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ON REGULAR SUPERALGEBRAS AND OBSTRUCTED CATEGORIES

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Regular and higher n-regular superalgebras generated by purely odd elements are introduced and their properties are investigated. They are described in terms of obstructed categories with invertible and noninvertible morphisms for which regular n-cocycle obstruction structure is defined. The n-regular functors are introduced and corresponding natural transformations are considered. In monoidal categories regular n-cocycle almost bialgebras and Hopf algebras are defined. A 3-cocycle example is given.
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The concept of higher regularization was introduced in the abstract way in the supermanifold theory [1, 2] and then considered for general morphisms [3, 4], which leaded to regularization of categories and Yang-Baxter equation [5, 6, 7]. This concept is close to the some generalizations of category theory [8, 9, 10] and connected with weak bialgebras [11, 12, 13] and plays an important role in topological quantum field theories [14, 15]. Here we introduce higher regular superalgebras and study them in terms of obstructed categories, which can clear and to concrete understanding of the concept of semisupermanifold [1].

Let $\mathcal{A}$ be a superalgebra [16]. We use the notation $m\,(a\otimes b)\equiv ab$ for the multiplication $m:\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ in $\mathcal{A}$. Denote by $|a|$ the parity of $a\in\mathcal{A}$. Then we have the relation $|ab|=|a|+|b|\,(mod2)$ for the multiplication in $\mathcal{A}$. This means that the multiplication is an even map. Let $a,b\in\mathcal{A}$ be odd elements. $|a|=|b|=1$, then the product ab is an even element of $\mathcal{A}$, $|ab|=0(mod2)$. We are going to consider certain superalgebras generated by purely odd elements. Let us consider a simple example.

Let us consider an associative superalgebra $\mathcal{A}=\Lambda\,(\Theta,\Theta^*)$ generated by two noncommuting odd generators Θ and Θ^* satisfying the following relations

$$\Theta^2=\Theta^{*2}=0. \tag{1}$$

and

$$\Theta\Theta^*\Theta=\Theta,\quad \Theta^*\Theta\Theta^*=\Theta^*. \tag{2}$$

Definition 1. The superalgebra $\mathcal{A}=\Lambda\,(\Theta,\Theta^*)$ is said to be regular.

Note that in this algebra $\Theta\Theta^*$ and $\Theta^*\Theta$ are idempotents, because $\Theta\Theta^*\Theta\Theta^*=(\Theta\Theta^*\Theta)\Theta^*=\Theta\Theta^*$ and $\Theta^*\Theta\Theta^*\Theta=(\Theta^*\Theta\Theta^*)\,\Theta=\Theta^*\Theta$.

Let us introduce two–dimensional linear spaces X_1 and X_2 over a field $\mathbb{K}$

$$X_1:=lin_{\mathbb{K}}\{\Theta,\Theta^*\Theta\},\quad X_2:=lin_{\mathbb{K}}\{\Theta^*,\Theta\Theta^*\}. \tag{3}$$

Define two linear mappings $f_1:X_1\to X_2$ and $f_2:X_2\to X_1$ as a right multiplication by Θ^* and Θ, respectively. We obtain

$$\begin{array}{ll} f_1(\Theta)=\Theta\Theta^*, & f_1(\Theta^*\Theta)=\Theta^*\Theta\Theta^*=\Theta^*,\\ f_2(\Theta^*)=\Theta^*\Theta, & f_2(\Theta\Theta^*)=\Theta\Theta^*\Theta=\Theta. \end{array} \tag{4}$$

Obviously we have

$$f_1\circ f_2\circ f_1=f_1,\quad f_2\circ f_1\circ f_2=f_2. \tag{5}$$

Define two self-mappings $X_1\to X_1$ and $X_2\to X_2$ by

$$e_{X_1}:=f_1\circ f_2,\quad e_{X_2}:=f_2\circ f_1. \tag{6}$$

It is obvious that e_{X_1} and e_{X_2} are idempotents

$$e_{X_1}\circ e_{X_1}=e_{X_1},\quad e_{X_2}\circ e_{X_2}=e_{X_2}. \tag{7}$$

Observe that there is a category $\mathcal{C}_\Lambda$ whose objects are $\mathcal{C}_0:=\{\mathbb{K},X_1,X_2,\ X_1\oplus X_2\}$ and whose morphisms are given as all compositions of mappings $\{f_1,f_2,e_{X_1},e_{X_2}\}$ and identity morphisms.

