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# Superconformal-Like Transformations and Nonlinear Realizations

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**Abstract**

We consider various properties of $N = 1$ superconformal-like transformations which generalize conformal transformations to supersymmetric and noninvertible case. Alternative tangent space reduction in $N = 1$ superspace leads to some new transformations which are similar to the anti-holomorphic ones of the complex function theory, which gives new odd $N = 1$ superanalog of complex structure. They are dual to the ordinary superconformal transformations subject to the Berezinian addition formula presented, noninvertible, highly degenerated and twist parity of the tangent space in the standard basis, and they also lead to some "mixed cocycle condition". A new parametrization for the superconformal group is presented which allows us to extend it to a semigroup and to unify the description of old and new transformations. The nonlinear realization of invertible and noninvertible $N = 1$ superconformal-like transformations is studied by means of the odd curve motion technique and introduced clear diagrammatic method.

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# 1 Introduction

More than 25 years have passed after discovery of supersymmetry [1, 2], nevertheless its influence on abstract algebraic properties of the physical theory was mostly symbolical. The main constructions of the theory after several trivial and clear modifications were equipped by the suffix "super". Then the building of quite supersymmetric model, excluding some negligible moments, was copied step by step from the corresponding nonsupersymmetric version, and the latter had to be its continuous limit in certain extent. In spite of the great success in the finite field models and superstring unified theories [3, 4], the supersymmetry by itself did not lead to significant changings or generalizations of the theory in abstract algebraic sense. In particular, the concept of superspace allowing to unify description of the bosonic and fermionic sectors of the theory was based on introduction of the additional nilpotent coordinates [5, 6]. They were used on a par with the ordinary "bosonic" coordinates, ignoring or avoiding in some way the fact that noninvertible objects and zero divisors inevitably arisen among the main variables of the theory. Therefore, a lot of mappings and functions turned out to be noninvertible, and indeed because of this fact, however it is strange and paradoxical from the mathematical viewpoint, they were artificially excluded by hand. So it was assumed that supergroups [7] are sufficient to be a super generalization of corresponding groups. This approach was named "factorizing by nilpotents" in physics [8, 9, 10] and applied in all previous supersymmetrizing approaches [11].

However, as a matter of fact, all transformations on a set into itself or all maps of a topological space preserving a definite structure form (without invertibility demands) indeed a semigroup[1] under composition [13, 14]. The above "factorizing" procedure in the semigroup theory is well-known and is named a Rees factorization [13].

In this paper we propose to reconsider the above ansatz and try to "factorize by non-nilpotents", i.e. to study "non-group" properties of some supersymmetric models. We suggest that a consistent from abstract algebraic viewpoint way is simultaneous transition from space to superspace and from transformation or topological groups to corresponding supersemigroups: "su-

[1]Moreover, in physical applications "... impossibility to restrict ourselves by only invertible transformations is clear"([12], p.40)

per" generalization of the physical theory should be accompanied by "semi" generalization of its mathematics as a whole. Therefore, necessary super analogs of groups are really supersemigroups, and not supergroups. Thus we can paraphrase: super-nature implies semi-symmetry (cf. [4], p. 4).

In global sense supersymmetric theory should have semigroup structure, while the observable sector at present energies can be satisfactorily described by its invertible group part [3]. Nevertheless, it should not be restricted with the investigations of the latter[2], because its properties are connected with ones of the rest (ideal) part [13, 12].

¿From another side in superstring unified theories [3, 4] a special class of reduced mappings of two-dimensional $(1|1)$ complex superspace, namely superconformal transformations [16, 17] play main and fundamental role [18, 19]. In the local approach to super Riemann surfaces represented as collections of open superdomains the superconformal transformations are used as gluing transition functions [16, 20]. They also can be defined as a result of the special reduction of the structure supergroup [21, 22]. Here using the invertibility weakening ansatz [23] we consider an alternative tangent space reduction, which leads to possible extensions of $N = 1$ superconformal transformations and to new transformations (see also [24]). We present a new parametrization of superconformal group is which allows us to extend it to a semigroup and to unify the description of old and new transformations.

We use the functional approach to superspace [25, 26] which admits existence of nontrivial topology in odd directions [27] and can be suitable for physical applications.

# 2 Preliminaries

A $(p|q)$-dimensional superspace $\mathbb{K}^{p|q}$ over $\Lambda$ (in the sense of [25]) is the even sector of the direct product $\Lambda_0^p \times \Lambda_1^q$, where $\Lambda$ be a commutative Banach $\mathbb{Z}_2$-graded superalgebra [7, 9] over a field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}_p$) with a decomposition into the direct sum: $\Lambda = \Lambda_0 \oplus \Lambda_1$. The elements $a$ from $\Lambda_0$ and $\Lambda_1$ are homogeneous and have the fixed even and odd parity defined as $|a| \stackrel{def}{=} \{i \in \{0,1\} = \mathbb{Z}_2 | \, a \in \Lambda_i\}$. The elements $X := (x_1 \ldots x_p) \in \Lambda_0^p$ and $\Theta := (\theta_1 \ldots \theta_q) \in \Lambda_0^q$ serve as coordinates in the superspace $\mathbb{K}^{p|q}$.

[2]For mathematical argumentation of replies to the question "why study semigroups" see [15].

The even homomorphism $\mathfrak{m}_b : \Lambda \to \mathbb{B}$, where $\mathbb{B}$ is a purely even algebra over $\mathbb{K}$, is called a body map [25]. If there exists an embedding $\mathfrak{n} : \mathbb{B} \hookrightarrow \Lambda$ such that $\mathfrak{m}_b \circ \mathfrak{n} = id$, then $\Lambda$ admits the body and soul decomposition $\Lambda = \mathbb{B} \oplus \mathbb{S}$, and a soul map can be defined as $\mathfrak{m}_s : \Lambda \to \mathbb{S}$. Usually the isomorphism $\mathbb{B} \cong \mathbb{K}$ is implied (which is not necessary in general and can lead to very nontrivial behavior of the body).

In case that $\Lambda$ is a Banach algebra (with a norm $||\cdot||$) soul elements are quasinilpotent [28], which means $\forall a \in \mathbb{S}$, $\lim\limits_{n\to\infty} ||a||^{1/n} = 0$. But quasinilpotency of the soul elements does not necessarily lead to their nilpotency ($\forall a \in \mathbb{S}\, \exists n,\, a^n = 0$) for the infinite-dimensional case [29]. In particular, if $\Lambda = \Lambda(N)$ is a Grassmann algebra with $N$ anticommuting generators $\xi_i$ such that $\xi_i \xi_k = -\xi_k \xi_i$ $i, k \in N$, then $l_1$norm of an element $a \in \Lambda$ , $a = \sum\limits_{i\in N} c_i \xi_i$ is defined by the expression $||a|| := \sum\limits_{i\in N} |c_i|$.

These facts allow us to consider noninvertible morphisms on a par with invertible ones (in some sense), which gives, in proper conditions, many interesting and nontrivial results (see [23, 30]).

# 3 Superanalytic $N = 1$ transformations

Locally $(1|1)$-dimensional superspace $\mathbb{C}^{1|1}$ in the coordinate language is described by the pair $Z = (z, \theta)$, where $z$ is an even coordinate and $\theta$ is an odd one. In above definition of superspace there exist soul parts in the even coordinate $z = z_{body} + z_{soul}$, $z_{body} = \mathfrak{m}_b(z)$, $z_{soul} \stackrel{def}{=} z - z_{body}$, where $\mathfrak{m}_b$ is the body map [25] vanishing all nilpotent generators. The body map acts on the coordinates as follows $\mathfrak{m}_b(z) = z_{body}$, $\mathfrak{m}_b(\theta) = 0$. This allows one to consider non-trivial soul topology in even directions on a par with odd ones [27].

Using holomorphy $\tilde{z} = \tilde{z}(z, \theta)$, conditions a general superanalytic (SA) transformation $\mathrm{T}_{SA} : \mathbb{C}^{1|1} \to \mathbb{C}^{1|1}$ can be presented as

$$\left\{ \begin{array}{rcl} \tilde{z} & = & \tilde{z}(z, \theta), \\ \tilde{\theta} & = & \tilde{\theta}(z, \theta) \end{array} \right. \tag{1}$$

which does not contain dependence of complex conjugated coordinates. Taking into account nilpotence of the odd coordinate $\theta^2 = 0$ we obtain

$$\left\{ \begin{array}{rcl} \tilde{z} & = & f(z) + \theta \cdot \chi(z), \\ \tilde{\theta} & = & \psi(z) + \theta \cdot g(z), \end{array} \right. \tag{2}$$

where four component functions $f(z), g(z) : \mathbb{C}^{1|0} \to \mathbb{C}^{1|0}$ and $\psi(z), \chi(z) : \mathbb{C}^{1|0} \to \mathbb{C}^{0|1}$ satisfy supersmooth conditions generalizing $C^{\infty}$ ([25]). Here and in the following we denote even functions and variables by Latin letters and odd ones by Greek letters, and remind that point $(\cdot)$ denotes a product in underlying Grassmann algebra.

By definition the odd functions $\psi(z), \chi(z)$ are noninvertible. So the invertibility of the whole superanalytic transformations transformation (1) is controlled by the even functions $f(z), g(z)$. Usually they are chosen invertible [16]. Here we will not restrict their invertibility and consider both cases on a par.

**Definition 1** *Sets of invertible and noninvertible transformations $\mathbb{C}^{1|1} \to \mathbb{C}^{1|1}$ (2) form a semigroup of superanalytic transformations $\mathcal{T}_{SA}$.*

The invertible transformations are in its subgroup, while the noninvertible ones are in an ideal [23, 30].

**Definition 2** *Invertible SA transformations are defined by the conditions $\mathfrak{m}_b[f(z)] \neq 0,\ \mathfrak{m}_b[g(z)] \neq 0$.*

**Definition 3** *Halfinvertible SA transformations are defined by $\mathfrak{m}_b[f(z)] = 0,\ \mathfrak{m}_b[g(z)] \neq 0$.*

**Definition 4** *Noninvertible SA transformations are defined by $\mathfrak{m}_b[f(z)] = 0$.*

*Remark.* The halfinvertible SA transformations can be resolved only under $\theta$ and not under $z$.

Obviously we can use the component functions from (2) for parametrization of the semigroup of SA transformations $\mathcal{T}_{SA}$.

