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Regular obstructed categories and topological quantum field theory

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A proposal of the concept of *n*-regular obstructed categories is given. The corresponding regularity conditions for mappings, morphisms, and related structures are considered. An *n*-regular topological quantum field theory is introduced. The connection of time nonivertibility with the regularity is shown. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1473681]

I. INTRODUCTION

In the generalized histories approach[1] to quantum theory the whole universe is represented by a class of ''histories.'' In this approach the standard Hamiltonian time evolution is replaced by a partial semigroup called a ''temporal support.'' A possible realization of such program can be described in terms of cobordism manifolds and corresponding categories.[2] The temporal support arises naturally as a cobordism M, where the boundary ∂M of M is a disjoint sum of the ''incoming'' boundary manifold Σ_0 and the ''outgoing'' one Σ_1. This means that the cobordism M represents certain quantum process transforming Σ_0 into Σ_1. In other words, Σ_1 is a time consequence of Σ_0. Obviously, we have two opposite possibilities to declare which boundary is the initial one.

Let N be a cobordism with the ''outgoing'' boundary of M as its ''incoming boundary'' and Σ_2 as the ''outgoing boundary.'' Then there is a cobordism $N\circ M$ whose incoming boundary is Σ_0, and the outgoing one is Σ_3. In this case we say that these two cobordisms are glued along Σ_1. Such gluing of cobordisms up to diffeomorphisms define a partial semigroup operation. One can consider cobordism with several incoming and outgoing boundary manifolds. The class of possible histories can be represented by gluing of cobordisms in several different ways. Hence there is the corresponding coherence problem for such description.

Let Cob be a category of cobordisms, where the boundary ∂M of $M\in\mathrm{Cob}$ is a disjoint sum of the incoming boundary manifold Σ_0 and the outgoing one Σ_1. There is also the cylinder cobordism $\Sigma\times[0,1]$ such that $\partial(\Sigma\times[0,1])=\Sigma\amalg\Sigma^*$. The class of boundary components is denoted by Cob_0. According to Atiyah,[3] Baez and Dolan,[4] the topological quantum field theory (TQFT) is a functor $\mathcal{F}$ from the category Cob to the category Vect of finite-dimensional vector spaces. This means that $\mathcal{F}$ sends every manifold $\Sigma\in\mathrm{Cob}_0$ into vector space $\mathcal{F}(\Sigma)$ such that

$$\mathcal{F}(\Sigma^*)=(\mathcal{F}(\Sigma))^*,\quad \mathcal{F}(\Sigma_0\amalg\Sigma_1)=(\mathcal{F}\Sigma_0)\otimes(\mathcal{F}\Sigma_1),\quad \mathcal{F}(\varnothing)=I, \tag{1}$$

and a cobordism $M(\Sigma_0,\Sigma_1)$ to a mapping $\Phi(M)\in\mathrm{lin}_I(\mathcal{F}\Sigma_0,\mathcal{F}\Sigma_1)$ such that $\mathcal{F}(\Sigma\times[0,1])=id_{\mathcal{F}\Sigma}$, where I is a field, and Σ^* is the same manifold Σ but with the opposite orientation. Kerler[5] found examples of categories formed by some classes of cobordism manifolds preserving some operations like the disjoint sum or surgery. It was discussed by Baez and Dolan[4] that it is not easy to describe such categories in a coherent way. Crane[6,7] applied the category theory to an algebraic structure of the quantum gravity.

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The idea of regularity as generalized invertibility was first introduced by von Neumann[8] and applied by Penrose for matrices.[9] Let $\mathbb{R}$ be a ring. If for an element $a\in\mathbb{R}$ there is an element $a^{\star}$ such that

$$aa^{\star}a=a,\quad a^{\star}aa^{\star}=a^{\star},\tag{2}$$

then a is said to be regular and $a^{\star}$ is called a generalized inverse inverse of a. Generalizing transition from invertibility to regularity is a widely used method of abstract extension of various algebraic structures. The intensive study of such regularity and related directions was developed in many different fields, e.g., generalized inverses theory,[10–12] semigroup theory,[13–19] supermanifold theory,[20–22] Yang–Baxter equation in endomorphism semigroup and braided almost bialgebras,[23–25] weak bialgebras, week Hopf algebras,[26] and category theory.[27]

In this paper we are going to study certain class of categories which can be useful for the study of quantum histories with noninvertible time, quantum, gravity, and field theory. The regularity concept for linear mappings and morphisms in categories are studied. Higher order regularity conditions are described. Commutative diagrams are replaced by "semicommutative" ones. The distinction between commutative and semicommutative cases is measured by a nonzero obstruction proportional to the difference of some self-mappings $e^{(n)}$ from the identity. This allows one to regularize the notion of categories, functions, and related algebraic structures. It is interesting that this procedure is unique up to an equivalence defined by invertible morphisms. Our regularity concept is nontrivial for equivalence classes of nonivertible morphisms. The regular version of TQFT is a natural application of the formalism presented here. In this case the n-regularity means that a time evolution is noninvertible, although repeated after n steps, but up to a classes of obstructions. Our considerations are based on the concepts of generalized inverse,[12,27] and semisupermanifolds.[20]

The paper is organized as follows. In Sec. II we consider linear mappings without requirement of "invertibility." If $f:X\to Y$ is a linear mapping, then instead of the inverse mapping $f^{-1}:Y\to X$ we use less restricted "regular" $f^{\star}$ one by extending "invertibility" to "regularity" according to

$$f\circ f^{\star}\circ f=f,\quad f^{\star}\circ f\circ f^{\star}=f^{\star}.\tag{3}$$

We also propose some higher regularity conditions. In Sec. III the higher regularity notion is extended to morphisms of categories. Commutative diagrams are replaced by semicommutative ones. The concept of regular cocycles of morphisms in a category is described. An existence theorem for these cocycles is given. The corresponding generalization of certain categorical structures as tensor operation, algebras and coalgebras, etc., to our higher regularity case is given in Sec. IV. Regular equivalence classes of cobordism manifolds and the corresponding structures are considered in Sec. V. An n-regular TQFT is introduced as an n-regular obstructed category represented by some special classes of cobordisms called "interactions." Our study is not complete, it is only a proposal for new algebraic structures related to topological quantum theories.