Note that the superalgebra $\mathcal{A}=\Lambda\,(\Theta,\Theta^*)$ can be described as the so-called free product of two one dimensional Grassmann algebras $\Lambda(\Theta)$ and $\Lambda(\Theta^*)$ modulo relations (2). Recall that the free product of algebras $\mathcal{A}$ and $\mathcal{B}$ is the algebra $\mathcal{A}*\mathcal{B}$ formed by all formal finite sums of monomials of the form $a_1*b_1*a_2*\ldots$ or $b_1*a_1*b_2*\ldots$, where $a_i\in\mathcal{A}$, $b_i\in\mathcal{B}$, $i=1,2,\ldots$ are non-scalar elements. In other words $\mathcal{A}*\mathcal{B}$ is the algebra generated by two algebras $\mathcal{A}$ and $\mathcal{B}$ with no relations.

Example 2. Let us consider n copies of one–dimensional Grassmann algebra $\Lambda(\Theta)$. The i–th copy is denoted by $\Lambda(\Theta^{\overbrace{**\cdots*}^{i}})$. Let us define a superalgebra $\Lambda(\Theta, \Theta^*, \ldots, \Theta^{\overbrace{**\cdots*}^{n-1}})$ as a free product of n copies of one–dimensional Grassmann algebras subject to the following relation

$$\Theta\Theta^*\Theta^{**}\cdots\Theta^{\overbrace{**\cdots*}^{n-1}}\,\Theta = \Theta \tag{8}$$

and its all cyclic permutations.

Definition 3. The superalgebra $\Lambda(\Theta, \Theta^*, \ldots, \Theta^{\overbrace{**\cdots*}^{n-1}})$ is said to be n–regular.

We define n–dimensional linear spaces over a field $\mathbb{K}$

$$\begin{aligned}
X_1 &:= lin_{\mathbb{K}}\{\Theta, \Theta^{\overbrace{*\cdots*}^{n}}\,\Theta, \ldots, \Theta^*\cdots\Theta^{\overbrace{*\cdots*}^{n}}\,\Theta\},\\
X_2 &:= lin_{\mathbb{K}}\{\Theta\Theta^*, \Theta^{\overbrace{*\cdots*}^{n}}\Theta\Theta^*, \ldots, \Theta^*\},\\
&\quad\vdots \qquad\qquad\qquad \vdots\\
X_n &:= lin_{\mathbb{K}}\{\Theta\Theta^*\cdots\Theta^{\overbrace{*\cdots*}^{n}}, \Theta^*\cdots\Theta^{\overbrace{*\cdots*}^{n-1}}\Theta^{\overbrace{*\cdots*}^{n}}, \ldots, \Theta^{\overbrace{*\cdots*}^{n}}\}
\end{aligned} \tag{9}$$

Define linear mapping $f_i : X_i \to X_{i+1}$ as a right multiplication by $\Theta^{\overbrace{**\cdots*}^{i}}$ for $i = 1, \ldots, n$. For f_1 we obtain

$$\begin{aligned}
&f_1(\Theta) = \Theta\Theta^*, f_1(\Theta^{\overbrace{*\cdots*}^{n}}\,\Theta) = \Theta^{\overbrace{*\cdots*}^{n}}\,\Theta\Theta^*, \ldots,\\
&f_1(\Theta^*\Theta^{\overbrace{*\cdots*}^{n}}\,\Theta) = \Theta^*\Theta^{\overbrace{*\cdots*}^{n}}\,\Theta\Theta^* = \Theta^*.
\end{aligned} \tag{10}$$

We can calculate that we have the relation

$$f_1 \circ f_n \circ \cdots \circ f_2 \circ f_1 = f_1 \tag{11}$$

and corresponding cyclic permutations. In this case there is also a specific category which contains all spaces and mappings considered above.

