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TOPOSES AND CATEGORIES IN QUANTUM THEORY AND GRAVITY

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The topos theory is discussed from the physical point of view. Basic ideas of topos are presented and explained. The connection with algebras of classical and quantum observables, alternative concepts of space-time, theory of relativity and quantum gravity, the generalized histories approach to a quantum theory of the whole universe are reviewed. Using developed by the authors formalism of n-regular obstructed categories the concept of a topos is properly generalized.

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The most important and interesting question in fundamental physics is how quantum mechanics and general relativity can be reconciled in a theory of 'quantum gravity' (for review see e.g. [1, 2]). It is well-known that the classical physics is based on the continuum concept of space-time, on the contrary the quantum gravity needs a discrete concept of space-time. The concept of smooth manifolds, points and coordinate systems are fundamental objects of classical theories. Quantum physics involve objects of quite different nature, namely operators acting on Hilbert space.

Unified description of classical gauge theories and general relativity and the corresponding space-time quantization leads to some generalization of physical ideas and corresponding mathematical structures. In physics, it is the sudden changes in viewpoint that go on to inspire progress shifting usually generalizations not involving total abandonment of the original ideas. The mathematically motivated shift from the use of the magnetic field to the more commonly discussed vector potential was one such change. Another was the decision of the earlier natural philosophers to accept the use of the number zero in physics. Such changes in viewpoint are often resisted by aficionados of the old ways. First, working with a new viewpoint may slow the prediction process of observed phenomena (ask an engineer to work with the full relativistic vector potential when using magnets). Second, the new viewpoint may predict exactly the same set of results for any experiment. Thirdly, the new viewpoint requires a greater repertoire of concepts, not all of which immediately sound physically plausible (how does one observe no bananas? Or empty space?). Those trained to think in the old reliable ways see little point in learning a new way to think, a way that simply slows one's ability to predict merely the same results. The development of gauge field theory threw new physical fields into the picture. Fields that were not directly observable. Physicists now treat these fields as more real than their more accessible predecessors. The real gain in having more than one viewpoint is that a more general feel for physics is obtained. It is this which inspires new ideas. There are various approaches to the notion of a topos [3], but we will focus here on one that emphasizes the underlying logical structure and we will only discuss one, albeit crucial, clause of the definition of a topos: the requirement that a topos contain a 'subobject classifier'. This is a generalization of the idea, familiar in set-theory, of characteristic functions. The generalization will turn out to have a particularly interesting logical structure in the case of the kind of topos: a topos of presheaves [4, 5].

A topos is a particular type of category. Very roughly, it is a category that behaves much like the category of sets; indeed, this category, which we will call **Set**, is itself a topos.

The goal of the paper is to discuss the topos theory from physical point of view (see e.g. [2, 6, 5]). We give some basic ideas of topos [3, 7, 8, 4] and concentrate our attention on the connection with algebras of classical and quantum observables, alternative concept of space-time, theory of relativity and quantum gravity, the generalized histories approach [9, 10, 11] to a quantum theory of the whole universe [12, 13, 14]. Then using formalism of *n*-regular obstructed categories [15, 16] we generalize the concept of a topos in a similar way.

TOPOSES AND CATEGORIES

We recall that a category (see e.g. [17]) consists of a collection of *objects*, and a collection of *arrows* (or *morphisms*), with the following three properties. (1) Each arrow f is associated with a pair of objects, known as its *domain* (dom f) and the *codomain* (cod f), and is written in the form $f : B \to A$ where $B = \operatorname{dom} f$ and $A = \operatorname{cod} f$. (2) Given two arrows $f : B \to A$ and $g : C \to B$ (so that the codomain of g is equal to the domain of f), there is a composite arrow $f \circ g : C \to A$; and this composition of arrows obeys the associative law. (3) Each object A has an identity arrow, $\mathrm{id}_A : A \to A$, with the properties that for all $f : B \to A$ and all $g : A \to C$, $\mathrm{id}_A \circ f = f$ and $g \circ \mathrm{id}_A = g$.

We have already mentioned the prototype category (indeed, topos) **Set**, in which the objects are sets and the arrows are ordinary functions between them (set-maps). In many categories, the objects are sets equipped with some type of additional structure, and the arrows are functions that preserve this structure (hence the word 'morphism'). An obvious algebraic example is the category of groups, where an object is a group, and an arrow $f : G_1 \to G_2$ is a group homomorphism from G_1 to G_2. (More generally, one often defines one category in terms of another; and in such a case, there is often only one obvious way of defining composition and identity maps for the new category.) However, a category need not have 'structured sets' as its objects. An example is given by any partially-ordered set ('poset') $\mathcal{P}$. It can be regarded as a category in which (i) the objects are the elements of $\mathcal{P}$; and (ii) if $p, q \in \mathcal{P}$, an arrow from p to q is defined to exist if, and only if, $p \leq q$ in the poset structure. Thus, in a poset regarded as a category, there is at most one arrow between any pair of objects $p, q \in \mathcal{P}$.

In any category, an object T is called *a terminal* (resp. *initial*) object if for every object A there is exactly one arrow $f : A \to T$ (resp. $f : T \to A$). Any two terminal (resp. initial) objects are isomorphic (two objects A and B in a category are said to be *isomorphic* if there exists arrows $f : A \to B$ and $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$). So we normally fix on one such object; and we write 'the' terminal (resp. initial) object as $\mathbf{1}$ (resp. $\mathbf{0}$). An arrow $\mathbf{1} \to A$ is called a *point*, or a *global element*, of A. For example, applying these definitions to our example **Set** of a category, we find that (i) each singleton set is a terminal object; (ii) the empty set $\emptyset$ is initial; and (iii) the points of A give a 'listing' of the elements of A.

We now introduce a very special kind of category called a 'topos' [7, 8]. We will discuss only one clause of the definition of a topos: the requirement that a topos contain a generalization of the set-theoretic concept of a characteristic function; this generalization is closely related to what is called a 'subobject classifier'.

Recall that characteristic functions classify whether an element x is in a given subset A of a set X by mapping x to 1 if $x \in A$, and to 0 if $x \notin A$. More fully: for any set X, and any subset $A \subseteq X$, there is a characteristic function $\chi_A : X \to \{0, 1\}$, with $\chi_A(x) = 1$ or 0 according as $x \in A$ or $x \notin A$. One thinks of $\{0, 1\}$ as the truth-values; and χ_A classifies the various x for the set-theoretically natural question, "$x \in A$?". Furthermore, the structure of **Set**—the category of sets—secures the existence of this set of truth-values and the various functions χ_A: in particular, $\{0, 1\}$ is itself a set, *i.e.* an object in the category **Set**, and for each A, X with $A \subseteq X$, χ_A is an arrow from X to $\{0, 1\}$. It is possible to formulate this 'classifying action' of the various χ_A in general category-theoretic terms, so as to give a fruitful generalization.

In any category, one can define a categorical analogue of the set-theoretic idea of subset: it is called a 'subobject'. More precisely, one generalizes the idea that a subset A of X has a preferred injective (*i.e.*, one-to-one) map $A \to X$ sending $x \in A$ to $x \in X$. For category theory provides a generalization of injective maps, called 'monic arrows' or 'monics'; so that in any category one defines a subobject of any object X to be a monic with codomain X.

Any topos is required to have an analogue, written Ω, of the set $\{0, 1\}$ of truth-values. That is to say: just as $\{0, 1\}$ is itself a set—*i.e.*, an object in the category **Set** of sets—so also in any topos, Ω is an object in the topos. And just as the set of subsets of a given set X corresponds to the set of characteristic functions from subsets of X to $\{0, 1\}$; so also in any topos, there is a one-to-one correspondence between subobjects of an object X, and arrows from X to Ω. In a topos, Ω acts as an object of generalized truth-values, just as $\{0, 1\}$ does in set-theory; (though Ω typically has more than two global elements). Intuitively, the elements of Ω are the answers to a natural 'multiple-choice question' about the objects in the topos, just as "$x \in X$?" is natural for sets. An example: A set X equipped with a given function $\alpha : X \to X$ is called an *endomap*, written $(X; \alpha)$; and the family of all endomaps forms a category—indeed, a topos—when one defines an arrow from $(X; \alpha)$ to $(Y; \beta)$ to be an ordinary set-function f between the underlying sets, from X to Y, that preserves the endomap structure, *i.e.*, $f \circ \alpha = \beta \circ f$. Applying the definition of a subobject, it turns out that a subobject of $(X; \alpha)$ is a subset of X that is closed under α, equipped with the restriction of α: *i.e.*, a subobject is $(Z, \alpha\mid_Z)$, with $Z \subseteq X$ and such that $\alpha(Z) \subset Z$. So a natural question, given $x \in X$ and a subendomap $(Z, \alpha\mid_Z)$, is: "How many iterations of α are needed to send x (or rather its descendant, $\alpha(x)$ or $\alpha^2(x)$ or $\alpha^3(x) \ldots$) into Z?" The possible answers are '0 (*i.e.*, $x \in Z$)', '1', '2',..., and 'infinity (*i.e.*, the descendants never enter Z)'; and if the answer for x is some natural number N (resp. 0, infinity), then the answer for $\alpha(x)$ is $N - 1$ (resp. 0, infinity). So the possible answers can be presented as an endomap, with the elements of the base-set labelled as '0', '1', '2', ..., and '∞', and with the map α acting as follows: $\alpha : N \mapsto N - 1$ for $N = 1, 2, ...$, and $\alpha : 0 \mapsto 0$, $\alpha : \infty \mapsto \infty$. And it turns out that this endomap is exactly the object Ω in the category of endomaps! Recall that in any topos Ω is an object in the topos, so that here Ω must itself be an endomap, a set equipped with a function to itself. This example suggests that Ω is fixed by the structure of the topos concerned. And indeed, this is so in the precise sense that, although the clause in the definition of a topos that postulates the existence of Ω characterizes Ω solely in terms of conditions on the topos' objects and arrows, Ω is provably unique (up to isomorphism).

In any topos, Ω has a natural logical structure. More exactly, Ω has the internal structure of a Heyting algebra object: the algebraic structure appropriate for intuitionistic logic [18]. In addition, in any topos, the collection of subobjects of any given object X is a complete Heyting algebra (a locale). This sort of Heyting algebra structure in more detail below, for the case that concerns us—presheaves. For the moment we note only the general point, valid for any topos, that because Ω is fixed by the structure of the topos concerned, and has a natural Heyting structure, a major traditional objection to multi-valued logics—that the exact structure of the logic, or associated algebras, seems arbitrary—does not apply here.

TOPOSES IN QUANTUM THEORY

Quantum theory has several interpretative problems, about such topics as measurement and non-locality; each of which can be formulated in several ways. But workers in the field would probably agree that all the problems center around the relation between—on the one hand—the values of physical quantities, and—on the other—the results of measurement. For our purposes, it will be helpful to put this in terms of statements: so the issue is the relation between "The quantity A has a value, and that value is r", (where r is a real number) and "If a measurement of A is made, the result will be r".

