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ON HIGHER REGULARITY AND MONOIDAL CATEGORIES

S. A. Duplij[1)], **W. Marcinek**[2)]

[1)] *Department of Physics and Technology, V. N. Karazin Kharkov National University, Kharkov 61077, Ukraine*
E-mail: Steven.A.Duplij@univer.kharkov.ua. Internet: http://gluon.physik.uni-kl.de/~duplij

[2)] *Institute of Theoretical Physics, University of Wrocław, Pl. Maxa Borna 9, 50-204 Wrocław, Poland*

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In general abstract algebraic language we extend "invertibility" to "regularity" for categories. Higher regularity conditions and "semicommutative" diagrams are introduced. Distinction between commutative and "semicommutative" cases is measured by non-zero obstruction proportional to the difference of some self-mappings (obstructors) $e^{(n)}$ from the identity, which allows us to generalize the notion of functor and to "regularize" braidings and related structures in monoidal categories. Also we propose a "noninvertible" analog of the Yang-Baxter equation.

KEYWORDS : morphism, regularity, obstruction, monoidal category, functor, Yang-Baxter equation

The concept of regularity was introduced by von Neumann [1] and applied by Penrose to matrices [2]. After that study of regularity was developed in many different fields, e.g. generalized inverses theory [3] and semigroup theory [4]. We consider here this concept in categorical language [5] and introduce the most abstract form of higher regularity conditions (firstly introduced in [6]).

Let X and Y be two arbitrary sets. A mapping f from X to Y is defined by a prescription which assigns an element of Y to each element of X, i.e. $f : X \to Y$. Injective mapping (injection) assigns different images to different elements, and in surjective mapping (surjection) every image has at least one pre-image. Bijection has both properties. Usually inverse mapping f^{-1} is defined as a new mapping $g : Y \to X$ which assigns to each $y \in Y$ such $x \in X$ that $f(x) = y$ and so $f^{-1} = g$. For injective f and any $A \subset X$ it is imposed the following "invertibility" condition

$$f^{-1}(f(A)) = A. \tag{1}$$

For surjective f and $B \subset Y$ the standard "invertibility" condition is

$$f\left(f^{-1}(B)\right) = B. \tag{2}$$

These conditions are strong, because they imply possibility to solve the equation $f(x) = y$ for all elements. In many cases, especially while considering supersymmetric theories, there naturally appear noninvertible morphisms [6, 7] and semigroups [8, 9]. That obviously needs extending some general assumptions. We propose to extend the "invertibility" conditions (1)–(2) in the following way (which comes from analogy of regularity in semigroup theory [4]). We introduce less restricted "regular" f^* mapping by extending "invertibility" to "regularity" in following way

$$f(f^*(f(A))) = f(A). \tag{3}$$

For the second equation (2) we have the "reflexive regularity" condition

$$f^*(f(f^*(B))) = f^*(B). \tag{4}$$

REGULAR MORPHISMS

We distinguish among all mappings $X \to Y$ the morphisms satisfying closure and associativity. That defines a category $\mathfrak{C}$ with objects $\mathrm{Ob}\,\mathfrak{C}$ as sets X, Y, Z and morphisms $\mathrm{Mor}\,\mathfrak{C}$ as mappings $f : X \to Y$ between them (or $f = \mathrm{Mor}\,(X, Y)$) [5]. For composition $h : X \to Z$ of morphisms $f : X \to Y$ and $g : Y \to Z$ instead of $h(x) = g(f(x))$ for mappings we use the notation $h = g \circ f$. Associativity implies that $h \circ (g \circ f) = (h \circ g) \circ f = h \circ g \circ f$. Let us consider "invertibility" properties of morphisms in general. If f satisfies the "right invertibility" condition $f \circ f^{-1} = Id_Y$ for some $f^{-1} : Y \to X$ then f is called a *retraction*, and if f satisfies the "left invertibility" condition $f^{-1} \circ f = Id_X$, then it is called a *coretraction*, where Id_X and Id_Y are identity mappings $Id_X : X \to X$ and $Id_Y : Y \to Y$ for which $\forall x \in X$, $Id_X(x) = x$ and $\forall y \in Y$, $Id_Y(y) = y$. These requirements sometimes are very strong to be fulfilled (see e.g. [10]). To obtain more weak conditions one has to introduced the following "regularity" conditions

