Lagrangian and non-Lagrangian approaches to electrodynamics including supersymmetry

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Abstract. We take a general approach to nonlinear electrodynamics that includes non-Lagrangian as well as Lagrangian theories. We introduce the *constitutive tensors* which, together with Maxwell's equations, describe nonlinear electrodynamics in an extremely general way. We show how this approach specializes to particular cases that were previously considered, and indicate how it generalizes to supersymmetric electrodynamics.

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Motivated by the desire to explore the existence of a $c \to \infty$ limit of Maxwell's equations in nonlinear electrodynamics, two of us earlier considered a formulation based on generalized constitutive equations consistent with the Lorentz symmetry [1, 2]. We wrote the condition that characterizes the subclass of theories in which the equations of motion derive from a Lagrangian, and considered some interesting variations on Born-Infeld and nonabelian Born-Infeld Lagrangians that actually have Galilean limits. Subsequently, Goldin, Mavromatos and Szabo proposed an application to the tensionless limit of relativistic strings [3], reinforcing the importance of generalizing the approach to encompass supersymmetric electrodynamics.

Here we propose a much more general form for the constitutive equations, describing Lagrangian and non-Lagrangian theories of nonlinear electrodynamics that include (for example) electromagnetic fields in anisotropic media, and piroelectric or ferromagnetic materials. We also indicate how to incorporate supersymmetric electrodynamics [4].

Following Refs. [1] and [2], we write Maxwell's equations for classical electromagnetic fields [5] so as to preserve their explicit dependence on the velocity of light c and allow a nontrivial Galilean-covariant limit as $c \to \infty$. We work in Minkowski space-time, with the flat metric tensor $\eta_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$, $x^\mu = \left(ct, x^i\right)$, $\mu,\nu,\dots = 0,1,2,3$, $i,j,\dots = 1,2,3$, with $\partial_\mu = \partial \big/ \partial x^\mu = \left[c^{-1}\partial \big/ \partial t, \nabla\right]$, and the antisymmetric Levi-Civita tensor $\varepsilon^{\mu\nu\rho\sigma}$, $\varepsilon^{0123} = 1$. The antisymmetric field strength tensors $F_{\mu\nu}$ and $G_{\mu\nu}$ are constructed as usual from the fields **E**, **B**, **D**, **H**: $F_{0i} = c^{-1}E_i$, $F_{ij} = \varepsilon_{ijk}B_k$, $G_{0i} = cD_i$, $G_{ij} = \varepsilon_{ijk}H_k$, where ε_{ijk} denotes the 3-dimensional antisymmetric tensor with $\varepsilon_{123} = 1$. Sometimes the forms corresponding to $F_{\mu\nu}$ and $G_{\mu\nu}$ are called the Faraday 2-form and Ampère 2-form respectively [6]. In these units, Maxwell's equations in $3+1$ dimensions

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contain no c:

$$\text{curl}\,\mathbf{E} = -\partial\mathbf{B}/\partial t, \quad \text{div}\,\mathbf{B} = 0, \quad \text{curl}\,\mathbf{H} = \partial\mathbf{D}/\partial t + \mathbf{j}, \quad \text{div}\,\mathbf{D} = \rho. \tag{1}$$

The covariant version is

$$\partial_\mu \tilde{F}^{\mu\nu} = 0, \quad \partial_\mu G^{\mu\nu} = j^\nu, \tag{2}$$

where $\tilde{F}^{\mu\nu} = (1/2)\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the Hodge dual field strength, and $j^\mu = (c\rho, \mathbf{j})$ is the 4-current. A solution to the first equation in (2) may be written $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where A_μ is the Abelian gauge field, while for $G_{\mu\nu}$ such a presentation is not in general possible. Note that our strategy is to keep $F_{\mu\nu}$ and $G_{\mu\nu}$ distinct as long as possible.