In classical physics, this relation is seen as unproblematic. One assumes that, at each moment of time:

(i) every physical quantity has a real number as a value (relative to an appropriate choice of units); and

(ii) one can measure any quantity A 'ideally', *i.e.* in such a way that the result obtained is the value that A possessed before the measurement was made; thus "epistemology models ontology".

Assumption **(i)** is implemented mathematically by the representation of quantities as real-valued functions on a state space Γ; so that, in particular, the statement "the value of A is r" ($r \in \mathbb{R}$) corresponds to $\bar{A}^{-1}\{r\}$, the subset of Γ that is the inverse image of the singleton set $\{r\} \subset \mathbb{R}$ under the function $\bar{A} : \Gamma \to \mathbb{R}$ that represents the physical quantity A. Thus, in particular, to any state $s \in \Gamma$ there is associated a 'valuation' (an assignment of values) on all quantities, defined by:

$$V^s(A) := \bar{A}(s). \tag{1}$$

More generally, the proposition "the value of A is in Δ" (where $\Delta \subset \mathbb{R}$) corresponds to the subset $\bar{A}^{-1}(\Delta)$ of Γ; these subsets form a Boolean lattice, which thus provides a natural representation of the 'logic' of propositions about the system. In particular, corresponding to the real-numbered valuation V^s on quantities, defined by a state $s \in \Gamma$, we have a $\{0, 1\}$-valued valuation (a truth-value assignment) to propositions:

$$V^s(A \in \Delta) := 1 \text{ if } \bar{A}(s) \in \Delta; \text{ otherwise } V^s(A \in \Delta) = 0. \tag{2}$$

Thus, in particular, in classical physics each proposition about the system at some fixed time is regarded as being either true or false.

Note that assumption **(ii)** is incorporated implicitly in the formalism—namely, in the absence of any explicit representation of measurement—by the fact that the function $\bar{A} : \Gamma \to \mathbb{R}$ suffices to represent the quantity A, since its values (in the sense of 'values of a function') are the possessed values (in the sense of 'values of a physical quantity'), and these would be revealed by an (ideal) measurement.

In quantum theory, on the other hand, the relation between values and results, and in particular assumptions **(i)** and **(ii)**, are notoriously problematic. The state-space is a Hilbert space $\mathcal{H}$; a quantity A is represented by a self-adjoint operator $\hat{A}$ (which, with no significant loss of generality, we can assume throughout to be bounded), and a statement about values "$A \in \Delta$" corresponds naturally to a linear subspace of $\mathcal{H}$ (or, equivalently, to a spectral projector, $\hat{E}[A \in \Delta]$, of $\hat{A}$).

Assumption **(i)** above (the existence of possessed values for all quantities) now fails by virtue of the famous Kochen-Specker theorem [19]; which says, roughly speaking, that provided $\dim(\mathcal{H}) > 2$, one cannot assign real numbers as values to all quantum-theory operators in such a way that for any operator $\hat{A}$ and any function of it $f(\hat{A})$ (f a function from $\mathbb{R}$ to $\mathbb{R}$), the value of $f(\hat{A})$ is the corresponding function of the value of $\hat{A}$. (On the other hand, in classical physics, this constraint, called *FUNC*, is trivially satisfied by the valuations V^s.) In particular, it is no longer possible to assign an unequivocal true-false value to each proposition of the form "$A \in \Delta$". In a strict instrumentalist approach to quantum theory, the non-existence of such valuations is of no great import, since this interpretation of the theory deals only with the counterfactual assertion of the probabilities of what values would be obtained *if* suitable measurements are made.

However, strict instrumentalism faces severe problems (not least in quantum gravity); and the question arises therefore of whether it may not after all be possible to retain some 'realist flavor' in the theory by, for example, changing the logical structure with which propositions about the values of physical quantities are handled. One of our claims is that this can indeed be done by introducing a certain topos perspective on the Kochen-Specker theorem. For the moment, we just remark that no-go theorems like that of Kochen and Specker depend upon the fact that the set of all spectral projectors of $\mathcal{H}$ form a non-Boolean, indeed non-distributive, lattice; suggesting a non-Boolean, indeed non-distributive, 'quantum logic'. This alluring idea, originated by Birkhoff and von Neumann [20], has been greatly developed in various directions. The Dalla Chiara and Giuntini's masterly recent survey [21] includes recent developments that generalize the basic correspondence between subspaces and propositions about values, so as to treat so-called 'unsharp' ('operational') quantum physics; on this see also [22] and other papers in this issue. The logic associated with the topos-theoretic proposals here is not non-distributive. On the contrary, *any* topos has an associated internal logical structure that *is* distributive. This retention of the distributive law marks a major departure from the dominant tradition of quantum logic stemming from Birkhoff and von Neumann.

On the other hand, the proposals do involve non-Boolean structure since the internal logic of a topos is ‘intuitionistic’, in the sense that the law of excluded middle may not hold (although for some toposes, such as the category of sets, it does apply). Some intuitionistic structures also arise in the dominant ‘non-distributive’ tradition in quantum logic; for example, in the Brouwer-Zadeh approach to unsharp quantum theory, cf. [23].

TOPOSES IN QUANTUM GRAVITY

The problem of realism becomes particularly acute in the case of quantum gravity. This field is notoriously problematic in comparison with other branches of theoretical physics, not just technically but also conceptually. In the first place, there is no clear agreement about what the aim of a quantum theory of gravity should be, apart from the broad goal of in some way unifying, or reconciling, quantum theory and general relativity. That these theories do indeed conflict is clear enough: general relativity is a highly successful theory of gravity and spacetime, which treats matter classically (both as a source of the gravitational field, and as influenced by it) and treats the structure of spacetime as dynamical; while quantum theory provides our successful theories of matter, and treats spacetime as a fixed, background structure.

Much has been written about the conceptual problems that arise in quantum gravity; (for review see [14]). But in the present context it suffices to say that these are sufficiently severe to cause a number of workers in the field to question many of the basic ideas that are implicit in most, if not all, of the existing programs. For example, there have been a number of suggestions that spatio-temporal ideas of classical general relativity such as topological spaces, continuum manifolds, space-time geometry, micro-causality, etc. are inapplicable in quantum gravity.

More iconoclastically, one may doubt the applicability of quantum theory itself, notwithstanding the fact that all current research programs in quantum gravity do adopt a more-or-less standard approach to quantum theory. In particular, as we shall discuss shortly, there is a danger of certain *a priori*, classical ideas about space and time being used unthinkingly in the very formulation of quantum theory; thus leading to a type of category error when attempts are made to apply this theory to domains in quantum gravity where such concepts may be inappropriate.

CONTINUUM IN PHYSICAL THEORY

We will now consider the use of the continuum—*i.e.*, of real and complex numbers—in the formulation of our physical theories in general. There are two natural alternative conceptions of space and time, which will involve the use of topos theory. We give this discussion before introducing toposes, since: (i) it is independent of the logical issues that will be emphasized in the rest of this paper; and accordingly, (ii) it can be understood without using details of the notion of a topos. So let us ask: why do we use the continuum, *i.e.*, the real numbers, in our physical theories? The three obvious answers are: (i) to be the values of physical quantities; (ii) to model space and time; and (iii) to be the values of probabilities. But let us pursue a little the question of what justifies these answers: we will discuss them in turn.

As to (i), the first point to recognize is of course that the whole edifice of physics, both classical and quantum, depends upon applying calculus and its higher developments (for example, functional analysis and differential geometry) to the values of physical quantities. But in the face of this, one could still take the view that the success of these physical theories only shows the ‘instrumental utility’ of the continuum—and not that physical quantities really have real-number values. This is not the place to enter the general philosophical debate between instrumentalist and realist views of scientific theories; or even the more specific question of whether an instrumentalist view about the continuum is committed to somehow rewriting all our physical theories without use of $\mathbb{R}$: for example, in terms of rational numbers (and if so, how he should do it!). Suffice it to say here that the issue whether physical quantities have real-number values leads into the issue whether space itself is modelled using $\mathbb{R}$. For not only is length one (obviously very important!) quantity in physics; also, one main, if not compelling, reason for taking other quantities to have real-number values is that results of measuring them can apparently always be reduced to the position of some sort of pointer in space—and space is modelled using $\mathbb{R}$.

We note that the formalism of elementary wave mechanics affords a good example of an *a priori* adoption of the idea of a continuum model of space: indeed, the x in $\psi(x)$ represents space, and in the theory this observable is modelled as having a continuous spectrum; in turn, this requires the underlying Hilbert space to be defined over the real or complex field.

So we turn to (ii): why should space be modelled using $\mathbb{R}$? More specifically, we ask, in the light of our remarks about (i): Can any reason be given apart from the (admittedly, immense) ‘instrumental utility’ of doing so, in the physical theories we have so far developed? In short, our answer is No. In particular, we believe there is no good *a priori* reason why space should be a continuum; similarly, *mutatis mutandis* for time. But then the crucial question arises of how this possibility of a non-continuum space should be reflected in our basic theories, in particular in quantum theory itself, which is one of the central ingredients of quantum gravity.

As to (iii), why should probabilities be real numbers? Admittedly, if probability is construed in terms of the relative frequency of a result in a sequence of measurements, then real numbers do arise as the limits of infinite sequences of finite relative frequencies (which are all rational numbers). But this limiting relative frequency interpretation of probability is disputable. In particular, it seems problematic in the quantum gravity regime where standard ideas of space and time might break down in such a way that the idea of spatial or temporal 'ensembles' is inappropriate.

On the other hand, for the other main interpretations of probability—subjective, logical, or propensity—there seems to us to be no compelling *a priori* reason why probabilities should be real numbers. For subjective probability (roughly: what a rational agent's minimum acceptable odds, for betting on a proposition, are or should be): many authors point out that the use of $\mathbb{R}$ as the values of probabilities is questionable, whether as an idealization of the psychological facts, or as a norm of rationality. For the logical and propensity interpretations—which are arguably more likely to be appropriate for the quantum gravity regime—the use of $\mathbb{R}$ as the values of probabilities is less discussed. But again, we see no *a priori* reason for $\mathbb{R}$.It seems to us that in the literature, the principal 'justification' given for $\mathbb{R}$ is the mathematical desideratum of securing a uniqueness claim in a representation theorem about axiom systems for qualitative probability; the claim is secured by imposing a continuity axiom that excludes number-fields other than $\mathbb{R}$ as the codomain of the representing probability-function. Indeed, we would claim that while no doubt in some cases, one 'degree of entailment' or 'propensity' is 'larger' than another, it also seems possible that in other cases two degrees of entailment, or two propensities, might be incomparable–so that the codomain of the probability-function should be, not a linear order, but some sort of partially ordered set (equipped with a sum-operation, so as to make sense of the additivity axiom for probabilities).