$$f \circ f^*_{in} \circ f = f, \tag{5}$$

where f^*_{in} is called an *inner inverse* [3], and such f is called *regular*. Similar "reflexive regularity" conditions

$$f^*_{out} \circ f \circ f^*_{out} = f^*_{out} \tag{6}$$

defines an *outer inverse* f^*_{out}. Notice that in general $f^*_{in} \neq f^*_{out} \neq f^{-1}$ or it can be that f^{-1} does not exist at all. If f^*_{in} is an inner inverse, then

$$f^* = f^*_{in} \circ f \circ f^*_{in} \tag{7}$$

is always both inner and outer inverse or *generalized inverse* (quasi-inverse) [3], and so for any regular f there exists (need not be unique) f^* from (7) for which both regularity conditions (5) and (6) hold. Let us consider a composition of two morphisms and its "invertibility" properties. It can be shown, that a retraction is an epimorphism, and a regular monomorphism is a coretraction [3]. If the composition $h = g \circ f$ belongs to the same class of functions (closure), then all such morphisms form a semigroup of such functions [4]. If for any $f : X \to Y$ there will be a unique $f^* : Y \to X$ satisfying (5)–(6), this semigroup is called an inverse semigroup [4] which we denote $\mathfrak{F}$.

Let us define two idempotent "projection operators" $\mathcal{P}_f = f \circ f^*$, $\mathcal{P}_f : Y \to Y$ and $\mathcal{P}_{f^*} = f^* \circ f$, $\mathcal{P}_{f^*} : X \to X$ satisfying $\mathcal{P}_f \circ \mathcal{P}_f = \mathcal{P}_f$, $\mathcal{P}_f \circ f = f \circ \mathcal{P}_{f^*} = f$ and $\mathcal{P}_{f^*} \circ \mathcal{P}_{f^*} = \mathcal{P}_{f^*}$, $\mathcal{P}_{f^*} \circ f^* = f^* \circ \mathcal{P}_f = f^*$. If we introduce the $*$-operation $(f)^* = f^*$ by formulas (5)–(6) and assume that this operation acts on the product of morphisms $f : X \to Y$ and $g : Y \to Z$ in the following way $(g \circ f)^* = f^* \circ g^*$, then commutativity of projectors $\mathcal{P}_f \circ \mathcal{P}_{g^*} = \mathcal{P}_{g^*} \circ \mathcal{P}_f$ leads to closure of the semigroup, i.e. the product $g \circ f$ also satisfies both regularity conditions (5)–(6).

HIGHER REGULARITY

Let us introduce higher analogs of regularity conditions (5)–(6) (they were proposed for some particular case (noninvertible analogs of supermanifolds) in [6,7]). Let we have two elements f and its regular f^* (in sense of (5)) of the semigroup $\mathfrak{F}$. Consider a third morphism $f^{**} : X \to Y$ and analyze the action $f \circ f^* \circ f^{**} : X \to Y$. This means the composition $f \circ f^* \circ f^{**}$ cannot be equal to identity Id_X. Therefore it is possible to "regularize" $f \circ f^* \circ f^{**}$ in the following way

$$f \circ f^* \circ f^{**} \circ f^* = f^*. \tag{8}$$

This formula can be called as 2-regularity condition and be considered as a definition of $**$-operation. For 3-regularity and $f^{***} : Y \to X$ we can obtain an analog of (5) in the form

$$f \circ f^* \circ f^{**} \circ f^{***} \circ f = f. \tag{9}$$

By recursive considerations we can propose the following formula of n-regularity

$$f \circ f^* \circ f^{**} \ldots \circ f^{\overbrace{**\cdots*}^{2k}} \circ f^* = f^*, \quad f \circ f^* \circ f^{**} \ldots \circ f^{\overbrace{**\cdots*}^{2k+1}} \circ f = f. \tag{10}$$

Note that for even number of stars $f^{\overbrace{**\cdots*}^{2k}} : X \to Y$ and for odd number of stars $f^{\overbrace{**\cdots*}^{2k+1}} : Y \to X$. We introduce "higher projector" by the formula

$$\mathcal{P}^{(n)}_f = f \circ f^* \circ f^{**} \ldots \circ f^{\overbrace{**\cdots*}^{n}}. \tag{11}$$

