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ON REGULAR SOLUTIONS OF QUANTUM YANG-BAXTER EQUATION AND WEAK HOPF ALGEBRAS

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Generalization of Hopf algebra $\mathfrak{sl}_q(2)$ by weakening the invertibility of the generator K, i.e. exchanging its invertibility $KK^{-1}=1$ to the regularity $K\overline{K}K=K$ is studied. Two weak Hopf algebras are introduced: a weak Hopf algebra $w\mathfrak{sl}_q(2)$ and a J-weak Hopf algebra $v\mathfrak{sl}_q(2)$ which are investigated in detail. The monoids of group-like elements of $w\mathfrak{sl}_q(2)$ and $v\mathfrak{sl}_q(2)$ are regular monoids, which supports the general conjucture on the connection betweek weak Hopf algebras and regular monoids. A quasi-braided weak Hopf algebra $\overline{U}_q^w$ is constructed from $w\mathfrak{sl}_q(2)$. It is shown that the corresponding quasi-R-matrix is regular $R^w\hat{R}^wR^w=R^w$.

KEYWORDS : Hopf algebra, regularity, Yang-Baxter equation, Noetherian ring, group-like element, quasi-R-matrix

A weak Hopf algebra as a generalization of a Hopf algebra [1, 2] was introduced in [3] and its characterizations and applications were studied in [4]. A k-bialgebra[1] $H=(H,\mu,\eta,\Delta,\varepsilon)$ is called a *weak Hopf algebra* if there exists $T\in\mathrm{Hom}_k(H,H)$ such that $id*T*id=id$ and $T*id*T=T$ where T is called a *weak antipode* of H. This concept also generalizes the notion of the left and right Hopf algebras [5, 6].

The first aim of this concept is to give a new sub-class of bialgebras which includes all of Hopf algebras such that it is possible to characterize this sub-class through their monoids of all group-like elements [3, 4]. It was known that for every regular monoid S, its semigroup algebra kS over k is a weak Hopf algebra as the generalization of a group algebra [7]. The second aim is to construct some singular solutions of the quantum Yang-Baxter equation (QYBE) and research QYBE in a larger scope. On this hand, in [4] a quantum quasi-double $D(H)$ for a finite dimensional cocommutative perfect weak Hopf algebra with invertible weak antipode was built and it was verified that its quasi-R-matrix is a regular solution of the QYBE. In particular, the quantum quasi-double of a finite Clifford monoid as a generalization of the quantum double of a finite group was derived [4].

Here we construct two weak Hopf algebras in the other direction as a generalization of the quantum algebra $\mathfrak{sl}_q(2)$ [8, 9]. We show that $w\mathfrak{sl}_2(q)$ possesses a quasi-R-matrix which becomes a singular (in fact, regular) solution of the QYBE, with a parameter q. In this reason, we want to treat the meaning of $w\mathfrak{sl}_q(2)$ and its quasi-R-matrix just as $\mathfrak{sl}_q(2)$ [10, 11]. It is interesting to note that $w\mathfrak{sl}_q(2)$ is a natural and non-trivial example of weak Hopf algebras.

WEAK QUANTUM ALGEBRAS

For completeness and consistency we remind the definition of the enveloping algebra $U_q=U_q(\mathfrak{sl}_q(2))$ (see e.g. [11]). Let $q\in\mathbb{C}$ and $q\neq\pm1,0$. The algebra U_q is generated by four variables(Chevalley generators) E, F, K, K^{-1} with the relations

$$K^{-1}K=KK^{-1}=1, \tag{1}$$
$$KEK^{-1}=q^2E, \tag{2}$$
$$KFK^{-1}=q^{-2}F, \tag{3}$$
$$EF-FE=\frac{K-K^{-1}}{q-q^{-1}}. \tag{4}$$

Now we try to generalize the invertibility condition (1). The first thought is weaken the invertibility to regularity, as it is usually made in semigroup theory [12] (see also [13, 14, 15] for higher regularity). So we will consider such weakening the algebra $U_q(\mathfrak{sl}_q(2))$, in which instead of the set $\{K,K^{-1}\}$ we introduce a pair $\{K_w,\overline{K}_w\}$ by means of the regularity relations

$$K_w\overline{K}_wK_w=K_w,\quad \overline{K}_wK_w\overline{K}_w=\overline{K}_w. \tag{5}$$

[1] In this paper, k always denotes a field.

If $\overline{K}_w$ satisfying (5) is unique for a given K_w, then it is called *inverse of* K_w (see e.g. [16, 17]). The regularity relations (5) imply that one can introduce the variables

$$J_w = K_w \overline{K}_w, \qquad \overline{J}_w = \overline{K}_w K_w. \tag{6}$$

In terms of J_w the regularity conditions (5) are

$$J_w K_w = K_w, \qquad \overline{K}_w J_w = \overline{K}_w, \tag{7}$$

$$\overline{J}_w \overline{K}_w = \overline{K}_w, \qquad K_w \overline{J}_w = K_w. \tag{8}$$

Since the noncommutativity of generators K_w and $\overline{K}_w$ very much complexifies the generalized construction, we first consider the commutative case and imply in what follow that

$$J_w = \overline{J}_w \tag{9}$$

Let us list some useful properties of J_w which will be needed below. First we note that commutativity of K_w and $\overline{K}_w$ leads to idempotency condition $J_w^2 = J_w$, which means that J_w is a projector (see e.g. [18]).

Conjecture 1. *In algebras satisfying the regularity conditions* (5) *there exists as minimum one zero divisor* $J_w - 1$.

Therefore, in addition with unity 1 we have an idempotent analog of unity J_w which makes the structure of weak algebras more complicated, but simultaneously more interesting. For any variable X we will define "J-conjugation" as

$$X_{J_w} \stackrel{def}{=} J_w X J_w \tag{10}$$

and the corresponding mapping will be written as $\mathbf{e}_w(X) : X \rightarrow X_{J_w}$. Note that the mapping $\mathbf{e}_w(X)$ is idempotent

$$\mathbf{e}_w^2(X) = \mathbf{e}_w(X). \tag{11}$$

In the invertible case $K_w = K, \overline{K}_w = K^{-1}$ we have $J_w = 1$ and $\mathbf{e}_w(X) = X = \mathrm{id}(X)$ for any X, so $\mathbf{e}_w = \mathrm{id}$. It is seen from (5) that the generators K_w and $\overline{K}_w$ are stable under "J_w-conjugation"

$$K_{J_w} = J_w K_w J_w = K_w, \qquad \overline{K}_{J_w} = J_w \overline{K}_w J_w = \overline{K}_w. \tag{12}$$

Obviously, for any X

$$K_w X \overline{K}_w = K_w X_{J_w} \overline{K}_w, \tag{13}$$

and for any X and Y

$$K_w X \overline{K}_w = Y \Rightarrow K_w X_{J_w} \overline{K}_w = Y_{J_w}, \tag{14}$$

Another definition connected with the idempotent analog of unity J_w is "J_w-product" for any two elements X and Y, viz.

$$X \odot_{J_w} Y \stackrel{def}{=} X J_w Y. \tag{15}$$

From (7) it follows that "J_w-product" coincides with usual product, if X ends with generators K_w and $\overline{K}_w$ on right side or Y starts with them on left side.

Let $J^{(ij)} = K_w^i \overline{K}_w^j$ then we will need a formula

$$J_w^{(ij)} = K_w^i \overline{K}_w^j = \begin{cases} K_w^{i-j}, & i > j, \\ J_w & i = j, \\ \overline{K}_w^{j-i} & i < j, \end{cases} \tag{16}$$

which follows from the regularity conditions (7). The variables $J^{(ij)}$ satisfy the regularity conditions

$$J_w^{(ij)} J_w^{(ji)} J_w^{(ij)} = J_w^{(ij)} \tag{17}$$

and stable under "J-conjugation" (10) $J_{wJ_w}^{(ij)} = J_w^{(ij)}$. The regularity conditions (7) lead to the noncancellativity: for any two elements X and Y the following relations hold valid

$$X = Y \Rightarrow K_w X = K_w Y \tag{18}$$

$$K_w X = K_w Y \nRightarrow X = Y \tag{19}$$

$$X = Y \Rightarrow \overline{K}_w X = \overline{K}_w Y \tag{20}$$

$$\overline{K}_w X = \overline{K}_w Y \nRightarrow X = Y \tag{21}$$

$$X = Y \Rightarrow X_{J_w} = Y_{J_w}, \tag{22}$$

$$X_{J_w} = Y_{J_w} \nRightarrow X = Y. \tag{23}$$

The generalization of $U_q(\mathfrak{sl}_q(2))$ by exploiting regularity (5) instead of invertibility (1) can be done in two different ways.