ALTERNATIVE CONCEPTIONS OF SPACETIME

Here we turn to briefly sketch two alternative conceptions of space and time. Both involve topos theory, and indeed raise the idea—even more iconoclastic than scepticism about the continuum— that the use of set theory itself may be inappropriate for modelling space and time.

In standard general relativity—and, indeed, in all classical physics—space (and similarly time) is modelled by a set, and the elements of that set are viewed as corresponding to points in space. However, if one is 'suspicious of points'—whether of spacetime, of space or of time (*i.e.* instants)—it is natural to try and construct a theory based on 'regions' as the primary concept; with 'points'—if they exist at all—being relegated to a secondary role in which they are determined by the 'regions' in some way (rather than regions being sets of points, as in the standard theories). For time, the natural word is 'intervals', not 'regions'; but we shall use only 'regions', though the discussion to follow applies equally to the one-dimensional case—and so to time—as it does to higher-dimensional cases, and so to space and spacetime.

So far as we know, the first rigorous development of this idea was made in the context of foundational studies in the 1920s and 1930s, by authors such as Tarski. The idea was to write down axioms for regions from which one could construct points, with the properties they enjoyed in some familiar theory such as three-dimensional Euclidean geometry. For example, the points were constructed in terms of sequences of regions, each contained in its predecessor, and whose 'widths' tended to zero; (more precisely, the point might be identified with an equivalence class of such sequences). The success of such a construction was embodied in a representation theorem, that any model of the given axiom system for regions was isomorphic to, for example, $\mathbb{R}^3$ equipped with a structured family of subsets, which corresponded to the axiom system's regions. In this sense, this line of work was 'conservative': one recovered the familiar theory with its points, from a new axiom system with regions as primitives. From the pure mathematical point of view, Stone's representation theorem for Boolean algebras of 1936 was a landmark for this sort of work.

The use of regions in place of points need not be 'conservative': one can imagine axiom systems for regions, whose models (or some of whose models) do not contain anything corresponding to points of which the regions are composed. Indeed, for any topological space Z, the family of all open sets can have algebraic operations of 'conjunction', 'disjunction' and 'negation' defined on them by: $O_1 \wedge O_2 := O_1 \cap O_2$; $O_1 \vee O_2 := O_1 \cup O_2$; and $\neg O := \mathrm{int}(Z - O)$; and with these operations, the open sets form a complete Heyting algebra, also known as a *locale*. Here, a Heyting algebra is defined to be a distributive lattice H, with null and unit elements, that is *relatively complemented*, which means that to any pair S_1, S_2 in H, there exists an element $S_1 \Rightarrow S_2$ of H with the property that, for all $S \in H$,

$$S \leq (S_1 \Rightarrow S_2) \text{ if and only if } S \wedge S_1 \leq S_2 \quad . \tag{3}$$

Heyting algebras are thus a generalization of Boolean algebras; they need not obey the law of excluded middle, and so provide natural algebraic structures for intuitionistic logic. A Heyting algebra is said to be *complete* if every family of elements has a least upper bound. Summing up: the open sets of any topological space form a Heyting algebra, when partially ordered by set-inclusion; indeed a complete Heyting algebra (a locale), since arbitrary unions of open sets are open. However, it turns out that not every locale is isomorphic to the Heyting algebra of open sets of some topological space; and in this sense, the theory of regions given by the definition of a locale is not 'conservative'—it genuinely generalizes the idea of a topological space, allowing families of regions that are not composed of underlying points.

A far-reaching generalization of this idea is given by topos theory: (i) in any topos, there is an analogue of the set-theoretic idea of the family of subsets of a given set—called the family of subobjects of a given object X; (ii) for any object X in any topos, the family of subobjects of X is a locale.

The idea of infinitesimals was heuristically valuable in the discovery and development of the calculus, and it was expunged in the nineteenth-century rigorization of analysis by authors such as Cauchy and Weierstrass—for surely no sense could be made of the idea of nilpotent real numbers, *i.e.*, d such that $d^2 = 0$, apart from the trivial case $d = 0$? But it turns out that sense *can* be made of this: indeed in two somewhat different ways.

In the first approach, called 'non-standard analysis', every infinitesimal (*i.e.*, every nilpotent $d \neq 0$) has a reciprocal, so that there are different infinite numbers corresponding to the different infinitesimals. There were attempts in the 1970s to apply this idea to quantum field theory: in particular, it was shown how the different orders of ultra-violet divergences that arise correspond to different types of infinite number in the sense of non-standard analysis [24]. However, we wish here to focus on the alternative approach in which we have infinitesimals, but without the corresponding infinite numbers. It transpires that this is possible provided we work within the context of a topos; for example, a careful study of the proof that the only real number d such that $d^2 = 0$ is 0, shows that it involves the principle of excluded middle, which in general does not hold in the characteristic intuitionistic logic of a topos [25].

So in this second approach, called 'synthetic differential geometry', infinitesimals do not have reciprocals. Applying this approach to elementary real analysis, 'all goes smoothly'. For example, all functions are differentiable, with the linear approximation familiar from Taylor's theorem, $f(x + d) = f(x) + d\, f'(x)$, being exact. And in the context of synthetic differential geometry, a tangent vector on a manifold $\mathcal{M}$ is a map (more precisely, a 'morphism') from the object $D := \{d \mid d^2 = 0\}$ to $\mathcal{M}$. Furthermore, one can go on to apply this approach to the higher developments of calculus. The crucial question is whether or not there are any *physically* natural applications of synthetic differential geometry to physics; (as against 'merely rewriting' standard theories in synthetic terms).

PRESHEAVES FROM TOPOS THEORY

We recall the idea of a 'functor' between a pair of categories $\mathcal{C}$ and $\mathcal{D}$: this is a arrow-preserving function from one category to the other. The precise definition is as follows: a *covariant functor* $\mathbf{F}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a function that assigns to each $\mathcal{C}$-object A, a $\mathcal{D}$-object $\mathbf{F}(A)$; to each $\mathcal{C}$-arrow $f : B \to A$, a $\mathcal{D}$-arrow $\mathbf{F}(f) : \mathbf{F}(B) \to \mathbf{F}(A)$ such that $\mathbf{F}(\mathrm{id}_A) = \mathrm{id}_{\mathbf{F}(A)}$; and, if $g : C \to B$, and $f : B \to A$ then

$$\mathbf{F}(f \circ g) = \mathbf{F}(f) \circ \mathbf{F}(g). \tag{4}$$

A *presheaf* (also known as a *varying set*) on the category $\mathcal{C}$ is defined to be a covariant functor $\mathbf{X}$ from the category $\mathcal{C}$ to the category '**Set**' of normal sets. We want to make the collection of presheaves on $\mathcal{C}$ into a category, and therefore we need to define what is meant by an 'arrow' between two presheaves $\mathbf{X}$ and $\mathbf{Y}$. The intuitive idea is that such an arrow from $\mathbf{X}$ to $\mathbf{Y}$ must give a 'picture' of $\mathbf{X}$ within $\mathbf{Y}$. Formally, such an arrow is defined to be a *natural transformation* $N : \mathbf{X} \to \mathbf{Y}$, by which is meant a family of maps (called the *components* of N) $N_A : \mathbf{X}(A) \to \mathbf{Y}(A)$, A an object in $\mathcal{C}$, such that if $f : A \to B$ is an arrow in $\mathcal{C}$, then the composite map $\mathbf{X}(A) \xrightarrow{N_A} \mathbf{Y}(A) \xrightarrow{\mathbf{Y}(f)} \mathbf{Y}(B)$ is equal to $\mathbf{X}(A) \xrightarrow{\mathbf{X}(f)} \mathbf{X}(B) \xrightarrow{N_B} \mathbf{Y}(B)$. The category of presheaves on $\mathcal{C}$ equipped with these arrows is denoted $\mathrm{Set}^{\mathcal{C}}$.

We say that $\mathbf{K}$ is a *subobject* of $\mathbf{X}$ if there is an arrow in the category of presheaves (*i.e.*, a natural transformation) $i : \mathbf{K} \to \mathbf{X}$ with the property that, for each A, the component map $i_A : \mathbf{K}(A) \to \mathbf{X}(A)$ is a subset embedding, *i.e.*, $\mathbf{K}(A) \subseteq \mathbf{X}(A)$. The category of presheaves on $\mathcal{C}$, $\mathrm{Set}^{\mathcal{C}}$, forms a topos. We will not need the full definition of a topos, but we do need the idea that a topos has a subobject classifier Ω, to which we now turn.

Among the key concepts in presheaf theory is that of a 'sieve', which plays a central role in the construction of the subobject classifier in the topos of presheaves on a category $\mathcal{C}$ [4]. A *sieve* on an object A in $\mathcal{C}$ is defined to be a collection S of arrows $f : A \to B$ in $\mathcal{C}$ with the property that if $f : A \to B$ belongs to S, and if $g : B \to C$ is any arrow, then $g \circ f : A \to C$ also belongs to S. In the simple case where $\mathcal{C}$ is a poset, a sieve on $p \in \mathcal{C}$ is any subset S of $\mathcal{C}$ such that if $r \in S$ then (i) $p \leq r$, and (ii) $r' \in S$ for all $r \leq r'$; in other words, a sieve is nothing but a *upper* set in the poset.

The presheaf $\mathbf{\Omega} : \mathcal{C} \to \mathrm{Set}$ is now defined as follows. If A is an object in $\mathcal{C}$, then $\mathbf{\Omega}(A)$ is defined to be the set of all sieves on A; and if $f : A \to B$, then $\mathbf{\Omega}(f) : \mathbf{\Omega}(A) \to \mathbf{\Omega}(B)$ is defined as

$$\mathbf{\Omega}(f)(S) := \{h : B \to C \mid h \circ f \in S\} \tag{5}$$

for all $S \in \mathbf{\Omega}(A)$. For our purposes in what follows, it is important to note that if S is a sieve on A, and if $f : A \to B$ belongs to S, then from the defining property of a sieve we have

$$\mathbf{\Omega}(f)(S) := \{h : B \to C \mid h \circ f \in S\} = \{h : B \to C\} =: \uparrow B, \tag{6}$$

where $\uparrow B$ denotes the *principal* sieve on B, defined to be the set of all arrows in $\mathcal{C}$ whose domain is B. If $\mathcal{C}$ is a poset, the associated operation on sieves corresponds to a family of maps $\Omega_{qp} : \Omega_p \to \Omega_q$ (where Ω_p denotes the set of all sieves on p in the poset) defined by $\Omega_{qp} = \mathbf{\Omega}(i_{pq})$ if $i_{pq} : p \to q$ (*i.e.*, $p \leq q$). It is straightforward to check that if $S \in \Omega_q$, then

$$\Omega_{qp}(S) := \uparrow p \cap S \tag{7}$$

where $\uparrow p := \{r \in C \mid p \leq r\}$.