II. GENERALIZED INVERTIBILITY AND REGULARITY

Let X and Y be two linear spaces over a field k. We use the following notation. Denote by Id_X and Id_Y the identity mappings $\mathrm{Id}_X:X\to X$ and $\mathrm{Id}_Y:Y\to Y$. If $f:X\to Y$ is a linear mapping, then the image of f is denoted by $\operatorname{Im} f$, and the kernel by $\operatorname{Ker} f$.

Here we are going to study some generalizations of the standard concept of invertibility properties of mappings. Our considerations are based on the article of Nashed.[12] Let $f:X\to Y$ be a linear mapping. If $f\circ f_r^{-1}=\mathrm{Id}_Y$ for some $f_r^{-1}:Y\to X$, then f is called a *retraction*, and f_r^{-1} is the right inverse. Similarly, if $f_l^{-1}\circ f=\mathrm{Id}_X$, then it is called a *coretraction*, f_l^{-1} is the left inverse of f. A mapping f^{-1} is called an inverse of f if and only if it is both right and left inverse of f.

This standard concept of invertibility is in many cases too strong to be fulfilled. To obtain more weak conditions one has to introduce the following "regularity" conditions

$$f \circ f^{\star}_{\mathrm{in}} \circ f = f, \tag{4}$$

where $f^{\star}_{\mathrm{in}}: Y \to X$ is called an *inner inverse*, and such f is called *regular*. Similar "reflexive regularity" conditions

$$f^{\star}_{\mathrm{out}} \circ f \circ f^{\star}_{\mathrm{out}} = f^{\star}_{\mathrm{out}} \tag{5}$$

define an *outer inverse* $f^{\star}_{\mathrm{out}}$. Notice that in general $f^{\star}_{\mathrm{in}} \neq f^{\star}_{\mathrm{out}} \neq f^{-1}$ or it can be that f^{-1} does not exist at all.

Definition 1: A mapping f satisfying one of the conditions (4) or (5) is said to be regular or three-regular. A generalized inverse of a mapping f is a mapping $f^{\star}$, which is both inner and outer inverse $f^{\star} = f^{\star}_{\mathrm{in}} = f^{\star}_{\mathrm{out}}$.

Lemma 2: If $f^{\star}_{\mathrm{in}}$ is an inner inverse of f, then a generalized inverse $f^{\star}$ exists, but need not be unique.

Proof: If $f^{\star}_{\mathrm{in}}$ is an inner inverse, then

$$f^{\star} = f^{\star}_{\mathrm{in}} \circ f \circ f^{\star}_{\mathrm{in}} \tag{6}$$

is always both inner and outer inverse i.e., generalized inverse. It follows from (6) that both regularity conditions (4) and (5) hold. □

Definition 3: Let us define two operators $\mathcal{P}_f : Y \to Y$ and $\mathcal{P}_{f^{\star}} : X \to X$ by

$$\mathcal{P}_f := f \circ f^{\star}, \quad \mathcal{P}_{f^{\star}} := f^{\star} \circ f. \tag{7}$$

Lemma 4: These operators satisfy

$$\begin{aligned} &\mathcal{P}_f \circ \mathcal{P}_f = \mathcal{P}_f, \quad \mathcal{P}_f \circ f = f \circ \mathcal{P}_{f^{\star}} = f \\ &\mathcal{P}_{f^{\star}} \circ \mathcal{P}_{f^{\star}} = \mathcal{P}_{f^{\star}}, \quad \mathcal{P}_{f^{\star}} \circ f^{\star} = f^{\star} \circ \mathcal{P}_f = f^{\star}. \end{aligned} \tag{8}$$

□

Lemma 5: If $f^{\star}$ is the generalized inverse of the mapping f, then the following properties are obvious:

$$\begin{aligned} &\mathrm{Im}\, f = \mathrm{Im}(f \circ f^{\star}), \quad \mathrm{Ker}(f \circ f^{\star}) = \mathrm{Ker}\, f^{\star}, \\ &\mathrm{Im}(f^{\star} \circ f) = \mathrm{Im}\, f^{\star}, \quad \mathrm{Ker}(f^{\star} \circ f) = \mathrm{Ker}\, f. \end{aligned} \tag{9}$$

In addition there are two decompositions

$$X = \mathrm{Im}\, f^{\star} \oplus \mathrm{Ker}\, f, \quad Y = \mathrm{Im}\, f \oplus \mathrm{Ker}\, f^{\star}. \tag{10}$$

The restriction $f|_{\mathrm{Im}\, f^{\star}} : \mathrm{Im}\, f^{\star} \to \mathrm{Im}\, f$ is one to one mapping, and operators P_f, $P_{f^{\star}}$ are projectors of Y, X onto $\mathrm{Im}\, f$, $\mathrm{Im}\, f^{\star}$, respectively.

Theorem 6: *Let $f: X \to Y$ be a linear mapping. If P and Q are projectors corresponding to the following two decompositions*

$$X = M \oplus \mathrm{Ker}\, f, \quad Y = \mathrm{Im}\, f \oplus N, \tag{11}$$

respectively, then there exist unique generalized inverse of f, and

$$f^{\star} := i \circ \tilde{f}^{-1} \circ Q, \tag{12}$$

where $\tilde{f} := f|_M$, and $i: M \hookrightarrow X$.