ALGEBRAS VIA CATEGORIES

Let us briefly recall the fundamental concept of the category theory for fixing the notation (or more details see e.g. [17]). A *category* $\mathcal{C} = (\mathcal{C}, c)$ contains

(i) a collection $\mathcal{C}_0$ of objects

(ii) a collection $\mathcal{C}_1$ of morphisms (arrows)

$$\mathcal{C}_1 = \bigcup_{\mathcal{U},\mathcal{V}\in\mathcal{C}_0} \mathcal{C}(\mathcal{U},\mathcal{V})$$

(iii) an associative composition c of morphisms

$$c : \mathcal{C}(\mathcal{U},\mathcal{V}) \times \mathcal{C}(\mathcal{V},\mathcal{W}) \to \mathcal{C}(\mathcal{U},\mathcal{W}) \tag{12}$$

The collection $\mathcal{C}_1$ is the union of mutually disjoint sets $\mathcal{C}(\mathcal{U},\mathcal{V})$ of morphisms $f : \mathcal{U} \to \mathcal{V}$ from $\mathcal{U}$ to $\mathcal{V}$ defined for every pair of objects $\mathcal{U},\mathcal{V} \in \mathcal{C}_0$. It may happen that for a pair $\mathcal{U},\mathcal{V} \in \mathcal{C}(\mathcal{C})$ the set $\mathcal{C}(\mathcal{U},\mathcal{V})$ is empty.

An *opposite* (or *dual*) category of a category $\mathcal{C} = (\mathcal{C}, c)$ is a category $\mathcal{C}^{op} = (\mathcal{C}^{op}, c^{op})$ equipped with the same collection of objects $\mathcal{C}_0$ as the category $\mathcal{C}$ but with reversed all arrows

$$\mathcal{C}^{op}(\mathcal{U},\mathcal{V}) \equiv \mathcal{C}(\mathcal{V},\mathcal{U}). \tag{13}$$

If D is a diagram built from objects and morphisms of the category $\mathcal{C}$, then the same diagram but with reversed all arrows is said to be *dual* to D.

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *functor* $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ of $\mathcal{C}$ into $\mathcal{D}$ is a pair of maps$\mathcal{F}_0 : \mathcal{C}_0 \to \mathcal{D}_0, \mathcal{F}_1 : \mathcal{C}_1 \to \mathcal{D}_1$ which sends objects of $\mathcal{C}$ into objects of $\mathcal{D}$ and morphisms of $\mathcal{C}$ into morphisms of $\mathcal{D}$ such that

$$\mathcal{F}_1(f \circ g) = \mathcal{F}_1(f) \circ \mathcal{F}_1(g) \tag{14}$$

for every morphisms $f : \mathcal{V} \longrightarrow \mathcal{W}$ and $g : \mathcal{U} \longrightarrow \mathcal{V}$ of $\mathcal{C}$. The generalization to multifunctors is obvious. For instance an n–ary functor $\mathcal{F} : \mathcal{C}^{\times n} \longrightarrow \mathcal{N}$ sends an n–tuple of objects of $\mathcal{C}$ into an object of $\mathcal{N}$. The corresponding condition for morphisms is evident.

Now we recall the concept of natural transformations [17]. A *natural transformation* $s : \mathcal{F} \to \mathcal{G}$ of $\mathcal{F}$ into $\mathcal{G}$ is a collection of morphisms

$$s = \{s_{\mathcal{U}} : \mathcal{F}(\mathcal{U}) \to \mathcal{G}(\mathcal{U}), \mathcal{U} \in \mathcal{C}_0\}$$

such that

$$s_{\mathcal{V}} \circ \mathcal{F}(f) = \mathcal{G}(f) \circ s_{\mathcal{U}} \tag{15}$$

for every morphism $f : \mathcal{U} \longrightarrow \mathcal{V}$ of $\mathcal{C}$. The set of all natural transformations of $\mathcal{F}$ into $\mathcal{G}$ is denoted by $\mathcal{N}at(\mathcal{F}, \mathcal{G})$. It is easy to see that the composition $t \circ f$ of natural transformation s of $\mathcal{F}$ into $\mathcal{G}$ and t of $\mathcal{G}$ into $\mathcal{H}$ is a natural transformation of $\mathcal{F}$ into $\mathcal{H}$. If $\mathcal{F} \equiv \mathcal{G}$, then we say that the natural transformation $s : \mathcal{F} \longrightarrow \mathcal{G}$ is a natural transformation of $\mathcal{F}$ into itself.