**Definition 5** *An element $\mathbf{s}$ of a superanalytic semigroup $\mathbf{S}_{SA}$ is parametrized by the four*

$$\left\{ \begin{array}{cc} f & \chi \\ \psi & g \end{array} \right\} \stackrel{def}{=} \mathbf{s} \in \mathbf{S}_{SA}, \tag{3}$$

*and the action in $\mathbf{S}_{SA}$ is*

$$\left\{ \begin{array}{cc} f_1 & \chi_1 \\ \psi_1 & g_1 \end{array} \right\} * \left\{ \begin{array}{cc} f_2 & \chi_2 \\ \psi_2 & g_2 \end{array} \right\} =$$

$$\left\{ \begin{array}{cc} f_1 \circ f_2 + \psi_2 \cdot \chi_1 \circ f_2 & f_1' \circ f_2 \cdot \chi_2 + g_2 \cdot \chi_1 \circ f_2 \\ & +\chi_1' \circ f_2 \cdot \chi_2 \cdot \psi_2 \\ \psi_1 \circ f_2 + \psi_2 \cdot g_1 \circ f_2 & \psi_1' \circ f_2 \cdot \chi_2 + g_2 \cdot g_1 \circ f_2 \\ & +g_1' \circ f_2 \cdot \chi_2 \cdot \psi_2 \end{array} \right\}. \tag{4}$$

*where*

$$f_1 \circ f_2 = f_1 \left( f_2 \left( z \right) \right) \tag{5}$$

*and the prime* ($'$) *denotes differentiation by argument.*

The associativity in $\mathbf{S}_{SA}$

$$\mathbf{s}_1 * (\mathbf{s}_2 * \mathbf{s}_3) = (\mathbf{s}_1 * \mathbf{s}_2) * \mathbf{s}_3 \tag{6}$$

is not trivial for (4) and needs to be proved.

**Proposition 6** *The multiplication law (4) is associative.*

*Proof.* The relation (6) consists of four relations corresponding to four entries in (3). Using (4) for 1-1 element we find

$$\begin{aligned} \mathbf{s}_1 * (\mathbf{s}_2 * \mathbf{s}_3) \left|_{1-1}\right. &= f_1 \circ (f_2 \circ f_3 + \psi_3 \cdot \chi_2 \circ f_3) \\ &\quad + (\psi_2 \circ f_3 + \psi_3 \cdot g_2 \circ f_3) \cdot \chi_1 \circ (f_2 \circ f_3 + \psi_3 \cdot \chi_2 \circ f_3) . \end{aligned}$$

Opening brackets and expanding in Taylor series and taking into account nilpotence of entries we derive

$$\begin{aligned} \mathbf{s}_1 * (\mathbf{s}_2 * \mathbf{s}_3) \left|_{1-1}\right. &= f_1 \circ f_2 \circ f_3 + \psi_3 \cdot \chi_2 \circ f_3 \cdot f_1' \circ f_2 \circ f_3 \\ &\quad +\psi_2 \circ f_3 \cdot \chi_1 \circ f_2 \circ f_3 \\ &\quad +\psi_3 \cdot g_2 \circ f_3 \cdot \chi_1 \circ f_2 \circ f_3 \\ &\quad +\psi_2 \circ f_3 \cdot \chi_1' \circ f_2 \circ f_3 \cdot \psi_3 \cdot \chi_2 \circ f_3. \end{aligned}$$

Then we group the elements in different manner and obtain

$$\begin{aligned} \mathbf{s}_1 * (\mathbf{s}_2 * \mathbf{s}_3) \left|_{1-1}\right. &= (f_1 \circ f_2 + \psi_2 \cdot \chi_1 \circ f_2) \circ f_3 \\ &\quad +\psi_3 \cdot (f_1' \circ f_2 \cdot \chi_2 + \chi_1' \circ f_2 \cdot \chi_2 \cdot \psi_2 + g_2 \cdot \chi_1 \circ f_2) \circ f_3 \\ &= (\mathbf{s}_1 * \mathbf{s}_2) * \mathbf{s}_3 |_{1-1}. \end{aligned}$$

Analogous computations can be performed for other elements, which proves the associativity of (4) and the fact that the parametrization (3) gives actually a semigroup. □

*Remark.* The multiplication (4) contains two products: superposition (5) and product in underlying Grassmann algebra $(\cdot)$. Therefore SA semigroup belongs nor to the class of continuous functions [31, 32, 33], neither to the class of multiplicative semigroups [34, 35].

The presence of two multiplications, zero divisors and nilpotents makes the analysis of abstract properties of SA semigroup (and superconformal semigroup considered below) much more complicated as against well studied function semigroups [31, 32, 33].

**Proposition 7** *A two side identity in* $\mathbf{S}_{SA}$ *is*

$$\mathbf{e} = \left\{ \begin{array}{cc} z & 0 \\ 0 & 1 \end{array} \right\} \tag{7}$$

*and a two side zero is the matrix (3) having zero entries.*

*Proof.* It can be easily checked using (4). □

Consider the homomorphism $\varphi$ of SA semigroup $\mathbf{S}_{SA}$ on the semigroup of superanalytic transformations $\mathcal{T}_{SA}$, i.e. $\varphi : \mathbf{S}_{SA} \to \mathcal{T}_{SA}$.

**Proposition 8** *As it should be* $\ker \varphi = \mathbf{e}$ *.*

In studies of supernumber systems containing zero divisors and nilpotents one usually says the magic words "factorizing by nilpotents" or "modulo nilpotents" and forget about additional exotic properties arising from thorough consideration of the latter. In function systems under study the situation is more delicate and needs additional abstract investigations.

For instance, in SA semigroup $\mathbf{S}_{SA}$ along the standard $\mathbf{e}$ and $\mathbf{z}$ we are able to introduce element dependent "local" identities and zeroes.

**Definition 9** *For a given element* $\mathbf{s}$ *of SA semigroup local left, right and two sided identities* $\mathbf{e}_{\mathbf{s}}^{left}, \mathbf{e}_{\mathbf{s}}^{right}, \mathbf{e}_{\mathbf{s}} \in \mathbf{S}_{SA}$ *are defined by*

$$\mathbf{e}_{\mathbf{s}}^{left} * \mathbf{s} = \mathbf{s}, \tag{8}$$

$$\mathbf{s} * \mathbf{e}_{\mathbf{s}}^{right} = \mathbf{s}, \tag{9}$$

$$\mathbf{e}_{\mathbf{s}} * \mathbf{s} * \mathbf{e}_{\mathbf{s}} = \mathbf{s}. \tag{10}$$

**Definition 10** *For a given element* $\mathbf{s}$ *of SA semigroup local left, right and two sided zeroes* $\mathbf{z}_{\mathbf{s}}^{left}, \mathbf{z}_{\mathbf{s}}^{right}, \mathbf{z}_{\mathbf{s}} \in \mathbf{S}_{SA}$ *are defined by*

$$\mathbf{z}_{\mathbf{s}}^{left} * \mathbf{s} = \mathbf{z}_{\mathbf{s}}^{left}, \tag{11}$$

$$\mathbf{s} * \mathbf{z}_{\mathbf{s}}^{right} = \mathbf{z}_{\mathbf{s}}^{right}, \tag{12}$$

$$\mathbf{z}_{\mathbf{s}} * \mathbf{s} * \mathbf{z}_{\mathbf{s}} = \mathbf{z}_{\mathbf{s}}. \tag{13}$$

The local identities and zeroes are sets of elements from $\mathbf{S}_{SA}$ and can be found from corresponding systems of functional differential equations, e.g. for $\mathbf{e}_{\mathbf{s}}^{left}$ from (8) in component form we have

$$\begin{array}{c} f_1 \circ f_2 + \psi_2 \cdot \chi_1 \circ f_2 = f_2, \\ \psi_1 \circ f_2 + \psi_2 \cdot g_1 \circ f_2 = \psi_2, \\ f_1' \circ f_2 \cdot \chi_2 + \chi_1' \circ f_2 \cdot \chi_2 \cdot \psi_2 + g_2 \cdot \chi_1 \circ f_2 = \chi_2, \\ \psi_1' \circ f_2 \cdot \chi_2 + g_1' \circ f_2 \cdot \chi_2 \cdot \psi_2 + g_2 \cdot g_1 \circ f_2 = g_2. \end{array} \tag{14}$$

*Example.* Let $\mathbf{s} = \left\{ \begin{array}{cc} z^2 & \beta \\ \alpha & z^{-1} \end{array} \right\}$, then $\mathbf{e}_{\mathbf{s}}^{left} = \left\{ \begin{array}{cc} z^2 & \beta \\ \alpha & z^{-1} \end{array} \right\}$.

To stress the difference from the function semigroup case we consider the left zeroes. From the multiplication law (5) it follows that for function semigroups the role of left zeroes play constant mappings

$$f_0(z) : z \to c_f = const. \tag{15}$$

because $\forall g(z), \ f_0 \circ g = f_0(g(z)) = c_f = f_0$. Let us take an element $\mathbf{s}_0$ of SA semigroup having analogous to (15), i.e.

$$\mathbf{s}_0 = \left\{ \begin{array}{cc} f_0 & \chi_0 \\ \psi_0 & g_0 \end{array} \right\}. \tag{16}$$

Then from (4) we have

$$\mathbf{s}_0 * \mathbf{s} = \left\{ \begin{array}{cc} f_0 & \chi_0 \\ \psi_0 & g_0 \end{array} \right\} * \left\{ \begin{array}{cc} f & \chi \\ \psi & g \end{array} \right\} = \left\{ \begin{array}{cc} c_f + c_\chi \cdot g & c_\chi \cdot g \\ c_\psi + c_g \cdot \psi & c_g \cdot g \end{array} \right\}, \tag{17}$$

and so $\mathbf{s}_0 * \mathbf{s} \neq const$ as opposite to the function semigroup case [36, 33].

While comparing the SA multiplication (4) with matrix semigroup multiplication [37, 38], we notice that the set of lower triangle supermatrices (3), i.e. elements having $\chi = 0$, form a subsemigroup as usual, however the set of upper triangle ones having $\psi = 0$ does not form any subsemigroup due to presence of the middle term in 2-2 element of (4).