Definition 2. Define $U_q^w = w\mathfrak{sl}_q(2)$ as the algebra generated by the four variables E_w, F_w, K_w, $\overline{K}_w$ with the relations:

$$K_w\overline{K}_w = \overline{K}_w K_w, \tag{24}$$
$$K_w\overline{K}_w K_w = K_w, \quad \overline{K}_w K_w \overline{K}_w = \overline{K}_w, \tag{25}$$
$$K_w E_w = q^2 E_w K_w, \quad \overline{K}_w E_w = q^{-2} E_w \overline{K}_w, \tag{26}$$
$$K_w F_w = q^{-2} F_w K_w, \quad \overline{K}_w F_w = q^2 F_w \overline{K}_w, \tag{27}$$
$$E_w F_w - F_w E_w = \frac{K_w - \overline{K}_w}{q - q^{-1}}. \tag{28}$$

We call $w\mathfrak{sl}_q(2)$ a *weak quantum algebra.*

Definition 3. Define $U_q^v = v\mathfrak{sl}_q(2)$ as the algebra generated by the four variables E_v, F_w, K_v, $\overline{K}_v$ with the relations ($J_v = K_v\overline{K}_v$):

$$K_v\overline{K}_v = \overline{K}_v K_v, \tag{29}$$
$$K_v\overline{K}_v K_v = K_v, \quad \overline{K}_v K_v \overline{K}_v = \overline{K}_v, \tag{30}$$
$$K_v E_v \overline{K}_v = q^2 E_v, \tag{31}$$
$$K_v F_v \overline{K}_v = q^{-2} F_v, \tag{32}$$
$$E_v J_v F_v - F_v J_v E_v = \frac{K_v - \overline{K}_v}{q - q^{-1}}. \tag{33}$$

We call $v\mathfrak{sl}_q(2)$ a *J-weak quantum algebra.*

In these definitions indeed the first two lines (24)–(25) and (29)–(30) are called to generalize the invertibility $KK^{-1} = K^{-1}K = 1$. Note that the EK and FK relations (31)–(32) can be written in the following form close to (26)–(27)

$$K_v E_v J_v = q^2 J_v E_v K_v, \quad \overline{K}_v E_v J_v = q^{-2} J_v E_v \overline{K}_v, \tag{34}$$
$$K_v F_v J_v = q^{-2} J_v F_v K_v, \quad \overline{K}_v F_v J_v = q^2 J_v F_v \overline{K}_v. \tag{35}$$

Using (15) and (7) in the case of J_v we can also present the $v\mathfrak{sl}_q(2)$ algebra as an algebra with "J_v-product"

$$K_v \odot_{J_v} \overline{K}_v = \overline{K}_v \odot_{J_v} K_v, \tag{36}$$
$$K_v \odot_{J_v} \overline{K}_v \odot_{J_v} K_v = K_v, \quad \overline{K}_v \odot_{J_v} K_v \odot_{J_v} \overline{K}_v = \overline{K}_v, \tag{37}$$
$$K_v \odot_{J_v} E_v \odot_{J_v} \overline{K}_v = q^2 E_v, \tag{38}$$
$$K_v \odot_{J_v} F_v \odot_{J_v} \overline{K}_v = q^{-2} F_v, \tag{39}$$
$$E_v \odot_{J_v} F_v - F_v \odot_{J_v} E_v = \frac{K_v - \overline{K}_v}{q - q^{-1}}. \tag{40}$$

Due to (7) the only relation where "J_w-product" is really plays its role is the last relation (40). From the following proposition, one can find the connection between $U_q^w = w\mathfrak{sl}_q(2), U_q^v = v\mathfrak{sl}_q(2)$ and the quantum algebra $\mathfrak{sl}_q(2)$.

Proposition 4. *$w\mathfrak{sl}_q(2)/(J_w - 1) \cong \mathfrak{sl}_q(2)$; $v\mathfrak{sl}_q(2)/(J_v - 1) \cong \mathfrak{sl}_q(2)$.*

Proof. For cancellative K_w and K_v it is obvious. □

Proposition 5. *Quantum algebras $w\mathfrak{sl}_q(2)$ and $v\mathfrak{sl}_q(2)$ possess zero divisors, one of which is*[2] $(J_{w,v} - 1)$ *which annihilates all generators.*

[2]We denote by $X_{w,v}$ one of the variables X_w or X_v.

Proof. From regularity (25) and (30) it follows $K_{w,v}\left(J_{w,v}-1\right)=0$ (see also (1)). Multiplying (26) on J_w gives $K_wE_wJ_w=q^2E_wK_wJ_w \Rightarrow K_w\left(E_w\overline{K}_w\right)K_w=q^2E_wK_w$. Using second equation in (26) for term in bracket we obtain $K_w\left(q^2\overline{K}_wE_w\right)K_w=q^2E_wK_w \Rightarrow \left(J_w-1\right)E_wK_w=0$. For F_w similarly, but using equation (27). By analogy, multiplying (31) on J_v we have $K_vE_v\overline{K}_vK_v\overline{K}_v=q^2E_vJ_v \Rightarrow K_vE_v\overline{K}_v=q^2E_vJ_v \Rightarrow q^2E_v=q^2E_vJ_v$, and so $E_v\left(J_v-1\right)=0$. For F_v similarly, but using equation (32). □

Since $\mathfrak{sl}_q(2)$ is an algebra without zero divisors, some properties of $\mathfrak{sl}_q(2)$ cannot be upgraded to $w\mathfrak{sl}_q(2)$ and $v\mathfrak{sl}_q(2)$, e.g. the standard theorem of Ore extensions and its proof (see Theorem I.7.1 in [11]). We conjecture that in U_q^w and U_q^v there are no other than $\left(J_{w,v}-1\right)$ zero divisors which annihilate *all* generators. In other case thorough analysis of them will be much more complicated and very different from the standard case of non-weak algebras. We can get some properties of U_q^w and U_q^v as follows.

Lemma 6. *The idempotent J_w is in the center of $w\mathfrak{sl}_q(2)$.*

Proof. For K_w it follows from (12). Multiplying first equation in (26) on $\overline{K}_w$ we derive $K_w\left(E_w\overline{K}_w\right)=q^2E_wJ_w$, and the applying second equation in (26) obtain $E_wJ_w=J_wE_w$. For F_w similarly, but using equation (27). □

Lemma 7. *There are unique algebra automorphism ω_w and ω_v of U_q^w and U_q^v respectively such that*

$$\begin{aligned} \omega_{w,v}(K_{w,v}) &= \overline{K}_{w,v}, \quad \omega_{w,v}(\overline{K}_{w,v}) = K_{w,v}, \\ \omega_{w,v}(E_{w,v}) &= F_{w,v}, \quad \omega_{w,v}(F_{w,v}) = E_{w,v}. \end{aligned} \tag{41}$$

Proof. The proof is obvious, if we note that $\omega_w^2=\mathrm{id}$ and $\omega_v^2=\mathrm{id}$. □

As in case of automorphism ω for $\mathfrak{sl}_q(2)$ [11], the mappings ω_w and ω_v can be called the *weak Cartan automorphisms.* Note that $\omega_w\neq\omega$ and $\omega_v\neq\omega$ in general case.

The connection between the algebras $w\mathfrak{sl}_q(2)$ and $v\mathfrak{sl}_q(2)$ can be seen from the following

Proposition 8. *There exist the following partial algebra morphism $\chi: v\mathfrak{sl}_q(2)\to w\mathfrak{sl}_q(2)$ such that*

$$\chi\left(X\right)=\mathbf{e}_v\left(X\right) \tag{42}$$

or more exactly: generators $X_w^{(v)}=J_vX_vJ_v=X_{vJ_v}$ for all $X_v=K_v,\overline{K}_v,E_v,F_v$ satisfy the same relations as X_w (24)–(28).

Proof. Multiplying the equation (31) on K_v we have $K_vE_v\overline{K}_vK_v=q^2E_vK_v$, and using (7) we obtain $K_vE_vJ_v=q^2E_vJ_vK_v \Rightarrow K_vJ_vE_vJ_v=q^2J_vE_vJ_vK_v$, and so

$$K_{vJ_v}E_{vJ_v}=q^2E_{vJ_v}K_{vJ_v}$$

which has shape of the first equation in (26). For F_v similarly using equation (32) we obtain

$$K_{vJ_v}F_{vJ_v}=q^{-2}F_{vJ_v}K_{vJ_v}.$$

The equation (33) can be modified using (7) and then applying (10), then we obtain

$$E_{vJ_v}F_{vJ_v}-F_{vJ_v}E_{vJ_v}=\frac{K_{vJ_v}-\overline{K}_{vJ_v}}{q-q^{-1}}$$

which coincides with (28).

For conjugated equations (second ones in (26)–(27)) after multiplication of (31) on $\overline{K}_v$ we have $\overline{K}_vK_vE_v\overline{K}_v=q^2\overline{K}_vE_v \Rightarrow J_vE_vJ_v\overline{K}_v=q^2\overline{K}_vJ_vE_vJ_v$ or using definition (10) and (7)

$$\overline{K}_{vJ_v}E_{vJ_v}=q^{-2}E_{vJ_v}\overline{K}_{vJ_v}.$$

By analogy from (32) it follows

$$\overline{K}_{vJ_v}F_{vJ_v}=q^2F_{vJ_v}\overline{K}_{vJ_v}.$$

□

Note that the generators $X_w^{(v)}$ coincide with X_w if $J_v=1$ only. Therefore, some (but not all) properties of $w\mathfrak{sl}_q(2)$ can be extended on $v\mathfrak{sl}_q(2)$ as well, and below we mostly will consider $w\mathfrak{sl}_q(2)$ in detail.