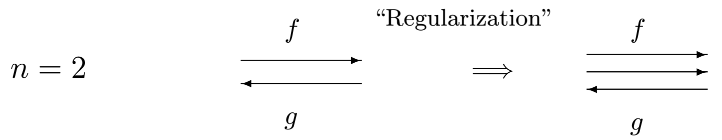
It is easy to check the following properties

$$\mathcal{P}^{(2k)}_f \circ f^* = f^*, \quad \mathcal{P}^{(2k+1)}_f \circ f = f. \tag{12}$$

and idempotence $\mathcal{P}^{(n)}_f \circ \mathcal{P}^{(n)}_f = \mathcal{P}^{(n)}_f$.

SEMICOMMUTATIVE DIAGRAMS AND OBSTRUCTORS

Obviously, that for two morphisms $f : X \to Y$ and $g : Y \to X$ instead of "invertibility" $g \circ f = Id_X$ we have the same generalization as regularity (5), i.e. $f \circ g \circ f = f$, where g plays the role of an inner inverse [3].

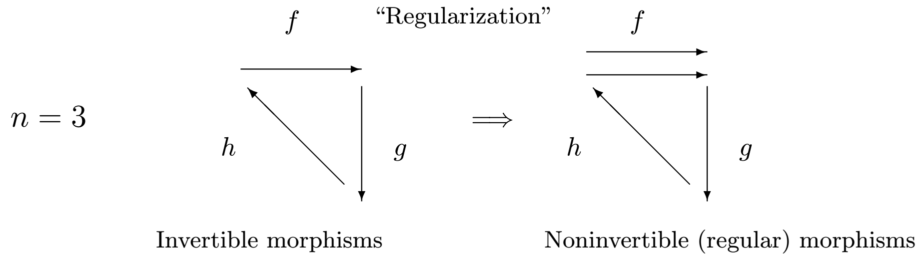


Invertible morphisms　　Noninvertible (regular) morphisms

Usually, for 3 objects X, Y, Z and 3 morphisms $f : X \to Y$ and $g : Y \to Z$ and $h : Z \to X$ one can have the "invertible" triangle commutative diagram $h \circ g \circ f = Id_X$. Its regular extension has the form

$$f \circ h \circ g \circ f = f. \tag{13}$$

Such a diagram (from the right)



can be called a *semicommutative diagram*. This triangle case can be expanded on any number of objects and morphisms. To measure difference between semicommutative and commutative cases let us introduce self-mappings $e_X^{(n)}: X \to X$ which are defined by

$$e_X^{(1)} = Id_X,\ e_X^{(2)} = g \circ f,\ e_X^{(3)} = h \circ g \circ f,\ \ldots \tag{14}$$

It is obvious that for commutative diagrams all $e_X^{(n)}$ are equal to identity $e_X^{(n)} = Id_X$. The deviation of $e_X^{(n)}$ from identity will give us measure of obstruction of commutativity, and therefore we call $e_X^{(n)}$ *obstructors.* The minimum number $n = n_{obstr}$ for which $e_X^{(n)} \neq Id_X$ occurs will define a quantitative measure of obstruction n_{obstr}. In terms of obstructors $e_X^{(n)}$ the n-regularity condition can be written in the short form $f \circ e_X^{(n)} = f$. From this equation and definitions (14) it simply follows that obstructors $e_X^{(n)}$ are idempotents.

REGULARIZATION OF MONOIDAL CATEGORIES

Let $\mathfrak{C}$ be a monoidal category equipped with a monoidal operation $\otimes : \mathfrak{C} \times \mathfrak{C} \longrightarrow \mathfrak{C}$. A triple of objects X, Y, Z is said to be a regular 3-cycle if and only if every sequence of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X$ define uniquely the morphism $e_X^{(3)} : X \longrightarrow X$ by the following relation $e_X^{(3)} := h \circ g \circ f$ and subjects to the relation $f \circ h \circ g \circ f = f$. The object Y is said to be a *(first) regular dual* of X, and the object Z is called the *second regular dual* of X. We denote by $C_3(\mathfrak{C})$ the collection of all regular 3-cycles on $\mathfrak{C}$. This collection is said to be *regularity* in $\mathfrak{C}$. The generalization to arbitrary $n \geq 4$ is obvious. Let X, Y, Z and X', Y', Z' be two regular 3-cycles in $\mathfrak{C}$. Then the morphism $f : X \longrightarrow X'$ such that $f \circ e_X^{(3)} = e_{X'}^{(3)} \circ f$ is said to be a *3-cycle morphism.* If $f : X \longrightarrow X'$ and $g : X' \longrightarrow X''$ are two 3-cycle morphisms, then the composition $g \circ f : X \to X''$ is also a 3-cycle morphism. Moreover the regularity $C_3(\mathfrak{C})$ forms a monoidal category with 3-cycles as objects and 3-cycle morphisms. The monoidal product of two regular 3-triples X, Y, Z and X', Y', Z' is the triple $X \otimes X', Y \otimes Y', Z \otimes Z'$ which is also a regular 3-cycle. The category $C_3(\mathfrak{C})$ is said to be a *regularization* of $\mathfrak{C}$.

