 41\star 12} = \mathbf{T},$$
$$\mathbf{T}^{\star 34\star 41\star 12\star 23} = \mathbf{T},$$
$$\mathbf{T}^{\star 41\star 12\star 23\star 34} = \mathbf{T},$$

2) *Non-Post-like multistar cycles*

$$\mathbf{T}^{\star 14\star 43\star 32\star 21} = \mathbf{T},$$
$$\mathbf{T}^{\star 43\star 32\star 21\star 14} = \mathbf{T},$$
$$\mathbf{T}^{\star 32\star 21\star 14\star 43} = \mathbf{T},$$
$$\mathbf{T}^{\star 21\star 14\star 43\star 32} = \mathbf{T}.$$

Thus, if the inner pairing has no symmetry, then both the Post-like and non-Post-like adjoints and corresponding multistar involutions are different.

Second, in the case of $N \geq 3$, any symmetry of the multiplace inner pairing restricts the polyadic adjoint sets and multistar involutions considerably.

Theorem 6.7. *If the inner pairing with $N \geq 3$ has the full permutation symmetry*

$$\langle\!\langle \mathsf{v}_1|\mathsf{v}_2|\ldots|\mathsf{v}_N\rangle\!\rangle = \langle\!\langle \sigma\mathsf{v}_1|\sigma\mathsf{v}_2|\ldots|\sigma\mathsf{v}_N\rangle\!\rangle, \quad \sigma \in \mathfrak{S}_N,$$

where $\mathfrak{S}_N$ is the symmetric group of N elements, then:

1. *All the multistars coincide $(\star_{ij}) = (\star_{kl}) := (*)$ for any allowed $i, j, k, l = 1, \ldots, N$;*
2. *All the operators are self-adjoint $\mathbf{T} = \mathbf{T}^*$.*

Proof. 1. In each adjointness relation from (6.15), we place the operator $\mathbf{T}$ in the l.h.s. to the first position and its multistar adjoint $\mathbf{T}^{\star_{ij}}$ to the second position, using the full permutation symmetry, which together with (6.14) gives the equality of all multistar operations.

2. We place the operator $\mathbf{T}$ in the l.h.s. to the first position and apply the derivation of the involution in the binary case to increasing cycles of size $i \leq N$ recursively, that is:

For $i = 2$,

$$\langle\!\langle \mathbf{T}\mathsf{v}_1|\mathsf{v}_2|\mathsf{v}_3|\ldots|\mathsf{v}_N\rangle\!\rangle = \langle\!\langle \mathsf{v}_1|\mathbf{T}^*\mathsf{v}_2|\mathsf{v}_3|\ldots|\mathsf{v}_N\rangle\!\rangle = \langle\!\langle \mathbf{T}^*\mathsf{v}_2|\mathsf{v}_1|\mathsf{v}_3|\ldots|\mathsf{v}_N\rangle\!\rangle$$
$$= \langle\!\langle \mathsf{v}_2|\mathbf{T}^{**}\mathsf{v}_1|\mathsf{v}_3|\ldots|\mathsf{v}_N\rangle\!\rangle = \langle\!\langle \mathbf{T}^{**}\mathsf{v}_1|\mathsf{v}_2|\mathsf{v}_3|\ldots|\mathsf{v}_N\rangle\!\rangle,$$

then, using (6.14), we get

$$\mathbf{T} = \mathbf{T}^{**}, \tag{6.17}$$

as in the standard binary case. However, for $N \geq 3$ we have N higher cycles in addition.

For $i = 3$,

$$\begin{aligned}\langle\!\langle \mathbf{T}\mathsf{v}_1|\mathsf{v}_2|\mathsf{v}_3|\dots|\mathsf{v}_N\rangle\!\rangle &= \langle\!\langle \mathsf{v}_1|\mathbf{T}^*\mathsf{v}_2|\mathsf{v}_3|\dots|\mathsf{v}_N\rangle\!\rangle = \langle\!\langle \mathbf{T}^*\mathsf{v}_2|\mathsf{v}_3|\mathsf{v}_1|\dots|\mathsf{v}_N\rangle\!\rangle \\ &= \langle\!\langle \mathsf{v}_2|\mathbf{T}^{**}\mathsf{v}_3|\mathsf{v}_1|\dots|\mathsf{v}_N\rangle\!\rangle = \langle\!\langle \mathbf{T}^{**}\mathsf{v}_3|\mathsf{v}_1|\mathsf{v}_2|\dots|\mathsf{v}_N\rangle\!\rangle \\ &= \langle\!\langle \mathsf{v}_3|\mathbf{T}^{***}\mathsf{v}_1|\mathsf{v}_2|\dots|\mathsf{v}_N\rangle\!\rangle = \langle\!\langle \mathbf{T}^{***}\mathsf{v}_1|\mathsf{v}_2|\mathsf{v}_3|\dots|\mathsf{v}_N\rangle\!\rangle ,\end{aligned}$$

which together with (6.14) gives

$$\mathbf{T} = \mathbf{T}^{***},$$

and after using (6.17),

$$\mathbf{T} = \mathbf{T}^*. \tag{6.18}$$

Similarly, for an arbitrary length of the cycle i we obtain $\mathbf{T} = \mathbf{T}^{\overbrace{*\,*\cdots *}^{i}}$, which should be valid for *each* cycle recursively with $i = 2, 3, \dots, N$. Therefore, for any $N \geq 3$ all the operators $\mathbf{T}$ are self-adjoint (6.18), while $N = 2$ is an exceptional case when we have $\mathbf{T} = \mathbf{T}^{**}$ (6.17) only. □

Now we show that imposing a partial symmetry on the polyadic inner pairing will give more interesting properties to the adjoint operators. Recall that one of the possible binary commutativity generalizations of (semi)groups to the polyadic case is the semicommutativity concept, when in the multiplication only the first and last elements are exchanged. Similarly, we introduce

Definition 6.8. The polyadic inner pairing is called *semicommutative* if

$$\langle\!\langle \mathsf{v}_1|\mathsf{v}_2|\mathsf{v}_3|\dots|\mathsf{v}_N\rangle\!\rangle = \langle\!\langle \mathsf{v}_N|\mathsf{v}_2|\mathsf{v}_3|\dots|\mathsf{v}_1\rangle\!\rangle , \quad \mathsf{v}_i \in \mathsf{V}. \tag{6.19}$$

Proposition 6.9. *If the polyadic inner pairing is semicommutative, then for any operator* $\mathbf{T}$ *(satisfying Post-like adjointness (6.15)) the last multistar operation* $(\star_{N,1})$ *is a binary involution and is a composition of all the previous multistars*

$$\mathbf{T}^{\star_{N,1}} = \mathbf{T}^{\star_{12}\star_{23}\star_{34}\cdots\star_{N-1,N}}, \tag{6.20}$$

$$\mathbf{T}^{\star_{N,1}\star_{N,1}} = \mathbf{T}. \tag{6.21}$$

Proof. It follows from (6.15) and (6.19), that

$$\begin{aligned}\langle\!\langle \mathsf{v}_1|\mathsf{v}_2|\mathsf{v}_3|\dots|\mathbf{T}\mathsf{v}_N\rangle\!\rangle &= \langle\!\langle \mathbf{T}\mathsf{v}_N|\mathsf{v}_2|\mathsf{v}_3|\dots|\mathsf{v}_1\rangle\!\rangle \\ &= \langle\!\langle \mathsf{v}_N|\mathsf{v}_2|\mathsf{v}_3|\dots|\mathbf{T}^{\star_{12}\star_{23}\star_{34}\cdots\star_{N-1,N}}\mathsf{v}_1\rangle\!\rangle \\ &= \langle\!\langle \mathbf{T}^{\star_{12}\star_{23}\star_{34}\cdots\star_{N-1,N}}\mathsf{v}_1|\mathsf{v}_2|\mathsf{v}_3|\dots|\mathsf{v}_N\rangle\!\rangle \\ &= \langle\!\langle \mathbf{T}^{\star_{N,1}}\mathsf{v}_1|\mathsf{v}_2|\mathsf{v}_3|\dots|\mathsf{v}_N\rangle\!\rangle ,\end{aligned}$$

which after using (6.14) gives (6.20), (6.21) follows from the first multistar cocycle in (6.16). □

The adjointness relations (6.15) (of all kinds) together with (6.12) and (6.13) allow us to fix the arity shape of the polyadic operator algebra $\mathcal{A}_T$. We will assume that the arity of the operator multiplication in $\mathcal{A}_T$ coincides with the number of places of the inner pairing N (5.1),

$$n_T = N, \tag{6.22}$$

because it is in agreement with (6.15). Thus, the arity shape of the polyadic operator algebra becomes

$$\mathcal{A}_T = \left\langle K; \{\mathbf{T}\} \mid \sigma_{m_K}, \kappa_{n_K}; \eta_{m_T=m_V}, \omega_{n_T=N} \mid \theta_{k_F=k_\rho=1} \right\rangle .$$

Definition 6.10. We call the operator algebra $\mathcal{A}_T$, which has the arity $n_T = N$, a *nonderived polyadic operator algebra*.

Let us investigate some structural properties of $\mathcal{A}_T$ and types of polyadic operators.

Remark 6.11. We can only *define*, but not derive as in the binary case, the action of any multistar $(\star_{ij})$ on the product of operators, because in the nonderived n_T-ary algebra we have a fixed number of operators in a product and sum, that is, $\ell' (n_T - 1) + 1$ and $\ell'' (m_T - 1) + 1$, correspondingly, where ℓ' is the number of n_T-ary multiplications and ℓ' is the number of m_T-ary additions. Therefore, we cannot transfer (one at a time) all the polyadic operators from one place in the inner pairing to another place, as in the standard proof for the binary case.

Taking this into account, as well as consistency under the multistar cycles (6.16), we arrive at the following definition

Definition 6.12. The fixed multistar operation acts on the $\ell = 1$ product of n_T polyadic operators, depending on the *sequential number of the multistar* $(\star_{ij})$ (for the Post-like adjointness relations (6.15))

$$s_{ij} := \begin{cases} \dfrac{i+j-1}{2} & \text{if } 3 \leq i+j \leq 2N-1 \\ N & \text{if } i\ j = N \end{cases}, \quad s_{ij} = 1, 2, \ldots, N-1, N, \tag{6.23}$$

in the following way:

$$\left(\omega_{n_T}\left[\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_{n_T-1}, \mathbf{T}_{n_T}\right]\right)^{\star_{ij}} = \begin{cases} \omega_{n_T}\left[\mathbf{T}_{n_T}^{\star_{ij}}, \mathbf{T}_{n_T-1}^{\star_{ij}}, \ldots, \mathbf{T}_2^{\star_{ij}}, \mathbf{T}_1^{\star_{ij}}\right], & s_{ij} \text{ odd}, \\ \omega_{n_T}\left[\mathbf{T}_1^{\star_{ij}}, \mathbf{T}_2^{\star_{ij}}, \ldots, \mathbf{T}_{n_T-1}^{\star_{ij}}, \mathbf{T}_{n_T}^{\star_{ij}}\right], & s_{ij} \text{ even}. \end{cases} \tag{6.24}$$

A rule similar to (6.24) holds also for non-Post-like adjointness relations, but their concrete form depends on the corresponding non-Post-like associative quiver.