Lemma 9. *Let $m \geq 0$ and $n \in \mathbb{Z}$. The following relations hold in U_q^w:*

$$E_w^m K_w^n = q^{-2mn} K_w^n E_w^m, \qquad F_w^m K_w^n = q^{2mn} K_w^n F_w^m, \tag{43}$$
$$E_w^m \overline{K}_w^n = q^{2mn} \overline{K}_w^n E_w^m, \qquad F_w^m \overline{K}_w^n = q^{-2mn} \overline{K}_w^n F_w^m, \tag{44}$$

$$[E_w, F_w^m] = [m] F_w^{m-1} \frac{q^{-(m-1)} K_w - q^{m-1} \overline{K}_w}{q - q^{-1}} \tag{45}$$
$$= [m] \frac{q^{m-1} K_w - q^{-(m-1)} \overline{K}_w}{q - q^{-1}} F_w^{m-1},$$
$$[E_w^m, F_w] = [m] \frac{q^{-(m-1)} K_w - q^{m-1} \overline{K}_w}{q - q^{-1}} E_w^{m-1} \tag{46}$$
$$= [m] E_w^{m-1} \frac{q^{m-1} K_w - q^{-(m-1)} \overline{K}_w}{q - q^{-1}}.$$

Proof. The first two relations can be resulted easily from Definition 2. The third one follows by induction using Definition 2 and

$$[E_w, F_w^m] = [E_w, F_w^{m-1}] F_w + F_w^{m-1} [E_w, F_w] = [E_w, F_w^{m-1}] F_w + F_w^{m-1} \frac{K_w - \overline{K}_w}{q - q^{-1}}.$$

Applying the automorphism ω_w (41) to (45), one gets (46). □

Note that the commutation relations (43)–(46) coincide with $\mathfrak{sl}_q(2)$ case. For $v\mathfrak{sl}_q(2)$ the situation is more complicated, because the equations (31)–(32) cannot be solved under $\overline{K}_v$ due to noncancellativity (see also (18)–(23)). Nevertheless, some analogous relations can be derived. Using the morphism (42) one can conclude that the similar as (43)–(46) relations hold for $X_w^{(v)} = J_v X_v J_v$, from which we obtain for $v\mathfrak{sl}_q(2)$

$$J_v E_v^m K_v^n = q^{-2mn} K_v^n E_v^m J_v, \qquad J_v F_v^m K_v^n = q^{2mn} K_v^n F_v^m J_v, \tag{47}$$
$$J_v E_v^m \overline{K}_v^n = q^{2mn} \overline{K}_v^n E_v^m J_v, \qquad J_v F_v^m \overline{K}_v^n = q^{-2mn} \overline{K}_v^n F_v^m J_v, \tag{48}$$

$$J_v E_v J_v F_v^m J_v - J_v F_v^m J_v E_v J_v = [m] J_v F_v^{m-1} \frac{q^{-(m-1)} K_v - q^{m-1} \overline{K}_v}{q - q^{-1}} \tag{49}$$
$$= [m] \frac{q^{m-1} K_v - q^{-(m-1)} \overline{K}_v}{q - q^{-1}} F_v^{m-1} J_v,$$
$$J_v E_v^m J_v F_v J_v - J_v F_v J_v E_v^m J_v = [m] \frac{q^{-(m-1)} K_v - q^{m-1} \overline{K}_v}{q - q^{-1}} E_v^{m-1} J_v \tag{50}$$
$$= [m] J_v E_v^{m-1} \frac{q^{m-1} K_v - q^{-(m-1)} \overline{K}_v}{q - q^{-1}}.$$

It is important to stress that due to noncancellativity of weak algebras we cannot cancel these relations on J_v (see (18)–(23)).

In order to discuss the basis of $U_q^w = w\mathfrak{sl}_q(2)$, we need to generalize some properties of Ore extensions (see [11]).

WEAK ORE EXTENSIONS

Let R be an algebra over k and $\mathsf{R}[t]$ be the free left R-module consisting of all polynomials of the form $P = \sum_{i=0}^{n} a_i t^i$ with coefficients in R. If $a_n \neq 0$, define $\deg(P) = n$; say $\deg(0) = -\infty$. Let α be an algebra morphism of R. An *α-derivation* of R is a k-linear endomorphism δ of R such that $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b$ for all $a, b \in \mathsf{R}$. It follows that $\delta(1) = 0$.

Theorem 10. (i) *Assume that $\mathsf{R}[t]$ has an algebra structure such that the natural inclusion of R into $\mathsf{R}[t]$ is a morphism of algebras and $\deg(PQ) \leq \deg(P) + \deg(Q)$ for any pair (P, Q) of elements of $\mathsf{R}[t]$. Then there exists a unique injective algebra endomorphism α of R and a unique α-derivation δ of R such that $ta = \alpha(a)t + \delta(a)$ for all $a \in \mathsf{R}$;*

(ii) *Conversely, given an algebra endomorphism α of R and an α-derivation δ of R, there exists a unique algebra structure on $\mathsf{R}[t]$ such that the inclusion of R into $\mathsf{R}[t]$ is an algebra morphism and $ta = \alpha(a)t + \delta(a)$ for all $a \in \mathsf{R}$.*

Proof. (i) Take any $0 \neq a \in \mathsf{R}$ and consider the product ta. We have $\deg(ta) \leq \deg(t) + \deg(a) = 1$. By the definition of $\mathsf{R}[t]$, there exists uniquely determined elements $\alpha(a)$ and $\delta(a)$ of R such that $ta = \alpha(a)t + \delta(a)$. This defines maps α and δ in a unique fashion. The left multiplication by t being linear, so are α and δ. Expanding both sides of the equality $(ta)b = t(ab)$ in $\mathsf{R}[t]$ using $ta = \alpha(a)t + \delta(a)$ for $a, b \in \mathsf{R}$, we get

$$\alpha(a)\alpha(b)t + \alpha(a)\delta(b) + \delta(a)b = \alpha(ab)t + \delta(ab).$$

It follows that $\alpha(ab) = \alpha(a)\alpha(b)$ and $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b$. And, $\alpha(1)t + \delta(1) = t1 = t$. So, $\alpha(1) = 1$, $\delta(1) = 0$. Therefore, we know that α is an algebra endomorphism and δ is an α-derivation. The uniqueness of α and δ follows from the freeness of $\mathsf{R}[t]$ over R.

(ii) We need to construct the multiplication on $\mathsf{R}[t]$ as an extension of that on R such that $ta = \alpha(a)t + \delta(a)$. For this, it needs only to determine the multiplication ta for any $a \in \mathsf{R}$.

Let $M = \{(f_{ij})_{i,j\geq 1} : f_{ij} \in \mathrm{End}_k(\mathsf{R})$ and each row and each column has only finitely many $f_{ij} \neq 0\}$ and $I = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{pmatrix}$ is the identity of M.

For $a \in \mathsf{R}$, let $\widehat{a} : \mathsf{R} \to \mathsf{R}$ satisfying $\widehat{a}(r) = ar$. Then $\widehat{a} \in \mathrm{End}_k(\mathsf{R})$; and for $r \in \mathsf{R}$, $(\alpha\widehat{a})(r) = \alpha(ar) = \alpha(a)\alpha(r) = (\widehat{\alpha(a)}\alpha)(r)$, $(\delta\widehat{a})(r) = \delta(ar) = \alpha(a)\delta(r) + \delta(a)r = (\widehat{\alpha(a)}\delta + \widehat{\delta(a)})(r)$, thus $\alpha\widehat{a} = \widehat{\alpha(a)}\alpha$, $\delta\widehat{a} = \widehat{\alpha(a)}\delta + \widehat{\delta(a)}$ in $\mathrm{End}_k(\mathsf{R})$. And, obviously, for $a, b \in \mathsf{R}$, $\widehat{ab} = \widehat{a}\widehat{b}$; $\widehat{a+b} = \widehat{a} + \widehat{b}$. □

Let $T = \begin{pmatrix} \delta & & & \\ \alpha & \delta & & \\ & \alpha & \ddots & \\ & & \ddots & \end{pmatrix} \in M$ and define $\Phi : \mathsf{R}[t] \to M$ satisfying $\Phi(\sum_{i=0}^n a_i t^i) = \sum_{i=0}^n (\widehat{a_i} I)T^i$. It is seen that Φ is a k-linear map.

Lemma 11. *The map Φ is injective.*

Proof. Let $p = \sum_{i=0}^n a_i t^i$. Assume $\Phi(p) = 0$. For e_i having 1 on i-th place and others zeroes, obviously, $\{e_i\}_{i\geq 1}$ are linear independent. Since $\delta(1) = 0$ and $\alpha(1) = 1$, we have $Te_i = e_{i+1}$ and $T^i e_1 = e_{i+1}$ for any $i \geq 0$. Thus, $0 = \Phi(P)e_1 = \sum_{i=0}^n (\widehat{a_i} I)T^i e_1 = \sum_{i=0}^n \widehat{a_i} e_{i+1}$. It means that $\widehat{a_i} = 0$ for all i, then $a_i = a_i 1 = \widehat{a_i} 1 = 0$. Hence $P = 0$. □

Lemma 12. *The following relation holds $T(\widehat{a}I) = (\widehat{\alpha(a)}I)T + \widehat{\delta(a)}I$.*

Proof. We have $T(\widehat{a}I) = \widehat{\alpha(a)}T + \widehat{\delta(a)}I = (\widehat{\alpha(a)}I)T + \widehat{\delta(a)}I$. □

Now, we complete the proof of Theorem 10.