Here we try to construct higher analogs of generalized invertibility and regularity conditions (4) and (5). Let us consider two mappings $f:X\to Y$ and $f^{\star}:Y\to X$ and introduce two additional mappings $f^{\star\star}:X\to Y$ and $f^{\star\star\star}:Y\to X$. We propose here the following higher regularity condition:

$$f\circ f^{\star}\circ f^{\star\star}\circ f^{\star\star\star}\circ f=f. \tag{13}$$

This equation defines a four-regularity condition. By cyclic permutations we obtain

$$\begin{aligned} f^{\star}\circ f^{\star\star}\circ f^{\star\star\star}\circ f\circ f^{\star}&=f^{\star},\\ f^{\star\star}\circ f^{\star\star\star}\circ f\circ f^{\star}\circ f^{\star\star}&=f^{\star\star},\\ f^{\star\star\star}\circ f\circ f^{\star}\circ f^{\star\star}\circ f^{\star\star\star}&=f^{\star\star\star}. \end{aligned} \tag{14}$$

By recursive considerations we can propose the following formula of n-regularity:

$$\underbrace{f\circ f^{\star}\circ f^{\star\star}\ldots\circ f^{\overbrace{\star\star\ldots\star}^{n=2k+1}}\circ f}_{n+1}=f, \tag{15}$$

where $n=2k,\ k=1,2,\ldots$ and their cyclic permutations.

For the $\overbrace{\star\star\ldots\star}^{2k+1}$-operation we have the following formula:

$$(g\circ f)^{\overbrace{\star\star\ldots\star}^{2k+1}}=f^{\overbrace{\star\star\ldots\star}^{2k+1}}\circ g^{\overbrace{\star\star\ldots\star}^{2k+1}}. \tag{16}$$

We can introduce "higher projector" by

$$\mathcal{P}_f^{(n)}=f\circ f^{\star}\circ f^{\star\star}\cdots\circ f^{\overbrace{\star\star\ldots\star}^{2k+1}}. \tag{17}$$

It is easy to check the following properties:

$$\mathcal{P}_f^{(2k)}\circ f=f \tag{18}$$

and idempotence $\mathcal{P}_f^{(2k)}\circ\mathcal{P}_f^{(2k)}=\mathcal{P}_f^{(2k)}$.

In general case for a given $n=2k$ all $f^{\star}$, $f^{\star\star}$,$\ldots f^{\overbrace{\star\star\ldots\star}^{n}}$ are different, and, for instance, $(f^{\star})^{\star}\neq f^{\star\star}$. The existence of analogous conditions for n odd is a problem.

Theorem 7: *Let $f:X\to Y$ be a linear mapping. If P and Q are projectors corresponding to the following two decompositions*

$$X=M\oplus\operatorname{Ker}f,\quad `Y=\operatorname{Im}f\oplus N, \tag{19}$$

respectively, and

$$f^{\star}|\operatorname{Im}f=f^{\star\star\star}|_{\operatorname{Im}f}, \tag{20}$$

then the five-regularity condition of f can be reduced to the two three-regularity conditions

$$f\circ f^{\star}\circ f=f,\quad f^{\star}\circ f^{\star\star}\circ f^{\star}=f^{\star}. \tag{21}$$

III. SEMICOMMUTATIVE DIAGRAMS AND REGULAR OBSTRUCTED CATEGORIES

In Sec. II we considered mappings and regularity properties for two given spaces X and Y, because we studied various types of inverses. Now we will extend these considerations to any number of spaces and introduce semicommutative diagrams (first introduced in Ref. 20).

A *directed graph* $\mathfrak{C}$ is a pair $\{\mathfrak{C}_0,\mathfrak{C}_1\}$ and a pair of functions

$$\mathfrak{C}_0 \underset{t}{\overset{s}{\leftleftarrows}} \mathfrak{C}_1, \tag{22}$$

where elements of $\mathfrak{C}_0$ are said to be *objects*, elements of $\mathfrak{C}_1$ are said to be *arrows* or *morphisms*, *sf* is said to be a *domain (or source)* of *f*, and *tf* is a *codomain (or target)* of $f\in\mathfrak{C}_1$. If $sf=X\in\mathfrak{C}_0$, and $tf=Y\in\mathfrak{C}_0$, then we use the following notation $X\xrightarrow{f}Y$ and

$$\mathfrak{C}(X,Y):=\{f\in\mathfrak{C}_1:sf=X,tf=Y\}. \tag{23}$$

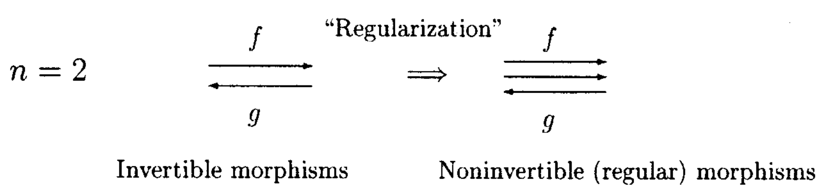
We denote by $\mathrm{End}(X)$ the collection of all morphisms defined on X into itself, i.e., $\mathrm{End}(X):=\mathfrak{C}(X,X),X\in\mathfrak{C}_0$.

Two arrows $f,g\in\mathfrak{C}_1$ such that $tf=sg$ are said to be composable. If in addition $sf=X$, $sg=tf=Y$, and $tg=Z$, then we use the notation $X\xrightarrow{f}Y\xrightarrow{g}Z$. In this case a composition $f\circ g$ of two arrows $f:X\to Y$ and $g:Y\to Z$ can be defined as an arrow $f:X\to Y$. The associativity means that $h\circ(g\circ f)=(h\circ g)\circ f=h\circ g\circ f$. An identity id in $\mathfrak{C}$ is an inclusion $X\in\mathfrak{C}_0\rightarrowtail\mathrm{id}_X\in\mathrm{End}(X)$ such that

$$f\circ\mathrm{id}_X=\mathrm{id}_Y\circ f=f \tag{24}$$

for every X, $Y\in\mathfrak{C}_1$, and $X\xrightarrow{f}Y$.