But the system (1) is underdetermined, and one now needs *constitutive* equations to relate the four fields $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$, $\mathbf{H}$. In the simplest (vacuum) case, these are $\mathbf{D} = \varepsilon_0\mathbf{E}$, $\mathbf{B} = \mu_0\mathbf{H}$, where ε_0 and μ_0 are the permittivity and permeability respectively. Nonlinear Lorentz-covariant constitutive equations, considered generally in [7], take the form

$$\mathbf{D} = M\mathbf{B} + c^{-2}N\mathbf{E}, \qquad \mathbf{H} = N\mathbf{B} - M\mathbf{E}, \tag{3}$$

or equivalently [2]

$$G_{\mu\nu} = NF_{\mu\nu} + cM\tilde{F}_{\mu\nu}, \tag{4}$$

where M and N are *scalar* functions of the two Lorentz invariants X and Y,

$$X = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\left(\mathbf{B}^2 - c^{-2}\mathbf{E}^2\right), \quad Y = \frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} = -c^{-1}\mathbf{B}\cdot\mathbf{E}. \tag{5}$$

In the classical vacuum case, we have $M = 0$ and $N = \mu_0^{-1}$, with $\varepsilon_0\mu_0 = c^{-2}$. In one version of Born-Infeld electrodynamics (see e.g. [8]), $M = -Y/\sqrt{1+2X-Y^2}$ and $N = 1/\sqrt{1+2X-Y^2}$.

It is well known [9] that a static gravitational field acts as a gyrotropic medium, with some permeability ε_{grav} and permittivity μ_{grav}. Here we shall consider only the flat space-time situation.

Although the relations (3) and (4) are fairly general, they do not take into account a number of physically important situations; for example, anisotropic media [10], chiral materials where derivatives are important [11], and other possibilities. Therefore we propose to generalize (3)–(4) by introducing three (or more) *constitutive tensors* $S_{\mu\nu}$, $R_{\mu\nu}^{\rho\sigma}$, $Q_{\mu\nu}^{\rho\sigma\lambda}$ as follows:

$$G_{\mu\nu} = S_{\mu\nu} + R_{\mu\nu}^{\rho\sigma}F_{\rho\sigma} + Q_{\mu\nu}^{\rho\sigma\lambda}\frac{\partial F_{\rho\sigma}}{\partial x^\lambda}. \tag{6}$$

The formula (6), together with Maxwell's equations (2), is intended to describe general, classical electrodynamics in possibly nonhomogeneous and/or nonisotropic media, as well as possibly nonhomogeneous and/or nonisotropic spacetime, while incorporating earlier examples. In general, $S_{\mu\nu} = S_{\mu\nu}(x)$, $R_{\mu\nu}^{\rho\sigma} = R_{\mu\nu}^{\rho\sigma}(x,F)$, and $Q_{\mu\nu}^{\rho\sigma\lambda} = Q_{\mu\nu}^{\rho\sigma\lambda}(x,F,\partial F)$. Additional, higher constitutive tensors following this pattern may be included to incorporate higher-derivative terms of finite order.

In the case of Lorentz-invariant constitutive equations, the constitutive tensors will depend on F only through the invariants X and Y, so that $S_{\mu\nu}$ is constant, while $R_{\mu\nu}^{\rho\sigma} = R_{\mu\nu}^{\rho\sigma}(X,Y)$. Evidently $S_{\mu\nu}$ is antisymmetric, $R_{\mu\nu}^{\rho\sigma}$ is antisymmetric in its upper and lower indices separately, and $Q_{\mu\nu}^{\rho\sigma\lambda}$ is antisymmetric in its lower and first two upper indices. Various further symmetries and properties of the constitutive tensors provide the possibility of studying these theories in unifying way.

Let us consider some examples. In the vacuum case $S_{\mu\nu} = 0$, $R_{\mu\nu}^{\rho\sigma} = \mu_0^{-1}\delta_{[\mu}^{\rho}\delta_{\nu]}^{\sigma}$, and $Q_{\mu\nu}^{\rho\sigma\lambda} = 0$, where by square brackets we denote antisymmetrization with a factor of $1/2$; i.e., $x_{[\mu\nu]} \equiv \left(x_{\mu\nu} - x_{\nu\mu}\right)/2$. Then the only nonvanishing constitutive tensor $R_{\mu\nu}^{\rho\sigma}$ is 'diagonal'. For the Born-Infeld theory mentioned above, we have $R_{\mu\nu}^{\rho\sigma} = \left(\delta_{[\mu}^{\rho}\delta_{\nu]}^{\sigma} - cY\varepsilon_{[\mu\nu]\lambda\delta}\eta^{\lambda\rho}\eta^{\delta\sigma}\right)\big/\sqrt{1+2X-Y^2}$, while $S_{\mu\nu} = 0$ and $Q_{\mu\nu}^{\rho\sigma\lambda} = 0$.