By means of SA transformations (2) one can construct a superanalytic supermanifold $M_{SA}$ in the standard manner ([25, 8]). The component functions play the role of gluing transition functions. Thus, let $M_{SA} = \cup U_\alpha$ , where $U_\alpha$ are superdomains covering $M_{SA}$. Its structure is fully determined by four transition functions $f_{\alpha\beta}(z_\beta), \chi_{\alpha\beta}(z_\beta), g_{\alpha\beta}(z_\beta), \psi_{\alpha\beta}(z_\beta)$ describing SA transformation $Z_\beta \to Z_\alpha$ on the intersection $U_\alpha \cap U_\beta$.

**Proposition 11** *On triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$ the transition functions of SA supermanifold satisfy the consistency conditions*

$$\begin{array}{rcl} f_{\alpha\gamma} & = & f_{\alpha\beta} \circ f_{\beta\gamma} + \psi_{\beta\gamma} \cdot \chi_{\alpha\beta} \circ f_{\beta\gamma}, \\ \chi_{\alpha\gamma} & = & f'_{\alpha\beta} \circ f_{\beta\gamma} \cdot \chi_{\beta\gamma} + g_{\beta\gamma} \cdot \chi_{\alpha\beta} \circ f_{\beta\gamma} + \chi'_{\alpha\beta} \circ f_{\beta\gamma} \cdot \chi_{\beta\gamma} \cdot \psi_{\beta\gamma}, \\ g_{\alpha\gamma} & = & f'_{\alpha\beta} \circ f_{\beta\gamma} \cdot \chi_{\beta\gamma} + g_{\beta\gamma} \cdot g_{\alpha\beta} \circ f_{\beta\gamma} + g'_{\alpha\beta} \circ f_{\beta\gamma} \cdot \chi_{\beta\gamma} \cdot \psi_{\beta\gamma}, \\ \psi_{\alpha\gamma} & = & \psi_{\alpha\beta} \circ f_{\beta\gamma} + \psi_{\beta\gamma} \cdot g_{\alpha\beta} \circ f_{\beta\gamma}. \end{array} \tag{18}$$

*Proof.* Immediately follows from (4). $\square$

### 3.1 Super Jacobian

Here we introduce an analog of Berezinian (super Jacobian) for noninvertible transformations.

Let us express the SA transformation (1) in the form of composition

$$1) \left\{ \begin{array}{rcl} \tilde{z} & = & F\left(z, \tilde{\theta}\right), \\ \tilde{\theta} & = & \tilde{\theta}, \end{array} \right. \qquad 2) \left\{ \begin{array}{rcl} z & = & z, \\ \tilde{\theta} & = & \tilde{\theta}(z, \theta), \end{array} \right. \tag{19}$$

where $F\left(z, \tilde{\theta}\right) = \tilde{z}(z, \theta)$. The super Jacobian of the first transformation is simply $J_1 = \partial F / \partial z$. If

$$\epsilon \left[ \frac{\partial \tilde{\theta}}{\partial \theta} \right] \neq 0, \tag{20}$$

then, taking into account that $\theta$ is odd, we find $J_2 = \left(\partial \tilde{\theta} / \partial \theta\right)^{-1}$ [7]. So the total super Jacobian is

$$J_{SA} = J_1 J_2 = \frac{\partial F}{\partial z} \cdot \left( \frac{\partial \tilde{\theta}}{\partial \theta} \right)^{-1}. \tag{21}$$

To derive $J_1$ we can write $F\left(z,\tilde{\theta}\right)=\tilde{z}\left(z,\theta\left(z,\tilde{\theta}\right)\right)$, then we differentiate $\tilde{z}\left(z,\theta\left(z,\tilde{\theta}\right)\right)$ as a composite function

$$\frac{\partial F}{\partial z}=\frac{\partial \tilde{z}}{\partial z}+\frac{\partial \tilde{z}}{\partial \theta}\cdot\frac{\partial \theta}{\partial \tilde{\theta}}\cdot\frac{\partial \tilde{\theta}}{\partial z}. \tag{22}$$

Thus we obtain the full super Jacobian

$$J_{SA}=\frac{\frac{\partial \tilde{z}}{\partial z}-\frac{\partial \tilde{\theta}}{\partial z}\cdot\frac{\partial \theta}{\partial \tilde{\theta}}\cdot\frac{\partial \tilde{z}}{\partial \theta}}{\frac{\partial \tilde{\theta}}{\partial \theta}} \tag{23}$$

without the condition of invertibility of the transformation, i.e. without the standard requirement $\mathfrak{m}_b\left[\partial\tilde{z}/\partial z\right]\neq 0$ [9]. Nevertheless, in [7] it was shown that the expression of kind 23 (in matrix algebra) can be extended on the case $\mathfrak{m}_b\left[\partial\tilde{z}/\partial z\right]=0$ (halfinvertible in our classification).

**Proposition 12** *Formula (23) gives a super Jacobian for invertible and halfinvertible SA transformations.*

*Proof.* From (2) we obtain

$$\frac{\partial \tilde{z}}{\partial z}=f'\left(z\right)+\theta\cdot\chi'\left(z\right), \tag{24}$$

$$\frac{\partial \tilde{\theta}}{\partial \theta}=g\left(z\right), \tag{25}$$

and therefore

$$\mathfrak{m}_b\left[\frac{\partial \tilde{z}}{\partial z}\right]=\mathfrak{m}_b\left[f'\left(z\right)\right]=\mathfrak{m}_b\left[f\left(z\right)\right],$$

$$\mathfrak{m}_b\left[\frac{\partial \tilde{\theta}}{\partial \theta}\right]=\mathfrak{m}_b\left[g\left(z\right)\right],$$

so according to the Definitions 2 and 3 the condition (20) covers invertible and halfinvertible transformations. $\square$

**Corollary 13** *For invertible and halfinvertible SA transformations we have*

$$J_{SA}^{inv,halfinv}=\mathrm{Ber}\left(\tilde{Z}/Z\right) \tag{26}$$

*with*

$$\mathrm{Ber}\,\left(\tilde{Z}/Z\right)=\mathrm{Ber}\,P_{SA}^{0}, \tag{27}$$

*where*

$$P_{SA}^{0}=\begin{pmatrix} \frac{\partial\tilde{z}}{\partial z} & \frac{\partial\tilde{\theta}}{\partial z} \\ \frac{\partial\theta}{\partial z} & \frac{\partial\theta}{\partial\theta} \end{pmatrix}. \tag{28}$$

In the noninvertible case when (20) does not satisfied we cannot use (21) and (22), and the relation (26) is not more valid. So we are forced to extend the definitions (see also [39] for Jacobians of nonsupersymmetric nilpotent mappings). The Jacobian $J_1$ should be found from

$$J_1^{noninv}\cdot\frac{\partial\tilde{\theta}}{\partial\theta}=\frac{\partial\tilde{z}}{\partial z}\cdot\frac{\partial\tilde{\theta}}{\partial\theta}+\frac{\partial\tilde{z}}{\partial\theta}\cdot\frac{\partial\tilde{\theta}}{\partial z}, \tag{29}$$

and therefore instead of (23) and (26) we have

**Definition 14** *The super Jacobian of noninvertible SA transformations is defined by the equation*

$$J_{SA}^{noninv}\cdot\left(\frac{\partial\tilde{\theta}}{\partial\theta}\right)^2=\frac{\partial\tilde{z}}{\partial z}\cdot\frac{\partial\tilde{\theta}}{\partial\theta}+\frac{\partial\tilde{z}}{\partial\theta}\cdot\frac{\partial\tilde{\theta}}{\partial z}. \tag{30}$$

Here the condition (20) is not necessary more . To find $J_1^{noninv}$ and $J_{SA}^{noninv}$ one should solve thoroughly the equations (29) and (30) (i.e. by expanding both sides in series of Grassmann algebra generators).

In terms of the component functions the super Jacobian of halfinvertible SA transformations (i.e. if $\mathfrak{m}_b\left[g\left(z\right)\right]\neq 0$) has the form

$$J_{SA}=\frac{f'\left(z\right)}{g\left(z\right)}+\frac{\chi\left(z\right)\cdot\psi'\left(z\right)}{g^2\left(z\right)}+\theta\left(\frac{\chi\left(z\right)}{g\left(z\right)}\right)' \tag{31}$$

which coincides with the Berezinian for the invertible and halfinvertible transformations. In case of noninvertible transformations we should use the following equation

$$\begin{aligned} J_{SA}^{noninv}\cdot g^2\left(z\right) &= f'\left(z\right)\cdot g\left(z\right)+\chi\left(z\right)\cdot\psi'\left(z\right) \\ &\quad +\theta\left(\chi'\left(z\right)\cdot g\left(z\right)-\chi\left(z\right)\cdot g'\left(z\right)\right) \end{aligned} \tag{32}$$

which can be solved by special methods dealing with nilpotents ([28, 39]).

**Corollary 15** *For invertible SA transformations the Berezinian exists and invertible (*$\mathfrak{m}_b[f(z)] \neq 0,\ \mathfrak{m}_b[g(z)] \neq 0$*), for halfinvertible the Berezinian exists and noninvertible, for noninvertible SA transformations (*$\mathfrak{m}_b[f(z)] = 0$*) we are able to exploit the super Jacobian* $J_{SA}$ *only.*

To classify all SA transformations we should introduce some numerical characteristic of noninvertibility.

**Definition 16** *Noninvertibility index of SA transformation is defined by*

$$\operatorname{ind} J_{SA} \stackrel{def}{=} \left\{ n \in \mathbb{N} \mid J_{SA}^n = 0,\ J_{SA}^{n-1} \neq 0 \right\}. \tag{33}$$

Obviously that indeed the inverse variable gives numerical measure of noninvertibility.

**Definition 17** *Noninvertibility degree of SA transformation is*

$$m \stackrel{def}{=} \frac{1}{\operatorname{ind} J_{SA}}. \tag{34}$$

**Corollary 18** *Invertible SA transformations have* $\operatorname{ind} J_{SA} = \infty$ *and* $m = 0$*.*

We exclude from consideration the trivial case $J_{SA} = 0$.