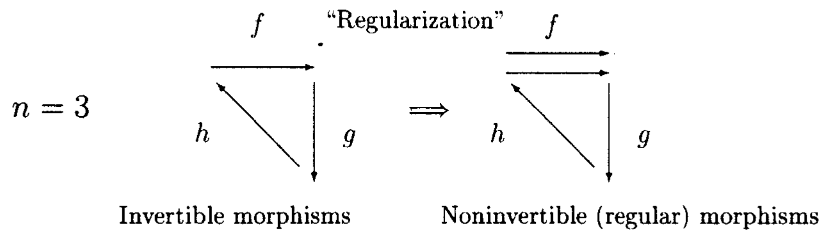
A directed graph $\mathfrak{C}$ equipped with associative composition of composable arrows and identity satisfying some natural axioms is said to be a *category*.[28,29] If $\mathfrak{C}$ is a category, then right cancellative morphisms are *epimorphisms* which satisfy $g_1\circ f=g_2\circ f\Rightarrow g_1=g_2$, where $g_{1,.2}:Y\to Z$ and left cancellative morphisms are *monomorphisms* which satisfy $f\circ h_1=f\circ h_2\Rightarrow h_1=h_2$, where $h_{1,.2}:Z\to X$. A morphism $X\xrightarrow{f}Y$ is invertible means that there is a morphism $Y\xrightarrow{g}X$ such that $f\circ g=\mathrm{id}_Y$ and $g\circ f=\mathrm{id}_X$. Instead of such invertibility we can use the regularity condition (4), i.e., $f\circ g\circ f=f$, where g plays the role of an inner inverse.[12]



Usually, for three objects X, Y, Z and three morphisms $f:X\to Y$ and $g:Y\to Z$ and $h:Z\to X$ one can have the "invertible" triangle commutative diagram $h\circ g\circ f=\mathrm{Id}_X$. Its regular extension has the form

$$f\circ h\circ g\circ f=f. \tag{25}$$

Such a diagram



can be called a *semicommutative diagram*. By cyclic permutations of (25) we obtain

$$\begin{aligned} h\circ g\circ f\circ h &= h, \\ g\circ f\circ h\circ g &= g. \end{aligned} \tag{26}$$

These formulas define the concept of three-regularity.

Definition 8: A mapping $f:X\to Y$ satisfying conditions (25) and (26) is said to be three-regular. The mapping $h:Z\to X$ is called the first three-inversion and the mapping $g:Y\to Z$ the second one.

The above-given concept can be expanded to any number of objects and morphisms.

Definition 9: Let $\mathfrak{C}=(\mathfrak{C}_0,\mathfrak{C}_1)$ be a directed graph. An *n-regular cocycle* (X,f) in $\mathfrak{C}$, $n=1,2,...$, is a sequence of composable arrows in $\mathfrak{C}$,

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_1, \tag{27}$$

such that

$$\begin{aligned} f_1\circ f_n\circ\cdots\circ f_2\circ f_1 &= f_1, \\ f_2\circ f_1\circ\cdots\circ f_3\circ f_2 &= f_2, \\ f_n\circ f_{n-1}\circ\cdots\circ f_1\circ f_n &= f_n, \end{aligned} \tag{28}$$

and

$$\begin{aligned} e_{X_1}^{(n)} &:= f_n\circ\cdots\circ f_2\circ f_1 \in \mathrm{End}(X_1), \\ e_{X_2}^{(n)} &:= f_1\circ\cdots\circ f_3\circ f_2 \in \mathrm{End}(X_2), \\ e_{X_n}^{(n)} &:= f_{n-1}\circ\cdots\circ f_1\circ f_n \in \mathrm{End}(X_n). \end{aligned} \tag{29}$$

Definition 10: Let (X,f) be an n-regular cocycle in $\mathfrak{C}$, then the correspondence $e_X^{(n)}:X_i\in\mathfrak{C}_0\mapsto e_{X_i}^{(n)}\in\mathrm{End}(X_i)$, $i=1,2,...,n$, is called an n-regular cocycle obstruction structure on (X,f) in $\mathfrak{C}$.

Lemma 11: We have the following relations

$$f_i\circ e_{X_i}^{(n)} = f_i, \quad e_{X_{i+1}}^{(i)}\circ f_i = f_i, \quad e_{X_i}^{(n)}\circ e_{X_i}^{(n)} = e_{X_i}^{(n)} \tag{30}$$

for $i=1,2,...,n(\mathrm{mod}\, n)$.

Proof: The lemma simply follows from relations (28) and (29). □

Definition 12: An n-regular obstructed category is a directed graph $\mathfrak{C}$ with an associative composition and such that every object is a component of an n-regular cocycle.

Example 1: If all obstructions are equal to the identity $e_{X_i}^{(n)}=\mathrm{id}_{X_i}$, and

$$f_n \circ \cdots \circ f_2 \circ f_1 = \mathrm{id}_{X_1},$$
$$f_1 \circ \cdots \circ f_3 \circ f_2 = \mathrm{id}_{X_2}, \tag{31}$$
$$f_{n-1} \circ \cdots \circ f_1 \circ f_n = \mathrm{id}_{X_n},$$

then the sequence (27) is trivially n-regular. Observe that the trivial two-regularity is just the usual invertibility, hence every grupoid G is a trivially two-regular obstructed category. We are interested with obstructed categories equipped with some obstruction different from the identity.

Definition 13: The minimum number $n = n_{\mathrm{obstr}}$ such that $e_X^{(n)} \neq \mathrm{id}_X$ is called the obstruction degree.

Example 2: Every inverse semigroup S is a nontrivial two-regular obstructed category. It has only one object, morphisms are the elements of S.