A crucial property of sieves is that the set $\mathbf{\Omega}(A)$ of sieves on A has the structure of a Heyting algebra. Recall that this is defined to be a distributive lattice, with null and unit elements, that is relatively complemented—which means that for any pair S_1, S_2 in $\mathbf{\Omega}(A)$, there exists an element $S_1 \Rightarrow S_2$ of $\mathbf{\Omega}(A)$ with the property that, for all $S \in \mathbf{\Omega}(A)$,

$$S \leq (S_1 \Rightarrow S_2) \text{ if and only if } S \wedge S_1 \leq S_2 \quad . \tag{8}$$

Specifically, $\mathbf{\Omega}(A)$ is a Heyting algebra where the unit element $1_{\mathbf{\Omega}(A)}$ in $\mathbf{\Omega}(A)$ is the principal sieve $\uparrow A$, and the null element $0_{\mathbf{\Omega}(A)}$ is the empty sieve $\emptyset$. The partial ordering in $\mathbf{\Omega}(A)$ is defined by $S_1 \leq S_2$ if, and only if, $S_1 \subseteq S_2$; and the logical connectives are defined as:

$$S_1 \wedge S_2 := S_1 \cap S_2 \tag{9}$$

$$S_1 \vee S_2 := S_1 \cup S_2 \tag{10}$$

$$S_1 \Rightarrow S_2 := \{ f : A \to B \mid \text{ for all } g : B \to C \text{ if } g \circ f \in S_1 \text{ then } g \circ f \in S_2 \ \}. \tag{11}$$

As in any Heyting algebra, the negation of an element S (called the *pseudo-complement* of S) is defined as $\neg S := S \Rightarrow 0$; so that

$$\neg S := \{ f : A \to B \mid \text{for all } g : B \to C, g \circ f \notin S \ \}. \tag{12}$$

The main distinction between a Heyting algebra and a Boolean algebra is that, in the former, the negation operation does not necessarily obey the law of excluded middle: instead, all that be can said is that, for any element S,

$$S \vee \neg S \leq 1. \tag{13}$$

It can be shown that the presheaf $\mathbf{\Omega}$ is a subobject classifier for the topos Set^{C}. That is to say, subobjects of any object $\mathbf{X}$ in this topos (*i.e.*, any presheaf on C) are in one-to-one correspondence with arrows $\chi : \mathbf{X} \to \mathbf{\Omega}$. This works as follows. First, let $\mathbf{K}$ be a subobject of $\mathbf{X}$. Then there is an associated *characteristic* arrow $\chi^{\mathbf{K}} : \mathbf{X} \to \mathbf{\Omega}$, whose 'component' $\chi^{\mathbf{K}}_A : \mathbf{X}(A) \to \mathbf{\Omega}(A)$ at each 'stage of truth' A in C is defined as

$$\chi^{\mathbf{K}}_A(x) := \{ f : A \to B \mid \mathbf{X}(f)(x) \in \mathbf{K}(B)\} \tag{14}$$

for all $x \in \mathbf{X}(A)$. That the right hand side of (14) actually *is* a sieve on A follows from the defining properties of a subobject.

Thus, in each 'branch' of the category C going 'upstream' from the stage A, $\chi^{\mathbf{K}}_A(x)$ picks out the first member B in that branch for which $\mathbf{X}(f)(x)$ lies in the subset $\mathbf{K}(B)$, then guarantees that $\mathbf{X}(h \circ f)(x)$ will lie in $\mathbf{K}(C)$ for all $h : B \to C$. Thus each 'stage of truth' A in C serves as a possible context for an assignment to each $x \in \mathbf{X}(A)$ of a generalized truth-value: which is a sieve, belonging to the Heyting algebra $\mathbf{\Omega}(A)$, rather than an element of the Boolean algebra $\{0, 1\}$ of normal set theory. This is the sense in which contextual, generalized truth-values arise naturally in a topos of presheaves.

There is a converse to (14): namely, each arrow $\chi : \mathbf{X} \to \mathbf{\Omega}$ (*i.e.*, a natural transformation between the presheaves $\mathbf{X}$ and $\mathbf{\Omega}$) defines a subobject $\mathbf{K}^{\chi}$ of $\mathbf{X}$ via

$$\mathbf{K}^{\chi}(A) := \chi_A^{-1}\{1_{\mathbf{\Omega}(A)}\} \tag{15}$$

at each stage of truth A. For the category of presheaves on C, a terminal object $\mathbf{1} : C \to \mathrm{Set}$ can be defined by $\mathbf{1}(A) := \{*\}$ at all stages A in C; if $f : A \to B$ is an arrow in C then $\mathbf{1}(f) : \{*\} \to \{*\}$ is defined to be the map $* \mapsto *$. This is indeed a terminal object since, for any presheaf $\mathbf{X}$, we can define a unique natural transformation $N : \mathbf{X} \to \mathbf{1}$ whose components $N_A : \mathbf{X}(A) \to \mathbf{1}(A) = \{*\}$ are the constant maps $x \mapsto *$ for all $x \in \mathbf{X}(A)$.

A global element (or point) of a presheaf $\mathbf{X}$ is also called a *global section*. As an arrow $\gamma : \mathbf{1} \to \mathbf{X}$ in the topos Set^{C}, a global section corresponds to a choice of an element $\gamma_A \in \mathbf{X}(A)$ for each stage of truth A in C, such that, if $f : A \to B$, the 'matching condition'

$$\mathbf{X}(f)(\gamma_A) = \gamma_B \tag{16}$$

is satisfied. As we shall see, the Kochen-Specher theorem can be read as asserting the non-existence of any global sections of certain presheaves that arises naturally in any quantum theory.

PRESHEAVES IN QUANTUM THEORY AND QUANTUM GRAVITY

We wish now to consider some possible applications of the idea of a topos in quantum physics. There are several natural orders in which to present these examples, but we will in fact proceed by first giving several examples involving space, time or spacetime, since: (i) in these examples, it is especially natural to think of the objects of the presheaf's base-category C as 'contexts' or 'stages' relative to which generalized truth-values are assigned; and (ii) these examples will serve as prototypes, in various ways, for later examples.

Throughout classical and quantum physics, we are often concerned with reference frames (or coordinate systems), the transformations between them, and the corresponding transformations on states of a physical system, and on physical quantities. Our first example will present in terms of presheaves some familiar material about reference frames in the context of non-relativistic wave mechanics.

Define the category of contexts C to have as its objects global Cartesian reference frames $e := \{e^1, e^2, e^3\}$ (where e^i, $i = 1, 2, 3$, are vectors in Euclidean 3-space E^3 such that $e^i \cdot e^j = \delta^{ij}$), all sharing a common origin; and define C to have as its arrows the orthogonal transformations $O(e, e')$ from one reference frame $\{e^i\}$ to another $\{e'^{\,i}\}$, *i.e.*, with a matrix representation $e'^{\,i} = \sum_{j=1}^{3} e^j O(e, e')^i_j$; (so that between any two objects, there is a unique arrow). Define a presheaf $\mathbf{H}$ as assigning to each object e in C, a copy $\mathbf{H}(e)$ of the Hilbert space $L^2(\mathbb{R}^3)$; and to each arrow $O(e, e')$, the unitary map $U(e, e') : \mathbf{H}(e) \to \mathbf{H}(e')$ defined by $(U(e, e')\psi)(x) := \psi(O(e, e')^{-1}(x))$ (so that $U(e, e')$ represents the action of $O(e, e')$ as a map from one copy, $\mathbf{H}(e)$, of the (pure) state-space $L^2(\mathbb{R}^3)$, to the other copy $\mathbf{H}(e')$). Any given $\psi \in L^2(\mathbb{R}^3)$, together with its transforms under the various unitary maps $U(e, e')$, defines a global section of $\mathbf{H}$. Discussions of the transformation of the wave-function under spatial rotations etc. normally identify the different copies of the state-space $L^2(\mathbb{R}^3)$; and from the viewpoint of those discussions, the above definition of $\mathbf{H}$ may seem at first sight to make a mountain out of a molehill, particularly since the category of contexts in this example is so trivial (for example, the internal logic is just the standard 'true-false' logic). But it is a helpful prototype to have in mind when we come to more complex or subtle examples. This definition has the advantage of clearly distinguishing the quantum state at the given time from its representing vectors ψ in various reference frames. We need to allow for the fact that the quantum state is a yet more abstract notion, also occurring in other representations than wave mechanics (position-representation). So the point is: this definition of $\mathbf{H}$ distinguishes the Schrödinger-picture, wave-mechanical representative of the quantum state at the given time—which it takes as a global section of $\mathbf{H}$—from its representing vectors ψ (elements of the global section at the various 'stages' e).

The example above illustrates a contextual aspect of standard quantum theory whereby the concrete representation of an abstract state depends on the observer; at least, this is so if we identify reference frames with observers. This contextual aspect is not emphasized in standard quantum theory since the different Hilbert spaces associated with different observers are all naturally isomorphic (via the unitary operators $U(e, e') : \mathbf{H}(e) \to \mathbf{H}(e')$). From a physical perspective, the fact that different observers, related by a translation or a rotation of reference frame, see 'equivalent' physics is a reflection of the homogeneity and isotropy of physical space. However, the situation might well be different in cosmological situations, since the existence of phenomena like event and particle horizons means that the physics perceptible from the perspective of one observer may be genuinely different from that seen by another. This suggests that any theory of quantum cosmology (or even quantum field theory in a fixed cosmological background) may require the use of more than one Hilbert space, in a way that cannot be 'reduced' to a single space.

It is well known that quantum field theory on a curved spacetime often requires more than one Hilbert space, associated with the unavoidable occurrence of inequivalent representations of the canonical commutation relations: this is one of the reasons for preferring a C^*-algebra approach. But what we have in mind is different—for example, our scheme could easily be adapted to involve a presheaf of C^*-algebras, each associated with an 'observer'. A key question in this context is what is meant by an 'observer'; or, more precisely, how this idea should be represented mathematically in the formalism. One natural choice might be a time-like curve (in the case of quantum field theory in a curved background with horizons), although this does suggest that a 'history' approach to quantum theory would be more appropriate than any of the standard ones. In the case of quantum cosmology proper, these issues become far more complex since—for example—even what is meant by a 'time-like curve' presumably becomes the subject of quantum fluctuations!