We can use functors and natural transformations in order to describe certain algebraic structures in categories [17]. Let $\mathcal{C}$ be a category and we define a functor $\mathcal{F}^n : \mathcal{C} \to \mathcal{C}$ which sends an object $\mathcal{A}$ into $\underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{n}$ and similarly for morphisms. A binary multiplication is a natural transformation $m : \mathcal{F}^2 \to \mathcal{F}$ satisfying the associativity condition $m \circ (m \times \mathcal{F}) = m \circ (\mathcal{F} \times m)$. Similarly, a comultiplication is a natural transformation $\triangle : \mathcal{F} \to \mathcal{F}^2$ satisfying the corresponding coassociativity condition $(\triangle \times \mathcal{F}) \circ \triangle = (\mathcal{F} \times \triangle) \circ \triangle$.

OBSTRUCTED CATEGORIES

Let $\mathcal{C}$ be a category with invertible and noninvertible morphisms [4] and equivalence. By an equivalence in $\mathcal{C}$ we mean a class of morphisms $\mathcal{C}^{inv} = \bigcup_{X,Y \in \mathcal{C}_0} \mathcal{C}^{inv}(X,Y)$, where $\mathcal{C}^{inv}(X,Y)$ is a subset of $\mathcal{C}(X,Y)$. Two objects X, Y of the category $\mathcal{C}$ is equivalent if and only if there is an morphism $X \xrightarrow{s} Y$ in $\mathcal{C}^{inv}(X,Y)$ such that $s^{-1} \circ s = id_X$ and $s \circ s^{-1} = id_Y$.

Definition 4. A sequence of noninvertible morphisms

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_1 \tag{16}$$

such that there is an (endo-)morphism $e^{(3)}_{X_1} : X_1 \longrightarrow X_1$ defined uniquely by the following equation

$$e^{(n)}_{X_1} := f_n \circ \cdots \circ f_2 \circ f_1 \tag{17}$$

and subjects to the following relation

$$f_1 \circ f_n \circ \cdots \circ f_2 \circ f_1 = f_1 \tag{18}$$

is said to be a *regular n-cocycle* on $\mathcal{C}$ and it is denoted by (X, f).

Definition 5. The morphism $e^{(n)}_{X_1}$ is said to be an obstruction of X_1 corresponding to the regular n–cocycle $f = (f_1, \ldots f_n)$ on $\mathcal{C}$. The obstruction (endo-)morphisms $e^{(n)}_{X_i} : X_i \longrightarrow X_i$ corresponding for $i = 2, \ldots, n$ are defined by a suitable cyclic permutation of above sequence.

Definition 6. If the obstruction (endo-)morphisms $e^{(n)}_X : X \longrightarrow X$ is defined for every object $X \in \mathcal{C}_0$, then the mapping $e^{(n)} : X \in \mathcal{C}_0 \to e^{(n)}_X \in \mathcal{C}(X,X)$ is called a regular n-cocycle obstruction structure on $\mathcal{C}$.

It is obvious that for usual category all $e^{(n)}_X$ are equal to identity $e^{(n)}_X = Id_X$. We are interested with categories for which the obstruction $e^{(n)}_X$ differs from the identity.

Definition 7. A category $\mathcal{C}$ equipped with a regular n-cocycle obstruction structure $e^{(n)} : X \in \mathcal{C}_0 \to e^{(n)}_X \in \mathcal{C}(X,X)$ such that $e^{(n)}_X \neq id_X$ for some $X \in \mathcal{C}_0$ is called an obstructed category. The minimum number $n = n_{obstr}$ for which it occurs will define a quantitative measure of obstruction n_{obstr}.

Let (Y, g) a regular n–cocycle, i. e. a sequence of morphisms $Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} Y_n \xrightarrow{g_n} Y_1$ such that $e^{(n)}_{Y_1} := g_n \circ \cdots \circ g_2 \circ g_1$.

Definition 8. A sequence of morphisms $\alpha := (\alpha_1, \ldots, \alpha_n)$ such that the diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & X_1 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_n & & \downarrow \alpha_1 \\ Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} \cdots \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & Y_1 \end{array} \tag{19}$$

is commutative, is said to be a regular n-cocycle morphism from (X, f) to (Y, g) and is denoted by $\alpha : (X, f) \to (Y, g)$.