Proof. Let S denote the subalgebra generated by T and $\widehat{a}I$ (all $a \in \mathsf{R}$) in M. From Lemma 12, we see that every element of S can be generated linearly by some elements in the form as $(\widehat{a}I)T^n$ ($a \in \mathsf{R}$, $n \geq 0$). But $\Phi(at^n) = (\widehat{a}I)T^n$, so $\Phi(\mathsf{R}[t]) = S$, i.e. Φ is surjective. Then by Lemma 11, Φ is bijective. It follows that $\mathsf{R}[t]$ and S are linearly isomorphic.

Define $ta = \Phi^{-1}(T(\widehat{a}I))$, then we can extend this formula to define the multiplication of $\mathsf{R}[t]$ with $fg = \Phi^{-1}(xy)$ for any $f, g \in \mathsf{R}[t]$ and $x = \Phi(f)$, $y = \Phi(g)$. Under this definition, $\mathsf{R}[t]$ becomes an algebra and Φ is an algebra isomorphism from $\mathsf{R}[t]$ to S. And, $ta = \Phi^{-1}(T(\widehat{a}I)) = \Phi^{-1}((\widehat{\alpha(a)}I)T + \widehat{\delta(a)}I) = \alpha(a)t + \delta(a)$ for all $a \in \mathsf{R}$. Obviously, the inclusion of R into $\mathsf{R}[t]$ is an algebra morphism. □

Definition 13. We call the algebra constructed from α and δ a *weak Ore extension* of R, denoted as $\mathsf{R}_w[t, \alpha, \delta]$.

Let $S_{n,k}$ be the linear endomorphism of R defined as the sum of all $\binom{n}{k}$ possible compositions of k copies of δ and of $n-k$ copies of α. By induction n, from $ta = \alpha(a)t + \delta(a)$ under the condition of Theorem 10(ii), we get $t^n a = \sum_{k=0}^n S_{n,k}(a)t^{n-k}$ and moreover, $(\sum_{i=0}^n a_i t^i)(\sum_{i=0}^m b_i t^i) = \sum_{i=0}^{n+m} c_i t^i$ where $c_i = \sum_{p=0}^i a_p \sum_{k=0}^p S_{p,k}(b_{i-p+k})$.

Corollary 14. Under the condition of Theorem 10(ii), the following statements hold:

(i) As a left R-module, $\mathsf{R}_w[t, \alpha, \delta]$ is free with basis $\{t^i\}_{i\geq 0}$;

(ii) If α is an automorphism, then $\mathsf{R}_w[t, \alpha, \delta]$ is also a right free R-module with the same basis $\{t^i\}_{i\geq 0}$.

Proof. (i) It follows from the fact that $\mathsf{R}_w[t,\alpha,\delta]$ is just $\mathsf{R}[t]$ as a left R-module.

(ii) Firstly, we can show that $\mathsf{R}_w[t,\alpha,\delta]=\sum_{i\geq 0}t^i\mathsf{R}$, i.e. for any $p\in\mathsf{R}_w[t,\alpha,\delta]$, there are $a_0,a_1,\cdots,a_n\in\mathsf{R}$ such that $p=\sum_{i=0}^n t^ia_i$. Equivalently, we show by induction on n that for any $b\in\mathsf{R}$, bt^n can be in the form $\sum_{i=0}^n t^ia_i$ for some a_i. When $n=0$, it is obvious. Suppose that for $n\leq k-1$ the result holds. Consider the case $n=k$. Since α is surjective, there is $a\in\mathsf{R}$ such that $b=\alpha^n(a)=S_{n,0}(a)$. But $t^na=\sum_{k=0}^n S_{n,k}(a)t^{n-k}$, we get $bt^n=t^na-\sum_{k=1}^n S_{n,k}(a)t^{n-k}=\sum_{i=0}^n t^ia_i$ by the hypothesis of induction for some a_i with $a_n=a$. For any i and $a,b\in\mathsf{R}$, $(t^ia)b=t^i(ab)$ since $\mathsf{R}_w[t,\alpha,\delta]$ is an algebra. Then $\mathsf{R}_w[t,\alpha,\delta]$ is a right R-module. Suppose $f(t)=t^na_n+\cdots+ta_1+A_0=0$ for $a_i\in\mathsf{R}$ and $a_n\neq 0$. Then $f(t)$ can be written as an element of $\mathsf{R}[t]$ by the formula $t^na=\sum_{k=0}^n S_{n,k}(a)t^{n-k}$ whose highest degree term is just that of $t^na_n=\sum_{k=0}^n S_{n,k}(a_n)t^{n-k}$, i.e. $\alpha^n(a_n)t^n$. From (i), we get $\alpha^n(a_n)=0$. It implies $a_n=0$. It is a contradiction. Hence $\mathsf{R}_w[t,\alpha,\delta]$ is a free right R-module. □

We will need the following:

Lemma 15. *Let R be an algebra, α be an algebra automorphism and δ be an α-derivation of R. If R is a left (resp. right) Noetherian, then so is the weak Ore extension $\mathsf{R}_w[t,\alpha,\delta]$.*

The proof can be made as similarly as for Theorem I.8.3 in [11].

Theorem 16. *The algebra $w\mathfrak{sl}_q(2)$ is Noetherian with the basis*

$$\mathsf{P}_w=\{E_w^iF_w^jK_w^l,E_w^iF_w^j\overline{K}_w^m,E_w^iF_w^jJ_w\},\tag{51}$$

where i,j,l are any non-negative integers, m is any positive integer.

Proof. As is well known, the two-variable polynomial algebra $k[K_w,\overline{K}_w]$ is Noetherian (see e.g. [18]). Then $A_0=k[K_w,\overline{K}_w]/(J_wK_w-K_w,\overline{K}_wJ_w-\overline{K}_w)$ is also Noetherian. For any $i,j\geq 0$ and $a,b,c\in k$, if at least one element of a,b,c does not equal 0, $aK_w^i+b\overline{K}_w^j+cJ_w$ is not in the ideal $(J_wK_w-K_w,\overline{K}_wJ_w-\overline{K}_w)$ of $k[K_w,\overline{K}_w]$. So, in A_0, $aK_w^i+b\overline{K}_w^j+cJ_w\neq 0$. It follows that $\{K_w^i,\overline{K}_w^j,J_w:i,j\geq 0\}$ is a basis of A_0.

Let α_1 satisfies $\alpha_1(K_w)=q^2K_w$ and $\alpha_1(\overline{K}_w)=q^{-2}\overline{K}_w$. Then α_1 can be extended to an algebra automorphism on A_0 and $A_1=A_0[F_w,\alpha_1,0]$ is a weak Ore extension of A_0 from $\alpha=\alpha_1$ and $\delta=0$. By Corollary 14, A_1 is a free left A_0-module with basis $\{F_w^j\}_{i\geq 0}$. Thus, A_1 is a k-algebra with basis $\{K_w^lF_w^j,\overline{K}_w^mF_w^j,J_wF_w^j: l$ and j run respectively over all non-negative integers, m runs over all positive integers$\}$. But, from the definition of the weak Ore extension, we have $K_w^lF_w^j=q^{-2lj}F_w^jK_w^l$, $\overline{K}_w^mF_w^j=q^{2mj}F_w^j\overline{K}_w^m$, $J_wF_w^j=F_w^jJ_w$. Thus, we can conclude that $\{F_w^jK_w^l,F_w^j\overline{K}_w^m,F_w^jJ_w: l$ and j run respectively over all non-negative integers, m runs over all positive integers$\}$ is a basis of A_1.

Let α_2 satisfies $\alpha_2(F_w^jK_w^l)=q^{-2l}F_w^jK_w^l$, $\alpha_2(F_w^j\overline{K}_w^m)=q^{2m}F_w^j\overline{K}_w^m$, $\alpha_2(F_w^jJ_w)=F_w^jJ_w$. Then α_2 can be extended to an algebra automorphism on A_1. Let δ satisfies

$$\begin{aligned}\delta(1)&=\delta(K_w)=\delta(\overline{K}_w)=0,\\ \delta(F_w^jK_w^l)&=\sum_{i=0}^{j-1}F_w^{j-1}\frac{q^{-2i}K_w-q^{2i}\overline{K}_w}{q-q^{-1}}K_w^l,\\ \delta(F_w^j\overline{K}_w^l)&=\sum_{i=0}^{j-1}F_w^{j-1}\frac{q^{-2i}K_w-q^{2i}\overline{K}_w}{q-q^{-1}}\overline{K}_w^l,\\ \delta\left(F_w^jJ_w\right)&=\sum_{i=0}^{j-1}F_w^{j-1}\frac{q^{-2i}K_w-q^{2i}\overline{K}_w}{q-q^{-1}}J_w\end{aligned}$$

for $j>0$ and $l\geq 0$. Then just as in the proof of Lemma VI.1.5 in [11], it can be shown that δ can be extended to an α_2-derivation of A_1 such that $A_2=A_1[E_w,\alpha_2,\delta]$ is a weak Ore extension of A_1. Then in A_2,

$$\begin{aligned}E_wK_w&=\alpha_2(K_w)E_w+\delta(K_w)=q^{-2}K_wE_w,\quad E_w\overline{K}_w=q^2\overline{K}_wE_w,\\ E_wF_w&=\alpha_2(F_w)E_w+\delta(F_w)=F_wE_w+\frac{K_w-\overline{K}_w}{q-q^{-1}}.\end{aligned}$$

From these, we conclude that $A_2\cong U_q^w$ as algebras. Thus, from Lemma 15, U_q^w is Noetherian. By Corollary 14, U_q^w is free with basis $\{E_w^i\}_{i\geq 0}$ as a left A_1-module. Thus, as a k-linear space, U_q^w has the basis $\mathsf{Q}_w=\{F_w^jK_w^lE_w^i,F_w^j\overline{K}_w^mE_w^i,F_w^jJ_wE_w^i: i,j,l$ run over all non-negative integers, m runs over all positive integers$\}$. By

Lemma 9 any $x \in \mathsf{P}_w$ (resp. Q_w) can be k-linearly generated by some elements of Q_w (resp. P_w), and therefore P_w and Q_w generate the same space U_q^w. $\square$

The similar theorem can be proved for $v\mathfrak{sl}_q(2)$ as well.