Let us fix once for all a global Cartesian reference frame in E^3, and define the base-category of contexts C to be the real line $\mathbb{R}$, representing time. That is to say, let the objects of C be instants $t \in \mathbb{R}$; and let there be an C-arrow from t to t', $f : t \to t'$, if and only if $t \leq t'$; so there is at most one arrow between any pair of objects t, t' in C. Define the presheaf, called $\mathbf{H}$, as assigning to each t a copy of the system's Hilbert space $\mathcal{H}$; ($\mathcal{H}$ need not be $L^2(\mathbb{R}^3)$—here we generalize from wave mechanics). Writing this copy as $\mathcal{H}_t$, we have $\mathbf{H}(t) := \mathcal{H}_t$. The action of $\mathbf{H}$ on C-arrows is defined by the Hamiltonian $\hat{H}$, via its one-parameter family of unitary exponentiations U_t. If $f : t \to t'$, then $\mathbf{H}(f) : \mathcal{H}_t \to \mathcal{H}_{t'}$ is defined by $U_{t'-t}$. The action of $U_{t'-t}$, then represents the Schrödinger-picture evolution of the system from time t to t'; and a total history of the system (as described in the given spatial coordinate system) is represented by a global section of the presheaf $\mathbf{H}$. We could similarly express in terms of presheaves Heisenberg-picture evolution: we would instead define a presheaf that assigned to each C-object t a copy of the set $B(\mathcal{H})$ of bounded self-adjoint operators on $\mathcal{H}$ (or say, a copy of some other fixed set taken as the algebra of observables), and then have the maps U_t induce Heisenberg-picture evolution on the elements of the copies of $B(\mathcal{H})$. A parallel discussion could be given for time evolution in classical physics: we would attach a copy of the phase-space Γ to each t, and a total history of the system (as described in the given spatial coordinate system) would be represented by a global section of the corresponding presheaf. It transpires that the development of such a 'history' approach to classical physics provides a very illuminating perspective on the mathematical structures used in the consistent-histories approach to quantum theory [26].

Now we will present in terms of presheaves some ideas that are currently being pursued in research on foundations of quantum theory and quantum gravity. The previous example admits an immediate generalization to the theory of causal sets. By a *causal set* we mean a partially-ordered set $\mathcal{P}$ whose elements represent spacetime points in a discrete, non-continuum model, and in which $p \leq q$, with $p, q \in \mathcal{P}$, means that q lies in the causal future of p. The set $\mathcal{P}$ is a natural base category for a presheaf of Hilbert spaces in which the Hilbert space at a point $p \in \mathcal{P}$ represents the quantum degrees of freedom that are 'localized' at that point/context. From another point of view, the Hilbert space at a point p could represent the history of the system (thought of now in a cosmological sense) as viewed from the perspective of an observer localized at that point (see [27]). The sieve, and hence logical, structure in this example is distinctly non-trivial.

PRESHEAVES FOR TOPOLOGICAL QUANTUM FIELD THEORY

Topological quantum field theory (TQFT) has a very well-known formulation in terms of category theory, and it is rather straightforward to see that this extends naturally to give a certain topos perspective.

Recall that in differential topology, two closed n-dimensional manifolds Σ_1 and Σ_2 are said to be *cobordant* if there is a compact $n+1$-manifold, M say, whose boundary ∂M is the disjoint union of Σ_1 and Σ_2. In TQFT, the n-dimensional manifolds are interpreted as possible models for physical space (so that spacetime has dimension $n+1$), and an interpolating $n+1$-manifold is thought of as describing a form of 'topology change' in the context of a (euclideanised) type of quantum gravity theory. In the famous Atiyah axioms for TQFT, a Hilbert space $\mathcal{H}_\Sigma$ is attached to each spatial n-manifold Σ, and to each cobordism from Σ_1 to Σ_2 there is associated a unitary map from $\mathcal{H}_{\Sigma_1}$ to $\mathcal{H}_{\Sigma_2}$.

The collection of all compact n-dimensional manifolds can be regarded as the set of objects in a category C, in which the arrows from an object Σ_1 to another Σ_2 are given by cobordisms from Σ_1 to Σ_2. The Atiyah axioms for TQFT can be viewed as a statement of the existence of a functor from C to the category of Hilbert spaces; indeed, this is how these axioms are usually stated. However, from the perspective being developed in the present paper, we see that we can also think of C as a 'category of contexts', in which case we have a natural presheaf reformulation of TQFT.

CONSISTENT HISTORIES FORMALISM FOR QUANTUM THEORY AND CONTINUOUS TIME

In the 'History Projection Operator' (HPO) version of the consistent-histories approach to quantum theory, propositions about the history of the system at a finite set of time points $(t_1, t_2, \ldots, t_n)$ are represented by projection operators on the tensor product $\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \cdots \otimes \mathcal{H}_{t_n}$ of n copies of the Hilbert space $\mathcal{H}$ associated with the system by standard quantum theory. The choice of this particular Hilbert space can be motivated in several different ways. The original motivation [9] was a desire to find a concrete representation of the temporal logic of such history propositions. This Hilbert space can be seen as the carrier of an irreducible representation of the 'history group' whose Lie algebra is (on the simplifying assumption that the system is a non-relativistic point particle moving in one dimension)

$$[x_{t_i}, x_{t_j}] = 0, \;\; [p_{t_i}, p_{t_j}] = 0, \;\; [x_{t_i}, p_{t_j}] = i\hbar\delta_{ij}, \tag{17}$$

where $i, j = 1, 2, \ldots, n$, and x_{t_i} (resp. p_{t_i}) is the Schrödinger-picture operator whose spectral projectors represent propositions about the position (resp. momentum) of the system at the time t_i. One advantage of the approach based on equations (17) is that it suggests an immediate generalization to the case of *continuous*-time histories: namely, the use of the history algebra

$$[x_t, x_{t'}] = 0, \;\; [p_t, p_{t'}] = 0, \;\; [x_t, p_{t'}] = i\hbar\tau\delta(t' - t), \tag{18}$$

where τ is a constant with the dimensions of time. This continuous-time history algebra has been studied by a variety of authors but here we will concentrate on Savvidou's observation [26] that the notion of 'time' appears in two ways that differ in certain significant respects. The main idea is to introduce a new time coordinate $s \in \mathbb{R}$, and to associate with it a Heisenberg picture defined from the time-averaged Hamiltonian $H = \int dt H_t$. Thus, in particular, one defines for the time-indexed position operator x_t

$$x_t(s) := \exp(isH/\hbar)\, x_t \, \exp(-isH/\hbar). \tag{19}$$

This new time s is *not* a difference in values of t. Rather, if one thinks of assigning a copy $\mathcal{H}_t$ of the system's (usual) Hilbert space $\mathcal{H}$ to each time t, then s parametrizes a Heisenberg-picture motion of quantities *within* $\mathcal{H}_t$. Accordingly, t is called 'external time', and s is called 'internal time'.

This formalism has been developed in various ways: in particular, there is a natural, dynamics-*independent* 'Liouville' operator that generates translations in the external time parameter. From our topos-theoretic perspective, we note that external time is more singular than internal time—as hinted by the delta-functions in t that occur in the history algebra's canonical commutation relations. This suggests modelling external time, not by the usual real numbers $\mathbb{R}$, but by the reals 'enriched' with infinitesimals in the sense of synthetic differential geometry, and which are related in some way to the action of the Liouville operator. This requires a non-standard model of the real line: in fact, we have to use a real number object in a topos. This use of a topos is quite different from, and in addition to, any development of a consistent-histories analogue of the temporal presheaf. In the latter case, the presheaf structure in the consistent-histories theory can arguably be related to ideas of state reduction of the kind discussed by von Neumann and Lüders (see [26]).

PRESHEAVES OF PROPOSITIONS AND VALUATIONS IN QUANTUM THEORY

In quantum theory, assumption (i), *i.e.*, that all quantities have real-number values, fails by virtue of the Kochen-Specker theorem; and assumption (ii), that one can measure any quantity ideally, is very problematic, involving as it does the notion of measurement. Standard quantum theory, with its 'eigenvalue-eigenstate link'—that in state ψ there is a value only for a quantity of which ψ is an eigenstate, viz. the eigenvalue—retains assumption (ii) only in the very limited sense that *if* the quantity A has a value, r say, according to the theory, *i.e.*, the (pure) state ψ is an eigenvector of $\hat{A}$ for eigenvalue r, then an ideal measurement of A would have result r. But setting aside this very special case, the theory faces the notorious 'measurement problem': the scarcity of values in the microrealm, due to the eigenvalue-eigenstate link, threatens to make the macrorealm indefinite ('Schrödinger's cat'). It is worth distinguishing two broad approaches to it, which we are called 'Literalism' and 'Extra Values'. For our topos-theoretic proposal will combine aspects of these approaches. They are:

1. **Literalism**. This approach aims to avoid the instrumentalism of standard quantum theory, and yet retain its scarcity of values (the eigenvalue-eigenstate link), while solving the measurement problem: not by postulating a non-unitary dynamics, but by a distinctively *interpretative* strategy. So far, there are two main forms of this approach: Everettian views (where the eigenvalue-eigenstate link is maintained 'within a branch'); and those based on quantum logic.

2. **Extra Values**. This approach gives up the eigenvalue-eigenstate link; but retains standard quantum theory's unitary dynamics for the quantum state. It postulates extra values (and equations for their time-evolution) for some quantities. The quantities getting these extra values are selected either *a priori*, as in the pilot-wave program, or by the quantum state itself, as in (most) modal interpretations.

The topos-theoretic proposal combines aspects of Literalism and Extra Values. Like both these approaches, the proposal is 'realist', not instrumentalist; (though it also shares with standard quantum theory, at least in its Bohrian or 'Copenhagen' version, an emphasis on contextuality). Like Extra Values (but unlike Literalism), it attributes values to quantities beyond those ascribed by the eigenvalue-eigenstate link. Like Literalism (but unlike Extra Values), these additional values are naturally defined by the orthodox quantum formalism. More specifically: *all* quantities get additional values (so no quantity is somehow 'selected' to get such values); any quantum state defines such a valuation, and any such valuation obeys an appropriate version of the *FUNC*. The 'trick', whereby such valuations avoid no-go theorems like the Kochen-Specker theorem [19], is that the truth value ascribed to a proposition about the value of a physical quantity is not just 'true' or 'false'!

Thus consider the proposition "$A \in \Delta$", saying that the value of the quantity A lies in a Borel set $\Delta \subseteq \mathbb{R}$. Roughly speaking, any such proposition is ascribed as a truth-value a set of coarse-grainings, $f(\hat{A})$, of the operator $\hat{A}$ that represents A. Exactly which coarse-grainings are in the truth-value depends in a precise and natural way on Δ and the quantum state ψ: in short, $f(\hat{A})$ is in the truth-value iff ψ is in the range of the spectral projector $\hat{E}[f(A) \in f(\Delta)]$. Note the contrast with the eigenstate-eigenvalue link: our requirement is not that ψ be in the range of $\hat{E}[A \in \Delta]$, but a weaker requirement. For $\hat{E}[f(A) \in f(\Delta)]$ is a larger spectral projector; *i.e.*, in the lattice $\mathcal{L}(H)$ of projectors on the Hilbert space $\mathcal{H}$, $\hat{E}[A \in \Delta] < \hat{E}[f(A) \in f(\Delta)]$. So the new proposed truth-value of "$A \in \Delta$" is given by the set of weaker propositions "$f(A) \in f(\Delta)$" that are true in the old (*i.e.*, eigenstate-eigenvalue link) sense. To put it a bit more exactly: the new proposed truth-value of "$A \in \Delta$" is given by the set of quantities $f(A)$ for which the corresponding weaker proposition "$f(A) \in f(\Delta)$" is true in the old (*i.e.*, eigenstate-eigenvalue link) sense. To put it *less* exactly, but more memorably: the new truth-value of a proposition is given by the set of its consequences that are true in the old sense.