The morphisms $e_X^{(n)}$ can be used to extend the notion of a functor $\mathsf{F} : \mathfrak{C}_1 \to C_2$. All the standard definitions of a functor (as a mapping of one category to another with preserving composition of morphisms [5]) do not changed, but preservation of identity $\mathsf{F}(Id_{X_1}) = Id_{X_2}$, where $X_2 = \mathsf{F}X_1$, $X_1 \in \mathrm{Ob}\mathfrak{C}_1$, $X_2 \in \mathrm{Ob}\mathfrak{C}_2$, can be replaced by requirement of preservation of morphisms $e_X^{(n)}$ as

$$\mathsf{F}^{(n)}\left(e_{X_1}^{(n)}\right) = e_{X_2}^{(n)}, \tag{15}$$

where $e_{X_1}^{(n)} \in \mathrm{Mor}\mathfrak{C}_1$, $e_{X_2}^{(n)} \in \mathrm{Mor}\mathfrak{C}_2$ defined in (14) for two categories. Then the generalized functor $\mathsf{F}^{(n)}$ becomes n-dependent. From (14) it follows that $n = 1$ corresponds to the standard functor, i.e. $\mathsf{F}^{(1)} = \mathsf{F}$.

HIGHER REGULAR YANG-BAXTER EQUATION

Let us consider a symmetric monoidal category $\mathfrak{C}$ [5] playing an important role in quantum groups [11] and quantum statistics [12].In $\mathfrak{C}$ for any two objects X and Y and the operation $X \otimes Y$ one usually defines a natural isomorphism ("braiding" [13]) by $\mathrm{B}_{X,Y} : X \otimes Y \to Y \otimes X$ satisfying the symmetry condition ("invertibility")

$$\mathrm{B}_{Y,X} \circ \mathrm{B}_{X,Y} = Id_{X \otimes Y} \tag{16}$$

which formally defines $\mathrm{B}_{Y,X} = \mathrm{B}_{X,Y}^{-1} : Y \otimes X \to X \otimes Y$. Note that possible nonsymmetric braiding in context of the noncommutative geometry was considered in [14]. Here we introduce a "regular" extension of the symmetry condition (16) in the form

$$\mathrm{B}_{X,Y} \circ \mathrm{B}_{X,Y}^{*} \circ \mathrm{B}_{X,Y} = \mathrm{B}_{X,Y}, \tag{17}$$

where in general $\mathrm{B}_{X,Y}^{*} \neq \mathrm{B}_{X,Y}^{-1}$. Such a category can be called a "regular" category to distinct from symmetric and "braided" categories [13].

In categorical sense the prebraiding relations usually are defined as [13, 14]

$$\mathrm{B}_{X\otimes Y,Z} = \mathbf{B}^{R}_{X,Z,Y} \circ \mathbf{B}^{L}_{X,Y,Z}, \quad \mathrm{B}_{Z,X\otimes Y} = \mathbf{B}^{L}_{X,Z,Y} \circ \mathbf{B}^{R}_{X,Y,Z}, \tag{18}$$

$$\mathbf{B}^{L}_{X,Y,Z} = Id_X \otimes \mathrm{B}_{Y,Z}, \quad \mathbf{B}^{R}_{X,Y,Z} = \mathrm{B}_{X,Y} \otimes Id_Z, \tag{19}$$