Observe that for a regular n-cocycle morphism $\alpha : (X, f) \to (Y, g)$ we have the relation

$$\alpha_1 \circ e^{(n)}_{X_1} = e^{(n)}_{Y_1} \circ \alpha_1. \tag{20}$$

Definition 9. A regular n-cocycle obstruction morphism $s : (X, f) \to (X', g)$ which sends the object X_i into equivalent object X'_i and morphism f_i into g_i is said to be obstruction n-cocycle equivalence. The corresponding obstructions $e^{(n)}_X$ and $e^{(n)}_{X'}$ are also said to be equivalent.

Let $\mathcal{C}$ and $\mathcal{D}$ be two obstructed categories. The morphisms $e^{(n)}_X$ can be used to extend the notion of functors. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a functor defined as usual as a pair of mappings $(\mathcal{F}_0, \mathcal{F}_1)$.

Definition 10. A new functor $\mathcal{F}^{(n)} : \mathcal{C} \to \mathcal{D}$ defined as a pair of mappings $(\mathcal{F}^{(n)}_0, \mathcal{F}^{(n)}_1)$ such that

$$\mathcal{F}^{(n)}_0 \equiv \mathcal{F}_0 \quad F^{(n)}_1 \left(e^{(n)}_X \right) = e^{(n)}_{\mathcal{F}_0(X)}, \tag{21}$$

where $X \in \mathcal{C}_0$, is said to be n-regular.

All the standard definitions of functor do not changed, but preservation of identity $F\left(Id_{X_1}\right) = Id_{X_2}$, where $X_2 = FX_1$, $X_1 \in \mathcal{C}_0$, $X_2 \in \mathcal{D}_0$, is be replaced by requirement of preservation of morphisms $e^{(n)}_X$. Then the generalized functor $F^{(n)}$ becomes n-dependent. Note that $n = 1$ corresponds to the standard functor, i.e. $F^{(1)} = F$.

Lemma 11. *Let the sequence*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_1 \tag{22}$$

be a regular n–cocycle in the category $\mathcal{C}$. If $F^{(n)} : \mathcal{C} \to \mathcal{D}$ is n-regular functor, then

$$F^{(n)}(X_1) \xrightarrow{F^{(n)}(f_1)} F^{(n)}(X_2) \xrightarrow{F^{(n)}(f_2)} \cdots \xrightarrow{F^{(n)}(f_{n-1})} F^{(n)}(X_n) \xrightarrow{F^{(n)}(f_n)} F^{(n)}(X_1) \tag{23}$$

is a regular n-cocycle in the category $\mathcal{D}$.

Proof. We need to prove that $F^{(n)}(f) \circ e^{(n)}_{X_2} = F^{(n)}(f)$. Indeed, we have

$$F^{(n)}(f) = F^{(n)} \left(f \circ e^{(n)}_X \right) = F^{(n)}\left(f\right) \circ F^{(n)} \left(e^{(n)}_{X_1} \right) = F^{(n)}(f) \circ e^{(n)}_{X_2}.$$

□

Multifunctors can be regularized in a similar way. One can also "regularize" natural transformations in the similar manner. Let $F^{(n)}$ and $G^{(n)}$ be two n–regular functors of the category $\mathcal{C}$ into the category $\mathcal{D}$.

Definition 12. A natural transformation $s : F^{(n)} \to G^{(n)}$ of $F^{(n)}$ into $G^{(n)}$ is a collection of morphisms $s = \{s_X : F_0(X) \to G_0(X), X \in \mathcal{C}\}$ such that

$$s_Y \circ F^{(n)}_1(\alpha) = G^{(n)}_1(\alpha) \circ s_X, \tag{24}$$

for every regular morphism $\alpha : X \to Y$ is said to be n-regular natural transformation.

A monoidal category $\mathcal{C} \equiv \mathcal{C}(\otimes, \mathbb{K})$ is a category $\mathcal{C}$ equipped with a monoidal operation (a bifunctor) $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object $\mathbb{K}$ satisfying some known axioms [18].

Definition 13. A monoidal category $\mathcal{C} \equiv \mathcal{C}(\otimes, \mathbb{K})$ equipped with a family of obstruction morphisms $e^{(n)} = \{e^{(n)}_X : X \in \mathcal{C}_0; n = 1, 2, ...\}$ satisfying the condition

$$e^{(n)}_{X \otimes Y} = e^{(n)}_X \otimes e^{(n)}_Y. \tag{25}$$

is said to be an obstructed monoidal category.