Sometimes, to shorten notation, it is more convenient to mark a multistar by the sequential number (6.23) such that $(\star_{ij}) \Rightarrow \left(\star_{s_{ij}}\right)$, e.g. $(\star_{23}) \Rightarrow (\star_2)$, $(\star_{N,1}) \Rightarrow (\star_N)$, etc. Also, in the examples, for the ternary multiplication we will use the square brackets without the name of operation if it is clear from the context, e.g., $\omega_3\left[\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3\right] \Rightarrow \left[\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3\right]$, etc.

Example 6.13. In the lowest ternary case, $N=3$, we have:

1) *Post-like adjointness relations*

$$\langle\langle \mathbf{T}\mathsf{v}_1|\mathsf{v}_2|\mathsf{v}_3\rangle\rangle = \langle\langle \mathsf{v}_1|\mathbf{T}^{\star_1}\mathsf{v}_2|\mathsf{v}_3\rangle\rangle,$$
$$\langle\langle \mathsf{v}_1|\mathbf{T}\mathsf{v}_2|\mathsf{v}_3\rangle\rangle = \langle\langle \mathsf{v}_1|\mathsf{v}_2|\mathbf{T}^{\star_2}\mathsf{v}_3\rangle\rangle,$$
$$\langle\langle \mathsf{v}_1|\mathsf{v}_2|\mathbf{T}\mathsf{v}_3\rangle\rangle = \langle\langle \mathbf{T}^{\star_3}\mathsf{v}_1|\mathsf{v}_2|\mathsf{v}_3\rangle\rangle,$$

2) *Non-Post-like adjointness relations*

$$\langle\langle \mathbf{T}\mathsf{v}_1|\mathsf{v}_2|\mathsf{v}_3\rangle\rangle = \langle\langle \mathsf{v}_1|\mathsf{v}_2|\mathbf{T}^{\star_3}\mathsf{v}_3\rangle\rangle,$$
$$\langle\langle \mathsf{v}_1|\mathsf{v}_2|\mathbf{T}\mathsf{v}_3\rangle\rangle = \langle\langle \mathsf{v}_1|\mathbf{T}^{\star_2}\mathsf{v}_2|\mathsf{v}_3\rangle\rangle,$$
$$\langle\langle \mathsf{v}_1|\mathbf{T}\mathsf{v}_2|\mathsf{v}_3\rangle\rangle = \langle\langle \mathbf{T}^{\star_1}\mathsf{v}_1|\mathsf{v}_2|\mathsf{v}_3\rangle\rangle,$$

and the corresponding multistar cycles:

1) *Post-like multistar cycles*

$$\mathbf{T}^{\star_1\star_2\star_3} = \mathbf{T},$$
$$\mathbf{T}^{\star_2\star_3\star_1} = \mathbf{T},$$
$$\mathbf{T}^{\star_3\star_1\star_2} = \mathbf{T},$$

2) *Non-Post-like multistar cycles*

$$\mathbf{T}^{\star_3\star_2\star_1} = \mathbf{T},$$
$$\mathbf{T}^{\star_2\star_1\star_3} = \mathbf{T},$$
$$\mathbf{T}^{\star_1\star_3\star_2} = \mathbf{T}.$$

Using (6.24), we obtain the ternary conjugation rules:

$$\left([\mathbf{T}_1,\mathbf{T}_2,\mathbf{T}_3]\right)^{\star_1} = [\mathbf{T}_3^{\star_1},\mathbf{T}_2^{\star_1},\mathbf{T}_1^{\star_1}],$$
$$\left([\mathbf{T}_1,\mathbf{T}_2,\mathbf{T}_3]\right)^{\star_2} = [\mathbf{T}_1^{\star_2},\mathbf{T}_2^{\star_2},\mathbf{T}_3^{\star_2}],$$
$$\left([\mathbf{T}_1,\mathbf{T}_2,\mathbf{T}_3]\right)^{\star_3} = [\mathbf{T}_3^{\star_3},\mathbf{T}_2^{\star_3},\mathbf{T}_1^{\star_3}],$$

which are common for both Post-like and non-Post-like adjointness relations 1), 2).

Definition 6.14. A polyadic operator $\mathbf{T}$ is called *self-adjoint* if all multistar operations are identities, i.e., $(\star_{ij}) = \mathrm{id}$ for all i, j.

6.2. Polyadic isometry and projection. Now we introduce polyadic analogs for the following important types of operator: isometry, unitary, and (orthogonal) projection. Taking into account Remark 6.11, we again cannot move operators singly, and instead of proving the operator relations, as is usually done in the binary case, we can only exploit some mnemonic rules to *define* the corresponding relations between polyadic operators.

If the polyadic operator algebra $\mathcal{A}_T$ contains a unit $\mathbf{I}$ and zero $\mathbf{Z}$ (see (6.2), (6.3)), we define the conditions of polyadic isometry and orthogonality:

Definition 6.15. A polyadic operator $\mathbf{T}$ is called a *polyadic isometry* if it preserves the polyadic inner pairing

$$\langle\langle \mathbf{T}\mathsf{v}_1|\mathbf{T}\mathsf{v}_2|\mathbf{T}\mathsf{v}_3|\ldots|\mathbf{T}\mathsf{v}_N\rangle\rangle = \langle\langle \mathsf{v}_1|\mathsf{v}_2|\mathsf{v}_3|\ldots|\mathsf{v}_N\rangle\rangle, \tag{6.25}$$

and satisfies

$$\omega_{n_T}[\mathbf{T}^{\star_{N-1,N}},\mathbf{T}^{\star_{N-2,N-1}\star_{N-1,N}},\ldots \\ \mathbf{T}^{\star_{23}\star_{34}\cdots\star_{N-2,N-1}\star_{N-1,N}},\mathbf{T}^{\star_{12}\star_{23}\star_{34}\cdots\star_{N-2,N-1}\star_{N-1,N}},\mathbf{T}] = \mathbf{I},$$
$$+\,(N-1)\text{ cycle permutations of multistars in the first }(N-1)\text{ terms.} \tag{6.26}$$

Remark 6.16. If the multiplication in $\mathcal{A}_T$ is derived and all multistars are equal, then the polyadic isometry operators satisfy some kind of N-regularity [20] or regular N-cocycle condition [19].

Proposition 6.17. *The polyadic isometry operator* $\mathbf{T}$ *preserves the polyadic norm*

$$\|\mathbf{T}\mathsf{v}\|_N = \|\mathsf{v}\|_N, \quad \mathsf{v} \in \mathsf{V}. \tag{6.27}$$

Proof. It follows from (5.8) and (6.25) that

$$\kappa_{n_K}\big[\overbrace{\|\mathbf{T}\mathsf{v}\|_N, \|\mathbf{T}\mathsf{v}\|_N, \ldots, \|\mathbf{T}\mathsf{v}\|_N}^{n_K}\big] = \kappa_{n_K}\big[\overbrace{\|\mathsf{v}\|_N, \|\mathsf{v}\|_N, \ldots, \|\mathsf{v}\|_N}^{n_K}\big],$$

which gives (6.27) when $n_K = N$. □

Definition 6.18. If for N polyadic operators $\mathbf{T}_i$ we have

$$\langle\!\langle \mathbf{T}_1\mathsf{v}_1|\mathbf{T}_2\mathsf{v}_2|\mathbf{T}_3\mathsf{v}_3|\ldots|\mathbf{T}_N\mathsf{v}_N\rangle\!\rangle = \mathsf{z}_K, \quad \mathsf{v}_i \in \mathsf{V},$$

where $\mathsf{z}_K \in \mathsf{V}$ is the zero of the underlying polyadic field $\mathbb{K}_{m_K,n_K}$, then we say that $\mathbf{T}_i$ are *(polyadically) orthogonal*, and they satisfy

$$\begin{aligned}&\omega_{n_T}\big[\mathbf{T}_1^{\star N-1,N}, \mathbf{T}_2^{\star N-2,N-1\star N-1,N}, \ldots\\ &\qquad \mathbf{T}_3^{\star 23\star 34\cdots\star N-2,N-1\star N-1,N}, \mathbf{T}_{N-1}^{\star 12\star 23\star 34\cdots\star N-2,N-1\star N-1,N}, \mathbf{T}_N\big] = \mathbf{Z},\\ &+ (N-1) \text{ cycle permutations of multistars in the first } (N-1) \text{ terms.}\end{aligned} \tag{6.28}$$

The polyadic analog of projection is given by

Definition 6.19. If a polyadic operator $\mathbf{P} \in \mathcal{A}_T$ satisfies the polyadic idempotency condition

$$\omega_{n_T}\big[\overbrace{\mathbf{P}, \mathbf{P}, \ldots \mathbf{P}}^{n_T}\big] = \mathbf{P}, \tag{6.29}$$

then it is called a *polyadic projection*.

By analogy with the binary case, polyadic projections can be constructed from polyadic isometry operators in a natural way.

Proposition 6.20. *If* $\mathbf{T} \in \mathcal{A}_T$ *is a polyadic isometry, then*

$$\begin{aligned}&\mathbf{P}_{\mathbf{T}}^{(1)} = \omega_{n_T}\big[\mathbf{T}, \mathbf{T}^{\star N-1,N}, \mathbf{T}^{\star N-2,N-1\star N-1,N}, \ldots\\ &\qquad \mathbf{T}^{\star 23\star 34\cdots\star N-2,N-1\star N-1,N}, \mathbf{T}^{\star 12\star 23\star 34\cdots\star N-2,N-1\star N-1,N}\big],\\ &+ (N-1) \text{ cycle permutations of multistars in the last } (N-1) \text{ terms,}\end{aligned} \tag{6.30}$$

are the corresponding polyadic projections $\mathbf{P}_{\mathbf{T}}^{(k)}$, $k = 1, \ldots, N$ *satisfying* (6.29).

Definition 6.21. A polyadic operator $\mathbf{T} \in \mathcal{A}_T$ is called *normal* if

$$\omega_{n_T}\left[\mathbf{T}^{\star N-1,N}, \mathbf{T}^{\star N-2,N-1\star N-1,N}, \ldots \mathbf{T}^{\star 23\star 34\cdots\star N-2,N-1\star N-1,N}, \mathbf{T}^{\star 12\star 23\star 34\cdots\star N-2,N-1\star N-1,N}, \mathbf{T}\right] =$$
$$\omega_{n_T}\left[\mathbf{T}, \mathbf{T}^{\star N-1,N}, \mathbf{T}^{\star N-2,N-1\star N-1,N}, \ldots \mathbf{T}^{\star 23\star 34\cdots\star N-2,N-1\star N-1,N}, \mathbf{T}^{\star 12\star 23\star 34\cdots\star N-2,N-1\star N-1,N}\right],$$
$$+\,(N-1) \text{ cycle permutations of multistars in the } (N-1) \text{ terms.}$$

Proof. Insert (6.30) into (6.29) and use (6.26) together with n_T-ary associativity. □

Definition 6.22. If all the polyadic projections (6.30) are equal to unity $\mathbf{P}_{\mathbf{T}}^{(k)} = \mathbf{I}$, then the corresponding polyadic isometry operator $\mathbf{T}$ is called a *polyadic unitary operator*.

It can be shown that each polyadic unitary operator is querable ("polyadically invertible") such that it has a querelement in $\mathcal{A}_T$.

6.3. Towards a polyadic analog of C^*-algebras. Let us, first, generalize the operator binary norm (6.1) to the polyadic case. This can be done provided that a binary ordering on the underlying polyadic field $\mathbb{K}_{m_K,n_K}$ can be introduced.