Let us introduce the set $\mathcal{O}$ of all bounded self-adjoint operators $\hat{A}, \hat{B}, \ldots$ on the Hilbert space $\mathcal{H}$ of a quantum system. We turn $\mathcal{O}$ into a category by defining the objects to be the elements of $\mathcal{O}$, and saying that there is an arrow from $\hat{A}$ to $\hat{B}$ if there exists a real-valued function f on $\sigma(\hat{A}) \subset \mathbb{R}$, the spectrum of $\hat{A}$, such that $\hat{B} = f(\hat{A})$ (with the usual definition of a function of a self-adjoint operator, using the spectral representation). If $\hat{B} = f(\hat{A})$, for some $f : \sigma(\hat{A}) \to \mathbb{R}$, then the corresponding arrow in the category $\mathcal{O}$ will be denoted $f_{\mathcal{O}} : \hat{A} \to \hat{B}$. Define two presheaves on the category $\mathcal{O}$, called the *dual presheaf* and the *coarse-graining presheaf* respectively. The former affords an elegant formulation of the Kochen-Specker theorem, namely as a statement that the dual presheaf does not have global sections. The latter is at the basis of our proposed generalized truth-value assignments. The dual presheaf on $\mathcal{O}$ is the covariant functor $\mathbf{D} : \mathcal{O} \to \mathrm{Set}$ defined as follows:

1. On objects: $\mathbf{D}(\hat{A})$ is the *dual* of W_A, where W_A is the spectral algebra of the operator $\hat{A}$; *i.e.* W_A is the collection of all projectors onto the subspaces of $\mathcal{H}$ associated with Borel subsets of $\sigma(\hat{A})$. That is to say: $\mathbf{D}(\hat{A})$ is defined to be the set $\mathrm{Hom}(W_A, \{0, 1\})$ of all homomorphisms from the Boolean algebra W_A to the Boolean algebra $\{0, 1\}$.

2. On arrows: If $f_O : \hat{A} \to \hat{B}$, so that $\hat{B} = f(\hat{A})$, then $\mathbf{D}(f_O) : D(W_A) \to D(W_B)$ is defined by $\mathbf{D}(f_O)(\chi) := \chi|_{W_{f(A)}}$ where $\chi|_{W_{f(A)}}$ denotes the restriction of $\chi \in D(W_A)$ to the subalgebra $W_{f(A)} \subseteq W_A$.

A global element (global section) of the functor $\mathbf{D} : O \to$ Set is then a function γ that associates to each $\hat{A} \in O$ an element γ_A of the dual of W_A such that if $f_O : \hat{A} \to \hat{B}$ (so $\hat{B} = f(\hat{A})$ and $W_B \subseteq W_A$), then $\gamma_A|_{W_B} = \gamma_B$. Thus, for all projectors $\hat{\alpha} \in W_B \subseteq W_A$,

$$\gamma_B(\hat{\alpha}) = \gamma_A(\hat{\alpha}). \tag{20}$$

Since each $\hat{\alpha}$ in the lattice $\mathcal{L}(\mathcal{H})$ of projection operators on $\mathcal{H}$ belongs to at least one such spectral algebra W_A (for example, the algebra $\{\hat{0}, \hat{1}, \hat{\alpha}, \hat{1} - \hat{\alpha}\}$) it follows from (20) that a global section of $\mathbf{D}$ associates to each projection operator $\hat{\alpha} \in \mathcal{L}(\mathcal{H})$ a number $V(\hat{\alpha})$ which is either 0 or 1, and is such that, if $\hat{\alpha} \wedge \hat{\beta} = \hat{0}$, then $V(\hat{\alpha} \vee \hat{\beta}) = V(\hat{\alpha}) + V(\hat{\beta})$. In other words, a global section γ of the presheaf $\mathbf{D}$ would correspond to an assignment of truth-values $\{0, 1\}$ to all propositions of the form "$A \in \Delta$", which obeyed the *FUNC* condition (20). These are precisely the types of valuation prohibited, provided that $\dim \mathcal{H} > 2$, by the Kochen-Specker theorem. So an alternative way of expressing the Kochen-Specker theorem is that, if $\dim \mathcal{H} > 2$, the dual presheaf $\mathbf{D}$ has no global sections.

However, we *can* use the subobject classifier $\mathbf{\Omega}$ in the topos Set^{O} of all presheaves on O to assign *generalized* truth-values to the propositions "$A \in \Delta$". These truth-values will be sieves; and since they will be assigned relative to each 'context' or 'stage of truth' $\hat{A}$ in O, these truth-values will be contextual as well as generalized. Because in any topos the subobject classifier $\mathbf{\Omega}$ is fixed by the structure of the topos, $\mathbf{\Omega}$ is unique up to isomorphism. Thus the family of associated truth-value assignments is fixed, and the traditional objection to multi-valued logics—that their structure often seems arbitrary—does not apply to these generalized, contextual truth-values. Define the appropriate presheaf of propositions. The *coarse-graining presheaf* over O is the covariant functor $\mathbf{G} : O \to$ Set defined as follows:

1. *On objects in O:* $\mathbf{G}(\hat{A}) := W_A$, where W_A is the spectral algebra of $\hat{A}$.
2. *On arrows in O:* If $f_O : \hat{A} \to \hat{B}$ (*i.e.*, $\hat{B} = f(\hat{A})$), then $\mathbf{G}(f_O) : W_A \to W_B$ is defined as $\mathbf{G}(f_O)(\hat{E}[A \in \Delta]) := \hat{E}[f(A) \in f(\Delta)]$, where, if $f(\Delta)$ is not Borel, the right hand side is to be understood in the sense of Theorem 4.1 of [12]—a measure-theoretic nicety that we shall not discuss here.

We call a function ν that assigns to each choice of object $\hat{A}$ in O and each Borel set $\Delta \subseteq \sigma(\hat{A})$, a sieve of arrows in O on $\hat{A}$ (*i.e.*, a sieve of arrows with $\hat{A}$ as domain), a *sieve-valued valuation* on $\mathbf{G}$. We write the values of this function as $\nu(A \in \Delta)$. One could equally well write $\nu(\hat{E}[A \in \Delta])$, provided one bears in mind that the value depends not only on the projector $\hat{E}[A \in \Delta]$, but also on the operator (context) $\hat{A}$ of whose spectral family the projector is considered to be a member. A natural desideratum for any kind of valuation on a presheaf of propositions such as $\mathbf{G}$ is that the valuation should specify a subobject of $\mathbf{G}$. For in logic one often thinks of a valuation as specifying the 'selected' or 'winning' propositions: in this case, the 'selected' elements $\hat{E}[A \in \Delta]$ in each $\mathbf{G}(\hat{A})$. So it is natural to require that the elements that a valuation 'selects' at the various contexts $\hat{A}$ together define a subobject of $\mathbf{G}$. Subobjects are in one-one correspondence with arrows, *i.e.*, natural transformations, $N : \mathbf{G} \to \mathbf{\Omega}$. So it is natural to require a sieve-valued valuation ν to define such a natural transformation by the equation $N^{\nu}_{A}(\hat{E}[A \in \Delta]) := \nu(A \in \Delta)$. This desideratum leads directly to the analogue for presheaves of the famous functional composition condition of the Kochen-Specker theorem [19], called *FUNC* above: and which we will again call *FUNC* in the setting of presheaves. A sieve-valued valuation defines such a natural transformation iff it obeys (the presheaf version of) *FUNC*.

Let us recall that the subobject classifier $\mathbf{\Omega}$ 'pushes along' sieves, according to (5). For the category O, this becomes: if $f_O : \hat{A} \to \hat{B}$, then $\mathbf{\Omega}(f_O) : \mathbf{\Omega}(\hat{A}) \to \mathbf{\Omega}(\hat{B})$ is defined by

$$\mathbf{\Omega}(f_O)(S) := \{h_O : B \to C \mid h_O \circ f_O \in S\} \tag{21}$$

for all sieves $S \in \mathbf{\Omega}(\hat{A})$. Accordingly, we say that a sieve-valued valuation ν on $\mathbf{G}$ satisfies *generalized functional composition*—for short, *FUNC*—if for all $\hat{A}, \hat{B}$ and $f_O : \hat{A} \to \hat{B}$ and all $\hat{E}[A \in \Delta] \in \mathbf{G}(\hat{A})$, the valuation obeys

$$\nu(B \in \mathbf{G}(f)(\hat{E}[A \in \Delta])) \equiv \nu(f(A) \in f(\Delta)) = \mathbf{\Omega}(f_O)(\nu(A \in \Delta)). \tag{22}$$

The *FUNC* is exactly the condition a sieve-valued valuation must obey in order to thus define a natural transformation, *i.e.*, a subobject of $\mathbf{G}$, by the natural equation $N^{\nu}_{A}(\hat{E}[A \in \Delta]) := \nu(A \in \Delta)$. That is: A sieve-valued valuation ν on $\mathbf{G}$ obeys *FUNC* if and only if the functions at each 'stage of truth' $\hat{A}$

$$N^{\nu}_{\hat{A}}(\hat{E}[A \in \Delta]) := \nu(A \in \Delta) \tag{23}$$

define a natural transformation N^{ν} from $\mathbf{G}$ to $\mathbf{\Omega}$. With any quantum state there is associated such a *FUNC*-obeying sieve-valued valuation. Furthermore, this valuation gives the natural generalization of the eigenvalue-eigenstate link, that is, a quantum state ψ induces a sieve on each $\hat{A}$ in O by the requirement that an arrow $f_O : \hat{A} \to \hat{B}$ is in the sieve iff ψ is in the range of the spectral projector $\hat{E}[B \in f(\Delta)]$. To be precise, we define for any ψ, and any Δ a Borel subset of the spectrum $\sigma(\hat{A})$ of $\hat{A}$:

$$\begin{aligned} \nu^{\psi}(A \in \Delta) &:= \{ f_O : \hat{A} \to \hat{B} \mid \hat{E}[B \in f(\Delta)]\psi = \psi \} \\ &= \{ f_O : \hat{A} \to \hat{B} \mid \mathrm{Prob}(B \in f(\Delta); \psi) = 1 \}, \end{aligned} \tag{24}$$

where $\mathrm{Prob}(B \in f(\Delta); \psi)$ is the usual Born-rule probability that the result of a measurement of B will lie in $f(\Delta)$, given the state ψ. This definition generalizes the eigenstate-eigenvalue link, in the sense that we require not that ψ be in the range of $\hat{E}[A \in \Delta]$, but only that it be in the range of the larger projector $\hat{E}[f(A) \in f(\Delta)]$. One can check that the definition satisfies *FUNC*, and also has other properties that it is natural to require of a valuation discussed in [12, 13, 28].