Theorem 14: *Let $\mathfrak{C}$ be a category, and*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{m-1}} X_n \xrightarrow{f_n} X_1 \tag{32}$$

be a sequence of morphisms of category $\mathfrak{C}$. Assume that there is a sequence

$$Y_1 \xrightarrow{\tilde{f}_1} Y_2 \xrightarrow{\tilde{f}_2} \cdots \xrightarrow{\tilde{f}_{n-1}} Y_n \xrightarrow{\tilde{f}_n} Y_1, \tag{33}$$

where Y_i is a subobject of X_i such that there is a collection of mappings $\pi_i : X_i \to Y_i$ and $\iota : Y_i \to X_i$ satisfying the condition $\pi_i \circ \iota_i = \mathrm{id}_{Y_i}$ for $i = 1,2,...,n$. If in addition

$$\tilde{f}_n \circ \cdots \tilde{f}_2 \circ \tilde{f}_1 = \mathrm{id}_{Y_1},$$
$$\tilde{f}_1 \circ \cdots \tilde{f}_3 \circ \tilde{f}_2 = \mathrm{id}_{Y_2},$$
$$\cdots \tag{34}$$
$$\tilde{f}_{n-1} \circ \cdots \tilde{f}_1 \circ \tilde{f}_n = \mathrm{id}_{Y_n},$$

and

$$f_i := t_{i+1} \circ \tilde{f}_i \circ \pi_i \tag{35}$$

then the sequence (62) *is an n-regular cocycle.*

Proof: The corresponding obstruction structure is given by

$$e_{X_i}^{(n)} = \iota_i \circ \pi_i\,. \tag{36}$$

If $x \in \mathrm{Ker}\, f_1$, then the theorem is trivial, if $x \in X_i \backslash \mathrm{Ker}\, f_1$, then we obtain

$$(f_1 \circ f_n \circ \cdots \circ f_2 \circ f_1)(x) = \iota_2 \circ \tilde{f}_1 \circ \pi_1 \circ \iota_1 \circ \tilde{f}_n \circ \cdots \circ \tilde{f}_2 \circ \tilde{f}_1 \circ \pi_1(x) = \iota_2 \circ \tilde{f}_1 \circ \pi_1 = f_1(x),$$

where conditions (34) and (35) have been used. We can calculate all cyclic permutations in a similar way. □

Example 3: There is an n-regular obstructed category $\mathfrak{C} = (\mathfrak{C}_0, \mathfrak{C}_1)$, where $\mathfrak{C}_0 = \{X_i : i = 1,...,n(\mathrm{mod}\, n+1)\}$ and $\mathfrak{C}_1 = \{f_i : i = 1,...,n(\mathrm{mod}\, n+1)\}$ are described in the above-mentioned theorem.

Definition 15: Let (X,f), (Y,g) be two n-regular cocycles in $\mathfrak{C}$. An n-regular cocycle morphism α: $(X,f) \to (Y,g)$ is a sequence of morphisms $\alpha := (\alpha_1, \ldots, \alpha_n)$ such that the diagram

$$\begin{array}{ccccccccc} X_1 & \stackrel{f_1}{\rightarrow} & X_2 & \stackrel{f_2}{\rightarrow}\cdots\stackrel{f_{n-1}}{\rightarrow} & X_n & \stackrel{f_n}{\rightarrow} & X_1 \\ \downarrow\alpha_1 & & \downarrow\alpha_2 & & \downarrow\alpha_n & & \downarrow\alpha_1 \\ Y_1 & \stackrel{g_1}{\rightarrow} & Y_2 & \stackrel{g_2}{\rightarrow}\cdots\stackrel{g_{n-1}}{\rightarrow} & Y_n & \stackrel{g_n}{\rightarrow} & Y_1 \end{array} \tag{37}$$

is commutative. If every component α_i of α is invertible, then α is said to be an n-regular cocycle equivalence.

It is obvious that the n-regular cocycle equivalence is an equivalence relation.

Definition 16: Let $\mathfrak{C}$ be an n-regular obstructed category. A collection of all equivalence classes of n-regular cocycles in $\mathfrak{C}$ and corresponding n-regular cocycle morphisms is denoted by $\mathrm{Reg}^{(n)}(\mathfrak{C})$ and is said to be an n-regularization of $\mathfrak{C}$.

Comment 17: It is obvious that the n-regular cocycle equivalence is an equivalence relation. Equivalence classes of this relation are just elements of $\mathrm{Reg}^{(n)}(\mathfrak{C})$. Our n-regular cocycles and obstruction structures are unique up to an invertible n-regular cocycle morphism. If $[(X,f)]$ is an equivalence class of n-regular cocycles, then there is the corresponding class of n-regular obstruction structures $e_X^{(n)}$ on it. The correspondence is a one to one.

IV. REGULARIZATION OF FUNCTORS AND RELATED STRUCTURES

We are going to introduce the concepts of n-regular functors, natural transformations, involution, duality, and so on. All of our definitions are in the general case the same as in the usual category theory,[29] but the preservation of the identity id_X is replaced by the requirement of preservation of obstructions $e_X^{(n)}$ up to the n-regular cocycle equivalence.

It is known that for two usual categories $\mathfrak{C}$ and $\mathfrak{D}$ a functor $\mathcal{F}:\mathfrak{C}\rightarrow\mathfrak{D}$ is defined as a pair of mappings $(\mathcal{F}_0,\mathcal{F}_1)$, where $\mathcal{F}_0$ sends objects of $\mathfrak{C}$ into objects of $\mathfrak{D}$, and $\mathcal{F}_1$ sends morphisms of $\mathfrak{C}$ into morphisms of $\mathfrak{D}$

$$\mathcal{F}_1(f\circ g)=\mathcal{F}_1(f)\circ\mathcal{F}_1(g),\quad \mathcal{F}_1\mathrm{id}_X=\mathrm{id}_{F_0X}, \tag{38}$$

for $X\in\mathfrak{C}_0,\ \mathcal{F}X\in\mathcal{D}_0$.

Let $\mathfrak{C}$ and $\mathfrak{D}$ be two n-regular obstructed categories. We postulate that all definitions are formulated on every n-regular cocycle (X,f) in $\mathfrak{C}$ up to the n-regular cocycle equivalence, and $i=1,2,\ldots$ (mod n).

Definition 18: An or n-regular cocycle functor $\mathcal{F}^{(n)}:\mathfrak{C}\rightarrow\mathfrak{D}$ is a pair of mappings $(\mathcal{F}_0^{(n)},\mathcal{F}_1^{(n)})$, where $\mathcal{F}_0^{(n)}$ sends objects of $\mathfrak{C}$ into objects of $\mathfrak{D}$, and $\mathcal{F}_1^{(n)}$ sends morphisms of $\mathfrak{C}$ into morphisms of $\mathfrak{D}$ such that

$$\mathcal{F}_1^{(n)}(f_i\circ f_{i+1})=\mathcal{F}_1^{(n)}(f_i)\circ\mathcal{F}_1^{(n)}(f_{i+1}),\quad \mathcal{F}_1^{(n)}(e_{X_i}^{(n)})=e_{\mathcal{F}_0(X_i)}^{(n)}, \tag{39}$$

where $X\in\mathfrak{C}_0$.