Theorem 17. *The algebra $v\mathfrak{sl}_q(2)$ is Noetherian with the basis*

$$\mathsf{P}_v = \{J_vE_v^iJ_vF_v^jK_v^l, J_vE_v^iJ_vF_v^j\overline{K}_v^m, J_vE_v^iJ_vF_v^jJ_v\}, \tag{52}$$

where i, j, l are any non-negative integers, m is any positive integer.

$q = 1$ CASE

Let $q \in \mathbb{C}$ and $q \neq \pm 1, 0$. Define $U_q^{w\prime}$ as the algebra generated by the five variables E_w, F_w, K_w, $\overline{K}_w$, L_v with the relations (for $U_q^{v\prime}$ the equations (55) and (56) should be exchanged with (31) and (32) respectively):

$$K_w\overline{K}_w = \overline{K}_wK_w, \tag{53}$$
$$K_w\overline{K}_wK_w = K_w, \quad \overline{K}_wK_w\overline{K}_w = \overline{K}_w, \tag{54}$$
$$K_wE_w = q^2E_wK_w, \quad \overline{K}_wE_w = q^{-2}E_w\overline{K}_w, \tag{55}$$
$$K_wF_w = q^{-2}F_wK_w, \quad \overline{K}_wF_w = q^2F_w\overline{K}_w, \tag{56}$$
$$[L_w, E_w] = q(E_wK_w + \overline{K}_wE_w), \tag{57}$$
$$[L_w, F_w] = -q^{-1}(F_wK_w + \overline{K}_wF_w). \tag{58}$$
$$E_wF_w - F_wE_w = L_w, \quad (q - q^{-1})L_w = (K_w - \overline{K}_w), \tag{59}$$

For $v\mathfrak{sl}_q(2)$ we can similarly define the algebra $U_q^{v\prime}$

$$K_v\overline{K}_v = \overline{K}_vK_v, \tag{60}$$
$$K_v\overline{K}_vK_v = K_v, \quad \overline{K}_vK_v\overline{K}_v = \overline{K}_v, \tag{61}$$
$$K_vE_v\overline{K}_v = q^2E_v, \tag{62}$$
$$K_vF_v\overline{K}_v = q^{-2}F_v, \tag{63}$$
$$L_vJ_vE_v - E_vJ_vL_v = q(E_vK_v + \overline{K}_vE_v), \tag{64}$$
$$L_vJ_vF_v - F_vJ_vL_v = -q^{-1}(F_vK_v + \overline{K}_vF_v). \tag{65}$$
$$E_vJ_vF_v - F_vJ_vE_v = L_v, \ (q - q^{-1})L_v = (K_v - \overline{K}_v), \tag{66}$$

Note that contrary to U_q^w and U_q^v, the algebras $U_q^{w\prime}$ and $U_q^{w\prime}$ are defined for all *invertible* values of the parameter q, in particular for $q = 1$.

Proposition 18. *The algebra U_q^w is isomorphic to the algebra $U_q^{w\prime}$ with φ_w satisfying $\varphi_w(E_w) = E_w$, $\varphi_w(F_w) = F_w$, $\varphi_w(K_w) = K_w$, $\varphi_w(\overline{K}_w) = \overline{K}_w$.*

The proof is similar to that of Proposition VI.2.1 in [11] for $\mathfrak{sl}_q(2)$. On the otherwise, we can give the following relationship between $U_q^{w\prime}$ and $U(\mathfrak{sl}(2))$ whose proof is easy.

Proposition 19. *For $q = 1$*

(i) the algebra isomorphism $U(\mathfrak{sl}(2)) \cong U_1^{w\prime}/(K_w - 1)$ holds;

(ii) there exists an injective algebra morphism π from U_1^w to $U(\mathfrak{sl}(2))[K_w]/(K_w^3 - K_w)$ satisfying $\pi(E_w) = XK_w$, $\pi(F_w) = Y$, $\pi(K_w) = K_w$, $\pi(L) = HK_w$.

REMARK. In Proposition 19(ii), π is only injective, but not surjective since $K^2 \neq 1$ in $U(\mathfrak{sl}(2))[K]/(K^3 - K)$ and then X does not lie in the image of π.

STRUCTURE OF WEAK HOPF ALGEBRAS

Here we define weak analogs in $w\mathfrak{sl}_q(2)$ and $v\mathfrak{sl}_q(2)$ for the standard Hopf algebra structures Δ, ε, S — comultiplication, counit and antipod, which should be algebra morphisms.

For the weak quantum algebra $w\mathfrak{sl}_q(2)$ we define the maps $\Delta_w : w\mathfrak{sl}_q(2) \to w\mathfrak{sl}_q(2) \otimes w\mathfrak{sl}_q(2)$, $\varepsilon_w : w\mathfrak{sl}_q(2) \to k$ and $T_w : w\mathfrak{sl}_q(2) \to w\mathfrak{sl}_q(2)$ satisfying respectively

$$\Delta_w(E_w) = 1 \otimes E_w + E_w \otimes K_w,\ \Delta(F_w) = F_w \otimes 1 + \overline{K}_w \otimes F_w, \tag{67}$$
$$\Delta_w(K_w) = K_w \otimes K_w,\ \Delta_w(\overline{K}_w) = \overline{K}_w \otimes \overline{K}_w, \tag{68}$$
$$\varepsilon_w(E_w) = \varepsilon_w(F_w) = 0,\ \varepsilon_w(K_w) = \varepsilon_w(\overline{K}_w) = 1, \tag{69}$$
$$T_w(E_w) = -E_w\overline{K}_w,\ T_w(F_w) = -K_wF_w,\ T(K_w) = \overline{K}_w,\ T_w(\overline{K}_w) = K_w. \tag{70}$$

The difference with the standard case (we follow notations of [11]) is in substitution K^{-1} with $\overline{K}_w$ and the last line, where instead of antipod S the weak antipod T_w is introduced [3].

Proposition 20. *The relations (67)–(70) endow $w\mathfrak{sl}_q(2)$ with a bialgebra structure.*

Proof. It can be shown by direct calculation that, through the basis in Theorem 16, Δ and ε_w can be extended to algebra morphisms from $w\mathfrak{sl}_q(2)$ to $w\mathfrak{sl}_q(2) \otimes w\mathfrak{sl}_q(2)$ and from $w\mathfrak{sl}_q(2)$ to k, T_w can be extended to an anti-algebra morphism from $w\mathfrak{sl}_q(2)$ to $w\mathfrak{sl}_q(2)$ respectively. Using (67)–(70) it can be shown that

$$(\Delta_w \otimes \mathrm{id}\,)\Delta_w(X) = (\mathrm{id}\, \otimes \Delta_w)\Delta_w(X), \tag{71}$$
$$(\varepsilon_w \otimes \mathrm{id}\,)\Delta_w(X) = (\mathrm{id}\, \otimes \varepsilon_w)\Delta_w(X) = X \tag{72}$$

for any $X = E_w, F_w, K_w$ or $\overline{K}_w$. Let μ_w and η_w be the product and the unit of $w\mathfrak{sl}_q(2)$ respectively. Hence $(w\mathfrak{sl}_q(2), \mu_w, \eta_w, \Delta_w, \varepsilon_w)$ becomes into a bialgebra. □

Next we introduce the star product in the bialgebra $(w\mathfrak{sl}_q(2), \mu_w, \eta_w, \Delta_w, \varepsilon_w)$ in the similar to the standard way (see e.g. [11])

$$(A \star_w B)(X) = \mu_w [A \otimes B]\, \Delta_w(X). \tag{73}$$

Proposition 21. *T_w satisfies the regularity conditions*

$$(\mathrm{id}\, \star_w T_w \star_w \mathrm{id}\,)(X) = X, \tag{74}$$
$$(T_w \star_w \mathrm{id}\, \star_w T_w)(X) = T_w(X) \tag{75}$$

for any $X = E_w, F_w, K_w$ or $\overline{K}_w$. It means that T_w is a weak antipode