Definition 6.23. The polyadic operator norm $\|\bullet\|_T : \{\mathbf{T}\} \to K$ is defined by

$$\|\mathbf{T}\|_T = \inf\left\{M \in K \mid \|\mathsf{v}\|_N \;\; \|\mathbf{T}\mathsf{v}\|_N \leq \mu_{n_K}\big[\overbrace{M,\ldots,M}^{n_K-1}\big], \forall \mathsf{v} \in \mathsf{V}\right\}, \tag{6.31}$$

where $\|\bullet\|_N$ is the polyadic norm in the inner pairing space $\mathcal{H}_{m_K,n_K,m_V,k_\rho=1,N}$ and μ_{n_K} is the n_K-ary multiplication in $\mathbb{K}_{m_K,n_K}$.

Definition 6.24. The polyadic operator norm is called *submultiplicative* if

$$\|\omega_{n_T}[\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_{n_T}]\|_T \leq \mu_{n_K}\left[\|\mathbf{T}_1\|_T, \|\mathbf{T}_2\|_T, \ldots, \|\mathbf{T}_{n_K}\|_T\right],$$
$$n_T = n_K.$$

Definition 6.25. The polyadic operator norm is called *subadditive* if

$$\|\eta_{m_T}[\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_{n_T}]\|_T \leq \nu_{m_K}\left[\|\mathbf{T}_1\|_T, \|\mathbf{T}_2\|_T, \ldots, \|\mathbf{T}_{m_K}\|_T\right],$$
$$m_T = m_K.$$

By analogy with the binary case, we have

Definition 6.26. The polyadic operator algebra $\mathcal{A}_T$ equipped with the submultiplicative norm $\|\bullet\|_T$ is a *polyadic Banach algebra* of operators $\mathcal{B}_T$.

The connection between the polyadic norms of operators and their polyadic adjoints is given by

Proposition 6.27. *For polyadic operators in the inner pairing space* $\mathcal{H}_{m_K,n_K,m_V,k_\rho=1,N}$*:*

1. *The following* N *multi-*C^**-relations*

$$\begin{aligned}&\|\omega_{n_T}[\mathbf{T}^{\star_{N-1,N}},\mathbf{T}^{\star_{N-2,N-1}\star_{N-1,N}},\ldots\\&\qquad\mathbf{T}^{\star_{23}\star_{34}\cdots\star_{N-2,N-1}\star_{N-1,N}},\mathbf{T}^{\star_{12}\star_{23}\star_{34}\cdots\star_{N-2,N-1}\star_{N-1,N}},\mathbf{T}_N]\|\\&=\mu_{n_K}[\overbrace{\|\mathbf{T}\|_T,\|\mathbf{T}\|_T,\ldots,\|\mathbf{T}\|_T}^{n_K}],\\&+(N-1)\ \textit{cycle permutations of}\ (N-1)\ \textit{terms with multistars},\end{aligned}\tag{6.32}$$

take place if $n_T = n_K$.

2. *The polyadic norms of operator and its all adjoints coincide*

$$\|\mathbf{T}^{\star_{i,j}}\|_T = \|\mathbf{T}\|_T, \quad i,j \in 1,\ldots,N.$$

Proof. Both statements follow from (6.15) and the definition of the polyadic operator norm (6.31). □

Therefore, we arrive at

Definition 6.28. The operator Banach algebra $\mathcal{B}_T$ satisfying the multi-C^*-relations is called a polyadic operator *multi-*C^**-algebra*.

The first example of a multi-C^*-algebra (as in the binary case) can be constructed from one isometry operator (see Definition 6.15).

Definition 6.29. A polyadic algebra generated by one isometry operator $\mathbf{T}$ satisfying (6.26) on the inner pairing space $\mathcal{H}_{m_K,n_K,m_V,k_\rho=1,N}$ represents a *polyadic Toeplitz algebra* $\mathcal{T}_{m_T,n_T}$ and has the arity shape $m_T = m_V$, $n_T = N$.

Example 6.30. The ternary Toeplitz algebra $\mathcal{T}_{3,3}$ is represented by the operator $\mathbf{T}$ and the relations:

$$\begin{aligned}[\mathbf{T}^{\star_1},\mathbf{T}^{\star_3\star_1},\mathbf{T}]&=\mathbf{I},\\ [\mathbf{T}^{\star_2},\mathbf{T}^{\star_1\star_2},\mathbf{T}]&=\mathbf{I},\\ [\mathbf{T}^{\star_3},\mathbf{T}^{\star_2\star_3},\mathbf{T}]&=\mathbf{I}.\end{aligned}$$

Example 6.31. If the inner pairing is semicommutative (6.19), then $(\star_3)$ can be eliminated by

$$\mathbf{T}^{\star_3} = \mathbf{T}^{\star_1\star_2}, \tag{6.33}$$

$$\mathbf{T}^{\star_3\star_3} = \mathbf{T}, \tag{6.34}$$

and the corresponding relations representing $\mathcal{T}_{3,3}$ become

$$\begin{aligned}[\mathbf{T}^{\star_1},\mathbf{T}^{\star_1},\mathbf{T}]&=\mathbf{I},\\ [\mathbf{T}^{\star_2},\mathbf{T}^{\star_1\star_2},\mathbf{T}]&=\mathbf{I},\\ [\mathbf{T}^{\star_1\star_2},\mathbf{T}^{\star_2},\mathbf{T}]&=\mathbf{I}.\end{aligned}$$

Let us consider M polyadic operators $\mathbf{T}_1\mathbf{T}_2\ldots\mathbf{T}_M \in \mathcal{B}_T$ and the related partial (in the usual sense) isometries (6.29) which are mutually orthogonal (6.28). In the binary case, the algebra generated by M operators, such that the sum of the related orthogonal partial projections is unity, represents the Cuntz algebra $\mathcal{O}_M$ [12].

Definition 6.32. A polyadic algebra generated by M polyadic isometric operators $\mathbf{T}_1\mathbf{T}_2\ldots\mathbf{T}_M \in \mathcal{B}_T$ satisfying

$$\eta_{m_T}^{(\ell_a)}\left[\mathbf{P}_{\mathbf{T}_1}^{(k)},\mathbf{P}_{\mathbf{T}_2}^{(k)}\ldots\mathbf{P}_{\mathbf{T}_M}^{(k)}\right]=\mathbf{I},\quad k=1,\ldots,N,$$

where $\mathbf{P}_{\mathbf{T}_i}^{(k)}$ are given by (6.30) and $\eta_{m_T}^{(\ell_a)}$ is a "long polyadic addition" (6.10), represents a *polyadic Cuntz algebra* $p\mathcal{O}_{M|m_T,n_T}$, which has the arity shape

$$M=\ell_a\left(m_T-1\right)+1,$$

where ℓ_a is the number of "m_T-ary additions".

Below we will use the same notations as in *Example* 6.13, also the ternary addition will be denoted by $(+_3)$ as follows: $\eta_3\left[\mathbf{T}_1,\mathbf{T}_2,\mathbf{T}_3\right]\equiv\mathbf{T}_1+_3\mathbf{T}_2+_3\mathbf{T}_3$.

Example 6.33. In the ternary case $m_T=n_T=3$ and one ternary addition $\ell_a=1$, we have $M=3$ mutually orthogonal isometries $\mathbf{T}_1,\mathbf{T}_2,\mathbf{T}_3\in\mathcal{B}_T$ and $N=3$ multistars $(\star_i)$. In the case of the Post-like multistar cocycles 1), 2), they satisfy:

Isometry *conditions*	*Orthogonality conditions*
$\left[\mathbf{T}_i^{\star_1},\mathbf{T}_i^{\star_3\star_1},\mathbf{T}_i\right]=\mathbf{I},$	$\left[\mathbf{T}_i^{\star_1},\mathbf{T}_j^{\star_3\star_1},\mathbf{T}_k\right]=\mathbf{Z},$
$\left[\mathbf{T}_i^{\star_2},\mathbf{T}_i^{\star_1\star_2},\mathbf{T}_i\right]=\mathbf{I},$	$\left[\mathbf{T}_i^{\star_2},\mathbf{T}_j^{\star_1\star_2},\mathbf{T}_k\right]=\mathbf{Z},$
$\left[\mathbf{T}_i^{\star_3},\mathbf{T}_i^{\star_2\star_3},\mathbf{T}_i\right]=\mathbf{I},$	$\left[\mathbf{T}_i^{\star_3},\mathbf{T}_j^{\star_2\star_3},\mathbf{T}_k\right]=\mathbf{Z},$
$i=1,2,3,$	$i,j,k=1,2,3,\ i\neq j\neq k,$

and the (sum of projections) relations:

$$\left[\mathbf{T}_1,\mathbf{T}_1^{\star_1},\mathbf{T}_1^{\star_3\star_1}\right]+_3\left[\mathbf{T}_2,\mathbf{T}_2^{\star_1},\mathbf{T}_2^{\star_3\star_1}\right]+_3\left[\mathbf{T}_3,\mathbf{T}_3^{\star_1},\mathbf{T}_3^{\star_3\star_1}\right]=\mathbf{I},$$
$$\left[\mathbf{T}_1,\mathbf{T}_1^{\star_2},\mathbf{T}_1^{\star_1\star_2}\right]+_3\left[\mathbf{T}_2,\mathbf{T}_2^{\star_2},\mathbf{T}_2^{\star_1\star_2}\right]+_3\left[\mathbf{T}_3,\mathbf{T}_3^{\star_2},\mathbf{T}_3^{\star_1\star_2}\right]=\mathbf{I},$$
$$\left[\mathbf{T}_1,\mathbf{T}_1^{\star_3},\mathbf{T}_1^{\star_2\star_3}\right]+_3\left[\mathbf{T}_2,\mathbf{T}_2^{\star_3},\mathbf{T}_2^{\star_2\star_3}\right]+_3\left[\mathbf{T}_3,\mathbf{T}_3^{\star_3},\mathbf{T}_3^{\star_2\star_3}\right]=\mathbf{I},$$

which represent the ternary Cuntz algebra $p\mathcal{O}_{3|3,3}$.

Example 6.34. In the case where the inner pairing is semicommutative (6.19), we can eliminate the multistar $(\star_3)$ by (6.33) and represent the *two-multistar ternary analog of the Cuntz algebra* $p\mathcal{O}_{3|3,3}$ by

$$\left[\mathbf{T}_i^{\star_1},\mathbf{T}_i^{\star_2},\mathbf{T}_i\right]=\mathbf{I},\qquad\left[\mathbf{T}_i^{\star_1},\mathbf{T}_j^{\star_2},\mathbf{T}_k\right]=\mathbf{Z},$$

$$\left[\mathbf{T}_i^{\star_2},\mathbf{T}_i^{\star_1\star_2},\mathbf{T}_i\right]=\mathbf{I},\qquad\left[\mathbf{T}_i^{\star_1},\mathbf{T}_j^{\star_1\star_2},\mathbf{T}_k\right]=\mathbf{Z},$$

$$[\mathbf{T}_i^{\star_1\star_2}, \mathbf{T}_i^{\star_2}, \mathbf{T}_i] = \mathbf{I}, \qquad \left[\mathbf{T}_i^{\star_1\star_2}, \mathbf{T}_j^{\star_2}, \mathbf{T}_k\right] = \mathbf{Z},$$
$$i = 1,2,3, \qquad i,j,k = 1,2,3, \;\; i \neq j \neq k,$$

and

$$[\mathbf{T}_1, \mathbf{T}_1^{\star_1}, \mathbf{T}_1^{\star_1}] +_3 [\mathbf{T}_2, \mathbf{T}_2^{\star_1}, \mathbf{T}_2^{\star_1}] +_3 [\mathbf{T}_3, \mathbf{T}_3^{\star_1}, \mathbf{T}_3^{\star_1}] = \mathbf{I},$$
$$[\mathbf{T}_1, \mathbf{T}_1^{\star_2}, \mathbf{T}_1^{\star_1\star_2}] +_3 [\mathbf{T}_2, \mathbf{T}_2^{\star_2}, \mathbf{T}_2^{\star_1\star_2}] +_3 [\mathbf{T}_3, \mathbf{T}_3^{\star_2}, \mathbf{T}_3^{\star_1\star_2}] = \mathbf{I},$$
$$[\mathbf{T}_1, \mathbf{T}_1^{\star_1\star_2}, \mathbf{T}_1^{\star_2}] +_3 [\mathbf{T}_2, \mathbf{T}_2^{\star_1\star_2}, \mathbf{T}_2^{\star_2}] +_3 [\mathbf{T}_3, \mathbf{T}_3^{\star_1\star_2}, \mathbf{T}_3^{\star_2}] = \mathbf{I}.$$

7. Congruence classes as polyadic rings

Here we will show that the inner structure of the residue classes (congruence classes) over integers is naturally described by polyadic rings [8,10,31], and then study some special properties of them.