The case $S_{\mu\nu} \neq 0$ corresponds to piroelectric and ferromagnetic materials.

In an anisotropic medium with tensorial permeability ε_{ij} and permittivity μ_{ij}, the constitutive equations relating **D** to **E** and **H** to **B** become $D_i = \varepsilon_{ij}E_j$ and $B_i = \mu_{ij}H_j$. This situation is not described by (3) or (4), but the more general constitutive tensor $R_{\mu\nu}^{\rho\sigma}$ is easily calculated.

Next consider a Lagrangian $L(X,Y)$ (a scalar function of invariants) describing a nonlinear theory. From the usual definitions, we have

$$G_{\mu\nu} = \frac{\partial L}{\partial X}F_{\mu\nu} + \frac{\partial L}{\partial Y}\tilde{F}_{\mu\nu}\,. \tag{7}$$

Comparing (6) and (7) gives us the constitutive tensors for a general covariant Lagrangian theory,

$$R_{\mu\nu}^{\rho\sigma} = \frac{\partial L}{\partial X}\delta_{[\mu}^{\rho}\delta_{\nu]}^{\sigma} + \frac{\partial L}{\partial Y}\varepsilon_{[\mu\nu]\lambda\delta}\eta^{\lambda\rho}\eta^{\delta\sigma}, \quad S_{\mu\nu} = 0, \quad Q_{\mu\nu\lambda_i}^{\rho\sigma} = 0. \tag{8}$$

Here $N = \partial L/\partial X$ and $M = c^{-1}\partial L/\partial Y$, so that consistent with Ref. [2], the compatibility condition for M and N to describe a Lagrangian theory is $\partial N/\partial Y = c\partial M/\partial X$.

Finally, consider the duality transformations [12]

$$\delta F_{\mu\nu} = \tilde{G}_{\mu\nu}, \quad \delta G_{\mu\nu} = \tilde{F}_{\mu\nu}, \tag{9}$$

where $\tilde{G}_{\mu\nu} = (1/2)\varepsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}$ is the Hodge dual of $G_{\mu\nu}$. Then the natural (anti-)self-duality condition is $F_{\mu\nu} = \varepsilon\tilde{G}_{\mu\nu}$, with $\varepsilon = \pm 1$, and $X = \varepsilon Y$. Using (6), we then write the (anti-)self-duality condition for the constitutive tensor $R_{\mu\nu}^{\rho\sigma}$ as

$$R_{\mu\nu}^{\rho\sigma}\varepsilon_{\rho\sigma\lambda\delta}\eta^{\lambda[\mu}\eta^{\delta\nu]} = 2\varepsilon\,. \tag{10}$$

We remark that the equations of motion can be obtained directly by applying the method of [12], Section 3.1.

More detailed consideration of various properties of the constitutive tensors $S_{\mu\nu}$, $R_{\mu\nu}^{\rho\sigma}$, and $Q_{\mu\nu}^{\rho\sigma\lambda}$, with application to concrete systems, will appear elsewhere.

We conclude this report by indicating briefly how our general formulation of the constitutive equations (6) extends to superfields in $\mathscr{N}=1$ four-dimensional "supermedia", (where $\mathscr{N}$ is the number of supersymmetries). Our goal is to to write a plausible "super" analogue of the constitutive equations, without any commitment to a Lagrangian.