Proof. Follows from (67)–(70) by tedious calculations. For $X = K_w$,$\overline{K}_w$ it is easy, and so we consider $X = E_w$, as an example. We have

$$\begin{aligned}
&(\mathrm{id}\, \star_w T_w \star_w \mathrm{id}\,)(E_w) = \mu_w [(\mathrm{id}\, \star_w T_w) \otimes \mathrm{id}\,]\, \Delta_w(E_w) \\
&= \mu_w [(\mathrm{id}\, \star_w T_w) \otimes \mathrm{id}\,]\, (1 \otimes E_w + E_w \otimes K_w) \\
&= (\mathrm{id}\, \star_w T_w)\,(1)\,\mathrm{id}\,(E_w) + (\mathrm{id}\, \star_w T_w)\,(E_w)\,\mathrm{id}\,(K_w) \\
&= \mu_w [\mathrm{id}\, \otimes T_w]\, \Delta_w(1)\mathrm{id}\,(E_w) + \mu_w [\mathrm{id}\, \otimes T_w]\, \Delta_w(E_w)\mathrm{id}\,(K_w) \\
&= \mu_w [\mathrm{id}\, \otimes T_w]\, (1 \otimes 1)\,\mathrm{id}\,(E_w) + \mu_w [\mathrm{id}\, \otimes T_w]\, (1 \otimes E_w + E_w \otimes K_w)\,\mathrm{id}\,(K_w) \\
&= T_w\,(1)\,\mathrm{id}\,(E_w) + \mathrm{id}\,(1)\, T_w\,(E_w)\,\mathrm{id}\,(K_w) + \mathrm{id}\,(E_w)\, T_w\,(K_w)\,\mathrm{id}\,(K_w) \\
&= E_w - E_w\overline{K}_w \cdot K_w + E_w \cdot \overline{K}_w \cdot K_w = E_w = \mathrm{id}\,(E_w)\,.
\end{aligned}$$

By analogy, for (75) and $X = E_w$ we obtain

$$\begin{aligned}
&(T_w \star_w \mathrm{id}\, \star_w T_w)(E_w) = \mu_w [(T_w \star_w \mathrm{id}\,) \otimes T_w]\, \Delta_w(E_w) \\
&= \mu_w [(T_w \star_w \mathrm{id}\,) \otimes T_w]\, (1 \otimes E_w + E_w \otimes K_w) \\
&= (T_w \star_w \mathrm{id}\,)\,(1) T_w\,(E_w) + (T_w \star_w \mathrm{id}\,)\,(E_w) T_w\,(K_w) \\
&= \mu_w [T_w \otimes \mathrm{id}\,]\, (1 \otimes 1)\, T_w\,(1E_w1) + \mu_w [T_w \otimes \mathrm{id}\,]\, (1 \otimes E_w + E_w \otimes K_w)\, T_w\,(K_w) \\
&= T_w\,(1)\, T_w\,(E_w) + T_w\,(1)\,\mathrm{id}\,(E_w)\, T_w\,(K_w) + T_w\,(E_w)\,\mathrm{id}\,(K_w)\, T_w\,(K_w) \\
&= -E_w\overline{K}_w + E_w\overline{K}_w - E_w\overline{K}_wK_w\overline{K}_w = -E_w\overline{K}_w = T_w(E_w).
\end{aligned}$$

□

Corollary 22. The bialgebra $w\mathfrak{sl}_q(2)$ is a weak Hopf algebra with the weak antipode T_w.

We can get an inner endomorphism as follows.

Proposition 23. *T_w^2 is an inner endomorphism of the algebra $w\mathfrak{sl}_q(2)$ satisfying for any $X \in w\mathfrak{sl}_q(2)$*

$$T_w^2(X) = K_w X \overline{K}_w, \tag{76}$$

especially

$$T_w^2(K_w) = \mathrm{id}\,(K_w), \quad T_w^2(\overline{K}_w) = \mathrm{id}\,(\overline{K}_w). \tag{77}$$

Proof. Follows from (70). □

Assume that with the operations $\mu_w, \eta_w, \Delta_w, \varepsilon_w$ the algebra $w\mathfrak{sl}_q(2)$ would possess an antipode S so as to become a Hopf algebra, which should satisfy $(S \star_w \mathrm{id}\,)(K_w) = \eta_w\varepsilon_w(K_w)$, and so it should follow that $S(K_w)K_w = 1$. But, it is not possible to hold since $S(K_w)$ can be written as a linearly sum of the basis in Theorem 16. It implies that $w\mathfrak{sl}_q(2)$ is impossible to become a Hopf algebra about the operations above.

Corollary 24. $w\mathfrak{sl}_q(2)$ is an example for a non-commutative and non-cocommutative weak Hopf algebra which is *not a Hopf algebra.*

In order to become $U_q^{w\prime}$ into a weak Hopf algebra, it is enough to define $\Delta_w(E_w)$, $\Delta_w(F_w)$, $\Delta_w(K_w)$, $\Delta_w(\overline{K}_w)$, $\varepsilon_w(E_w)$, $\varepsilon_w(F_w)$, $\varepsilon_w(K_w)$, $\varepsilon_w(\overline{K}_w)$, $T_w(E_w)$, $T_w(F_w)$, $T_w(K_w)$, $T_w(\overline{K}_w)$ just as in $w\mathfrak{sl}_q(2)$ and define

$$\Delta_w(L_w) = \frac{1}{q-q^{-1}}(K_w \otimes K_w - \overline{K}_w \otimes \overline{K}_w),\ \varepsilon_w(L_w) = 0,\ T_w(L_w) = \frac{\overline{K}_w - K_w}{q-q^{-1}}.$$

From Proposition 18 we conclude that $w\mathfrak{sl}_q(2)$ is isomorphic to the algebra $U_q^{w\prime}$ with φ_w. Moreover, one can see easily that φ_w is an isomorphism of weak Hopf algebras from $w\mathfrak{sl}_q(2)$ to $U_q^{w\prime}$.

For J-weak quantum algebra $v\mathfrak{sl}_q(2)$ we suppose that some additional J_v should appear even in the definition of comultiplication and antipod. A thorough analysis gives the following nontrivial definitions

$$\Delta_v(E_v) = J_v \otimes J_v E_v J_v + J_v E_v J_v \otimes K_v, \tag{78}$$
$$\Delta_v(F_v) = J_v F_v J_v \otimes J_v + \overline{K}_v \otimes J_v F_v J_v, \tag{79}$$
$$\Delta_v(K_v) = K_v \otimes K_v,\ \Delta_v(\overline{K}_v) = \overline{K}_v \otimes \overline{K}_v, \tag{80}$$
$$\varepsilon_v(E_v) = \varepsilon_v(F_v) = 0,\ \ \varepsilon_v(K_v) = \varepsilon_v(\overline{K}_v) = 1, \tag{81}$$
$$T_v(E_v) = -J_v E_v \overline{K}_v, \quad T_v(F_v) = -K_v F_v J_v, \tag{82}$$
$$T_v(K_v) = \overline{K}_v, \quad T_v(\overline{K}_v) = K_v. \tag{83}$$

Note that from (80) it follows that

$$\Delta_v(J_v) = J_v \otimes J_v, \tag{84}$$

and so J_v is a group-like element.

Proposition 25. *The relations (78)–(83) endow $v\mathfrak{sl}_q(2)$ with a bialgebra structure.*

Proof. First it is easy to check that Δ_v defines a morphism of algebras from $v\mathfrak{sl}_q(2) \otimes v\mathfrak{sl}_q(2)$ into $v\mathfrak{sl}_q(2)$. Then it can be shown that $\Delta_v(X)$ is coassociative

$$(\Delta_v \otimes \mathrm{id}\,)\,\Delta_v(X) = (\mathrm{id}\,\otimes \Delta_v)\,\Delta_v(X) \tag{85}$$

Proof that the counit ε defines a morphism of algebras from $v\mathfrak{sl}_q(2)$ onto k is straithforward. Moreover, it can be shown that $(\varepsilon_v \otimes \mathrm{id}\,)\Delta_v(X) = (\mathrm{id}\,\otimes \varepsilon_v)\Delta_v(X) = X$ for $X = E_v, F_v, K_v, \overline{K}_v$. Further it can be checked that T_v defines an anti-morphism of algebras from $v\mathfrak{sl}_q(2)$ to $v\mathfrak{sl}_q^{op}(2)$. Therefore, we conclude that $(v\mathfrak{sl}_q(2), \mu_v, \eta_v, \Delta_v, T_v)$ has a structure of a bialgebra. □

The following property of T_v is crucial for understanding the structure of the bialgebra $(v\mathfrak{sl}_q(2), \mu_v, \eta_v, \Delta_v, T_v)$.

Proposition 26. *For any $X \in v\mathfrak{sl}_q(2)$ we have (cf. (76)-(77))*

$$T_v^2(K_v) = \mathbf{e}_v(K_v),\ T_v^2(\overline{K}_v) = \mathbf{e}_v(\overline{K}_v), \tag{86}$$
$$T_v^2(E_v) = K_v E_v \overline{K}_v,\ T_v^2(F_v) = K_v F_v \overline{K}_v, \tag{87}$$

where $\mathbf{e}_v(X)$ is defined in (10).