Lemma 19: Let $\mathfrak{C}$ and $\mathfrak{D}$ be n-regular obstructed categories, and let

$$X_1\stackrel{f_1}{\rightarrow}X_2\stackrel{f_2}{\rightarrow}\cdots\stackrel{f_{n-1}}{\rightarrow}X_n\stackrel{f_n}{\rightarrow}X_1 \tag{40}$$

be an n-regular cocycle in $\mathfrak{C}$. If $\mathcal{F}^{(n)}:\mathfrak{C}\rightarrow\mathfrak{D}$ is n-regular cocycle functor, then

$$\mathcal{F}^{(n)}(f_i)\circ e_{X_i}^{(n)}=\mathcal{F}^{(n)}(f_i). \tag{41}$$

Proof: It is a simple calculation

$$\mathcal{F}^{(n)}(f_i)=\mathcal{F}^{(n)}(f\circ e_{X_i}^{(n)})=\mathcal{F}^{(n)}(f)\circ\mathcal{F}^{(n)}(e_{X_i}^{(n)})=\mathcal{F}(f_i)\circ e_{\mathcal{F}_0X_i}^{(n)}. \tag{42}$$

$\square$

Multifunctors can be regularized in a similar way.

Let $\mathcal{F}^{(n)}$ and $\mathcal{G}^{(n)}$ be two n-regular cocycle morphisms of the category $\mathfrak{C}$ into the category $\mathfrak{D}$.

Definition 20: An n-regular natural transformation $s:\mathcal{F}^{(n)}\rightarrow\mathcal{G}^{(n)}$ of $\mathcal{F}^{(n)}$ into $\mathcal{G}^{(n)}$ is a collection of functors $s=\{s_{X_i}:\mathcal{F}_0(X_i)\rightarrow\mathcal{G}_0(X_i)\}$ such that

$$s_{X_{i+1}}\circ\mathcal{F}_1^{(n)}(f_i)=\mathcal{G}_1^{(n)}(f_i)\circ s_{X_i}, \tag{43}$$

for $f_i:X_i\rightarrow X_{i+1}$.

Definition 21: An n-regular obstructed monoidal category $\mathfrak{C}\equiv\mathfrak{C}(\otimes,I)$ can be defined as usual, but we must remember that instead of the identity $\mathrm{id}_X\otimes\mathrm{id}_Y=\mathrm{id}_{X\otimes Y}$ we have an obstruction structure $e_X^{(n)}=\{e_{X_i}^{(n)}\in\mathrm{End}(X_i);n=1,2,...\}$ satisfying the condition

$$e_{X_i\otimes Y_i}^{(n)}=e_{X_i}^{(n)}\otimes e_{Y_i}^{(n)} \tag{44}$$

for every two n-regular cocycles (X,f) and (Y,f').

Let $\mathfrak{C}$ be an n-regular obstructed monoidal category. We introduce an *-operation in $\mathfrak{C}$ as a function which sends every object X_i into object X_i^* called the dual of X,

$$X_i^{**}=X_i\,,\quad (X_i\otimes Y_i)^*=X_i^*\otimes Y_i^*\,, \tag{45}$$

reverse all arrows

$$(f\circ g)^*=g^*\circ f^*. \tag{46}$$

The category $\mathfrak{C}$ equipped with such $*$-operation is called an *n-regular obstructed monoidal category with duals*.

Lemma 22: Let $\mathfrak{C}$ be an n-regular obstructed monoidal category with duals. If (X,f) is an n-regular cocycle in $\mathfrak{C}$, then there is a corresponding n-regular cocycle (X^*,f^*) in $\mathfrak{C}^*$, called the dual of (X,f).

Proof: If we reverse all arrows in (X,f) and replace all objects by the corresponding duals, then we obtain (X^*,f^*), where

$$X_1^*\xrightarrow{f_n^*}X_n^*\xrightarrow{f_{n-1}^*}\cdots\xrightarrow{f_2^*}X_2^*\xrightarrow{f_1^*}X_1^* \tag{47}$$

is a sequence such that

$$f_1^*\circ f_n^*\circ\cdots\circ f_2^*\circ f_1^*=f_1^*\,,\quad e_{X_1^*}^{(n)}:=f_n^*\circ\cdots\circ f_2^*\circ f_1^*\,, \tag{48}$$

where $f_i^*:X_{i+1}^*\rightarrow X_i^*$, $i=1,...,n$, and $X_{n+1}^*\equiv X_1^*$ is the dual. We have corresponding relations for all cyclic permutations. □

Definition 23: An n-regular pairing $g_{\mathfrak{C}}$ in an n-regular obstructed monoidal category $\mathfrak{C}$ can be defined in an analogy to the usual case as a collection of mappings

$$g_{\mathfrak{C}}=\{g_{X_i}\equiv\langle-|-\rangle_{X_i}:X_i^*\otimes X_i\rightarrow I\} \tag{49}$$

satisfying some natural consistency conditions and in addition the following regularity relations:

$$g_{X_{i+1}}\circ(f_i^*\otimes f_i)=g_{X_i}, \tag{50}$$

and

$$\langle e_{X_i^*}^{(n)}X_i^*|X_i\rangle_{X_i}=\langle X_i^*|e_{X_i}^{(n)}X_i\rangle_{X_i}, \tag{51}$$

where (X,f) is a regular n-cocycle in $\mathfrak{C}$, and let (X^*,f^*) be the corresponding duals.

It is known that an associative algebra in an ordinary category is an object $\mathcal{A}$ of this category such that there is a multiplication $m:\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ which is also a morphism of this category satisfying some axioms like the associativity, the existence of the unity.