With standard notations (following mostly [13]), the supermanifold is described by coordinates x^μ, θ^α, and $\bar{\theta}^{\dot{\alpha}}$, where θ^α and $\bar{\theta}^{\dot{\alpha}}$ (with $\alpha, \dot{\alpha}=1,2$) are Grassmann variables (2-component, complex Majorana spinors). The Abelian gauge field $A_\mu(x)$ is a component of the gauge superfield (vector multiplet) $\mathsf{V}(x,\theta,\bar{\theta})=\mathsf{V}^+(x,\theta,\bar{\theta})$, where $+$ denotes super-Hermitian conjugation). In the Wess-Zumino gauge, half of the component fields are gauged away with supergauge transformations, and V takes the form

$$\mathsf{V}_{WZ}\left(x,\theta,\bar{\theta}\right)=-\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}A_\mu\left(x\right)-i\bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\theta^\alpha\psi_\alpha\left(x\right)+i\theta^\alpha\theta_\alpha\bar{\theta}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}\left(x\right) \qquad (11)$$

$$+\frac{1}{2}\theta^\alpha\theta_\alpha\bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}D\left(x\right),$$

where the σ^μ are Pauli spin matrices, $\psi_\alpha(x)$ is a Majorana fermionic field, and $D(x)$ is an auxiliary (nonphysical) field needed for supersymmetry invariance.

The actual super analogues of the field strength $F_{\mu\nu}$ are mixed spin-vector (odd valued) superfields $\mathsf{F}_{\alpha\mu}$ and $\bar{\mathsf{F}}_{\dot{\alpha}\mu}$ in superspace, which may be expressed directly in terms of a prepotential V: $\mathsf{F}_{\alpha\mu}=-(1/2)\mathrm{D}_\alpha\mathrm{D}_\beta\sigma_\mu^{\beta\dot{\beta}}\bar{\mathrm{D}}_{\dot{\beta}}\mathsf{V}$, where D_α, $\bar{\mathrm{D}}_{\dot{\beta}}$ denote supercovariant derivatives. Alternatively, $\mathsf{F}_{\alpha\mu}$ and $\bar{\mathsf{F}}_{\dot{\alpha}\mu}$ can be expressed in terms of chiral spinor superfields W and $\bar{\mathsf{W}}$ depending on only one spinorial coordinate:

$$\mathsf{F}_{\alpha\mu}\left(x,\bar{\theta}\right)=-i\varepsilon_{\alpha\beta}\sigma_\mu^{\beta\dot{\beta}}\bar{\mathsf{W}}_{\dot{\beta}}\left(x,\bar{\theta}\right),\quad \bar{\mathsf{F}}_{\dot{\alpha}\mu}\left(x,\theta\right)=-i\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}_\mu^{\dot{\beta}\beta}\mathsf{W}_\beta\left(x,\theta\right), \qquad (12)$$

where the antisymmetric tensor ε has components $\varepsilon^{12}=-\varepsilon_{12}=\varepsilon_{\dot{1}\dot{2}}=-\varepsilon^{\dot{1}\dot{2}}=1$; and where $\mathsf{W}_\beta(x,\theta)=(1/2)\bar{\mathrm{D}}_{\dot{\alpha}}\bar{\mathrm{D}}^{\dot{\beta}}\mathrm{D}_\beta\mathsf{V}(x,\theta,\bar{\theta})$ and $\bar{\mathsf{W}}_{\dot{\beta}}(x,\bar{\theta})=(1/2)\mathrm{D}^\alpha\mathrm{D}_\beta\bar{\mathrm{D}}_{\dot{\beta}}\mathsf{V}(x,\theta,\bar{\theta})$, with $\bar{\mathrm{D}}_{\dot{\alpha}}\mathsf{W}_\beta(x,\theta)=0$ and $\mathrm{D}_\alpha\bar{\mathsf{W}}_{\dot{\beta}}\left(x,\bar{\theta}\right)=0$. The chiral superfields satisfy the additional constraints $\mathrm{D}^\alpha\mathsf{W}_\alpha(x,\theta)=\bar{\mathrm{D}}_{\dot{\alpha}}\bar{\mathsf{W}}^{\dot{\alpha}}\left(x,\bar{\theta}\right)$. In the Wess-Zumino gauge (11),

$$\mathsf{W}_\alpha\left(x,\theta\right)=-i\psi_\alpha\left(x\right)+\left(\varepsilon_{\alpha\gamma}D\left(x\right)-\frac{i}{2}\sigma^\mu_{\alpha\dot{\alpha}}\varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\sigma}^\nu_{\dot{\beta}\gamma}F_{\mu\nu}\left(x\right)\right)\theta^\gamma \qquad (13)$$

$$-\theta^\beta\theta_\beta\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\bar{\psi}^{\dot{\alpha}}\left(x\right).$$