**Corollary 19** *The "most noninvertible" SA transformations have* $\operatorname{ind} J_{SA} = 2$ *and* $m = 1/2$*.*

## 3.2 Tangent superspace and its reduction

Let we consider action of noninvertible transformation in some analog of tangent superspace and consequences following from that. Among the latter there are new noninvertible transformations which are dual in some sense to the superconformal ones [23, 40]. The invertible case is studied in detail [16, 41, 4], therefore we concentrate our attention on new features connected with noninvertibility, trying simplify the consideration for clarity.

The tangent superspace in $\mathbb{C}^{1|1}$ is defined by the standard supersymmetric basis $\{\partial,\ D\}$, where $D = \partial_\theta + \theta\partial,\ \partial_\theta = \partial/\partial\theta,\ \partial = \partial/\partial z$. The dual cotangent space is spanned by 1-forms $\{dZ,\ d\theta\}$, where $dZ = dz + \theta d\theta$ (the signs as in [16]). In these notations the supersymmetry relations are $D^2 = \partial,\ dZ^2 = dz$.

The semigroup of SA transformations $\mathcal{T}_{SA}$ acts in tangent and cotangent superspaces by means of the tangent space matrix $P_{SA}$ as

$$\begin{pmatrix} \partial \\ D \end{pmatrix} = P_{SA} \begin{pmatrix} \tilde{\partial} \\ \tilde{D} \end{pmatrix}$$

and

$$\begin{pmatrix} d\tilde{Z}, & d\tilde{\theta} \end{pmatrix} = \begin{pmatrix} dZ, & d\theta \end{pmatrix} P_{SA},$$

where

$$P_{SA} = \begin{pmatrix} \partial\tilde{z} - \partial\tilde{\theta}\cdot\tilde{\theta} & \partial\tilde{\theta} \\ D\tilde{z} - D\tilde{\theta}\cdot\tilde{\theta} & D\tilde{\theta} \end{pmatrix}. \tag{35}$$

**Proposition 20** *The exterior differential $d = dZ\partial + d\theta D$ is invariant under SA transformations.*

*Proof.* We have

$$\begin{aligned} d &= \begin{pmatrix} dZ, & d\theta \end{pmatrix} \begin{pmatrix} \partial \\ D \end{pmatrix} = \begin{pmatrix} dZ, & d\theta \end{pmatrix} P_{SA} \begin{pmatrix} \tilde{\partial} \\ \tilde{D} \end{pmatrix} \\ &= \begin{pmatrix} d\tilde{Z}, & d\tilde{\theta} \end{pmatrix} \begin{pmatrix} \tilde{\partial} \\ \tilde{D} \end{pmatrix} = \tilde{d} \end{aligned} \tag{36}$$

□

*Remark.* We note that in (36) the invertibility is not used.

**Proposition 21** Ber $\left(\tilde{Z}/Z\right) = \operatorname{Ber} P_{SA}$.

*Proof.* We observe that

$$\begin{pmatrix} \frac{\partial\tilde{z}}{\partial z} & \frac{\partial\tilde{\theta}}{\partial z} \\ \frac{\partial\tilde{\theta}}{\partial z} & \frac{\partial\tilde{\theta}}{\partial\theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\theta & 1 \end{pmatrix} \cdot \begin{pmatrix} \partial\tilde{z} - \partial\tilde{\theta}\cdot\tilde{\theta} & \partial\tilde{\theta} \\ D\tilde{z} - D\tilde{\theta}\cdot\tilde{\theta} & D\tilde{\theta} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tilde{\theta} & 1 \end{pmatrix}. \tag{37}$$

Then from (37), (27), (28) and (35) it follows

$$\begin{aligned} \operatorname{Ber}\left(\tilde{Z}/Z\right) &= \operatorname{Ber} P_{SA}^{0} = \operatorname{Ber}\left(\begin{pmatrix} 1 & 0 \\ -\theta & 1 \end{pmatrix} \cdot P_{SA} \cdot \begin{pmatrix} 1 & 0 \\ \tilde{\theta} & 1 \end{pmatrix}\right) \\ &= \operatorname{Ber}\begin{pmatrix} 1 & 0 \\ -\theta & 1 \end{pmatrix} \cdot \operatorname{Ber} P_{SA} \cdot \operatorname{Ber}\begin{pmatrix} 1 & 0 \\ \tilde{\theta} & 1 \end{pmatrix} \\ &= \operatorname{Ber} P_{SA}. \end{aligned}$$

$\square$

In case of invertible SA transformations the matrix $P_{SA}$ defines the structure of a supermanifold for which these transformations play the part of transition functions. Therefore different reductions of the matrix $P_{SA}$ give us various additional structures, but only one of them is usually considered [21], because only one of them can be invertible.

Taking into account noninvertibility we analyze *all* of them via vanishing *every* of the $P_{SA}$ entries in turn, which gives in general 4 possibilities:

$$1)\ D\tilde{\theta} = 0, \tag{38}$$

$$2)\ \partial\tilde{\theta} = 0, \tag{39}$$

$$3)\ \Delta \equiv D\tilde{z} - D\tilde{\theta}\cdot\tilde{\theta} = 0, \tag{40}$$

$$4)\ Q \equiv \partial\tilde{z} - \partial\tilde{\theta}\cdot\tilde{\theta} = 0. \tag{41}$$

which are arranged according to the increasing of their nontriviality. The first two cases (38) and (39) are most simple, but they have some interesting peculiarities and will be considered separately.

# 4 Superconformal-like transformations

We here consider two other possible reductions (40) and (41) together.

In [24] it was shown that there exist two nontrivial reductions of any supermatrix (not one, triangle, as in the invertible case). We apply this result to $P_{SA}$ (35).

**Assertion 22** *The condition*

$$\mathfrak{m}_b\left[D\tilde{\theta}\right] \neq 0 \tag{42}$$

*coincides with halfinvertibility of the SA transformation (3).*

*Proof.* Indeed we observe from (37) that the Berezinian can be presented in two additive parts

$$\operatorname{Ber} P_A = \frac{\partial\tilde{z} - \partial\tilde{\theta}\cdot\tilde{\theta}}{D\tilde{\theta}} + \frac{\left(D\tilde{z} - D\tilde{\theta}\cdot\tilde{\theta}\right)\partial\tilde{\theta}}{\left(D\tilde{\theta}\right)^2} = \frac{Q}{D\tilde{\theta}} + \frac{\Delta\cdot\partial\tilde{\theta}}{\left(D\tilde{\theta}\right)^2}. \tag{43}$$

only if $\mathfrak{m}_b\left[D\tilde{\theta}\right]\neq 0$. Then from the component form (2) we derive $D\tilde{\theta}=g\left(z\right)+\theta\cdot\psi\left(z\right)$ and so $\mathfrak{m}_b\left[D\tilde{\theta}\right]=\mathfrak{m}_b\left[g\left(z\right)\right]$, therefore $\epsilon\left[D\tilde{\theta}\right]\neq 0\Rightarrow\epsilon\left[g\left(z\right)\right]\neq 0$ which is really the halfinvertibility condition (3). $\square$

**Proposition 23** *The Berezinian of superanalytic transformations in case* $D\tilde{\theta}\neq 0$ *is described by the formula*

$$\mathrm{Ber}\,P_{SA}=D\left(\frac{D\tilde{z}}{D\tilde{\theta}}\right). \tag{44}$$

*Proof.* After differentiating of the right hand side and using $D^2=\partial$ we derive

$$D\left(\frac{D\tilde{z}}{D\tilde{\theta}}\right)=\frac{\partial\tilde{z}\cdot D\tilde{\theta}+D\tilde{z}\cdot\partial\tilde{\theta}}{\left(D\tilde{\theta}\right)^2}=\frac{\partial\tilde{z}\cdot D\tilde{\theta}+\tilde{\theta}\cdot\partial\tilde{\theta}\cdot D\tilde{\theta}-\tilde{\theta}\cdot\partial\tilde{\theta}\cdot D\tilde{\theta}+D\tilde{z}\cdot\partial\tilde{z}}{\left(D\tilde{\theta}\right)^2}=$$

$$\frac{\left(\partial\tilde{z}+\tilde{\theta}\cdot\partial\tilde{\theta}\right)\cdot D\tilde{\theta}-\left(D\tilde{z}-\tilde{\theta}\cdot D\tilde{\theta}\right)\cdot\partial\tilde{\theta}}{\left(D\tilde{\theta}\right)^2}=\frac{Q\cdot D\tilde{\theta}-\Delta\cdot\partial\tilde{\theta}}{\left(D\tilde{\theta}\right)^2},$$

which coincides with (43). $\square$

Using the Berezinian addition theorem [24] we obtain the formula

$$\mathrm{Ber}\,P_A=\mathrm{Ber}\,P_S+\mathrm{Ber}\,P_T, \tag{45}$$

where

$$P_S\stackrel{def}{=}\begin{pmatrix}\partial\tilde{z}-\partial\tilde{\theta}\cdot\tilde{\theta} & \partial\tilde{\theta}\\ 0 & D\tilde{\theta}\end{pmatrix}=\begin{pmatrix}Q & \partial\tilde{\theta}\\ 0 & D\tilde{\theta}\end{pmatrix}, \tag{46}$$

$$P_T\stackrel{def}{=}\begin{pmatrix}0 & \partial\tilde{\theta}\\ D\tilde{z}-D\tilde{\theta}\cdot\tilde{\theta} & D\tilde{\theta}\end{pmatrix}=\begin{pmatrix}0 & \partial\tilde{\theta}\\ \Delta & D\tilde{\theta}\end{pmatrix}. \tag{47}$$

Denote sets of the matrices (46) and (47) by $\mathbf{P}_S$ and $\mathbf{P}_T$ respectively. We stress that till now there are no conditions imposed on the transformations and they are superanalytic (1).

*Superconformal-like transformations* arise when we project the Berezinian on one of the terms in (43) or the set of superanalytic matrices $\mathbf{P}_{SA}$ on $\mathbf{P}_S$ or $\mathbf{P}_T$. It is seen that there exit two kinds of superconformal-like transformations.