TOPOSES AND THEORY OF RELATIVITY

The system of axioms for the Special theory of relativity contains fewer primary notions and relations, is simple, and lead directly to the ultimate goal (see review [29]). In the case of the General relativity it is difficult to introduce a smoothness (see [30, 31, 32]).

Does the unified way of axiomatization of these different physical theories exist? Does the unified way of axiomatization of these different physical theories exist? The language of topos theory [8, 7] gives the unified way of axiomatization of the Special and General Relativity, the axioms being the same in both cases. Selecting one or another physical theory amounts to selecting a concrete topos. Here we give a topos-theoretic causal theory of space-time.

Let $\mathcal{E}$ be an elementary topos with an object of natural numbers, and let R_T be the object of continuous real numbers [33]. *An affine morphism* $\alpha : R_T \to R_T$ is a finite composition of morphisms of the form 1_{R_T}, $\otimes \circ (\lambda \times 1_{R_T}) \circ j$, $\oplus \circ (1_{R_T} \times \mu) \circ j$, where $\oplus, \otimes$ are the operations of addition and multiplication in R_T respectively, λ, μ are arbitrary elements in R_T, and $j : R_T \simeq 1 \times R_T$ is an isomorphism. Let Γ be the set of all affine morphisms from R_T to R_T. An *affine object* in $\mathcal{E}$ is an object a together with two sets of morphisms:

$$\Phi \subset \mathrm{Hom}_{\mathcal{E}}(R_T, a), \quad \Psi \subset \mathrm{Hom}_{\mathcal{E}}(a, R_T)$$

such that the following conditions hold:

1) For any $\phi \in \Phi, \psi \in \Psi$ there is $\psi \circ \phi \in \Gamma$.
2) If $f \in \mathrm{Hom}_{\mathcal{E}}(R_T, a) \setminus \Phi$ then there exists $\psi \in \Psi$ such that $\psi \circ f \notin \Gamma$.
3) If $f \in \mathrm{Hom}_{\mathcal{E}}(a, R_T) \setminus \Psi$ then there exists $\phi \in \Phi$ such that $f \circ \phi \notin \Gamma$.
4) For any monomorphisms $f : \Omega \mapsto a, g : \Omega \mapsto R_T$ there exists $\phi \in \Phi$ such that $\phi \circ g = f$.
5) For any monomorphisms $f : \Omega \mapsto a, g : \Omega \mapsto R_T$ there exists $\psi \in \Psi$ such that $\psi \circ f = g$.

Here Ω is the subobject classifier in $\mathcal{E}$. An affine object in category **Set** is the set equipped with an affine structure [34]. In the topos $\mathbf{Bn}(M)$ and in the spatial topos $\mathbf{Top}(M)$ (see notations in [8]), an affine object is a fiber bundle with base M and affine space as fibers.

A categorical description of the Relativity means the introduction of the Lorentz structure either in an affine space or in a fiber bundle with affine spaces as fibers, which can be done by defining in the affine space a family of equal and parallel elliptic cones or a relativistic elliptic conal order [35] (we use the notations from [8]).

Let a be an affine object in the topos $\mathcal{E}$. *An order* in a is an object P together with a collection of subobjects $p_x : P \mapsto a$, where $x : 1 \to a$ is an arbitrary element, such that:

1) $x \in p_x$.
2) If $y \in p_x$, then $z \in p_y$ implies $z \in p_x$.

The order $\langle P, \{p_x\} \rangle$ is denoted as O. A morphism $f : a \to a$ is called *affine*, if $\psi \circ f \circ \phi \in \Gamma$ for any $\phi \in \Phi$ and $\psi \in \Psi$. We denote the set of all affine morphisms by Aff (a).

Let $\mathcal{A} \subset$ Aff (a) consist of all commuting morphisms. An order O is *invariant with respect to* $\mathcal{A}$ if for any p_x, p_y there exists $g_{xy} \in \mathcal{A}$ such that $g_{xy} \circ p_x = p_y$. A morphism $f : a \to a$ *preserves* an order O, if for each p_x there exists p_y such that $f \circ p_x = p_y$. The collection of all morphisms preserving an order O that is invariant with respect to $\mathcal{A}$ is denoted by $Aut(O)$. A *ray* is a morphism $\lambda : R_+ \mapsto R_T \xrightarrow{\varphi} a$, where $\phi \in \Phi_0 \subset \Phi$, and for any $\phi \in \Phi_0$ there is no $x : 1 \to a$ such that $\phi = x \circ !$. Here $! : R_T \to 1$ and R_+ is the subobject of object R_T consisting of those t for which $0 \leq t$ (see definition of order in R_T in [33]). An order O is called *conic* if 1) for every $y \in p_x$ there exists a ray $\lambda \subset p_x$ such that $x, y \in \lambda$, and 2) x is the origin of λ, i.e. if μ is a ray and $y \in \mu \subset \lambda$, $\mu \neq \lambda$, then $x \notin \mu$. An order O has *the acute vertex* or *pointed* one if for each p_x there does not exist $\phi_x \in \Phi_0$ such that $\phi_x \subset p_x$. An order O is *complete*, if for any element $z : 1 \to a$ and p_x there exist different elements $u_x, v_x : 1 \to a$ and $\phi \in \Phi_0$ such that $z, u_x, v_x \in \phi$ and $u_x, v_x \in p_x$. An element $u \in p_x$ is called *extreme* if there exists $\phi \in \Phi_0$ for which $u \in \phi$, but $y \notin \phi$ for all $y \in p_x, y \neq u$. A conic order O is said to be *strict* if, for each nonextreme element $u \in p_x$, and $v \in p_x, v \neq u$, and each ray λ with origin u such that $v \in \lambda$, there exists an extreme element $w \in \lambda$, and $w \in p_x$. An affine object a with an order O, which is complete, strict, conic, has an acute vertex, and is invariant with respect to $\mathcal{A}$ is said to be *Lorentz* if for each $x : 1 \to a$ and each extreme elements $u, v \in p_x$, where $u, v \neq x$ there exists a $f \in Aut(O)$ such that $f \circ u = v$, $f \circ x = x$. A Lorentz object in the topos **Set** is an affine space admitting a pseudo-Euclidean structure defined by a quadratic form $x_0^2 - \sum_{i=1}^n x_i^2$, where n is finite or equal to ∞, and $Aut(O)$ is the Poincaré group (see [35]). A Lorentz object in the topos **Top**(M) is a fiber bundle over M with fibers equipped with an affine structure and a continuous pseudo-Euclidean structure of finite or infinite dimension. It is quite possible to take not only the toposes **Set**, **Bn**(M), or **Top**(M), but also any others that have an affine object.

The existing categorical determination of the set theory and determination of **Top**(M) between elementary toposes gives the possibility to speak about the solution of problem of categorical description of the Theory of Relativity. If $\mathcal{E}$ is a well-pointed topos satisfying the axiom of partial transitivity with a Lorentz object a, then $\mathcal{E}$ is a model of set theory Z and a is a model of the Special Relativity. If $\mathcal{E}$ is a topos defined over **Set** that has enough points and satisfies the axiom (SG) (see [7]) with a Lorentz object a, then $\mathcal{E}$ is a topos **Top**(M) and a is a model of the General Relativity.

REGULAR OBSTRUCTED CATEGORIES AND TOPOSES

Let us describe the concept of n-regularity (introduced in [36, 37] for supermanifolds) in the topos theory. We use the previously developed notions of obstructed categories, functors, natural transformations (defined in [15, 38, 16, 39]) as a certain nonstandard topos. All definitions of are in general case the same like in the usual topos theory [7, 8, 4], but the preservation of the identity id_X, is replaced by the requirement of preservation of obstructions $\mathfrak{e}_X^{(n)}$ and certain compatibility conditions are added.

Let $\mathfrak{C}$ be a topos [7, 8]. An *n-regular cocycle* (X, f) in $\mathfrak{C}$ is a sequence of composable arrows in $\mathfrak{C}$

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_1, \tag{25}$$

such that

$$\begin{aligned} f_1 \circ f_n \circ \cdots \circ f_2 \circ f_1 &= f_1, \\ f_2 \circ f_1 \circ \cdots \circ f_3 \circ f_2 &= f_2, \\ &\\ f_n \circ f_{n-1} \circ \cdots \circ f_1 \circ f_n &= f_n, \end{aligned} \tag{26}$$

and

$$\begin{aligned} \mathfrak{e}_{X_1}^{(n)} &:= f_n \circ \cdots \circ f_2 \circ f_1 \in \mathrm{End}\,(X_1), \\ \mathfrak{e}_{X_2}^{(n)} &:= f_1 \circ \cdots \circ f_3 \circ f_2 \in \mathrm{End}\,(X_2), \\ &\\ \mathfrak{e}_{X_n}^{(n)} &:= f_{n-1} \circ \cdots \circ f_1 \circ f_n \in \mathrm{End}\,(X_n). \end{aligned} \tag{27}$$

Let $(X, f), (Y, g)$ be two n-regular cocycles in $\mathfrak{C}$. An n-regular cocycle morphism $\alpha : (X, f) \to (Y, g)$ is a sequence of morphisms $\alpha := (\alpha_1, \ldots, \alpha_n)$ such that we have the relation

$$\alpha_i \circ f_i = g_i \circ \alpha_i \tag{28}$$

for every $i = 1, ..., n, 1$. If every component α_i of α is invertible, then α is said to be an n-regular cocycle equivalence. It is obvious that the n-regular cocycle equivalence is an equivalence relation. We postulate that all definitions are formulated n-regular cocycles in $\mathfrak{C}$ up to the n-regular cocycle equivalence, and $i = 1, 2, ..., n$. Let (X, f) be an n-regular cocycle in $\mathfrak{C}$, then the correspondence $\mathfrak{e}_X^{(n)} : X_i \in \mathfrak{C}_0 \mapsto \mathfrak{e}_{X_i}^{(n)} \in \mathrm{End}\,(X_i)$, $i = 1, 2, \ldots, n$, is called an n-regular cocycle obstruction structure on (X, f) in $\mathfrak{C}$. We have the following relations

$$f_i \circ \mathfrak{e}_{X_i}^{(n)} = f_i, \quad \mathfrak{e}_{X_{i+1}}^{(i)} \circ f_i = f_i, \quad \mathfrak{e}_{X_i}^{(n)} \circ \mathfrak{e}_{X_i}^{(n)} = \mathfrak{e}_{X_i}^{(n)} \tag{29}$$

for $i = 1, 2, \ldots, n (\mathrm{mod}\ n + 1)$. Let (X, f) be an n-regular sequence in $\mathfrak{C}$. An n-regular subcocycle (Y, g) of (X, f) is an n-regular cocycle of the following forms

$$Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} Y_n \xrightarrow{g_n} Y_1, \tag{30}$$

where Y_i is a subobject of X_i. Let $(X, f), (Y, g)$ be two n-regular cocycles in $\mathfrak{C}$. An n-regular cocycle product $(X \times Y, f \times g)$ of (X, f) and (Y, g) is an n-regular cocycle

$$X_1 \times Y_1 \xrightarrow{f_1 \times g_1} X_2 \times Y_2 \xrightarrow{f_2 \times g_2} \cdots \xrightarrow{f_{n-1} \times g_{n-1}} X_n \times Y_n \xrightarrow{f_n \times g_n} X_1 \times Y_1. \tag{31}$$

An n-regular obstructed category is a directed graph $\mathfrak{C}$ with an associative composition and such that every object is a component of an n-regular cocycle [16, 39].