Definition 24: Let $\mathfrak{C}$ be an n-regular obstructed monoidal category. An n-regular cocycle algebra $\mathcal{A}$ in the category $\mathfrak{C}$ is an object of this category equipped with an associative multiplication $m:\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ such that

$$m\circ(e_A^{(n)}\otimes e_A^{(n)})=e_A^{(n)}\circ m. \tag{52}$$

Obviously such multiplication does not need to be unique.

One can define an n-regular cocycle coalgebra or bialgebra in a similar way. A comultiplication $\Delta:\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ can be regularized according to the relation

$$\Delta\circ e_{\mathcal{A}}^{(n)}=(e_{\mathcal{A}}^{(n)}\otimes e_{\mathcal{A}}^{(n)})\circ\Delta. \tag{53}$$

Definition 25: Let $\mathcal{A}$ be an n-regular cocycle algebra. If $\mathcal{A}$ is also regular coalgebra such that $\Delta(ab)=\Delta(a)\Delta(b)$, then it is said to be an n-regular cocycle almost bialgebra.

If $\mathcal{A}$ is an n-regular cocycle algebra, then we denote by $hom_m(\mathcal{A},\mathcal{A})$ the set of morphisms $s\in hom_C(\mathcal{A},\mathcal{A})$ satisfying the condition

$$s\circ m=m\circ(s\otimes s). \tag{54}$$

Let $\mathcal{A}$ be an n-regular cocycle almost bialgebra. We define the convolution product

$$s\star t:=m\circ(s\otimes t)\circ\Delta, \tag{55}$$

where $s,t\in hom_m(\mathcal{A},\mathcal{A})$. If $\mathcal{A}$ is a regular n-cocycle almost bialgebra, then the convolution product is regular.

Definition 26: A two-regular cocycle almost bialgebra $\mathcal{H}$ equipped with an element $S\in hom_m(\mathcal{H},\mathcal{H})$ such that

$$S\star\mathrm{id}_{\mathcal{H}}\star S=S,\quad \mathrm{id}_{\mathcal{H}}\star\mathrm{id}_{\mathcal{H}}=\mathrm{id}_{\mathcal{H}} \tag{56}$$

is said to be a two-regular cocycle almost Hopf algebra $\mathcal{H}$.

The above-given definition is a regular analogy of week Hopf algebras considered in Ref. 26. Similar algebras has been also considered in Ref. 30 and 31.

Lemma 27: If $\mathcal{A}$ is an n-regular cocycle algebra, then there is an n-regular cocycle coalgebra $\mathcal{A}^*$ such that[31]

$$\langle\Delta(\xi),x_1\otimes x_2\rangle=\langle\xi,m(x_1\otimes x_2)\rangle, \tag{57}$$

where $x_1,\ x_2\in\mathcal{A},\ \xi\in\mathcal{A}^*$.

Proof: Let us apply the regularity condition (52) to the above-given duality condition (57). Then the lemma follows from relations (44), (53), and (51). □

Lemma 28: Let $\mathcal{A}$ be an n-regular cocycle almost bialgebra. Then the dual $\mathcal{A}^*$ is also n-regular cocycle almost bialgebra

$$\begin{gathered}\langle\Delta(\xi),x_1\otimes x_2\rangle=\langle\xi,m(x_1\otimes x_2)\rangle,\\ \langle\hat{m}(\xi\otimes\zeta),x_1\otimes x_2\rangle=\langle\xi\otimes\zeta,\hat{\Delta}x\rangle.\end{gathered} \tag{58}$$

Let $\mathcal{A}$ be an n-regular cocycle algebra. Then we can define a left n-regular cocycle $\mathcal{A}$-module as an object equipped with an $\mathcal{A}$-module action $\rho_M:\mathcal{A}\otimes M\to M$ such that

$$\rho_M\circ(m\otimes \mathrm{id}_M)=\rho_M\circ(\mathrm{id}_{\mathcal{A}}\otimes\rho_M),$$
$$\rho_M\circ(e_{\mathcal{A}}^{(n)}\otimes e_M^{(n)})=e_M^{(n)}\circ\rho_M. \tag{59}$$

If $\mathcal{A}$ is an n-regular cocycle coalgebra, then one can define an n-regular cocycle comodule M in a similar way. For a coaction $\delta_M:\mathcal{A}\to\mathcal{A}\otimes M$ of $\mathcal{A}$ on M we have the following regularity condition:

$$\delta_M\circ(e_{\mathcal{A}}^{(n)}\otimes e_M^{(n)})=e_M^{(n)}\circ\varrho_M. \tag{60}$$

Remark 1: Observe that we have the following duality between $\mathcal{A}$-module action $\rho_M:\mathcal{A}\otimes M\to M$ and $\mathcal{A}^*$-comodule coactions $\delta_{M*}:\mathcal{A}^*\to M^*\otimes\mathcal{A}^*$,

$$\langle\,\delta_{M*}(\xi),a\otimes x\rangle=\langle\,\xi\varrho_M(a\otimes x)\rangle, \tag{61}$$

where $a\in\mathcal{A},\ x\in M,\ \xi\in\mathcal{A}^*$.

V. REGULAR COBORDISMS AND TQFT

Let Cob be a directed graph of cobordisms whose objects Cob_0 are d-dimensional compact smooth and oriented manifolds without boundary and whose arrows are classes of cobordism manifolds with boundaries. We would like to discuss the corresponding n-regular cocycles and their meaning. For this goal we use here a parametrization such that the boundary ∂M is a multiconnected space, a disjoint sum of the "incoming" boundary manifold Σ_{in} and the "outgoing" one Σ_{out}. We call them "physical." The empty boundary component is also admissible. Let $\Sigma_0,\ \Sigma_1\in\mathrm{Cob}_0$, then the disjoint sum is denoted by $\Sigma_0\amalg\Sigma_1$. For a manifold $\Sigma\in\mathrm{Cob}_0$ there is the corresponding manifold Σ^* with the opposite orientation.