**Definition 24** *Invertible, halfinvertible and noninvertible superconformal transformations (SCf) are defined by the condition*

$$\Delta = D\tilde{z} - D\tilde{\theta} \cdot \tilde{\theta} = 0. \tag{48}$$

**Definition 25** *Halfinvertible and noninvertible transformations twisting parity of tangent space*[3] *(TPt) are defined by*

$$Q = \partial\tilde{z} - \partial\tilde{\theta} \cdot \tilde{\theta} = 0. \tag{49}$$

If we apply the conditions (49) and (48) to the matrices $P_S$ and $P_T$ we derive

$$P_{SCf} \stackrel{def}{=} P_S|_{\Delta=0} = \begin{pmatrix} Q_{SCf} & \partial\tilde{\theta} \\ 0 & D\tilde{\theta} \end{pmatrix}, \tag{50}$$

$$P_{TPt} \stackrel{def}{=} P_T|_{Q=0} = \begin{pmatrix} 0 & \partial\tilde{\theta} \\ \Delta_{TPt} & D\tilde{\theta} \end{pmatrix}, \tag{51}$$

where $Q_{SCf} = Q|_{\Delta=0}$ and $\Delta_{TPt} = \Delta|_{Q=0}$.

The condition $\Delta = 0$ (48) gives us in the invertible case the ordinary superconformal (SCf ) transformations $\mathrm{T}_{SCf}$ [16] and the reduced matrix $P_{SCf}$ (50) is a result of the standard reduction of structure supergroup (see e.g. [21]). Another condition $Q = 0$ (49) leads to the degenerated noninvertible transformations $\mathrm{T}_{TPt}$ twisting parity of the standard tangent space (TPt ) (see [23] and below). The alternative reduction [24] of the tangent space supermatrix $P_A$ gives us the antitriangle supermatrix $P_{TPt}$ (51). The dual role of SCf and TPt transformations is clearly seen from the Berezinian addition theorem (45 ) (see [24]) and the projections (50) and (51). Since SCf transformations can be viewed as a superanalog of complex structure [42, 43] , we can treat TPt transformations as another odd $N = 1$ superanalog of complex structure in a certain extent.

*Remark.* It is more natural to call TPt transformations anti-SCf transformations due to the following analogy with the nonsupersymmetric case. For an ordinary $2 \times 2$ matrix $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we obviously have the following identity $\det P = \det \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + \det \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} = \det P_{Diag} + \det P_{Antidiag}$, which can

[3] *The reason of such name will be seen below.*

be called a "determinant addition formula". In the complex function theory the first matrix describes the tangent space matrix of holomorphic mappings and the second one—of antiholomorphic mappings. In supersymmetric case the supermatrices $P_S$ and $P_T$ play the role similar to one of the nonsupersymmetric diagonal and antidiagonal matrices in ordinary theory as it is seen from (45). Therefore, if $P_{SCf}$ generalizes the tangent space matrix of holomorphic mappings, supermatrices $P_{TPt}$ could be considered as respective generalization for antiholomorphic mappings.

**Corollary 26** *Evidently,*

$$\operatorname{Ber} P_S|_{Q=0} = \operatorname{Ber} P_T|_{\Delta=0} = \operatorname{Ber} \begin{pmatrix} 0 & \partial\tilde{\theta} \\ 0 & D\tilde{\theta} \end{pmatrix} = 0. \tag{52}$$

Using this relation together with (50) and (51) we can project the Berezinian addition equality (45) on the superconformal-like transformations $\mathrm{T}_{SCf}$ and $\mathrm{T}_{TPt}$ as follows

$$\begin{array}{rcl} \operatorname{Ber} P_A & = & \left\{ \begin{array}{ll} \operatorname{Ber} P_S + \operatorname{Ber} P_T, & \Delta = 0, \\ \operatorname{Ber} P_S + \operatorname{Ber} P_T, & Q = 0. \end{array} \right. = \\ \left\{ \begin{array}{l} \operatorname{Ber} P_{SCf} + 0, \\ 0 + \operatorname{Ber} P_{TPt}, \end{array} \right. & = & \left\{ \begin{array}{ll} \operatorname{Ber} P_{SCf}, & (\mathrm{SCf}\ ) \\ \operatorname{Ber} P_{TPt}, & (\mathrm{TPt}\ ) \end{array} \right. \end{array} \tag{53}$$

After corresponding projections for $Q$ and $\Delta$ we have

$$Q_{SCf} = \left(\partial\tilde{z} - \partial\tilde{\theta}\cdot\tilde{\theta}\right)|_{\Delta=0} = \left(D\tilde{\theta}\right)^2, \tag{54}$$

$$\Delta_{TPt} = \left(D\tilde{z} - D\tilde{\theta}\cdot\tilde{\theta}\right)|_{Q=0} = \partial_\theta\tilde{z} - \partial_\theta\tilde{\theta}\cdot\tilde{\theta}. \tag{55}$$

It is remarkable to notice the similarity of the formulas (54) and (55), which proves us once more the duality between SCf and TPt transformations.

Using (54) one obtains [21]

$$P_{SCf} = \begin{pmatrix} \left(D\tilde{\theta}\right)^2 & \partial\tilde{\theta} \\ 0 & D\tilde{\theta} \end{pmatrix}. \tag{56}$$

If $\mathfrak{m}_b\left[D\tilde{\theta}\right]\neq 0$ then $\operatorname{Ber} P_{SCf}$ can be simply determined from (56), and it is [16]

$$\operatorname{Ber} P_{SCf}=D\tilde{\theta}. \tag{57}$$

In noninvertible case $\mathfrak{m}_b\left[D\tilde{\theta}\right]=0$ the Berezinian cannot be defined, but we can accept (57) as a definition of the Jacobian of noninvertible SCf transformations (see [23, 44]).

**Definition 27** *The Berezinian of noninvertible SCf transformations is*

$$\operatorname{Ber} P_{SCf}^{noninv}=D\tilde{\theta}. \tag{58}$$

¿From (55) we derive

$$P_{TPt}=\begin{pmatrix} 0 & \partial\tilde{\theta} \\ \Delta_{TPt} & D\tilde{\theta} \end{pmatrix} \tag{59}$$

(cf. (46)). If $\epsilon\left[D\tilde{\theta}\right]\neq 0$ the Berezinian of $P_{TPt}$ can be determined as

$$\operatorname{Ber} P_{TPt}=\frac{\Delta_{TPt}\cdot\partial\tilde{\theta}}{\left(D\tilde{\theta}\right)^{2}}. \tag{60}$$

¿From (55) it follows that $D\Delta_{TPt}=-\left(D\tilde{\theta}\right)^{2}$, and therefore $\partial\Delta_{TPt}=-2\cdot D\tilde{\theta}\cdot\partial\tilde{\theta}$, which gives

$$\operatorname{Ber} P_{TPt}=\frac{\partial\Delta_{TPt}\cdot\Delta_{TPt}}{2\left(D\tilde{\theta}\right)^{3}}. \tag{61}$$

Since $\Delta_{TPt}$ is odd and so nilpotent, $\operatorname{Ber} P_{TPt}$ is also nilpotent and pure soul.

We observe that the even and odd superfunctions $Q=Q\left(z,\theta\right)$ and $\Delta=\Delta\left(z,\theta\right)$ play an important role in possible reductions of superanalytic structure, and therefore it is worth to study them in more detail. A general relation between $Q$ and $\Delta$ is

$$Q-D\Delta=\left(D\tilde{\theta}\right)^{2}. \tag{62}$$

¿From it and (43) we derive another useful formula for the Berezinian of general SA transformations (if $\epsilon\left[D\tilde{\theta}\right] \neq 0$)

$$\operatorname{Ber} P_{SA} = D\tilde{\theta} + D\left(\frac{\Delta}{D\tilde{\theta}}\right) = D\left(\tilde{\theta} + \frac{\Delta}{D\tilde{\theta}}\right) \tag{63}$$

in which the SCf condition $\Delta = 0$ is seen manifestly.

# 5 Degenerated transformations

Let us consider the intersection $\mathbf{P}_D = \mathbf{P}_S \cap \mathbf{P}_T$ which is a set of the degenerated matrices $P_D$ of the form

$$P_D \stackrel{def}{=} \begin{pmatrix} 0 & \partial\tilde{\theta} \\ 0 & D\tilde{\theta} \end{pmatrix}, \tag{64}$$

which depend on the odd coordinate $\theta$ transformation only. The degenerated matrix of the shape (64) can be obtained by projection from $P_S$ and $P_T$ matrices. It means that, if the transformation of the odd sector (second line in (2)) is given, i.e. the functions $\psi(z)$ and $g(z)$ are fixed, the conditions (49) and (48) determine behavior of the even sector (functions $f(z)$ and $\chi(z)$). In this case, since the degenerated matrix $P_D$ depends on the odd sector transformation only, we obtain

$$P_D = P_S|_{Q=0} = P_T|_{\Delta=0}. \tag{65}$$

If $D\tilde{\theta} = 0$, then $\tilde{\theta} = \alpha = const$ and also $\partial\tilde{\theta} = D\left(D\tilde{\theta}\right) = 0$, therefore the odd sector becomes degenerated being a left zero and constant mapping similarly to 15 (such mappings form restrictive semigroups, see e.g. [45]).

Nevertheless, the whole SA transformation (2) is not a left zero due to (17) and has the following form

$$\begin{cases} \tilde{z} &= f(z) + \theta \cdot \chi(z), \\ \tilde{\theta} &= \alpha. \end{cases} \tag{66}$$

These transformations are noninvertible (due to degenerated odd sector) and form a semigroup Deg

It is remarkable that under the degenerated (Deg) transformations defined by (65) the both cocycle relations hold valid simultaneously. Also, Deg transformations form a subsemigroup $\mathcal{T}_{Deg}$ in $\mathcal{T}_{SA}$, because of $\mathbf{P}_D \cdot \mathbf{P}_D \subseteq \mathbf{P}_D$. Moreover, $\mathcal{T}_{Deg}$ is an ideal in $\mathcal{T}_{SA}$, $\mathcal{T}_{SCf}$ and $\mathcal{T}_{TPt}$ since $\mathbf{P}_D \cdot \mathbf{P}_A \subseteq \mathbf{P}_D$, $\mathbf{P}_D \cdot \mathbf{P}_S \subseteq \mathbf{P}_D$ and $\mathbf{P}_D \cdot \mathbf{P}_T \subseteq \mathbf{P}_D$. The degenerated transformations are characterized by one odd function $\psi(z)$ only and by the absence of the $\theta$-dependence of the transformation $Z \to \tilde{Z}$ (see (55)), so that

$$\begin{cases} \tilde{z}_{Deg} = f(z), \\ \tilde{\theta}_{Deg} = \psi(z), \end{cases} \tag{67}$$

where $f'(z) = \psi'(z)\,\psi(z)$. The multiplication in $\mathcal{T}_{Deg}$ coincides with the second row of (69).