Proof. Follows from (7) and (82)–(83). As an example for E_v we have $T_v^2(E_v) = T_v\left(-J_vE_v\overline{K}_v\right) = -T_v\left(\overline{K}_v\right)T_v(E_v)T_v(J_v) = K_v\left(J_vE_v\overline{K}_v\right)J_v = K_vE_v\overline{K}_v$. □

The star product in $(v\mathfrak{sl}_q(2), \mu_v, \eta_v, \Delta_v, T_v)$ has the form

$$(A \star_v B)(X) = \mu_v [A \otimes B] \Delta_v(X). \tag{88}$$

Proposition 27. *T_v satisfies the regularity conditions*

$$(\mathbf{e}_v \star_v T_v \star_v \mathbf{e}_v)(X) = \mathbf{e}_v(X), \tag{89}$$

$$(T_v \star_v \mathbf{e}_v \star_v T_v)(X) = T_v(X) \tag{90}$$

for any $X = E_v, F_v, K_v$ or $\overline{K}_v$.

Proof. Follows from (78)–(83) and (88). For $X = K_v, \overline{K}_v$ it is easy, and so we consider $X = E_v$, as an example. We have

$$\begin{aligned}
&(\mathbf{e}_v \star_v T_v \star_v \mathbf{e}_v)(E_v) = \mu_v [(\mathbf{e}_v \star_v T_v) \otimes \mathbf{e}_v] \Delta_v(E_v)\\
&= \mu_v [(\mathbf{e}_v \star_v T_v) \otimes \mathbf{e}_v] (J_v \otimes J_vE_vJ_v + J_vE_vJ_v \otimes K_v)\\
&= (\mathbf{e}_v \star_v T_v)(J_v)\, \mathbf{e}_v(J_vE_vJ_v) + (\mathbf{e}_v \star_v T_v)(J_vE_vJ_v)\, \mathbf{e}_v(K_v)\\
&= \mu_v [\mathbf{e}_v \otimes T_v] \Delta_v(J_v) \mathbf{e}_v(J_vE_vJ_v) + \mu_v [\mathbf{e}_v \otimes T_v] \Delta_v(E_v) \mathbf{e}_v(K_v)\\
&= \mu_v [\mathbf{e}_v \otimes T_v] (J_v \otimes J_v)\, \mathbf{e}_v(E_v) + \mu_v [\mathbf{e}_v \otimes T_v] (J_v \otimes J_vE_vJ_v + J_vE_vJ_v \otimes K_v)\, \mathbf{e}_v(K_v)\\
&= \mathbf{e}_v(J_v)T_v(J_v)\, \mathbf{e}_v(E_v) + \mathbf{e}_v(J_v)T_v(J_vE_vJ_v)\, \mathbf{e}_v(K_v) + \mathbf{e}_v(E_v)T_v(K_v)\, \mathbf{e}_v(K_v)\\
&= J_v \cdot J_v \cdot J_vE_vJ_v - J_v \cdot J_vJ_vE_v\overline{K}_v \cdot J_vK_vJ_v + J_vE_vJ_v \cdot \overline{K}_v \cdot J_vK_vJ_v\\
&= J_vE_vJ_v = \mathbf{e}_v(E_v).
\end{aligned}$$

By analogy, for (90) and $X = E_v$ we obtain

$$\begin{aligned}
&(T_v \star_v \mathbf{e}_v \star_v T_v)(E_v) = \mu_v [(T_v \star_v \mathbf{e}_v) \otimes T_v] \Delta_v(E_v)\\
&= \mu_v [(T_v \star_v \mathbf{e}_v) \otimes T_v] (J_v \otimes J_vE_vJ_v + J_vE_vJ_v \otimes K_v)\\
&= (T_v \star_v \mathbf{e}_v)(J_v)T_v(J_vE_vJ_v) + (T_v \star_v \mathbf{e}_v)(E_v)T_v(K_v)\\
&= \mu_v [T_v \otimes \mathbf{e}_v] (J_v \otimes J_v)\, T_v(J_vE_vJ_v)\\
&+ \mu_v [T_v \otimes \mathbf{e}_v] (J_v \otimes J_vE_vJ_v + J_vE_vJ_v \otimes K_v)\, T_v(K_v)\\
&= T_v(J_v)\, \mathbf{e}_v(J_v)\, T_v(J_vE_vJ_v) + T_v(J_v)\, \mathbf{e}_v(J_vE_vJ_v)\, T_v(K_v)\\
&+ T_v(J_vE_vJ_v)\, \mathbf{e}_v(K_v)\, T_v(K_v) = -J_v \cdot J_v \cdot J_v\left(J_vE_v\overline{K}_v\right)J_v + J_v \cdot J_vE_vJ_v \cdot \overline{K}_v\\
&- J_v\left(J_vE_v\overline{K}_v\right)J_v \cdot J_vK_vJ_v \cdot \overline{K}_v = -J_vE_v\overline{K}_v = T_v(E_v).
\end{aligned}$$

□

From (89)–(90) it follows that $v\mathfrak{sl}_q(2)$ is not a weak Hopf algebra in the definition of [3]. So we will call it *J-weak Hopf algebra* and T_v a *J-weak antipode.* As it is seen from (74)–(75) and (89)–(90) the difference between them is in the exchange id with $\mathbf{e}_v$.

REMARK. The variable $\mathbf{e}_v$ can be treated as $n = 2$ example of the "tower identity" $e_{\alpha\beta}^{(n)}$ introduced for semisupermanifolds in [19, 13] or the "obstructor" $\mathbf{e}_X^{(n)}$ for general mappings, categories and Yang-Baxter equation in [14, 15, 20].

Comparing (67)–(70) with (78)–(83) we conclude that the connection of $\Delta_w, T_w, \varepsilon_w$ and $\Delta_v, T_v, \varepsilon_v$ can be written in the following way

$$\Delta_v(X) = \Delta_w(\mathbf{e}_v(X)), \tag{91}$$

$$T_v(X) = T_w(\mathbf{e}_v(X)), \tag{92}$$

$$\varepsilon_v(X) = \varepsilon_w(\mathbf{e}_v(X)), \tag{93}$$

which means that additionally to the partially algebra morphism (42) there exists a partial coalgebra morphism which is described by (91)–(93).

GROUP-LIKE ELEMENTS

Now, we discuss the set $G(w\mathfrak{sl}_q(2))$ of all group-like elements of $w\mathfrak{sl}_q(2)$. As is well-known (see e.g. [21]) a semigroup S is called an inverse semigroup if for every $x \in S$, there exists a unique $y \in S$ such that $xyx = x$ and $yxy = y$, and a monoid is a semigroup with identity. We will show the following

Proposition 28. *The set of all group-like elements $G(w\mathfrak{sl}_q(2)) = \{J^{(ij)} = K_w^i \overline{K}_w^j : i, j$ run over all non-negative integers$\}$, which forms a regular monoid under the multiplication of $w\mathfrak{sl}_q(2)$.*

Proof. Suppose $x \in w\mathfrak{sl}_q(2)$ is a group-like element, i.e. $\Delta_w(x) = x \otimes x$. By Theorem 16, x can be written as $x = \sum_{i,j,l,m} \alpha_{ijl} E_w^i F_w^j K_w^l + \beta_{ijm} E_w^i F_w^j \overline{K}_w^m + \gamma_{ij} E_w^i F_w^j J_w$. Here and in the sequel, every α, β and γ with subscripts is in the field k and does not equal zero. Then

$$\begin{aligned}
\Delta_w(x) &= \sum_{i,j,l,m} [\alpha_{ijl} \Delta_w(E_w^i F_w^j K_w^l) + \Delta_w(\beta_{ijm} E_w^i F_w^j \overline{K}_w^m) + \Delta_w(\gamma_{ij} E_w^i F_w^j J_w)] \\
&= \sum_{i,j,l,m} [\alpha_{ijl} (1 \otimes E_w + E_w \otimes K_w)^i (F_w \otimes 1 + \overline{K}_w \otimes F_w)^j (K_w \otimes K_w)^l \\
&+ \beta_{ijm} (1 \otimes E_w + E_w \otimes K_w)^i (F_w \otimes 1 + \overline{K}_w \otimes F_w)^j (\overline{K}_w \otimes \overline{K}_w)^m \\
&+ \gamma_{ij} (1 \otimes E_w + E_w \otimes K_w)^i (F_w \otimes 1 + \overline{K}_w \otimes F_w)^j J_w];
\end{aligned}$$

and

$$\begin{aligned}
x \otimes x = (&\sum_{i,j,l,m} \alpha_{ijl} E_w^i F_w^j K_w^l + \beta_{ijm} E_w^i F_w^j \overline{K}_w^m + \gamma_{ij} E_w^i F_w^j J_w) \\
&\otimes (\sum_{i,j,l,m} \alpha_{ijl} E_w^i F_w^j K_w^l + \beta_{ijm} E_w^i F_w^j \overline{K}_w^m + \gamma_{ij} E_w^i F_w^j J_w).
\end{aligned}$$