Denote a residue class (congruence class) of an integer a, modulo b by

$$[[a]]_b = \{\{a + bk\} \mid k \in \mathbb{Z}, \; a \in \mathbb{Z}_+, \; b \in \mathbb{N}, \; 0 \leq a \leq b-1\}. \tag{7.1}$$

For the residue class, we use the notation $[[a]]_b$, because the standard notations by one square bracket $[a]_b$ or $\bar{a}_b$ are already used for n-ary operations and querelements, respectively. A representative element of the class $[[a]]_b$ will be denoted by $x_k = x_k^{(a,b)} = a + bk$. Here we do not consider the addition and multiplication of the residue classes (congruence classes). Instead, we consider the *fixed* congruence class $[[a]]_b$, and note that for arbitrary a and b, it is *not closed* under binary operations. However, it can be *closed with respect to polyadic operations.*

7.1. Polyadic ring of integers. Let us introduce the m-ary addition and n-ary multiplication of representatives of the fixed congruence class $[[a]]_b$ by

$$\nu_m [x_{k_1}, x_{k_2}, \ldots, x_{k_m}] = x_{k_1} + x_{k_2} + \cdots + x_{k_m}, \tag{7.2}$$
$$\mu_n [x_{k_1}, x_{k_2}, \ldots, x_{k_n}] = x_{k_1} x_{k_2} \cdots x_{k_n}, \quad x_{k_i} \in [[a]]_b, \; k_i \in \mathbb{Z}, \tag{7.3}$$

where in the r.h.s. the operations are the ordinary binary addition and the binary multiplication in $\mathbb{Z}$.

Remark 7.1. The polyadic operations (7.2), (7.3) are *not derived* (see, e.g., [26,34]), because on the set $\{x_{k_i}\}$ one cannot define the *binary* semigroup structure with respect to ordinary addition and multiplication. Derived polyadic rings which consist of the repeated binary sums and binary products were considered in [31].

Lemma 7.2. *In the case*

$$(m-1)\frac{a}{b} = I^{(m)}(a,b) = I = \text{integer}, \tag{7.4}$$

the algebraic structure $\langle [[a]]_b \mid \nu_m \rangle$ is a commutative m-ary group.

Proof. The closure of the operation (7.2) can be written as $x_{k_1} + x_{k_2} + \ldots + x_{k_m} = x_{k_0}$, or $ma + b(k_1 + k_2 + \ldots + k_m) = a + bk_0$, and then $k_0 = (m-1)a/b + (k_1 + k_2 + \ldots + k_m)$ from (7.4). The (total) associativity and commutativity of ν_m follows from those of the addition in the binary $\mathbb{Z}$. Each element x_k has its *unique querelement* $\tilde{x} = x_{\tilde{k}}$ determined by the equation $(m-1)x_k + x_{\tilde{k}} = x_k$, which (uniquely, for any $k \in \mathbb{Z}$) gives

$$\tilde{k} = bk(2-m) - (m-1)\frac{a}{b}.$$

Thus, each element is "querable" (polyadic invertible), and so $\langle [[a]]_b \mid \nu_m \rangle$ is an m-ary group. □

Example 7.3. For $a = 2$, $b = 7$, we have an 8-ary group, and the querelement of x_k is $\tilde{x} = x_{(-2-12k)}$.

Proposition 7.4. *The m-ary commutative group $\langle [[a]]_b \mid \nu_m \rangle$:*

1) *has an infinite number of neutral sequences for each element;*

2) *if $a \neq 0$, it has no "unit" (which is actually zero, because ν_m plays the role of "addition");*

3) *in the case of the zero congruence class $[[0]]_b$, the zero is $x_k = 0$.*

Proof. 1) The (additive) neutral sequence $\tilde{\mathbf{n}}_{m-1}$ of the length $(m-1)$ is defined by $\nu_m[\tilde{\mathbf{n}}_{m-1}, x_k] = x_k$. Using (7.2), we take $\tilde{\mathbf{n}}_{m-1} = x_{k_1} + x_{k_2} + \ldots + x_{k_{m-1}} = 0$ and obtain the equation

$$(m-1)a + b(k_1 + k_2 + \ldots + k_{m-1}) = 0. \tag{7.5}$$

Because of (7.4), we obtain

$$k_1 + k_2 + \ldots + k_{m-1} = -I^{(m)}(a,b), \tag{7.6}$$

and so there is an infinite number of sums satisfying this condition.

2) The polyadic "unit"/zero $z = x_{k_0} = a + bk_0$ satisfies $\nu_m[\overbrace{z, z, \ldots, z}^{m-1}, x_k] = x_k$ for all $x_k \in [[a]]_b$ (the neutral sequence $\tilde{\mathbf{n}}_{m-1}$ consists of one element z only), which gives $(m-1)(a + bk_0) = 0$ having no solutions with $a \neq 0$ since $a < b$.
3) In the case of $a = 0$, the only solution is $z = x_{k=0} = 0$. □

Example 7.5. For the case $a = 1$, $b = 2$, we have $m = 3$ and $I^{(3)}(1,2) = 1$, and so from (7.5) we get $k_1 + k_2 = -1$, thus the infinite number of neutral sequences consists of 2 elements $\tilde{\mathbf{n}}_2 = x_k + x_{-1-k}$, with arbitrary $k \in \mathbb{Z}$.

Lemma 7.6. *If*

$$\frac{a^n - a}{b} = J^{(n)}(a,b) = J = \text{integer}, \tag{7.7}$$

then $\langle [[a]]_b \mid \mu_n \rangle$ is a commutative n-ary semigroup.

Proof. It follows from (7.3) that the closeness of the operation μ_n is $x_{k_1}x_{k_2}\ldots x_{k_n} = x_{k_0}$, which can be written as $a^n + b\,(\text{integer}) = a + bk_0$ leading to (7.7). The (total) associativity and commutativity of μ_n follows from those of the multiplication in $\mathbb{Z}$. □

Definition 7.7. A unique pair of integers (I, J) is called a (*polyadic*) *shape invariants* of the congruence class $[[a]]_b$.

Theorem 7.8. *The algebraic structure of the fixed congruence class* $[[a]]_b$ *is a polyadic* (m, n)*-ring*

$$\mathcal{R}^{[a,b]}_{m,n} = \langle [[a]]_b \mid \nu_m, \mu_n \rangle \,, \tag{7.8}$$

where the arities m *and* n *are minimal positive integers (more than or equal to* 2*), for which the congruences*

$$ma \equiv a \,(\mathrm{mod}\, b)\,, \tag{7.9}$$

$$a^n \equiv a \,(\mathrm{mod}\, b) \tag{7.10}$$

take place simultaneously, fixating its polyadic shape invariants (I, J)*.*

Proof. By Lemmas 7.2, 7.6, the set $[[a]]_b$ is an m-ary group with respect to "m-ary addition" ν_m and an n-ary semigroup with respect to "n-ary multiplication" μ_n, while the polyadic distributivity (2.1)–(2.3) follows from (7.2) and (7.3) and the binary distributivity in $\mathbb{Z}$. □

Remark 7.9. For a fixed $b \geq 2$, there are b congruence classes $[[a]]_b$, $0 \leq a \leq b - 1$, and therefore exactly b corresponding polyadic (m, n)-rings $\mathcal{R}^{[a,b]}_{m,n}$, each of them is *infinite-dimensional*.

Corollary 7.10. *For the case* $\gcd(a, b) = 1$ *and* b *is prime, there exists the solution* $n = b$*.*

Proof. The proof follows from (7.10) and Fermat's little theorem. □

Remark 7.11. We exclude from consideration the zero congruence class $[[0]]_b$, because the arities of operations ν_m and μ_n cannot be fixed up by (7.9), (7.10) becoming identities for any m and n. Since the arities are uncertain, their minimal values can be chosen $m = n = 2$, and therefore, it follows from (7.2) and (7.3) that $\mathcal{R}^{[0,b]}_{2,2} = \mathbb{Z}$. Thus, in what follows we always imply that $a \neq 0$ (without using a special notation, e.g., $\mathcal{R}^*$, etc.).

In Table 7.1, we present the allowed (by (7.9), (7.10)) arities of the polyadic ring $\mathcal{R}^{[a,b]}_{m,n}$ and the corresponding polyadic shape invariants (I, J) for $b \leq 10$.

Let us study the properties of $\mathcal{R}^{[a,b]}_{m,n}$ in more detail. First, we consider equal arity polyadic rings and find the relation between the corresponding congruence classes.

Proposition 7.12. *The residue (congruence) classes* $[[a]]_b$ *and* $[[a']]_{b'}$ *which are described by the polyadic rings of the same arities* $\mathcal{R}_{m,n}^{[a,b]}$ *and* $\mathcal{R}_{m,n}^{[a',b']}$ *are related by*

$$\frac{b'I'}{a'} = \frac{bI}{a}, \tag{7.11}$$

$$a' + b'J' = (a + bJ)^{\log_a a'}. \tag{7.12}$$

Proof. Follows from (7.4) and (7.7). □

For instance, in Table 7.1 the congruence classes $[[2]]_5$, $[[3]]_5$, $[[2]]_{10}$, and $[[8]]_{10}$ are $(6,5)$-rings. If, in addition, $a = a'$, then the polyadic shapes satisfy

$$\frac{I}{J} = \frac{I'}{J'}. \tag{7.13}$$

7.2. Limiting cases. The limiting cases $a \equiv \pm 1 \, (\operatorname{mod} b)$ are described by

Corollary 7.13. *The polyadic ring of the fixed congruence class* $[[a]]_b$ *is:*

1) *multiplicative binary if* $a = 1$*;*

2) *multiplicative ternary if* $a = b - 1$*;*

3) *additive* $(b+1)$*-ary in both cases.*

That is, the limiting cases contain the rings $\mathcal{R}_{b+1,2}^{[1,b]}$ *and* $\mathcal{R}_{b+1,3}^{[b-1,b]}$*. They correspond to the first row and the main diagonal of Table 7.1. Their intersection consists of the* $(3,2)$*-ring* $\mathcal{R}_{3,2}^{[1,2]}$*.*

Definition 7.14. The congruence classes $[[1]]_b$ and $[[b-1]]_b$ are called the *limiting classes*, and the corresponding polyadic rings are named the *limiting polyadic rings* of a fixed congruence class.