The role of the gauge invariants X and Y in (5) will be played by superinvariants X and Y, expressed using (12) by

$$\mathsf{X}\left(x,\theta\right)=\frac{1}{4}\bar{\mathsf{F}}_{\dot{\alpha}\mu}\left(x,\theta\right)\bar{\mathsf{F}}^{\dot{\alpha}\mu}\left(x,\theta\right)=\mathsf{W}^\alpha\left(x,\theta\right)\mathsf{W}_\alpha\left(x,\theta\right),$$

$$\mathsf{Y}\left(x,\bar{\theta}\right)=\frac{1}{4}\mathsf{F}^{\alpha\mu}\left(x,\bar{\theta}\right)\mathsf{F}_{\alpha\mu}\left(x,\bar{\theta}\right)=\bar{\mathsf{W}}_{\dot{\alpha}}\left(x,\bar{\theta}\right)\bar{\mathsf{W}}^{\dot{\alpha}}\left(x,\bar{\theta}\right). \qquad (14)$$

Now, by analogy with what we have written for F and $\bar{\mathsf{F}}$, we introduce $\mathsf{G}^{\alpha\mu}(x,\bar{\theta})$ and $\bar{\mathsf{G}}_{\dot{\alpha}\mu}(x,\theta)$ *without* expressing these through any prepotential analogue of V. Instead, we propose that they have a representation analogous to (12); i.e.,

$$\mathsf{G}_{\alpha\mu}\left(x,\bar{\theta}\right)=-i\varepsilon_{\alpha\beta}\sigma_{\mu}^{\beta\dot{\beta}}\bar{\mathsf{W}}_{\dot{\beta}}^{G}\left(x,\bar{\theta}\right),\quad \bar{\mathsf{G}}_{\dot{\alpha}\mu}\left(x,\theta\right)=-i\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\sigma}_{\mu}^{\dot{\beta}\beta}\mathsf{W}_{\beta}^{G}\left(x,\theta\right), \tag{15}$$

where the chiral superfields in media, $\mathsf{W}_{\beta}^{G}(x,\theta)$ and $\bar{\mathsf{W}}_{\dot{\beta}}^{G}\left(x,\bar{\theta}\right)$, are likewise not expressed through a prepotential. Rather $\mathsf{W}_{\beta}^{G}(x,\theta)$ is taken to have a component expansion analogous to (13), but with the field strength tensor $F_{\mu\nu}(x)$ in (13) replaced by the tensor $G_{\mu\nu}(x)$ satisfying (2).

Finally, we formulate the generalized superconstitutive equations analogous to (6), in terms of chiral superfields:

$$\begin{aligned}\mathsf{W}_{\alpha}^{G}(x,\theta) &= \mathsf{S}_{\alpha}+\mathsf{R}_{\alpha}^{\beta}\mathsf{W}_{\beta}+\ldots,\\ \bar{\mathsf{W}}_{\dot{\alpha}}^{G}(x,\bar{\theta}) &= \bar{\mathsf{S}}_{\dot{\alpha}}+\bar{\mathsf{R}}_{\dot{\alpha}}^{\dot{\beta}}\bar{\mathsf{W}}_{\dot{\beta}}+\ldots,\end{aligned} \tag{16}$$

where "…" represents the possibility of additional terms with superderivative contributions. Then $\mathsf{R}_{\alpha}^{\beta}$ can depend on (x,θ) through the gauge superinvariant $\mathsf{X}(x,\theta)$, and $\bar{\mathsf{R}}_{\dot{\alpha}}^{\dot{\beta}}$ can depend on $(x,\bar{\theta})$ through $\mathsf{Y}(x,\bar{\theta})$, with X and Y defined as in (14).

The expansion (13) and its analogue for $\mathsf{W}_{\beta}^{G}(x,\theta)$ lets us rewrite the superfield constitutive equations (16) in components.

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