# 6 Alternative parametrization

The reduction conditions (49) and (48) fix 2 of 4 component functions form (2) in each case. Usually [16] SCf transformations $\mathrm{T}_{SCf}$ are parametrized by $\begin{pmatrix} f \\ \psi \end{pmatrix}$, while other functions are found from (49) and (48). However, the latter can be done for invertible transformations only. To avoid this difficulty we introduce an alternative parametrization by the pair $\begin{pmatrix} g \\ \psi \end{pmatrix}$, which allows us to consider SCf and TPt transformations in a unified way and include noninvertibility. Indeed, fixing $g(z)$ and $\psi(z)$ we find for other component functions of (2 ) the equations

$$\begin{cases} f_n'(z) &= \psi'(z)\,\psi(z) + \frac{1+n}{2} g^2(z), \\ \chi_n'(z) &= g'(z)\,\psi(z) + n g(z)\,\psi'(z), \end{cases} \tag{68}$$

where $n = \begin{cases} +1, & \text{SCf}, \\ -1, & \text{TPt}, \end{cases}$ can be treated as a projection of some "reduction spin" switching the type of transformation. So the reduced transformation of the even coordinate (see (2)) should contain this additional index, i.e. $z \to \tilde{z}_n$ (at this point some additional to (45) analogy with complex structure is transparent). Since $f'_{-1}(z) = \psi'(z)\,\psi(z)$ is nilpotent, TPt transformations are always noninvertible and high degenerated after the body mapping. The unified multiplication law is

$$\left(\begin{array}{c} h \\ \varphi \end{array}\right)_n * \left(\begin{array}{c} g \\ \psi \end{array}\right)_m = \left(\begin{array}{c} g \cdot h \circ f_m + \chi_m \cdot \psi \cdot h' \circ f_m + \chi_m \cdot \varphi' \circ f_m \\ \varphi \circ f_m + \psi \cdot h \circ f_m \end{array}\right), \quad (69)$$

where $(*)$ is transformation composition and $(\circ)$ is function composition. For "reduction spin" projections we have only two definite products $(+1) * (+1) = (+1)$ and $(+1) * (-1) = (-1)$. The first formula is a consequence of $\mathbf{P}_S \cdot \mathbf{P}_S \subseteq \mathbf{P}_S$ (see (46)), which is simple manifestation of the fact that SCf transformations $\mathrm{T}_{SCf}$ form a substructure [21], i.e. a subsemigroup $\mathcal{T}_{SCf}$ of SA semigroup $\mathcal{T}_{SA}$ (in the invertible case—a subgroup [16]).

# 7 Nonlinear realizations of superconformal-like transformations

The study of "nonlinear" SCf and TPt transformations is interesting and worthwhile due to several reasons. ¿From one side, some of the first papers on supersymmetry [1, 46] were written in terms of its nonlinear realization (for nonsupersymmetric background of the method see [47, 48, 49]). Further, there were hopes that in the framework of nonlinear realizations one could solve the problems with superpartners and spontaneously supersymmetry breaking in realistic models [50, 51]. From another side, the nonlinearly realized two dimensional superconformal symmetry [52, 53] were used in the theory of superstrings [54]. In addition to these investigations we will study finite transformations and include noninvertibility. We also consider the connection between "linear" and "nonlinear" realizations [55, 56, 57], but from the pure kinematical viewpoint, and give a transparent diagram presentation for it in our special case.

## 7.1 The motion of odd curve in $\mathbb{C}^{1|1}$

According to the interpretation of [58] we can study the motion of the curve $\theta = \lambda(z)$ in $\mathbb{C}^{1|1}$. So that using the superanalytic transformations (2) we obtain

$$\tilde{z} = f(z) + \lambda(z) \cdot \chi(z), \quad (70)$$

$$\tilde{\lambda}(\tilde{z}) = \psi(z) + \lambda(z) \cdot g(z), \tag{71}$$

where the second equation reflects the Einstein style of transformations. In the next paragraph we will derive the equation (71) from the general diagrammatic approach.

In four dimensional case the function $\lambda(z)$ is usually called Akulov-Volkov field [50, 58] and in physical applications it plays a role of Nambu-Goldstone fermion [46, 59] (and therefore it is also called goldstino).

As it is seen from (71) the transformation of $\lambda(z)$ is highly nonlinear. The relations of such kind always appear in nonlinear group realizations, and the goldstino $\lambda(z)$ describes supersymmetry breaking [60, 61, 62].

To find the goldstino transformation we expand $\tilde{\lambda}(\tilde{z})$ in series and iterate exploiting nilpotency

$$\tilde{\lambda}(f(z)) = \psi(z) + \lambda(z) \cdot g(z) - \tilde{\lambda}'(f(z)) \cdot \lambda(z) \cdot \chi(z)\ . \tag{72}$$

In case $f^{-1}$ exists, we derive the finite superanalytic transformation of $\lambda(z)$

$$\tilde{\lambda} = \psi \circ f^{-1} + \lambda \circ f^{-1} \cdot g \circ f^{-1} - \tilde{\lambda}' \cdot \lambda \circ f^{-1} \cdot \chi \circ f^{-1}, \tag{73}$$

where $f \circ g = f(g(z))$.

It is not possible to find a general solution of the equation (72), and therefore we consider some particular cases.

*Example. (Global SUSY) The global supersymmetry in $\mathbb{C}^{1|1}$ corresponds to the following choice*

$$f(z) = z,\ g(z) = 1,\ \chi(z) = \varepsilon,\ \psi(z) = \varepsilon, \tag{74}$$

*where $\varepsilon$ is a constant odd parameter. Then from (70) and (71) we have*

$$\tilde{\lambda}_{Glob}(z) = \varepsilon + \lambda(z) - \tilde{\lambda}'_{Glob}(z) \cdot \lambda(z) \cdot \varepsilon\ . \tag{75}$$

*This equation is also difficult to solve manifestly without any additional requirements. But for infinitesimal transformations we obtain*

$$\delta_\varepsilon \lambda_{Glob}(z) = \tilde{\lambda}_{Glob}(z) - \lambda(z) = \varepsilon \cdot [1 + \lambda(z) \cdot \lambda'(z)] \tag{76}$$

*which satisfy the conventional supersymmetry algebra*

$$[\delta_\varepsilon, \delta_\eta]\, \lambda_{Glob}(z) = 2\varepsilon\eta \cdot \lambda(z) \cdot \lambda'(z) \tag{77}$$

*in accordance with [46, 59].*

*Remark.* In finite global case we put

$$\tilde{\lambda}^{fin}_{Glob}(z) = \tilde{\lambda}_{Glob}(z) + \Delta(z), \tag{78}$$

where $\tilde{\lambda}_{Glob}(z)$ is given by (76). Inserting (78) into (75) one derives the equation for $\Delta(z)$ as follows

$$\Delta'(z) \cdot \varepsilon \cdot \lambda(z) = \Delta(z) \tag{79}$$

which can be solved by expanding on nilpotents in a given underlying superalgebra.

Let us consider superconformal-like transformations parametrized by two functions $g(z)$, $\psi(z)$ (see Section 4). Then starting from the same function $\lambda(z)$ we can in general find $\tilde{\lambda}_n(z)$ from (71) as two separate solutions (corresponding to the "reduction spin" (68) projection $n$) of the following system of equations

$$\begin{cases} \tilde{\lambda}_n\left(f_n^{(g\psi)}(z)\right) & = & \psi(z) + \lambda(z) \cdot g(z) - \tilde{\lambda}'_n\left(f_n^{(g\psi)}(z)\right) \cdot \lambda(z) \cdot \chi_n^{(g\psi)}(z), \\ f_n^{(g\psi)\prime}(z) & = & \psi'(z)\,\psi(z) + \frac{1+n}{2} g^2(z), \\ \chi_n^{(g\psi)\prime}(z) & = & g'(z)\,\psi(z) + ng(z)\,\psi'(z), \end{cases} \tag{80}$$

where prime denotes derivative by argument, $n = +1$ corresponds to SCf transformations and $n = -1$ - to TPt transformations, and TPt are so called *noninvertible transformations twisting parity of tangent space* (see (68) and [23, 40]).

**Definition 28** *We call $\tilde{\lambda}_{SCf}(z) = \tilde{\lambda}_{n=+1}(z)$ a SCf goldstino, and $\tilde{\lambda}_{TPt}(z) = \tilde{\lambda}_{n=-1}(z)$ a TPt goldstino.*

As previously, it is not possible to solve the system (80) manifestly in general case.

*Remark.* It is necessary to stress that equations (80) do not depend on invertibility properties of superconformal-like transformations [44, 23] and only they can be used to find TPt goldstino evolution ($n = -1$ case).

*Example. (Infinitesimal SCf) Let we parametrize infinitesimal SCf transformations by*

$$f(z) = z + r(z), \; g(z) = 1 + \frac{1}{2} r'(z), \; \chi(z) = \varepsilon(z), \; \psi(z) = \varepsilon(z), \tag{81}$$

*where $r(z), \varepsilon(z)$ are infinitesimal. Then, from (80) we obtain*

$$\delta_{r,\varepsilon} \lambda_{SCf}(z) = \varepsilon(z) \cdot [1 + \lambda(z) \cdot \lambda'(z)] + \frac{1}{2} r'(z) \cdot \lambda(z) - r(z) \cdot \lambda'(z) \tag{82}$$

*in agreement with [53].*

# 8 Connection between linear and nonlinear realizations from diagrammatic viewpoint

The relationship between linear and nonlinear realizations [55, 57] plays an important role in understanding of the spontaneously supersymmetry breaking mechanisms [56]. The interest to the study of $N = 1$ superconformal-like transformations is stipulated for the fact that nonlinearly realized infinitesimal SCf transformations [53, 63] are widely used in superstring embeddings [54, 64] and hierarhies [65, 66]. Here we investigate them in the noninvertible finite case and from some another kinematical viewpoint using a clear diagrammatic approach (which is applicable to any general multidimensional case as well).