Proposition 7.15. *In the limiting cases* $a = 1$ *and* $a = b - 1$*, the* n*-ary semigroup* $\langle [[a]]_b \mid \mu_n \rangle$*:*

1) *has the neutral sequences of the form* $\bar{\mathbf{n}}_{n-1} = x_{k_1} x_{k_2} \dots x_{k_{n-1}} = 1$*, where* $x_{k_i} = \pm 1$*;*

2) *has*

   a) *the unit* $e = x_{k=1} = 1$ *for the limiting class* $[[1]]_b$*,*

   b) *the unit* $e^- = x_{k=-1} = -1$*, if* n *is odd, for* $[[b-1]]_b$*,*

   c) *the class* $[[1]]_2$ *contains both polyadic units* e *and* e^-*;*

3) *has the set of "querable" (polyadic invertible) elements which consists of* $\bar{x} = x_{\bar{k}} = \pm 1$*;*

4) *has in the "intersecting" case* $a = 1$*,* $b = 2$ *and* $n = 2$ *the binary subgroup* $\mathbb{Z}_2 = \{1, -1\}$*, while other elements have no inverses.*

Proof. 1) The (multiplicative) neutral sequence $\bar{\mathbf{n}}_{n-1}$ of length $(n-1)$ is defined by $\mu_n[\bar{\mathbf{n}}_{n-1}, x_k] = x_k$. It follows from (7.3) and cancellativity in $\mathbb{Z}$ that $\bar{\mathbf{n}}_{n-1} = x_{k_1} x_{k_2} \dots x_{k_{n-1}} = 1$, which is

$$(a + bk_1)(a + bk_2)\dots(a + bk_{n-1}) = 1. \tag{7.14}$$

The solution of this equation in integers is the following: a) all multipliers are $a + bk_i = 1$, $i = 1, \dots, n-1$; b) an even number of multipliers can be $a + bk_i = -1$, while the others are 1.

2) If the polyadic unit $e = x_{k_1} = a + bk_1$ exists, it should satisfy $\mu_m[\overbrace{e, e, \dots, e}^{n-1}, x_k] = x_k \; \forall x_k \in \langle [[a]]_b \mid \mu_n \rangle$, such that the neutral sequence $\bar{\mathbf{n}}_{n-1}$ consists of one element e only, and this leads to $(a + bk_1)^{n-1} = 1$. For any n, this equation has the solution $a+bk_1 = 1$, which uniquely gives $a = 1$ and $k_1 = 0$, thus $e = x_{k_1=0} = 1$. If n is odd, then there exists a "negative unit" $e^- = x_{k_1=-1} = -1$, such that $a + bk_1 = -1$, which can be uniquely solved by $k_1 = -1$ and $a = b - 1$. The neutral sequence becomes $\bar{\mathbf{n}}_{n-1} = \overbrace{e^-, e^-, \dots, e^-}^{n-1} = 1$ as a product of an even number of $e^- = -1$. The intersection of limiting classes consists of a single class $[[1]]_2$, and therefore it contains both polyadic units e and e^-.

3) An element x_k in $\langle [[a]]_b \mid \mu_n \rangle$ is "querable" if there exists its querelement $\bar{x} = x_{\bar{k}}$ such that $\mu_n[\overbrace{x_k, x_k, \dots, x_k}^{n-1}, \bar{x}] = x_k$. Using (7.3) and the cancellativity in $\mathbb{Z}$, we obtain the equation $(a + bk)^{n-2}\left(a + b\bar{k}\right) = 1$, which in integers has 2 solutions: a) $(a + bk)^{n-2} = 1$ and $\left(a + b\bar{k}\right) = 1$, the last relation fixes up the class $[[1]]_b$, and the arity of multiplication $n = 2$, and therefore the first relation is valid for all elements in the class, each of them has the same querelement $\bar{x} = 1$. This means that all elements in $[[1]]_b$ are "querable", but only one element $x = 1$ has an inverse which is also 1; b) $(a + bk)^{n-2} = -1$ and $\left(a + b\bar{k}\right) = -1$. The second relation fixes the class $[[b-1]]_b$, and from the first relation we conclude that the arity n should be odd. In this case, only one element -1 is "querable", which has $\bar{x} = -1$ as a querelement.

4) The "intersecting" class $[[1]]_2$ contains 2 "querable" elements ± 1 coinciding with their inverses, which means that $\{+1, -1\}$ is a binary subgroup (that is $\mathbb{Z}_2$) of the binary semigroup $\langle [[1]]_2 \mid \mu_2 \rangle$. □

Corollary 7.16. *In the non-limiting cases $a \neq 1, b-1$, the n-ary semigroup $\langle [[a]]_b \mid \mu_n \rangle$ contains no "querable" (polyadic invertible) elements at all.*

Proof. It follows from $(a + bk) \neq \pm 1$ for any $k \in \mathbb{Z}$ or $a \neq \pm 1 \,(\mathrm{mod}\, b)$. □

Basing on the above statements, consider the properties of the polyadic rings $\mathcal{R}_{m,n}^{[a,b]}$ $(a \neq 0)$ describing non-zero congruence classes (see *Remark* 7.11).

Definition 7.17. The infinite set of representatives of the congruence (residue) class $[[a]]_b$ having fixed arities and forming the (m,n)-ring $\mathcal{R}_{m,n}^{[a,b]}$ is called the set of *(polyadic) (m,n)-integers (numbers)* and denoted by $\mathbb{Z}_{(m,n)}$.

Table 7.1: The polyadic ring $\mathcal{R}_{m,n}^{\mathbb{Z}(a,b)}$ of the fixed residue class $[[a]]_b$: arity shape.

$a \backslash b$	2	3	4	5	6	7	8	9	10
1	$m=\mathbf{3}$ $n=\mathbf{2}$ $I=1$ $J=0$	$m=\mathbf{4}$ $n=\mathbf{2}$ $I=1$ $J=0$	$m=\mathbf{5}$ $n=\mathbf{2}$ $I=1$ $J=0$	$m=\mathbf{6}$ $n=\mathbf{2}$ $I=1$ $J=0$	$m=\mathbf{7}$ $n=\mathbf{2}$ $I=1$ $J=0$	$m=\mathbf{8}$ $n=\mathbf{2}$ $I=1$ $J=0$	$m=\mathbf{9}$ $n=\mathbf{2}$ $I=1$ $J=0$	$m=\mathbf{10}$ $n=\mathbf{2}$ $I=1$ $J=0$	$m=\mathbf{11}$ $n=\mathbf{2}$ $I=1$ $J=0$
2		$m=\mathbf{4}$ $n=\mathbf{3}$ $I=2$ $J=2$		$m=\mathbf{6}$ $n=\mathbf{5}$ $I=2$ $J=6$	$m=\mathbf{4}$ $n=\mathbf{3}$ $I=1$ $J=1$	$m=\mathbf{8}$ $n=\mathbf{4}$ $I=2$ $J=2$		$m=\mathbf{10}$ $n=\mathbf{7}$ $I=2$ $J=14$	$m=\mathbf{6}$ $n=\mathbf{5}$ $I=1$ $J=3$
3			$m=\mathbf{5}$ $n=\mathbf{3}$ $I=3$ $J=6$	$m=\mathbf{6}$ $n=\mathbf{5}$ $I=3$ $J=48$	$m=\mathbf{3}$ $n=\mathbf{2}$ $I=1$ $J=1$	$m=\mathbf{8}$ $n=\mathbf{7}$ $I=3$ $J=312$	$m=\mathbf{9}$ $n=\mathbf{3}$ $I=3$ $J=3$		$m=\mathbf{11}$ $n=\mathbf{5}$ $I=3$ $J=24$
4				$m=\mathbf{6}$ $n=\mathbf{3}$ $I=4$ $J=12$	$m=\mathbf{4}$ $n=\mathbf{2}$ $I=2$ $J=2$	$m=\mathbf{8}$ $n=\mathbf{4}$ $I=4$ $J=36$		$m=\mathbf{10}$ $n=\mathbf{4}$ $I=4$ $J=28$	$m=\mathbf{6}$ $n=\mathbf{3}$ $I=2$ $J=6$
5					$m=\mathbf{7}$ $n=\mathbf{3}$ $I=5$ $J=20$	$m=\mathbf{8}$ $n=\mathbf{7}$ $I=5$ $J=11160$	$m=\mathbf{9}$ $n=\mathbf{3}$ $I=5$ $J=15$	$m=\mathbf{10}$ $n=\mathbf{7}$ $I=5$ $J=8680$	$m=\mathbf{3}$ $n=\mathbf{2}$ $I=1$ $J=2$
6						$m=\mathbf{8}$ $n=\mathbf{3}$ $I=6$ $J=30$			$m=\mathbf{6}$ $n=\mathbf{2}$ $I=3$ $J=3$
7							$m=\mathbf{9}$ $n=\mathbf{3}$ $I=7$ $J=42$	$m=\mathbf{10}$ $n=\mathbf{4}$ $I=7$ $J=266$	$m=\mathbf{11}$ $n=\mathbf{5}$ $I=7$ $J=1680$
8								$m=\mathbf{10}$ $n=\mathbf{3}$ $I=8$ $J=56$	$m=\mathbf{6}$ $n=\mathbf{5}$ $I=4$ $J=3276$
9									$m=\mathbf{11}$ $n=\mathbf{3}$ $I=9$ $J=72$

Obviously, for ordinary integers $\mathbb{Z} = \mathbb{Z}_{(2,2)}$, and they form the binary ring $\mathcal{R}_{2,2}^{[0,1]}$.

Proposition 7.18. *The polyadic ring $\mathcal{R}_{m,n}^{[a,b]}$ is an (m,n)-integral domain.*

Proof. The proof follows from the definitions (7.2), (7.3), the condition $a \neq 0$, and the commutativity and cancellativity in $\mathbb{Z}$. $\square$

Lemma 7.19. *There are no such congruence classes which can be described by polyadic* (m,n)*-field.*

Proof. Follows from Proposition 7.15 and Corollary 7.16. $\square$

This statement for the limiting case $[[1]]_2$ appeared in [21] while studying the ideal structure of the corresponding $(3,2)$-ring.

Proposition 7.20. *In the limiting case* $a = 1$*, the polyadic ring* $\mathcal{R}_{b+1,2}^{[1,b]}$ *can be embedded into a* $(b+1,2)$*-ary field.*

Proof. Because the polyadic ring $\mathcal{R}_{b+1,2}^{[1,b]}$ of the congruence class $[[1]]_b$ is an $(b+1,2)$-integral domain by Proposition 7.18, we can construct in a standard way the corresponding $(b+1,2)$-quotient ring, which is a $(b+1,2)$-ary field up to isomorphism as was shown in [11]. By analogy, it can be called the field of *polyadic rational numbers* which has the form

$$x = \frac{1+bk_1}{1+bk_2}, \quad k_i \in \mathbb{Z}.$$

Indeed, they form a $(b+1,2)$-field, because each element has an inverse under multiplication (which is obvious) and is additively "querable", such that the equation for the querelement $\bar{x}$ becomes $\nu_{b+1}[\overbrace{x,x,\ldots,x}^{b},\bar{x}] = x$, which can be solved for any x, giving uniquely $\bar{x} = -(b-1)\dfrac{1+bk_1}{1+bk_2}$. $\square$

The introduced polyadic inner structure of the residue (congruence) classes allows us to extend various number theoretic problems by considering the polyadic (m,n)-integers $\mathbb{Z}_{(m,n)}$ instead of $\mathbb{Z}$.