We wish to represent quantum processes of certain physical system by cobordism manifolds M with the incoming boundary manifold Σ_0 (an "input"), and the outgoing one Σ_0 (an "output"). The incoming boundary manifold Σ_0 represents an initial condition of the system, the outgoing boundary represents the final configuration, and the cobordism manifolds represent possible interaction of the system. Note that the same cobordism manifold M but with different boundary parametrization represent different physical processes!

Definition 29: An "interaction" is a triple ${}_{\Sigma_0}\mathcal{M}_{\Sigma_1}$, where the incoming boundary manifold Σ_0 is multiconnected space with m components and the outgoing one Σ_1 is equipped with n components, and $\mathcal{M}$ is a class of cobordism manifolds up to parametrization preserving diffeomorfisms, $\Sigma_0,\ \Sigma_1\in\mathrm{Cob}_0,\ \mathcal{M}\in\mathrm{Cob}_1$.

Definition 30: The "opposite interaction" of ${}_{\Sigma_0}\mathcal{M}_{\Sigma_1}$ is the "interaction" ${}_{\Sigma_1}\mathcal{M}^{\mathrm{op}}_{\Sigma_0}$ with reversed boundary parametrization, i.e., the incoming boundary of $\mathcal{M}$ is the outgoing boundary of $\mathcal{M}^{\mathrm{op}}$ and vice versa.

Example 4: A "collapsion" of $\Sigma\in\mathrm{Cob}_0$ is an arbitrary "interaction" of the forms ${}_{\Sigma}\mathcal{M}_{\emptyset}$, this means the incoming boundary is Σ and the outgoing boundary is empty. The corresponding "expansion" of Σ is the opposite of the collapsion.

Definition 31: Let us denote by $\mathfrak{C}ob=(\mathfrak{C}ob_0,\mathfrak{C}ob_1)$ a directed graph whose objects are $\mathfrak{C}ob_0\equiv\mathrm{Cob}_0$ and arrows $\mathfrak{C}ob_1$ are "interactions." A composition of two interactions ${}_{\Sigma_1}\mathcal{M}_{1\Sigma_2}$ and ${}_{\Sigma_2}\mathcal{M}_{2\Sigma_3}$ is an interaction ${}_{\Sigma_1}(\mathcal{M}_{1\Sigma_2}\mathcal{M}_2)_{\Sigma_3}$, where $\mathcal{M}_{1\Sigma_2}\mathcal{M}_2$ is a result of gluing $\mathcal{M}_1$ and $\mathcal{M}_2$ along Σ_2.

The trivial gluing along the empty boundary component is also admissible. For instance we can glue a "collapsion" of Σ and the corresponding "expansion" in the trivial way. In this way we obtain an interaction ${}_{\Sigma}(\mathcal{M}\mathcal{M}^{\mathrm{op}})_{\Sigma}$. If we glue the expansion of Σ and the collapsion of Σ along Σ, then we obtain a class of manifolds with empty boundaries.

Example 5: Classes of two-dimensional surfaces with holes provide examples of string interactions.

We wish to build the temporal support semigroup as an arbitrary sequence

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \tag{62}$$

of objects and arrows of a directed graph $\mathfrak{C}$ indexed by a discrete time. We wish to represent an interaction ${}_{\Sigma_1}\mathcal{M}_{\Sigma_2}$ as an arrow $X_1 \xrightarrow{f} X_2$ of $\mathfrak{C}$. Obviously composable arrows $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3$ should represent the gluing ${}_{\Sigma_1}(\mathcal{M}_{1\Sigma_2}\mathcal{M}_2)_{\Sigma_3}$. Two interactions ${}_{\Sigma_1}\mathcal{M}_{\Sigma_2}$ and ${}_{\Sigma_1'}\mathcal{M}'{}_{\Sigma_2'}$ should be represented by the same arrow $X_1 \xrightarrow{f} X_2$ if and only if both interactions are "parallel (simultaneous) in the time."

Let us assume that the directed graph $\mathfrak{C}$ is an n-regular monoidal category with duals. Let $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n$ be an n-regular cocycle. If there is an equivalence $\cong$ in $\mathfrak{C}$ob such that objects of the n-regular cocycle represent equivalence classes of $\cong$ and arrows represent time consequences, then we say that we have an n-regular TQFT.

What does n-regularity mean here? It is natural to assume that the opposite ${}_{\Sigma_2}\mathcal{M}^{\mathrm{op}}_{\Sigma_1}$ of ${}_{\Sigma_1}\mathcal{M}_{\Sigma_2}$ should be represented by a reversed arrow $X_1 \xleftarrow{f} X_2$. The trivial two-regularity is clear, it means that the time is invertible. We postulate that the time is directed and always runs further, never back, never stops. In other words, "our time" is not invertible in general, but it can be n-regular, where the regularity is nontrivial.

Example 6: Let us consider for instance the two-regular "interactions." Let

$${}_{\Sigma_1}\mathcal{M}_{1\Sigma_2} \quad \text{and} \quad {}_{\Sigma_2}\mathcal{M}_{2\Sigma_1}$$

be two interactions, then ${}_{\Sigma_1}(\mathcal{M}_{1\Sigma_2}\mathcal{M}_2)_{\Sigma_1}$ and ${}_{\Sigma_2}(\mathcal{M}_{2\Sigma_1}\mathcal{M}_1)_{\Sigma_2}$ can be represented as arrows $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_1$, and $X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_2$, respectively. Interactions ${}_{\Sigma_1}\mathcal{M}_{1\Sigma_2}\mathcal{M}_{2\Sigma_1}\mathcal{M}_{1\Sigma_2}$ and ${}_{\Sigma_2}\mathcal{M}_{2\Sigma_1}\mathcal{M}_{1\Sigma_2}\mathcal{M}_{2\Sigma_1}$ should be represented by $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_2$, and $X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_1$, respectively. Now the two-regularity conditions are clear.

Observe that the regularity concept can be useful for the construction of quantum theory of the whole universe with nonivertible time evolution. In fact the nontrivial n-regularity conditions mean that all processes always go further, never back, never stop, but are cyclically repeating after n-steps up to an equivalence.

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