and prebraidings $\mathrm{B}_{X\otimes Y,Z}$ and $\mathrm{B}_{Z,X\otimes Y}$ satisfy (for symmetric case) the "invertibility" property $\mathrm{B}^{-1}_{X\otimes Y,Z} \circ \mathrm{B}_{X\otimes Y,Z} = Id_{X\otimes Y\otimes Z}$, where $\mathrm{B}^{-1}_{X\otimes Y,Z} = \mathrm{B}_{Z,X\otimes Y}$. In this notations the standard "invertible" Yang-Baxter equation is [11]

$$\mathbf{B}^{R}_{Y,Z,X} \circ \mathbf{B}^{L}_{Y,X,Z} \circ \mathbf{B}^{R}_{X,Y,Z} = \mathbf{B}^{L}_{Z,X,Y} \circ \mathbf{B}^{R}_{X,Z,Y} \circ \mathbf{B}^{L}_{X,Y,Z}. \tag{20}$$

Possible "noninvertible" (endomorphism semigroup) solutions of this equation without introduction of $e^{(n)}_X$ were studied in [15]. For "noninvertible" braidings satisfying regularity (17) it is naturally to exploit the obstructors $e^{(n)}_X$ instead of identity Id_X as

$$\mathbf{B}^{L(n)}_{X,Y,Z} = e^{(n)}_X \otimes \mathrm{B}_{Y,Z}, \quad \mathbf{B}^{R(n)}_{X,Y,Z} = \mathrm{B}_{X,Y} \otimes e^{(n)}_Z, \tag{21}$$

to weaken prebraiding construction in the following way

$$\mathrm{B}^{(n)}_{X\otimes Y,Z} = \mathbf{B}^{R(n)}_{X,Z,Y} \circ \mathbf{B}^{L(n)}_{X,Y,Z}, \quad \mathrm{B}^{(n)}_{Z,X\otimes Y} = \mathbf{B}^{L(n)}_{X,Z,Y} \circ \mathbf{B}^{R(n)}_{X,Y,Z}, \tag{22}$$

Then their "invertibility" can be also "regularized" as follows

$$\mathrm{B}^{(n)}_{X\otimes Y,Z} \circ \mathrm{B}^{(n)*}_{X\otimes Y,Z} \circ \mathrm{B}^{(n)}_{X\otimes Y,Z} = \mathrm{B}^{(n)}_{X\otimes Y,Z}, \tag{23}$$

where in general case $\mathrm{B}^{(n)*}_{X\otimes Y,Z} \neq \mathrm{B}^{-1}_{X\otimes Y,Z}$. Thus the corresponding n-"noninvertible" analog of the Yang-Baxter equation (20) is

$$\mathbf{B}^{R(n)}_{Y,Z,X} \circ \mathbf{B}^{L(n)}_{Y,X,Z} \circ \mathbf{B}^{R(n)}_{X,Y,Z} = \mathbf{B}^{L(n)}_{Z,X,Y} \circ \mathbf{B}^{R(n)}_{X,Z,Y} \circ \mathbf{B}^{L(n)}_{X,Y,Z}. \tag{24}$$

Its solutions can be found by application of the semigroup methods (see e.g. [15]). The introduced formalism can be used in analysis of categories with some weaken invertibility conditions, which can appear in nontrivial supersymmetric or noncommutative geometry constructions beyond the group theory.

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О ВЫСШЕЙ РЕГУЛЯРНОСТИ И МОНОИДАЛЬНЫХ КАТЕГОРИЯХ

С. А. Дуплий [1], **В. Марчинек** [2]

[1] *Харьковский национальный университет им. В. Н. Каразина, пл. Свободы, 4, г. Харьков, 61077, Украина*

[2] *Институт теоретической физики, университет Вроцлава, пл. Макса Борна 9, 50-204 Вроцлав, Польша*

Мы расширяем "обратимость" на "регулярность" для категорий в абстрактном алгебраическом подходе. Введены условия высшей регулярности и "полукоммутативные" диаграммы. Различие между коммутативным и "полукоммутативным" случаями измеряется отличием некоторых отображений $e^{(n)}$ от единичного, что позволяет обобщить понятие функтора и "регуляризовать" подобные структуры в моноидальных категориях. Предложен также "необратимый" аналог уравнения Янга-Бакстера.

КЛЮЧЕВЫЕ СЛОВА: морфизм, регулярность, препятствие, моноидальная категория, функтор, уравнение Янга-Бакстера