8. Equal sums of like powers Diophantine equation over polyadic integers

First, recall the standard binary version of the equal sums of like powers Diophantine equation [22, 30]. Take the fixed non-negative integers $p,q,l \in \mathbb{N}^0$, $p \leq q$, and the positive integer unknowns $u_i, v_j \in \mathbb{Z}_+$, $i = 1,\ldots p+1$, $j = 1,1,\ldots q+1$, then the Diophantine equation is

$$\sum_{i=1}^{p+1} u_i^{l+1} = \sum_{j=1}^{q+1} v_j^{l+1}. \tag{8.1}$$

The trivial case, when $u_i = 0$, $v_j = 0$ for all i,j, is not considered. In the binary case, the solutions of (8.1) are usually denoted by $(l+1 \mid p+1, q+1)_r$, which shows the number of summands on both sides and the powers of elements [30]. But in the polyadic case (see below), the number of summands and powers do not coincide with $l+1$, $p+1$, $q+1$. We mark the solutions of (8.1) by the triple $(l \mid p,q)_r$ showing the quantity of operations, where r (if it is used) is the order of the solution (ranked by the value of the sum) and the unknowns u_i, v_j are placed in ascending order $u_i \le u_{i+1}, v_j \le v_{j+1}$.

Let us recall the *Tarry–Escott problem* (or multigrades problem) [15]: to find the solutions to (8.1) for an equal number of summands on both sides of $p=q$ and s equations simultaneously, such that $l=0,\ldots,s$. Known solutions exist for powers until $s=10$, which are bounded such that $s \le p$ (in our notations), see also [39]. The solutions with highest powers $s=p$ are the most interesting and called the *ideal solutions* [5].

Theorem 8.1 (Frolov [24])**.** *If the set of s Diophantine equations (8.1) with $p=q$ for $l=0,\ldots,s$ has a solution $\{u_i, v_i, i=1,\ldots p+1\}$, then it has the solution $\{a+bu_i, a+bv_i, i=1,\ldots p+1\}$, where $a,b \in \mathbb{Z}$ are arbitrary and fixed.*

In the simplest case $(1 \mid 0,1)$, one term in the l.h.s., one addition in the r.h.s. and one multiplication, the (coprime) positive numbers satisfying (8.1) are called a (primitive) *Pythagorean triple*. For *Fermat's triple* $(l \mid 0,1)$ with one addition in the r.h.s. and more than one multiplication $l \ge 2$, there are no solutions of (8.1), which is known as *Fermat's Last Theorem* proved in [44]. There are many solutions known with more than one addition on both sides, where the highest number of multiplications till now is 31 (S. Chase, 2012).

Before generalizing (8.1) to the polyadic case we note the following.

Remark 8.2. The notations in (8.1) are chosen in such a way that p and q are the *numbers of binary additions* on both sides, while l is the *number of binary multiplications* in each term, which is natural for using polyadic powers [16].

8.1. Polyadic analog of the Lander–Parkin–Selfridge conjecture. In [30], a generalization of Fermat's Last Theorem was conjectured that the solutions of (8.1) exist for small powers only, which can be formulated in terms of the numbers of operations as

Conjecture 8.3 (Lander–Parkin–Selfridge [30])**.** *There exist solutions of (8.1) in positive integers if the number of multiplications is less than or equal to the total number of additions plus one*

$$3 \le l \le l_{LSP} = p+q+1,$$

where $p+q \ge 2$.

Remark 8.4. If equation (8.1) is considered over the *binary ring* of integers $\mathbb{Z}$ such that $u_i, v_j \in \mathbb{Z}$, it leads to a straightforward reformulation: for even powers it is obvious, but for odd powers all negative terms can be rearranged and placed on the other side.

Let us consider the Diophantine equation (8.1) over polyadic integers $\mathbb{Z}_{(m,n)}$ (i.e., over the polyadic (m,n)-ary ring $\mathcal{R}^{\mathbb{Z}}_{m,n}$) such that $u_i, v_j \in \mathcal{R}^{\mathbb{Z}}_{m,n}$. We use the "long products" $\mu_n^{(l)}$ and $\nu_m^{(l)}$ containing l operations and also the "polyadic power" for an element $x \in \mathcal{R}_{m,n}^{\mathbb{Z}}$ with respect to the n-ary multiplication [16],

$$x^{\langle l \rangle_n} = \mu_n^{(l)} [\overbrace{x, x, \ldots, x}^{l(n-1)+1}]. \tag{8.2}$$

In the binary case ($n = 2$), the polyadic power coincides with $(l + 1)$ power of an element $x^{\langle l \rangle_2} = x^{l+1}$, which explains Remark 8.2. In this notation the polyadic analog of the equal sums of like powers Diophantine equation has the form

$$\nu_m^{(p)} \left[u_1^{\langle l \rangle_n}, u_2^{\langle l \rangle_n}, \ldots, u_{p(m-1)+1}^{\langle l \rangle_n} \right] = \nu_m^{(q)} \left[v_1^{\langle l \rangle_n}, v_2^{\langle l \rangle_n}, \ldots, v_{q(m-1)+1}^{\langle l \rangle_n} \right], \tag{8.3}$$

where p and q are the numbers of m-ary additions in the l.h.s. and r.h.s., respectively. The solutions of (8.3) will be denoted by $\{u_1, u_2, \ldots, u_{p(m-1)+1}; v_1, v_2, \ldots, v_{q(m-1)+1}\}$. In the binary case $m = 2, n = 2$, (8.3) reduces to (8.1). Analogously, we mark the solutions of (8.3) by the *polyadic triple* $(l \mid p, q)_r^{(m,n)}$. Now the *polyadic Pythagorean triple* $(1 \mid 0, 1)^{(m,n)}$, having one term in the l.h.s., one m-ary addition in the r.h.s. and one n-ary multiplication (elements are in the first polyadic power $\langle 1 \rangle_n$), becomes

$$u_1^{\langle 1 \rangle_n} = \nu_m \left[v_1^{\langle 1 \rangle_n}, v_2^{\langle 1 \rangle_n}, \ldots, v_m^{\langle 1 \rangle_n} \right]. \tag{8.4}$$

Definition 8.5. Equation (8.4) solved by minimal $u_1, v_i \in \mathbb{Z}$, $i = 1, \ldots, m$ can be named the *polyadic Pythagorean theorem*.

The *polyadic Fermat's triple* $(l \mid 0, 1)^{(m,n)}$ has one term in the l.h.s., one m-ary addition in the r.h.s. and l (n-ary) multiplications

$$u_1^{\langle l \rangle_n} = \nu_m \left[v_1^{\langle l \rangle_n}, v_2^{\langle l \rangle_n}, \ldots, v_m^{\langle l \rangle_n} \right]. \tag{8.5}$$

One may be interested in whether the polyadic analog of Fermat's Last Theorem is valid, and if not, in which cases the analogy with the binary case can be sustained.

Conjecture 8.6 (Polyadic analog of Fermat's Last Theorem)**.** *The polyadic Fermat triple* (8.5) *has no solutions over the polyadic (m, n)-ary ring $\mathcal{R}_{m,n}^{\mathbb{Z}}$ if $l \geq 2$, i.e., there is more than one n-ary multiplication.*

Its straightforward generalization leads to the polyadic version of the Lander–Parkin–Selfridge conjecture.

Conjecture 8.7 (Polyadic Lander–Parkin–Selfridge conjecture)**.** *There exist solutions of the polyadic analog of the equal sums of like powers Diophantine equation* (8.3) *in integers if the number of n-ary multiplications is less than or equal to the total number of m-ary additions plus one*

$$3 \leq l \leq l_{pLPS} = p + q + 1. \tag{8.6}$$

Below we will see a counterexample to both of the above conjectures.

Example 8.8. Let us consider the $(3,2)$-ring $\mathcal{R}^{\mathbb{Z}}_{3,2} = \langle \mathbb{Z} \mid \nu_3, \mu_2 \rangle$, where

$$\nu_3 [x, y, z] = x + y + z + 2, \tag{8.7}$$
$$\mu_2 [x, y] = xy + x + y. \tag{8.8}$$

Note that this exotic polyadic ring is commutative and cancellative, having unit 0, no multiplicative inverses, and for any $x \in \mathcal{R}^{\mathbb{Z}}_{3,2}$ its additive querelement $\tilde{x} = -x-2$, therefore $\langle \mathbb{Z} \mid \nu_3 \rangle$ is a ternary group (as it should be). The polyadic power of any element is

$$x^{\langle l \rangle_2} = (x+1)^{l+1} - 1. \tag{8.9}$$

1) For $\mathcal{R}^{\mathbb{Z}}_{3,2}$, the polyadic Pythagorean triple $(1 \mid 0,1)^{(3,2)}$ in (8.4) now is

$$u^{\langle 1 \rangle_2} = \nu_3 \left[x^{\langle 1 \rangle_2}, y^{\langle 1 \rangle_2}, z^{\langle 1 \rangle_2} \right],$$

which, using (8.2), (8.8) and (8.9), becomes the (shifted) Pythagorean quadruple [42],

$$(u+1)^2 = (x+1)^2 + (y+1)^2 + (z+1)^2,$$

and it has infinite number of solutions, among which two minimal ones $\{u = 2; x = 0, y = z = 1\}$ and $\{u = 14; x = 1, y = 9, z = 10\}$ give $3^2 = 1^2 + 2^2 + 2^2$ and $15^2 = 2^2 + 10^2 + 11^2$, respectively.

2) For this $(3,2)$-ring $\mathcal{R}^{\mathbb{Z}}_{3,2}$, the polyadic Fermat triple $(l \mid 0,1)^{(3,2)}$ becomes

$$(u+1)^{l+1} = (x+1)^{l+1} + (y+1)^{l+1} + (z+1)^{l+1}. \tag{8.10}$$

If the polyadic analog of Fermat's Last Theorem 8.6 holds, then there are no solutions to (8.10) for more than one n-ary multiplication $l \geq 2$. But this is the particular case, $p = 0$, $q = 2$, of the *binary* Lander–Parkin–Selfridge Conjecture 8.3 which now takes the form: the solutions to (8.10) exist if $l \leq 3$. Thus, as a *counterexample* to the polyadic analog of Fermat's Last Theorem, we have two possible solutions with numbers of multiplications: $l = 2, 3$. In the case of $l = 2$, there exist two solutions: one well-known solution $\{u = 5; x = 2, y = 3, z = 4\}$ giving $6^3 = 3^3 + 4^3 + 5^3$ and another one giving $709^3 = 193^3 + 461^3 + 631^3$ (J.-C. Meyrignac, 2000), while for $l = 3$ there exists an infinite number of solutions, and one of them (minimal) gives $422481^4 = 95800^4 + 217519^4 + 414560^4$ [23].