Let us consider the following diagram

$$\begin{tikzcd} Z_A \arrow[r, "\mathcal{G}", "\text{W-Z}"'] & \tilde{Z} \\ Z \arrow[u, "\mathcal{A}"] \arrow[r, "\mathcal{H}", "\text{A-V}"'] & Z_H \arrow[u, "\mathcal{B}"'] \end{tikzcd} \tag{83}$$

where $\mathcal{A} : Z \to Z_A$, $\mathcal{G} : Z_A \to \tilde{Z}$, $\mathcal{B} : Z_H \to \tilde{Z}$, $\mathcal{H} : Z \to Z_H$ (and $Z = (z, \theta)$) are superanalytic transformations (1). The transformation $\mathcal{G}$ plays the role of the linear transformation of Wess-Zumino type and the nonlinear transformation $\mathcal{H}$ (from a subgroup) is of Akulov-Volkov type, while $\mathcal{A}$ and $\mathcal{B}$ correspond to the transformations with Goldstone fields as parameters (describing cosets) [49, 47].

## 8.1 Global 2D supersymmetry

According to the general prescriptions [55, 48] we can take $\mathcal{G}$ as a global linear supersymmetry transformation in two-dimensional case

$$\mathcal{G}:\left\{\begin{array}{rcl}\tilde{z} & = & z_A+\theta_A\cdot\varepsilon,\\ \tilde{\theta} & = & \varepsilon+\theta_A,\end{array}\right. \tag{84}$$

then we take $\mathcal{H}$ as an ordinary conformal transformation with composite parameters to be find and interpret $\mathcal{A}$ and $\mathcal{B}$ as coset transformations with the local odd parameters $\lambda(z)$ and $\tilde{\lambda}_{Glob}(z_H)$

*Example.*

$$\mathcal{A}:\left\{\begin{array}{rcl}z_A & = & z+\theta\cdot\lambda(z),\\ \theta_A & = & \lambda(z)+\theta,\end{array}\right. \quad \mathcal{B}:\left\{\begin{array}{rcl}\tilde{z} & = & z_H+\theta_H\cdot\tilde{\lambda}_{Glob}(z_H),\\ \tilde{\theta} & = & \tilde{\lambda}_{Glob}(z_H)+\theta_H,\end{array}\right. \tag{85}$$

Indeed, the commutativity of the diagram (83) gives us the equation of $\lambda(z)$ evolution similar to (71) and (75) and equations for parameters of $\mathcal{H}$ in the following way.

**Definition 29** *A "linear" transformation $\mathcal{G}$ is representable by a "nonlinear" transformation $\mathcal{H}$, iff the diagram (83) is commutative*

$$\mathcal{G}\circ\mathcal{A}=\mathcal{B}\circ\mathcal{H}. \tag{86}$$

*Remark.* In the group theory this construction is related to the induced representation [67]. But here we, in general, do not demand invertibility of the entries in (86) and consider finite transformations.

Using (86) we obtain the relations

$$\begin{array}{rcl}\tilde{z}_{\mathcal{G}\circ\mathcal{A}} & = & \tilde{z}_{\mathcal{B}\circ\mathcal{H}},\\ \tilde{\theta}_{\mathcal{G}\circ\mathcal{A}} & = & \tilde{\theta}_{\mathcal{B}\circ\mathcal{H}}\end{array} \tag{87}$$

which are the representability condition (86) in coordinate language (as 4 component equations after expanding in $\theta$).

In the particular case of global supersymmetry (84) the equations 87 are

$$\begin{array}{l}z_A+\theta_A\cdot\varepsilon=z_H+\theta_H\cdot\tilde{\lambda}_{Glob}(z_H),\\ \theta_A+\varepsilon=\tilde{\lambda}_{Glob}(z_H)+\theta_H.\end{array} \tag{88}$$

After exploiting (85) we derive parameters of the conformal transformation

$$\mathcal{H}:\left\{\begin{array}{lcl} z_H & = & z+\lambda\left(z\right)\cdot\varepsilon, \\ \theta_H & = & \theta, \end{array}\right. \tag{89}$$

and the evolution equation for

$$\tilde{\lambda}_{Glob}\left(z_H\right)=\varepsilon+\lambda\left(z\right). \tag{90}$$

then expanding on nilpotents

$$\varepsilon+\lambda\left(z\right)=\tilde{\lambda}_{Glob}\left(z\right)+\tilde{\lambda}'_{Glob}\left(z\right)\cdot\lambda\left(z\right)\cdot\varepsilon \tag{91}$$

which coincides with (75). Thus, the relations (86) and (87) are initial in determining Goldstone field evolution.

## 8.2 The $-\lambda$-rule in two dimensions

If $\mathcal{A}$ is invertible, the representability condition (86) becomes

$$\mathcal{G}=\mathcal{B}\circ\mathcal{H}\circ\mathcal{A}^{-1}. \tag{92}$$

In the global case invertibility of $\mathcal{A}$ is evident, then from (85) we derive

$$\mathcal{A}^{-1}:\left\{\begin{array}{lcl} z & = & z_A-\theta_A\cdot\lambda\left(z_A\right), \\ \theta & = & -\lambda\left(z_A\right)+\theta_A\left[1+\lambda\left(z_A\right)\cdot\lambda'\left(z_A\right)\right]. \end{array}\right. \tag{93}$$

This explains nature of the well-known "$-\lambda$ rule" [55, 68] while comparing superfields of linear and nonlinear realizations [69]. The relation (92) is a general form of the "splitting trick" [55, 56] according to which any linear superfield can be presented as a set of nonlinear transforming components. The analog of this trick for a noninvertible finite case is the representability condition (86), and it is not solved under $\mathcal{A}$. Thus, for a superfield $\Phi\left(z,\theta\right)$ we can write

$$\delta_{\mathcal{H}}\Phi\left(z,\theta\right)=\Phi\left(z+\lambda\left(z\right)\cdot\varepsilon,\theta\right)-\Phi\left(z,\theta\right)=\varepsilon\cdot\lambda\left(z\right)\cdot\frac{\partial\Phi\left(z,\theta\right)}{\partial z}, \tag{94}$$

where $\delta_{\mathcal{H}}$ is infinitesimal "nonlinear" transformation $\mathcal{H}$ corresponding to $\mathcal{G}$. If we use (93) and put

$$\begin{array}{rcl} \Phi\left(z,\theta\right) & = & \Phi\left(z_A-\theta_A\cdot\lambda\left(z_A\right),-\lambda\left(z_A\right)+\theta_A\left[1+\lambda\left(z_A\right)\cdot\lambda'\left(z_A\right)\right]\right) \\ & \stackrel{def}{=} & \Phi_A\left(z_A,\theta_A\right), \end{array} \tag{95}$$

then for infinitesimal "linear" transformation $\mathcal{G}$ we obtain the standard supersymmetry relation

$$\delta_{\mathcal{G}}\Phi_A(z_A,\theta_A)=\Phi(z_A+\varepsilon\cdot\theta_A,\theta_A+\varepsilon)-\Phi_A(z_A,\theta_A)=\varepsilon\cdot Q_A\Phi_A(z_A,\theta_A), \tag{96}$$

where $Q_A$ is an ordinary supertranslation (cf. [55]).

Now we are ready to prove the "reversed" splitting trick which manifestly follows from the representability condition (86) applied to global two dimensional supersymmetry.

**Proposition 30** *Any superfield $\Phi(z,\theta)$ transforming nonlinearly as in (94) together with $\lambda(z)$ transforming as in (76) give a linearly (globally) transformed superfield (96).*

*Proof.* We should prove that $\Delta\Phi(z,\theta)=\delta_{\mathcal{G}}\Phi_A(z_A,\theta_A)$, where

$$\Delta\Phi(z,\theta)\overset{def}{=}\delta_{\mathcal{H}}\Phi(z,\theta)+\delta_{\mathcal{B}}\Phi(z,\theta)-\delta_{\mathcal{A}}\Phi(z,\theta),$$

and $\delta_{\mathcal{H}}$ is given by (94). It follows from (85) that $\delta_{\mathcal{B}}-\delta_{\mathcal{A}}$ describes changing of $\lambda(z)$, therefore

$$\delta_{\mathcal{B}}\Phi(z,\theta)-\delta_{\mathcal{A}}\Phi(z,\theta)=\delta_\varepsilon\lambda_{Glob}(z)\cdot\frac{\partial\Phi(z,\theta)}{\partial\lambda}.$$

So that from (76) we have

$$\Delta\Phi(z,\theta)=\varepsilon\cdot\left(\lambda(z)\cdot\frac{\partial\Phi(z,\theta)}{\partial z}+(1+\lambda(z)\cdot\lambda'(z))\cdot\frac{\partial\Phi(z,\theta)}{\partial\lambda}\right).$$

Making change of variables $(z,\theta)\to(z_A,\theta_A)$ and using the relations

$$\frac{\partial\Phi(z,\theta)}{\partial z}=(1+\theta\cdot\lambda'(z))\cdot\frac{\partial\Phi_A(z_A,\theta_A)}{\partial z_A}+\lambda'(z)\cdot\frac{\partial\Phi_A(z_A,\theta_A)}{\partial\theta_A}$$

and

$$\frac{\partial\Phi(z,\theta)}{\partial\lambda}=-\theta\cdot\frac{\partial\Phi_A(z_A,\theta_A)}{\partial z_A}+\frac{\partial\Phi_A(z_A,\theta_A)}{\partial\theta_A}$$

following from (85), we obtain

$$\begin{aligned}\Delta\Phi(z,\theta) &= (\theta+\lambda(z))\cdot\varepsilon\cdot\frac{\partial\Phi_A(z_A,\theta_A)}{\partial z_A}+\varepsilon\cdot\frac{\partial\Phi_A(z_A,\theta_A)}{\partial\theta_A}\\ &= \delta_{\mathcal{G}}z_A\cdot\frac{\partial\Phi_A(z_A,\theta_A)}{\partial z_A}+\delta_{\mathcal{G}}\theta_A\cdot\frac{\partial\Phi_A(z_A,\theta_A)}{\partial\theta_A}\\ &= \delta_{\mathcal{G}}\Phi_A(z_A,\theta_A).\end{aligned}$$

□

# 9 Nonlinear realization of general finite $N = 1$ superconformal transformations

Let us consider the representability condition (86) for a general $N = 1$ superconformal-like transformations $Z_A \to \tilde{Z}$ which now play the role of "linear" ones. According to Section 4 they can be parametrized by two functions $g(z_A)$ and $\psi(z_A)$ and have the form

$$\mathcal{G}:\left\{\begin{array}{rcl}\tilde{z} &=& f_n^{(g\psi)}(z_A)+\theta_A\cdot\chi_n^{(g\psi)}(z_A),\\ \tilde{\theta} &=& \psi(z_A)+\theta_A\cdot g(z_A),\end{array}\right. \tag{97}$$

where

$$\begin{array}{rcl}f_n^{(g\psi)\prime}(z_A) &=& \psi'(z_A)\,\psi(z_A)+\frac{1+n}{2}\cdot g^2(z_A),\\ \chi_n^{(g\psi)\prime}(z_A) &=& g'(z_A)\,\psi(z_A)+n\cdot g(z_A)\,\psi'(z_A),\end{array} \tag{98}$$

where $n=\left\{\begin{array}{ll}+1, & \text{SCf transformation,}\\ -1, & \text{TPt transformation,}\end{array}\right.$ is a projection of "reduction spin" switching the type of transformation (see also [40] for more details).