REGULAR ALGEBRAS AND BIALGEBRAS

Let $\mathcal{C}$ be an obstructed monoidal category [5, 6].

Definition 14. An algebra $\mathcal{A}$ in the category $\mathcal{C}$ such that the multiplication $\mathrm{m} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is a regular n–cocycle morphism

$$\mathrm{m} \circ (e_{\mathcal{A}}^{(n)} \otimes e_{\mathcal{A}}^{(n)}) = e_{\mathcal{A}}^{(n)} \circ \mathrm{m}, \tag{26}$$

is said to be a regular n-cocycle algebra.

Obviously such multiplication not need to be unique. Denote by $\Re eg_n(\mathcal{C})(\mathcal{A}\otimes\mathcal{A}, \mathcal{A})$ a class of all such multiplications. We can see that a regular n-cocycle 2-morphisms $s : \mathrm{m} \Rightarrow \mathrm{m}'$ which send the multiplication m into a new one m' should be an algebra homomorphism. One can define regular n-cycle coalgebra or bialgebra in a similar way. A comultiplication $\triangle : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ can be regularized according to the relation

$$\triangle \circ e_{\mathcal{A}}^{(n)} = (e_{\mathcal{A}}^{(n)} \otimes e_{\mathcal{A}}^{(n)}) \circ \triangle. \tag{27}$$

In this case we obtain a class $\Re eg_n(\mathcal{C})(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$ of comultiplications.

Let $\mathcal{A}$ be a regular n-cocycle algebra. If $\mathcal{A}$ is also regular coalgebra such that $\Delta(ab) = \Delta(a)\Delta(b)$, then it is said to be a *regular n-cocycle almost bialgebra*. If $\mathcal{A}$ is a regular n-cocycle algebra, then we denote by $\hom_{\mathrm{m}}(\mathcal{A}, \mathcal{A})$ the set of morphisms $s \in \hom_{\mathcal{C}}(\mathcal{A}, \mathcal{A})$ satisfying the condition

$$s \circ \mathrm{m} = \mathrm{m} \circ (s \otimes s). \tag{28}$$

Let $\mathcal{A}$ be a regular n-cocycle almost bialgebra. We define the *convolution product*

$$s \star t := \mathrm{m} \circ (s \otimes t) \circ \triangle, \tag{29}$$

where $s, t \in \hom_{\mathrm{m}}(\mathcal{A}, \mathcal{A})$. If $\mathcal{A}$ is a regular n-cocycle almost bialgebra, then the convolution product is regular. A regular n-cocycle almost bialgebra $\mathcal{H}$ equipped with an element $S \in \hom_{\mathrm{m}}(\mathcal{H}, \mathcal{H})$ such that

$$S \star id_{\mathcal{H}} \star S = S, \quad id_{\mathcal{H}} \star S \star id_{\mathcal{H}} = id_{\mathcal{H}}. \tag{30}$$

is said to be a *regular n-cocycle almost Hopf algebra $\mathcal{H}$*. This is a regular analogy of week Hopf algebras considered in [11] (see also [12]).

Let ${}_{\mathcal{A}}\mathcal{C}$ be a category of all left $\mathcal{A}$ -modules, where $\mathcal{A}$ is a bialgebra. For the regularization $\Re eg_n({}_{\mathcal{A}}\mathcal{C})$ of the $\mathcal{A}$–module action $\rho_M : \mathcal{A} \otimes M \longrightarrow M$ we use the following formula

$$\rho_M \circ (e_{\mathcal{A}}^{(n)} \otimes e_M^{(n)}) = e_M^{(n)} \circ \rho_M, \tag{31}$$

where $\rho_M : \mathcal{A} \otimes M \longrightarrow M$ is the left module action of $\mathcal{A}$ on M. The class of all such module actions is denoted by $\Re eg_n({}_{\mathcal{A}}\mathcal{C})(\mathcal{A} \otimes \mathcal{M}, \mathcal{M})$. The monoidal operation in this category is given as the following tensor product of $\mathcal{A}$-modules

$$\rho_{M\otimes N} := (id_M \otimes \tau \otimes id_N) \circ (\rho_M \otimes \rho_N) \circ (\triangle \otimes id_{M\otimes N}), \tag{32}$$

where $\tau : \mathcal{A} \otimes M \to M \otimes \mathcal{A}$ is the twist, i. e. $\tau(a \otimes m) := m \otimes a$ for every $a \in \mathcal{A}$, $m \in M$.