Let $\mathfrak{C}$ and $\mathfrak{D}$ be two n-regular obstructed categories. We postulate that all definitions are formulated on every n-regular cocycle (X, f) in $\mathfrak{C}$ up to the n-regular cocycle equivalence, and $i = 1, 2, \ldots$(mod n).

An or n-regular cocycle functor $\mathcal{F}^{(n)} : \mathfrak{C} \to \mathfrak{D}$ is a pair of mappings $(\mathcal{F}_0^{(n)}, \mathcal{F}_1^{(n)})$, where $\mathcal{F}_0^{(n)}$ sends objects of $\mathfrak{C}$ into objects of $\mathfrak{D}$, and $\mathcal{F}_1^{(n)}$ sends morphisms of $\mathfrak{C}$ into morphisms of $\mathfrak{D}$ such that [15, 16]

$$\mathcal{F}_1^{(n)}(f_i \circ f_{i+1}) = \mathcal{F}_1^{(n)}(f_i) \circ \mathcal{F}_1^{(n)}(f_{i+1}), \quad \mathcal{F}_1^{(n)}\left(\mathfrak{e}_{X_i}^{(n)}\right) = \mathfrak{e}_{\mathcal{F}_0(X_i)}^{(n)}, \tag{32}$$

where $X \in \mathfrak{C}_0$. Let $\mathfrak{C}$ and $\mathfrak{D}$ be n-regular obstructed categories, and let

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_1 \tag{33}$$

be an n-regular cocycle in $\mathfrak{C}$. If $\mathcal{F}^{(n)} : \mathfrak{C} \to \mathfrak{D}$ is n-regular cocycle functor, then

$$\mathcal{F}^{(n)}(f_i) \circ \mathfrak{e}_{X_i}^{(n)} = \mathcal{F}^{(n)}(f_i). \tag{34}$$

It is a simple calculation $\mathcal{F}^{(n)}(f_i) = \mathcal{F}^{(n)}\left(f_i \circ \mathfrak{e}_{X_i}^{(n)}\right) = \mathcal{F}^{(n)}(f_i) \circ \mathcal{F}^{(n)}\left(\mathfrak{e}_{X_i}^{(n)}\right) = \mathcal{F}^{(n)}(f_i) \circ \mathfrak{e}_{\mathcal{F}_0 X_i}^{(n)}$. Let $\mathcal{F}^{(n)}$ and $\mathcal{G}^{(n)}$ be two n–regular cocycle morphisms of the category $\mathfrak{C}$ into the category $\mathfrak{D}$. An n-regular natural transformation $s : \mathcal{F}^{(n)} \to \mathcal{G}^{(n)}$ of $\mathcal{F}^{(n)}$ into $\mathcal{G}^{(n)}$ is a collection of functors $s = \{s_{X_i} : \mathcal{F}_0(X_i) \to \mathcal{G}_0(X_i)\}$ such that

$$s_{X_{i+1}} \circ \mathcal{F}_1^{(n)}(f_i) = \mathcal{G}_1^{(n)}(f_i) \circ s_{X_i}, \tag{35}$$

for $f_i : X_i \to X_{i+1}$. There is an obstructed topos $(\mathfrak{C}, \mathfrak{e}^{(n)})$ equipped with an obstruction structure $\mathfrak{e}_X^{(n)} : X \in \mathfrak{C}_0 \mapsto \mathfrak{e}_X \in \mathrm{End}\,(X)$ for every n-regular cocycle (X, f) in $\mathfrak{C}$. Let $\mathfrak{C}$ be an n-regular obstructed category. This means that for every object X in $\mathfrak{C}$, there is n-regular cocycle (X, f) and the corresponding obstruction structure $\mathfrak{e}_X^{(n)} : X_i \in \mathfrak{C}_0 \mapsto \mathfrak{e}_{X_i} \in \mathrm{End}\,(X_i), i = 1, 2, \ldots, n$. Let us describe the obstruction structure for subobjects and products. If (Y, g) is a subcocycle of (X, f), then the corresponding obstruction structure $\mathfrak{e}_Y^{(n)} : Y_i \in \mathfrak{C}_0 \mapsto \mathfrak{e}_{Y_i}^{(n)} \in \mathrm{End}\,(Y_i)$ is well defined if and only if $\mathfrak{e}_{Y_i}^{(n)}$ is the restriction of $\mathfrak{e}_{X_i}^{(n)}$ to Y_i. One can describe products and the terminal object.

An n-regularization $\mathfrak{Reg}^{(n)}(\mathfrak{C})$ of $\mathfrak{C}$ is a collection of all n-regular cocycles in $\mathfrak{C}$ and corresponding n-regular cocycle morphisms up to an n-regular cocycle equivalence. There is a topos $\mathfrak{Reg}^{(n)}(\mathfrak{C})$ called n-regular topos on $\mathfrak{C}$. Indeed, the n-regular cocycle equivalence is an equivalence relation. Equivalence classes of this relation are just elements of $\mathfrak{Reg}^{(n)}(\mathfrak{C})$. Our n-regular cocycles and obstruction structures are unique up to the equivalence. For every equivalence class of n-regular cocycles, there is the corresponding class of n-regular cocycle obstruction structure on it. The correspondence is a one to one. Subobjects are given by subcocycles, and product are the above cocycle product. One can describe the generalized truth object. All is up to an equivalence. Finally, one can introduce the notion of n-regular obstructed presheaves, Heyting algebras in an analogous way. It should be important to a study of the noninvertible histories approach to a quantum physics and related topics. It will be done explicitly in forthcoming publications.

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APPENDIX

Number systems in synthetic differential geometry

The Naturals, Rationals and Cauchy Reals. The naturals, $\mathbb{N}$, in any topos that possesses them, can be thought of in exactly the same way as the naturals we are all familiar with. Showing this is another matter, and is left to texts such as [7]. From the natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$ and rationals $\mathbb{Q}$ can be easily constructed. These objects also behave as expected. Classical reasoning is entirely permissible when considering the arithmetic of these number systems, as they are all decidable objects. So too is the object of Cauchy reals $\mathbb{R}_C$, constructed as equivalence classes of sequences of rationals. As an example, the topos of sheaves over $[0, 1]$ has a natural numbers object. It is given by

$$\mathbb{N}(U) := \{ f : U \longrightarrow \mathbb{N} \mid f \text{ is continuous} \}$$
$$\mathbb{N}(V \xrightarrow{i} U)(f) := f|_V$$

Note that in the topos of sheaves over $[a, b]$, where $a = b$, the natural numbers object is the same as that of set theory.

The Smooth Reals. The smooth real object was discovered inhabiting many toposes. Some of its properties were abstracted to provide the foundation for synthetic differential geometry, however the smooth real object of one topos generally behaves slightly differently to those of others. The axiomatic scheme we assume for the smooth reals is as follows. **R** is to model the axioms (A1) - (A15) proposed by I. Moerdijk and G. E. Reyes, listed on pages 295-298 of their work [40]. We use $U(\mathbf{R})$ to denote the subobject of invertible elements. The most relevant of those axioms for this paper are the following:

Axiom [A1]. **R** is a commutative ring with unit.

Axiom [A2]. **R** is local, such that $0 \neq 1$ $(\forall x)$.

Axiom [A3]. $(\mathbf{R}, <)$ is a Euclidean ordered local ring, such that

$$
\begin{aligned}
&0 < 1\\
&(0 < x) \wedge (0 < y) \Rightarrow (0 < x + y) \wedge (0 < xy)\\
&x \in U(\mathbf{R}) \Leftrightarrow 0 < x \vee x < 0\\
&(0 < x) \Rightarrow \exists y(x = y^2)
\end{aligned}
$$

We take the following additional field axiom, relating distinguishability and invertibility, to hold true for our ring **R**.

Axiom [Field]. **R** is a field in the following sense $(\forall x_1, \ldots, x_n \in \mathbf{R})$.

Also assumed is the following topological axiom

Axiom [Open Cover]. $(\forall x \in \mathbf{R})$.

This says that the object $\{(\leftarrow, 1), (0, \rightarrow)\}$ is an open cover. Fundamental to synthetic differential geometry are the differentiation and integration axioms

Axiom [Kock-Lawvere]. For each $f : \mathbf{D} \longrightarrow \mathbf{R}$, there exists a unique $b \in \mathbf{R}$, such that for every $d \in \mathbf{D}$ one has $f(d) = f(0) + d.b$.

Axiom [Integration]. For each $f : \mathbf{R} \longrightarrow \mathbf{R}$, there exists a unique $F : \mathbf{R} \longrightarrow \mathbf{R}$, such that $F'(x) = f(x)$, $F(0) = 0$.

The well-adapted models of synthetic differential geometry discovered are all Grothendieck toposes (categories of sheaves over sites), and extensive studies of the synthetic differential geometry aspects of these toposes (and others) have been made in [40]. Those readers who wish to examine the toposes are referred to this source. The studies have revealed some of the variable behaviors of the smooth reals as one moves from one topos to another. It is inevitable that there will be preferred choices of topos for the construction of quantum mechanics within.

It is found that the most basic topos modelling parts of synthetic differential geometry ($\mathbf{Sets}^{\mathbb{L}^{op}}$) does not possess the nicest of properties regarding **R**. For example, in this topos, the interval $[0, 1]$ is not compact, **R** is not a local ring, and many of the axioms of the previous section are not modelled. It is difficult to imagine this smooth real object modelling the space around us. The toposes $\mathcal{F}$ and $\mathcal{G}$ detailed in [40] validate all of the axioms required in the body of this paper.

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ТОПОСЫ И КАТЕГОРИИ В КВАНТОВОЙ ТЕОРИИ И ГРАВИТАЦИИ

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Рассмотрена теория топосов с физической точки зрения. Представлены и объяснены основные идеи понятия топос. Сделан обзор связей алгебр классических и квантовых наблюдаемых, альтернативных концепций пространства-времени, теории относительности и квантовой гравитации, приближение обобщенных историй к квантовой теории вселенной в целом. Концепция топоса обобщена, используя развитый авторами формализм n-регулярных препятственных категорий.

КЛЮЧЕВЫЕ СЛОВА: топос, категория, предпучок, интуиционистская логика, препятствие, n-регулярность