3) The general polyadic triple $(l \mid p, q)^{(3,2)}$, using (8.3), can be presented in the standard binary form (as (8.1)),

$$\sum_{i=1}^{2p+1} (u_i + 1)^{l+1} = \sum_{j=1}^{2q+1} (v_i + 1)^{l+1}, \quad u_i, v_j \in \mathbb{Z}. \tag{8.11}$$

Let us apply the *polyadic* Lander–Parkin–Selfridge Conjecture 8.7 for this case: the solutions to (8.11) exist if $3 \leq l \leq l_{pLSP} = p + q + 1$. But the *binary* Lander–Parkin–Selfridge Conjecture 8.3, applied directly, gives $3 \leq l \leq l_{LSP} =$

$2p+2q+1$. So, we should have *counterexamples* to the polyadic Lander–Parkin–Selfridge conjecture when $l_{pLSP} < l \le l_{LSP}$. For instance, for $p = q = 1$, we have $l_{pLSP} = 3$, while the (minimal) counterexample with $l = 5$ is $\{u_1 = 3, u_2 = 18, u_3 = 21, v_1 = 9, v_2 = 14, v_3 = 22\}$ giving $3^6 + 19^6 + 22^6 = 10^6 + 15^6 + 23^6$ [43].

As can be observed from Example 8.8, the arity shape of the polyadic ring $\mathcal{R}^{\mathbb{Z}}_{m,n}$ is crucial in constructing polyadic analogs of the equal sums of like powers conjectures. We can make some general estimations assuming a special (more or less natural) form of its operations over integers.

Definition 8.9. We call $\mathcal{R}^{\mathbb{Z}}_{m,n}$ the *standard polyadic ring* if the "leading terms" of its m-ary addition and n-ary multiplication are

$$\nu_m[\overbrace{x, x, \ldots, x}^{m}] \sim mx, \tag{8.12}$$

$$\mu_n[\overbrace{x, x, \ldots, x}^{n}] \sim x^n, \qquad x \in \mathbb{Z}. \tag{8.13}$$

The polyadic ring $\mathcal{R}^{\mathbb{Z}}_{3,2}$ from *Example* 8.8 and the congruence class polyadic ring $\mathcal{R}^{[a,b]}_{m,n}$ (7.8) are both standard.

Using (8.2), we obtain approximate behavior of the polyadic power in the standard polyadic ring

$$x^{\langle l \rangle_n} \sim x^{l(n-1)+1}, \quad x \in \mathbb{Z},\ l \in \mathbb{N},\ n \ge 2. \tag{8.14}$$

So, the increasing of the arity of multiplication leads to higher powers, while the increasing of the arity of addition gives more terms in sums. Thus, the estimation for the polyadic analog of the equal sums of like powers Diophantine equation (8.3) becomes

$$(p(m-1)+1)\, x^{l(n-1)+1} \sim (q(m-1)+1)\, x^{l(n-1)+1}, \quad x \in \mathbb{Z}. \tag{8.15}$$

Now we can apply the *binary* Lander–Parkin–Selfridge Conjecture 8.3 in the form: the solutions to (8.15) can exist if $3 \le l \le l_{LPS}$, where l_{LPS} is an integer solution of

$$(n-1)\, l_{LPS} = (p+q)(m-1)+1. \tag{8.16}$$

On the other hand, the *polyadic* Lander–Parkin–Selfridge Conjecture 8.7 gives: the solutions to (8.15) can exist if $3 \le l \le l_{pLPS} = p+q+1$. Note that $(p+q) \ge 2$ now.

An interesting question arises: which arities give the same limit, that is, when $l_{pLPS} = l_{LPS}$?

Proposition 8.10. *For any fixed number of additions in both sizes of the polyadic analog of the equal sums of like powers Diophantine equation* (8.3) $p+q \ge 2$, *there exist limiting arities* m_0 *and* n_0 *(excluding the trivial binary case,* $m_0 =$

$n_0 = 2$), *for which the binary and polyadic Lander–Parkin–Selfridge conjectures coincide* $l_{pLPS} = l_{LPS}$ *such that*

$$m_0 = 3 + p + q + (p + q + 1)\, k, \tag{8.17}$$

$$n_0 = 2 + p + q + (p + q)\, k, \qquad k \in \mathbb{N}^0. \tag{8.18}$$

Proof. To equate $l_{LPS} = l_{pLPS} = p+q+1$, we use (8.16) and solve in integers the equation

$$(n_0 - 1)(p + q + 1) = (p + q)(m_0 - 1) + 1.$$

In the trivial case, $m_0 = n_0 = 2$, this is an identity, while the other solutions can be found from $n_0\,(p + q + 1) = (p + q)\, m_0 + 2$, which gives (8.17), (8.18). ☐

Corollary 8.11. *In the limiting case,* $l_{pLPS} = l_{LPS}$*, the arity of multiplication always exceeds the arity of addition*

$$m_0 - n_0 = k + 1, \quad k \in \mathbb{N}^0,$$

and they start from $m_0 \geq 5$, $n_0 \geq 4$.

The first allowed arities m_0 and n_0 are presented in Table 8.1. Their meaning is the following.

Corollary 8.12. *For the polyadic analog of the equal sums of like powers equation over the standard polyadic ring* $\mathcal{R}^{\mathbb{Z}}_{m,n}$ *(with fixed* $p + q \geq 2$*) the polyadic Lander–Parkin–Selfridge conjecture becomes weaker than the binary one* $l_{pLPS} \geq l_{LPS}$ *if:*

1) *the arity of multiplication exceeds its limiting value* n_0 *with the fixed arity of addition;*

2) *the arity of addition is lower than its limiting value* m_0 *with the fixed arity of multiplication.*

Table 8.1: The limiting arities m_0 and n_0 which give $l_{pLPS} = l_{LPS}$ in (8.15).

$p+q=2$		$p+q=3$		$p+q=4$	
m_0	n_0	m_0	n_0	m_0	n_0
5	4	6	5	7	6
8	6	10	8	12	10
11	8	14	11	17	14
14	10	18	14	22	18

Example 8.13. Consider the standard polyadic ring $\mathcal{R}^{\mathbb{Z}}_{m,n}$ and fix the arity of addition $m_0 = 12$, then take in (8.15) the total number of additions $p + q = 4$ (the last column in Table 8.1). We observe that the arity of multiplication

$n = 16$, which exceeds the limiting arity $n_0 = 10$ (corresponding to m_0). Thus, we obtain $l_{pLPS} = 5$ and $l_{LPS} = 3$ by solving (8.16) in integers, and therefore the polyadic Lander–Parkin–Selfridge conjecture becomes now weaker than the binary one, and we do not obtain counterexamples to it as in Example 8.8 (where the situation was opposite, $l_{pLPS} = 3$ and $l_{LPS} = 5$, and they could not be equal).

A concrete example of the standard polyadic ring (Definition 8.9) is the polyadic ring of the fixed congruence class $\mathcal{R}_{m,n}^{[a,b]}$ considered in Section 7, because its operations (7.2), (7.3) have the same straightforward behavior (8.12), (8.13). Let us formulate the polyadic analog of equal sums of like powers Diophantine equation (8.3) over $\mathcal{R}_{m,n}^{[a,b]}$ in terms of operations in $\mathbb{Z}$. Using (7.2), (7.3) and (8.14), for (8.3) we obtain

$$\sum_{i=1}^{p(m-1)+1} (a + bk_i)^{l(n-1)+1} = \sum_{j=1}^{q(m-1)+1} (a + bk_j)^{l(n-1)+1}, \quad a, b, k_i \in \mathbb{Z}. \tag{8.19}$$

It is seen that the leading power behavior of both sides in (8.19) coincides with the general estimation (8.15). But now the arity shape (m, n) is fixed by (7.9), (7.10) and given in Table 7.1. Nevertheless, we can consider for (8.19) the polyadic analog of Fermat's Last Theorem 8.6, the Lander–Parkin–Selfridge Conjecture 8.3 (solutions exist for $l \le l_{LPS}$) and its polyadic version (Conjecture 8.7, solutions exist for $l \le l_{pLPS}$) as in the estimations above. Let us consider some examples of solutions to (8.19).

Example 8.14. Let $[[2]]_3$ be the congruence class, which is described by $(4,3)$-ring $\mathcal{R}_{4,3}^{[2,3]}$ (see Table 7.1), and we consider the polyadic Fermat's triple $(l \mid 0, 5)^{(4,3)}$ (8.5). Now the powers are $l_{LPS} = 8$, $l_{pLPS} = 6$ and, for instance, if $l = 2$, we have solutions, because $l < l_{pLPS} < l_{LPS}$, and one of them is

$$14^5 = 4 \cdot (-1)^5 + 7 \cdot 5^5 + 8^5 + 2 \cdot 11^5.$$

8.2. Frolov's theorem and the Tarry–Escott problem. A special set of solutions to the polyadic Lander–Parkin–Selfridge Conjecture 8.7 can be generated if we put $p = q$ in (8.19), which we call *equal-summand solutions* (the term "symmetric solution" is already taken and widely used [5]), by exploiting the Tarry–Escott problem approach [15] and Frolov's theorem 8.1.

Theorem 8.15. *If the set of integers $k_i \in \mathbb{Z}$ solves the Tarry–Escott problem*

$$\sum_{i=1}^{p(m-1)+1} k_i^r = \sum_{j=1}^{p(m-1)+1} k_j^r, \quad r = 1, \ldots, s = l(n-1) + 1, \tag{8.20}$$

then the polyadic equal sums of like powers equation with equal summands (8.3) has a solution over the polyadic (m,n)-ring $\mathcal{R}_{m,n}^{[a,b]}$ having the arity shape given by the following relations:

1. *Inequality*

$$l\,(n-1)+1 \leq p\,(m-1)\,; \tag{8.21}$$

2. *Equality*

$$p\,(m-1) = 2^{l(n-1)+1}. \tag{8.22}$$

Proof. Using Frolov's theorem 8.1 applied to (8.20), we state that

$$\sum_{i=1}^{p(m-1)+1} (a+bk_i)^r = \sum_{j=1}^{p(m-1)+1} (a+bk_j)^r\,, \quad r=1,\ldots,s=l\,(n-1)+1, \tag{8.23}$$

for any fixed integers $a,b \in \mathbb{Z}$. This means that (8.23) with k_i (satisfying (8.20)) corresponds to a solution to the polyadic equal sums of like powers equation (8.3) for any congruence class $[[a]]_b$. Nevertheless, the values a and b are fixed by the restrictions on the arity shape and the relations (7.4) and (7.7).

1. It is known that the Tarry–Escott problem can have a solution only when the powers are strongly less than the number of summands [5, 15], that is $(l\,(n-1)+1)+1 \leq p\,(m-1)+1$, which gives (8.21).

2. A special kind of solutions, when the number of summands is equal to 2 into the number of powers, was found using the *Thue–Morse sequence* [1], which always satisfies the bound (8.21), and in our notation it is (8.22).

In both cases the relations (8.21) and (8.22) should be solved in positive integers and with $m \geq 2$ and $n \geq 2$, which can lead to non-unique solutions. □

Let us consider some examples which give solutions to the polyadic equal sums of like powers equation (8.3) with $p=q$ over the polyadic (m,n)-ring $\mathcal{R}_{m,n}^{[a,b]}$ of the fixed congruence class $[[a]]_b$.