Lemma 15. *For the tensor product of module actions we have the following formula*

$$\rho_{M\otimes N} \circ (e_{\mathcal{A}} \otimes e_{M\otimes N}) = e_{M\otimes N} \circ \rho_{M\otimes N}. \tag{33}$$

Let $\mathcal{C}^{\mathcal{A}}$ be a category of right $\mathcal{A}$-comodules, where $\mathcal{A}$ is an algebra. The corresponding regularization can be given by the formulae

$$\begin{aligned} &\rho \circ e_{\mathcal{A}}^{(n)} = (e_M^{(n)} \otimes e_{\mathcal{A}}^{(n)}) \circ \rho_M, \\ &\rho_{M\otimes N} = (id_M \otimes \mathrm{m}_{\mathcal{A}}) \circ (id_M \otimes \tau \otimes id_N) \circ (\rho_M \otimes \rho_N), \end{aligned} \tag{34}$$

where $\tau : M \otimes N \to N \otimes M$ is the twist, $\mathrm{m}_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the multiplication in $\mathcal{A}$.

We consider $\Lambda(\Theta, \Theta^*)$ as an example and have the following

Lemma 16. *The superalgebra $\Lambda(\Theta, \Theta^*)$ is a regular 3-cocycle algebra in the category $\mathcal{C}_\Lambda$.*

Proof. Let $a, b \in \Lambda(\Theta, \Theta^*)$ have the following form

$$a = \lambda_0 + \lambda_1\Theta + \lambda_2\Theta^* + \lambda_{12}\Theta\Theta^* + \lambda_{21}\Theta^*\Theta, \quad b = \mu_0 + \mu_1\Theta + \mu_2\Theta^* + \mu_{12}\Theta\Theta^* + \mu_{21}\Theta^*\Theta. \tag{35}$$

The multiplication in $\Lambda(\Theta, \Theta^*)$ is given by

$$\begin{aligned} ab = \lambda_0\mu_0 + (\lambda_0\mu_1 + \lambda_1\mu_0 + \lambda_1\mu_{21} + \lambda_{12}\mu_1)\Theta + (\lambda_0\mu_2 + \lambda_2\mu_0 + \lambda_2\mu_{12} + \lambda_{21}\mu_2)\Theta^* \\ +(\lambda_0\mu_{12} + \lambda_1\mu_2 + \lambda_{12}\mu_0 + \lambda_{12}\mu_{12})\Theta\Theta^* + (\lambda_0\mu_{21} + \lambda_2\mu_1 + \lambda_{21}\mu_0 + \lambda_{21}\mu_{21})\Theta^*\Theta. \end{aligned} \tag{36}$$

The obstruction $e_{\mathcal{A}}^{(3)}$ is given by

$$e_{\mathcal{A}}^{(3)}(a) := \lambda_0 + \lambda_2\Theta + \lambda_1\Theta^* + \lambda_{21}\Theta\Theta^* + \lambda_{12}\Theta^*\Theta \tag{37}$$

We can calculate that the condition (26) holds. $\square$

We conclude that further study of regular superalgebras can lead to new structures in corresponding objects built from them and possible nontrivial features of resulting supersymmetric theories.

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О РЕГУЛЯРНЫХ СУПЕРАЛГЕБРАХ И ПРЕПЯТСТВЕННЫХ КАТЕГОРИЯХ

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Вводятся регулярные и высшие n-регулярные супералсягебры, генерируемые нечетными элементами, исследуются их свойства. Они описываются в терминах препятственных категорий с обратимыми и необратимыми морфизмами, для которых определена n-регулярная препятственная структура. Вводятся n-регулярные функторы и рассматриваются соответствующие естественные преобразования. Для моноидальных категорий определены регулярные n-коциклические биалгебры и алгебры Хопфа. Приведен пример 3-коцикла.

КЛЮЧЕВЫЕ СЛОВА: супералгебра, регулярность, категория, естественное преобразование, препятственность, функтор