Then while trying to represent $\mathcal{G}$ in terms of nonlinear composition similarly to the diagram (83) we face with the following restriction which is consequence of the $N = 1$ superconformal-like multiplication law [40]. If $\mathcal{T}$ is a superconformal-like transformation, then there are only two possibilities in the composition $z \xrightarrow{\mathcal{T}} \tilde{z} \xrightarrow{\tilde{\mathcal{T}}} \tilde{\tilde{z}}$

$$\begin{array}{rcl}\tilde{\mathcal{T}}_{SCf}*\mathcal{T}_{SCf} &=& \tilde{\tilde{\mathcal{T}}}_{SCf},\\ \tilde{\mathcal{T}}_{TPt}*\mathcal{T}_{SCf} &=& \tilde{\tilde{\mathcal{T}}}_{TPt}.\end{array} \tag{99}$$

Therefore, we have only two possibilities to include TPt transformations into the diagrammatic representation (83) as

$$\mathcal{G}_{SCf} \circ \mathcal{A}_{SCf} = \mathcal{B}_{SCf} \circ \mathcal{H}_{SCf}, \tag{100}$$

$$\mathcal{G}_{TPt} \circ \mathcal{A}_{SCf} = \mathcal{B}_{TPt} \circ \mathcal{H}_{SCf}. \tag{101}$$

The first one is the nonlinear representation of $N = 1$ superconformal group in analogy with the ordinary infinitesimal invertible four-dimensional case [55, 70] (and 86) in which $\mathcal{A}_{SCf}$ and $\mathcal{B}_{SCf}$ play the role of cosets.

Let us consider (100) in more detail. The exact shape of cosets $\mathcal{A}_{SCf}$ and $\mathcal{B}_{SCf}$ can be taken as

$$\mathcal{A}_{SCf} : \begin{cases} z_A &= z + \theta \cdot \lambda(z), \\ \theta_A &= \lambda(z) + \theta\sqrt{1 + \lambda(z) \cdot \lambda'(z)}, \end{cases} \tag{102}$$

$$\mathcal{B}_{SCf} : \begin{cases} \tilde{z} &= z_H + \theta_H \cdot \tilde{\lambda}(z_H), \\ \tilde{\theta} &= \tilde{\lambda}(z_H) + \theta_H\sqrt{1 + \tilde{\lambda}(z_H) \cdot \tilde{\lambda}'(z_H)}, \end{cases} \tag{103}$$

and for $\mathcal{H}$ we choose the following general parametrization

$$\mathcal{H}_{SCf} : \begin{cases} z_H &= p(z), \\ \theta_H &= \rho(z) + \theta \cdot q(z) \end{cases} \tag{104}$$

Then, expanding the coordinate form (87) into components we obtain 4 corresponding equations for 4 unknown functions $p(z), q(z), \rho(z), \tilde{\lambda}(z)$

$$p(z) + \rho(z) \cdot \tilde{\lambda}(p(z)) = f_{+1}^{(g\psi)}(z) + g(z) \cdot \lambda(z) \cdot \psi(z), \tag{105}$$

$$\tilde{\lambda}(p(z)) + \rho(z) \cdot \sqrt{1 + \tilde{\lambda}(p(z)) \cdot \tilde{\lambda}'(p(z))} = \psi(z) + g(z) \cdot \lambda(z), \tag{106}$$

$$\begin{aligned} q(z) \cdot \tilde{\lambda}(p(z)) &= \lambda(z) \cdot f_{+1}^{(g\psi)\prime}(z) + \\ &\quad g(z) \cdot \psi(z) \cdot \sqrt{1 + \lambda(z) \cdot \lambda'(z)}, \end{aligned} \tag{107}$$

$$\begin{aligned} q(z) \cdot \sqrt{1 + \tilde{\lambda}(p(z)) \cdot \tilde{\lambda}'(p(z))} &= \lambda(z) \cdot \psi'(z) + \\ &\quad g(z) \cdot \sqrt{1 + \lambda(z) \cdot \lambda'(z)}, \end{aligned} \tag{108}$$

where $f_{+1}^{(g\psi)}(z)$ is determined from (98).

In case $q(z)$ and $g(z)$ are invertible, these equations have the following solution for parameters of nonlinear $\mathcal{H}$ transformation in terms of parameters of "linear" $\mathcal{G}$ transformation as

$$\begin{aligned} p(z) &= f_{+1}^{(g\psi)}(z) + g(z)\cdot\lambda(z)\cdot\psi(z), &(109)\\ q(z) &= \sqrt{p'(z)}, &(110)\\ \rho(z) &= 0, &(111)\end{aligned}$$

and for goldstino transformation rule

$$\tilde{\lambda}(p(z)) = \psi(z) + g(z)\cdot\lambda(z), \tag{112}$$

that naturally coincides with the previous approach (72) with $f(z) = f_{+1}^{(g\psi)}(z)$ and $\chi(z) = g(z)\cdot\psi(z)$.

Therefore, $\mathcal{H}$ is the split $N=1$ SCf transformation [71, 20]

$$\mathcal{H}_{SCf}: \begin{cases} z_H &= p(z), \\ \theta_H &= \theta\cdot\sqrt{p'(z)} \end{cases} \tag{113}$$

with the composite parameter $p(z)$ from (109), which can be presented as the following commutative diagram

$$\begin{tikzcd} Z_A \arrow[r, "\mathcal{G}_{SCf}", "\text{full}"'] & \tilde{Z} \\ Z \arrow[u, "\mathcal{A}_{SCf}"] \arrow[r, "\mathcal{H}_{SCf}", "\text{split}"'] & Z_H \arrow[u, "\mathcal{B}_{SCf}"'] \end{tikzcd} \tag{114}$$

Thus, using the SCf goldstino field $\lambda(z)$ we have manifestly obtained a nonlinear realization of general finite SCf transformations.

Second relation (101) and the corresponding commutative diagram

$$\begin{tikzcd} Z_A \arrow[r, "\mathcal{G}_{TPt}"] & \tilde{Z} \\ Z \arrow[u, "\mathcal{A}_{SCf}"] \arrow[r, "\mathcal{H}_{SCf}"] & Z_H \arrow[u, "\mathcal{B}_{TPt}"'] \end{tikzcd} \tag{115}$$

have no such transparent meaning, because $\mathcal{B}_{TPt}$ is noninvertible, and so it cannot be a standard coset. Nevertheless, since the final answer for the nonlinear transformation $\mathcal{H}_{SCf}$ is known from the previous approach (80), the noninvertible analog of coset $\mathcal{B}_{TPt}$ can be found in principle from the system of equations analogous to (105)–(108).Let we write $\mathcal{B}_{TPt}$ in the form

$$\mathcal{B}_{SCf}:\begin{cases}\tilde{z} &= f_{-1}^{(\widetilde{b\lambda})}(z_H)+\theta_H\cdot\chi_{-1}^{(\widetilde{b\lambda})}(z_H),\\ \tilde{\theta} &= \tilde{\lambda}(z_H)+\theta_H\cdot b(z_H),\end{cases} \tag{116}$$

where

$$\begin{aligned} f_n^{(\widetilde{b\lambda})\prime}(z_H) &= \tilde{\lambda}'(z_H)\cdot\tilde{\lambda}(z_H)+\tfrac{1+n}{2}\cdot b^2(z_H),\\ \chi_n^{(\widetilde{b\lambda})\prime}(z_H) &= b'(z_H)\cdot\tilde{\lambda}(z_H)+n\cdot b(z_H)\cdot\tilde{\lambda}'(z_H),\end{aligned} \tag{117}$$

and prime denotes derivative by argument. So the corresponding system of equations now is

$$f_{-1}^{(\widetilde{b\lambda})}(p(z))+\rho(z)\cdot\chi_{-1}^{(\widetilde{b\lambda})}(p(z)) = f_{+1}^{(g\psi)}(z)+\lambda(z)\cdot\chi_{+1}^{(g\psi)}(z), \tag{118}$$

$$\tilde{\lambda}(p(z))+\rho(z)\cdot b(p(z)) = \psi(z)+g(z)\cdot\lambda(z), \tag{119}$$

$$\begin{aligned}\rho(z)\cdot f_{-1}^{(\widetilde{b\lambda})\prime}(p(z))+q(z)\cdot\chi_{-1}^{(\widetilde{b\lambda})}(p(z)) &= \lambda(z)\cdot f_{+1}^{(g\psi)\prime}(z)+\\ &\quad \chi_{+1}^{(g\psi)}(z)\cdot\sqrt{1+\lambda(z)\cdot\lambda'(z)},\end{aligned} \tag{120}$$

$$\begin{aligned}\rho(z)\cdot q(z)\cdot\tilde{\lambda}'(p(z))+q(z)\cdot b(p(z)) &= \lambda(z)\cdot\psi'(z)+\\ &\quad g(z)\cdot\sqrt{1+\lambda(z)\cdot\lambda'(z)}.\end{aligned} \tag{121}$$

In case $\mathcal{A}_{SCf}$ is invertible we can obtain

$$\mathcal{G}_{TPt}=\mathcal{B}_{TPt}\circ\mathcal{H}_{SCf}\circ\mathcal{A}_{SCf}^{-1} \tag{122}$$

which gives an analog of nonlinear realization for noninvertible TPt transformations.

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