Example 8.16. 1) One of the first ideal (non-symmetric) solutions to the Tarry–Escott problem has 6 summands and 5 powers (A. Golden, 1944),

$$0^r+19^r+25^r+57^r+62^r+86^r$$
$$=2^r+11^r+40^r+42^r+69^r+85^r, \quad r=1,\ldots,5. \tag{8.24}$$

By comparing it with (8.20), we obtain

$$p\,(m-1)=5, \qquad l\,(n-1)=4.$$

After ignoring binary arities, we get $m=6$, $p=1$ and $n=3$, $l=2$. From Theorem 7.8 and Table 7.1, we observe the minimal choice $a=4$ and $b=5$. It follows from Frolov's theorem 8.1 that all equations in (8.24) have symmetry $k_i \to a+bk_i = 4+5k_i$. Thus, we obtain the solution of the polyadic equal sums of like powers equation (8.3) for the fixed congruence class $[[4]]_5$ in the form

$$4^5+99^5+129^5+289^5+314^5+434^5 = 14^5+59^5+204^5+214^5+349^5+429^5.$$

It is seen from Table 7.1 that the arity shape $(m = 6, n = 3)$ corresponds, e.g., to the congruence class $[[4]]_{10}$ as well. Using Frolov's theorem, we substitute in (8.24) $k_i \to 4 + 10k_i$ to obtain the solution in the congruence class $[[4]]_{10}$,

$$4^5 + 194^5 + 254^5 + 574^5 + 624^5 + 864^5 = 24^5 + 114^5 + 404^5 + 424^5 + 694^5 + 854^5.$$

2) To obtain the special kind of solutions to the Tarry–Escott problem, we start with the known one with 8 summands and 3 powers (see, e.g., [32]),

$$0^r + 3^r + 5^r + 6^r + 9^r + 10^r + 12^r + 15^r$$
$$= 1^r + 2^r + 4^r + 7^r + 8^r + 11^r + 13^r + 14^r, \quad r = 1, 2, 3. \quad (8.25)$$

Thus we have the concrete solution to the system (8.20) with the condition (8.22) which now takes the form $8 = 2^3$, and therefore

$$p\,(m-1) = 7, \qquad l\,(n-1) = 2.$$

Excluding the trivial case containing binary arities, we have $m = 8$, $p = 1$ and $n = 3$, $l = 1$. It follows from Theorem 7.8 and Table 7.1 that $a = 6$ and $b = 7$, and so the polyadic ring is $\mathcal{R}^{[6,7]}_{8,3}$. Using Frolov's theorem 8.1, we can substitute the entries in (8.25) as $k_i \to a + bk_i = 6 + 7k_i$ in the equation with highest power $r = 3$ (which is relevant to our task) and obtain the solution of (8.3) for $[[6]]_7$ as follows:

$$6^3 + 27^3 + 41^3 + 48^3 + 69^3 + 76^3 + 90^3 + 111^3$$
$$= 13^3 + 20^3 + 34^3 + 55^3 + 62^3 + 83^3 + 97^3 + 104^3.$$

We conclude that consideration of the Tarry–Escott problem and Frolov's theorem over polyadic rings gives us the possibility to obtain many nontrivial solutions to the polyadic equal sums of like powers equation for fixed congruence classes.

Acknowledgment. The author would like to express his deep gratitude and sincere thankfulness to Joachim Cuntz, Christopher Deninger, Grigorij Kurinnoj, Mike Hewitt, Jim Stasheff, Alexander Voronov, and Wend Werner for discussions, and to Dara Shayda for Mathematica programming help.

References

[1] J.-P. Allouche and J. Shallit, *The ubiquitous Prouhet–Thue–Morse sequence*, Sequences and their Applications. Proceedings of the International Conference, SETA '98, Singapore, December, 1998, Springer, London, 1999, 1–16.

[2] V.D. Belousov, *n-Ary Quasigroups*, Shtintsa, Kishinev, 1972.

[3] C. Bergman, *Universal Algebra: Fundamentals and Selected Topics*, CRC Press, New York, 2012.

[4] D. Boccioni, *Caratterizzazione di una classe di anelli generalizzati*, Rend. Semin. Mat. Univ. Padova **35** (1965), 116–127.

[5] P. Borwein, *The Prouhet–Tarry–Escott problem*, Computational Excursions in Analysis and Number Theory, CMS books in mathematics: 10, Springer-Verlag, New York, 2002, 85–95.

[6] R. Carlsson, *Cohomology of associative triple systems*, Proc. Amer. Math. Soc. **60** (1976), 1–7.

[7] R. Carlsson, *N-ary algebras*, Nagoya Math. J. **78** (1980), 45–56.

[8] N. Celakoski, *On (F, G)-rings*, Prirod.-Mat. Fak. Univ. Kiril Metodij Skopje Godišen Zb. **28** (1977), 5–15.

[9] P.M. Cohn, *Universal Algebra*, Harper & Row, New York, 1965.

[10] G. Crombez, *On (n, m)-rings*, Abh. Math. Semin. Univ. Hamb. **37** (1972), 180–199.

[11] G. Crombez and J. Timm, *On (n, m)-quotient rings*, Abh. Math. Semin. Univ. Hamb. **37** (1972), 200–203.

[12] J. Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), 173–185.

[13] G. Čupona, *On $[m, n]$-rings*, Bull. Soc. Math. Phys. Macedoine **16** (1965), 5–9.

[14] K. Denecke and S.L. Wismath, *Universal Algebra and Coalgebra*, World Scientific, Singapore, 2009.

[15] H.L. Dorwart and O.E. Brown, *The Tarry–Escott problem*, Amer. Math. Monthly **44** (1937), 613–626.

[16] S. Duplij, *Polyadic systems, representations and quantum groups*, Visn. Kharkiv. Nats. Univ. **1017** (2012), No. 3(55), Yadra, Chastynky, Polya, 28–59. Expanded version available from: arXiv: 1308.4060.

[17] S. Duplij, *A "q-deformed" generalization of the Hosszú-Gluskin theorem*, Filomat **30** (2016), 2985–3005.

[18] S. Duplij, *Polyadic integer numbers and finite (m,n)-fields*, p-Adic Numbers, Ultrametric Analysis and Appl. **9** (2017), 257–281. Available from: arXiv: 1707.00719.

[19] S. Duplij and W. Marcinek, *Semisupermanifolds and regularization of categories, modules, algebras and Yang-Baxter equation*, Nucl. Phys. Proc. Suppl. **102** (2001), 293–297.

[20] S. Duplij and W. Marcinek, *Regular obstructed categories and topological quantum field theory*, J. Math. Phys. **43** (2002), 3329–3341.

[21] S. Duplij and W. Werner, *Structure of unital 3-fields, preprint*, arXiv: 1505.04393.

[22] R.L. Ekl, *New results in equal sums of like powers*, Math. Comp. **67** (1998), 1309–1315.

[23] N.D. Elkies, *On $A^4 + B^4 + C^4 = D^4$*, Math. Comp. **51** (1988), 825–835.

[24] M. Frolov, *Égalités à deux degrés*, Bull. Soc. Math. Fr. **17** (1889), 69–83.

[25] A.M. Gal'mak, *n-Ary Groups*, **Part 1**, Gomel University, Gomel, 2003.

[26] K. Głazek and J. Michalski, *On polyadic groups which are term-derived from groups*, Stud. Sci. Math. Hung. **19** (1984), 307–315.

[27] G. Grätser, *Universal Algebra*, D. Van Nostrand Co., Inc., New York–Toronto, 1968.

[28] P. Halmos, *Algebraic Logic*, Chelsea Publishing, New York, 1962.

[29] L. Iancu and M. S. Pop, *A Post type theorem for (m, n) fields*, Proceedings of the Scientific Communications Meeting of "Aurel Vlaicu" University, Arad, Romania, May 16–17, 1996, Edition III, **14A**, "Aurel Vlaicu" Univ. of Arad Publishing Centre, Arad, 1997, 13–18.

[30] L. Lander, T. Parkin, and J. Selfridge, *A survey of equal sums of like powers*, Math. Comput. **21** (1967), 446–459.

[31] J.J. Leeson and A.T. Butson, *On the general theory of (m, n) rings.*, Algebra Univers. **11** (1980), 42–76.

[32] D.H. Lehmer, *The Tarry–Escott problem*, Scripta Math. **13** (1947), 37–41.

[33] W.G. Lister, *Ternary rings*, Trans. Amer. Math. Soc. **154** (1971), 37–55.

[34] J. Michalski, *On J-derived polyadic groups*, Mathematica (Cluj) **30 (53)** (1988), 149–155.

[35] P.W. Michor and A.M. Vinogradov, *n-ary Lie and associative algebras*, Rend. Sem. Mat. Univ. Pol. Torino **54** (1996), 373–392.

[36] F. Mignot, *Contrôle dans les inéquations variationelles elliptiques*, J. Funct. Anal. **22** (1976), 130–185.

[37] A. Misiak, *n-Inner product spaces*, Math. Nachr. **140** (1989), 299–319.

[38] J.D. Monk and F. Sioson, *m-semigroups, semigroups and function representations*, Fundam. Math. **59** (1966), 233–241.

[39] H.D. Nguyen, *A new proof of the Prouhet–Tarry–Escott problem*, Integers **16** (2016), A01, 1–9.

[40] A. Pop and M.S. Pop, *Some embeddings theorems for $(n, 2)$-rings*, Bul. Ştiinţ. Univ. Baia Mare, Ser. B, Fasc. Mat.-Inform. **18** (2002), 311–316.

[41] S.A. Rusakov, *Some Applications of n-ary Group Theory*, Belaruskaya Navuka, Minsk, 1998.

[42] R. Spira, *The Diophantine equation $x^2 + y^2 + z^2 = m^2$*, Am. Math. Mon. **69** (1962), 360–365.

[43] K. Subba Rao, *On sums of sixth powers*, J. London Math. Soc. **9** (1934), No. 3, 172–173.

[44] A. Wiles, *Modular elliptic curves and Fermat's Last Theorem*, Ann. Math. **141** (1995), 443–551.

[45] D. Zupnik, *Polyadic semigroups*, Publ. Math. (Debrecen) **14** (1967), 273–279.

Received July 10, 2017, revised April 19, 2018.

Steven Duplij,

Mathematisches Institut, Universität Münster, Einsteinstrasse 62, D-48149 Münster, Deutschland,
E-mail: douplii@uni-muenster.de, sduplij@gmail.com

Форма арності поліадичних алгебраїчних структур

Steven Duplij

Конкретні двомножинні (модуль-подібні і алгебра-подібні) алгебраїчні структури досліджено з точки зору початкових арностей операцій, які вважаються довільними. Однак співвідношення між операціями, які є наслідками структури означень, призводять до обмежень, що визначаються формою можливих арностей і дозволяють нам сформулювати принцип свободи часткових арностей. Розглядаються поліадичні векторні простори та алгебри, двоїсті векторні простори, прямі суми, тензорні добутки, спарювання внутрішніх просторів. Окреслено елементи поліадичної теорії операторів: уведено мультизірки і поліадичні аналоги спряжених, операторних норм, ізометрій і проекцій, а також поліадичні C^*-алгебри, алгебри Тепліца і алгебри Кунца, представлені поліадичними операторами. Показано, що класи конгруенції є поліадичними кільцями спеціального виду. Уведено поліадичні числа (див. означення 7.17) та діофантові рівняння над поліадичними кільцями. Сформульовано поліадичні аналоги гіпотези Ландера–Паркіна–Селфріджа і останню теорему Ферма. Доведено, що для поліадичних чисел жодне із згаданих тверджень не виконується. Сформульовано поліадичні версії теореми Фролова та проблеми Таррі–Ескотта.

Ключові слова: поліадичне кільце, поліадичний векторний простір, мультидія, мультизірка, діофантове рівняння, остання теорема Ферма, гіпотеза Ландера–Паркіна–Селфріджа.