It is seen that if $i \neq 0$ or $j \neq 0$, $\Delta_w(x)$ is impossible to equal $x \otimes x$. So, $i = 0$ and $j = 0$. We get $x = \sum_{l,m} \alpha_l K_w^l + \beta_m \overline{K}_w^m + J_w$. Then

$$\begin{aligned}
\Delta_w(x) &= \sum_{l,m} [\alpha_l K_w^l \otimes K_w^l + \beta_m \overline{K}_w^m \otimes \overline{K}_w^m + J_w \otimes J_w]; \\
x \otimes x &= \sum_{l,l',m,m'} [\alpha_l \alpha_{l'} K_w^l \otimes K_w^{l'} + \alpha_l \beta_{m'} K_w^l \otimes \overline{K}_w^{m'} + \alpha_l K_w^l \otimes J_w \\
&+ \alpha_{l'} \beta_m \overline{K}_w^m \otimes K_w^{l'} + \beta_m \beta_{m'} \overline{K}_w^m \otimes \overline{K}_w^{m'} + \beta_m \overline{K}_w^m \otimes J_w \\
&+ \alpha_{l'} J_w \otimes K_w^{l'} + \beta_{m'} J_w \otimes \overline{K}_w^{m'} + J_w \otimes J_w].
\end{aligned}$$

If there exists $l \neq l'$, then $x \otimes x$ possesses the monomial $K_w^l \otimes K_w^{l'}$, which does not appear in $\Delta_w(x)$. It contradicts to $\Delta_w(x) = x \otimes x$. Hence we have only a unique l. Similarly, there exists a unique m. Thus $x = \alpha_l K_w^l + \beta_m \overline{K}_w^m + J_w$. Moreover, it is easy to see that $\alpha_l K_w^l$, $\beta_m \overline{K}_w^m$ and J_w can not appear simultaneously in the expression of x. Therefore, we conclude that $x = \alpha_l K_w^l$, $\beta_m \overline{K}_w^m$ or J_w (no summation) and we have

$$\Delta_w(J_w^{(ij)}) = J_w^{(ij)} \otimes J_w^{(ij)}. \tag{94}$$

It follows that $G(w\mathfrak{sl}_q(2)) = \{J_w^{(ij)} = K_w^i \overline{K}_w^j : i, j$ run over all non-negative integers$\}$.

For any $J^{(ij)} = K_w^i \overline{K}_w^j \in G(w\mathfrak{sl}_q(2))$, one can find $J^{(ji)} = K_w^j \overline{K}_w^i \in G(w\mathfrak{sl}_q(2))$ such that the regularity (17) takes place $J_w^{(ij)} J_w^{(ji)} J_w^{(ij)} = J_w^{(ij)}$, which means that $G(w\mathfrak{sl}_q(2))$ forms a regular monoid under the multiplication of $w\mathfrak{sl}_q(2)$. □

For $v\mathfrak{sl}_q(2)$ we have a similar statement.

Proposition 29. *The set of all group-like elements $G(v\mathfrak{sl}_q(2)) = \{J_v^{(ij)} = K_v^i \overline{K}_v^j : i, j$ run over all non-negative integers$\}$, which forms a regular monoid under the multiplication of $v\mathfrak{sl}_q(2)$.*

Proof. Suppose $x \in v\mathfrak{sl}_q(2)$ is a group-like element, i.e. $\Delta_v(x) = x \otimes x$. By Theorem 17, x can be written as $x = \sum_{i,j,l,m} \alpha_{ijl} J_v E_v^i J_v F_v^j K_v^l + \beta_{ijm} J_v E_v^i J_v F_v^j \overline{K}_v^m + \gamma_{ij} J_v E_v^i J_v F_v^j J_v$. Here and in the sequel, every α, β and γ

with subscripts is in the field k and does not equal zero. Then

$$\begin{aligned}\Delta_v(x) = &\sum_{i,j,l,m} [\alpha_{ijl}\Delta_v(J_vE_v^iJ_vF_v^jK_v^l) \\ &+ \Delta_v(\beta_{ijm}J_vE_v^iJ_vF_v^j\overline{K}_v^m) + \Delta_v(\gamma_{ij}J_vE_v^iJ_vF_v^jJ_v)] \\ = &\sum_{i,j,l,m} [\alpha_{ijl}(J_v\otimes J_v)(J_v\otimes J_vE_vJ_v + J_vE_vJ_v\otimes K_v)^i \\ &\times (J_v\otimes J_v)(J_vF_vJ_v\otimes J_v + \overline{K}_v\otimes J_vF_vJ_v)^j(K_v\otimes K_v)^l \\ &+ \beta_{ijm}(J_v\otimes J_v)(J_v\otimes J_vE_vJ_v + J_vE_vJ_v\otimes K_v)^i \\ &\times (J_v\otimes J_v)(J_vF_vJ_v\otimes J_v + \overline{K}_v\otimes J_vF_vJ_v)^j(\overline{K}_v\otimes \overline{K}_v)^m \\ &+ \gamma_{ij}(J_v\otimes J_v)(J_v\otimes J_vE_vJ_v + J_vE_vJ_v\otimes K_v)^i \\ &\times (J_v\otimes J_v)(J_vF_vJ_v\otimes J_v + \overline{K}_v\otimes J_vF_vJ_v)^jJ_v];\end{aligned}$$

and

$$\begin{aligned}x\otimes x = &(\sum_{i,j,l,m} \alpha_{ijl}J_vE_v^iJ_vF_v^jK_v^l + \beta_{ijm}J_vE_v^iJ_vF_v^j\overline{K}_v^m + \gamma_{ij}J_vE_v^iJ_vF_v^jJ_v) \\ &\otimes (\sum_{i,j,l,m} \alpha_{ijl}J_vE_v^iJ_vF_v^jK_v^l + \beta_{ijm}J_vE_v^iJ_vF_v^j\overline{K}_v^m + \gamma_{ij}J_vE_v^iJ_vF_v^jJ_v).\end{aligned}$$

It is seen that if $i \neq 0$ or $j \neq 0$, $\Delta_v(x)$ is impossible to equal $x\otimes x$. So, $i = 0$ and $j = 0$. We get $x = \sum_{l,m} \alpha_l K_v^l + \beta_m \overline{K}_v^m + J_v$. Then

$$\begin{aligned}\Delta_v(x) = &\sum_{l,m}[\alpha_lK_v^l\otimes K_v^l + \beta_m\overline{K}_v^m\otimes\overline{K}_v^m + J_v\otimes J_v]; \\ x\otimes x = &\sum_{l,l',m,m'} [\alpha_l\alpha_{l'}K_v^l\otimes K_v^{l'} + \alpha_l\beta_{m'}K_v^l\otimes\overline{K}_v^{m'} + \alpha_lK_v^l\otimes J_v \\ &+ \alpha_{l'}\beta_m\overline{K}_v^m\otimes K_v^{l'} + \beta_m\beta_{m'}\overline{K}_v^m\otimes\overline{K}_v^{m'} + \beta_m\overline{K}_v^m\otimes J_v \\ &+ \alpha_{l'}J_v\otimes K_v^{l'} + \beta_{m'}J_v\otimes\overline{K}_v^{m'} + J_v\otimes J_v].\end{aligned}$$

If there exists $l \neq l'$, then $x\otimes x$ possesses the monomial $K_v^l\otimes K_v^{l'}$, which does not appear in $\Delta_v(x)$. It contradicts to $\Delta_v(x) = x\otimes x$. Hence we have only a unique l. Similarly, there exists a unique m. Thus $x = \alpha_lK_v^l + \beta_m\overline{K}_v^m + J_v$ Moreover, it is easy to see that $\alpha_lK_v^l$, $\beta_m\overline{K}_v^m$ and J_v can not appear simultaneously in the expression of x. Therefore, we conclude that $x = \alpha_lK_v^l$, $\beta_m\overline{K}_v^m$ or J_v (no summation) and we have

$$\Delta_v(J_v^{(ij)}) = J_v^{(ij)} \otimes J_v^{(ij)}. \tag{95}$$

It follows that $G(v\mathfrak{sl}_q(2)) = \{J_v^{(ij)} = K_v^i\overline{K}_v^j : i, j$ run over all non-negative integers$\}$.

For any $J_v^{(ij)} = K_v^i\overline{K}_v^j \in G(v\mathfrak{sl}_q(2))$, one can find $J_v^{(ji)} = K_v^j\overline{K}_v^i \in G(v\mathfrak{sl}_q(2))$ such that the regularity (17) takes place $J_v^{(ij)}J_v^{(ji)}J_v^{(ij)} = J_v^{(ij)}$, which means that $G(v\mathfrak{sl}_q(2))$ forms a regular monoid under the multiplication of $v\mathfrak{sl}_q(2)$. □

These results show that $w\mathfrak{sl}_q(2)$ and $v\mathfrak{sl}_q(2)$ are examples of a weak Hopf algebra whose monoid of all group-like elements is a regular monoid. It incarnates further the corresponding relationship between weak Hopf algebras and regular monoids [7].

REGULAR QUASI-R-MATRIX

From Proposition 4 we have seen that $w\mathfrak{sl}_q(2)/(J_w-1) = \mathfrak{sl}_q(2)$. Now, we give another relationship between $w\mathfrak{sl}_q(2)$ and $\mathfrak{sl}_q(2)$ so as to construct a non-invertible universal R^w-matrix from $w\mathfrak{sl}_q(2)$.

Theorem 30. *$w\mathfrak{sl}_q(2)$ possesses an ideal W and a sub-algebra Y satisfying $w\mathfrak{sl}_q(2) = Y \oplus W$ and $W \cong \mathfrak{sl}_q(2)$ as Hopf algebras.*

Proof. Let W be the linear sub-space generated by $\{E_w^iF_w^jK_w^l, E_w^iF_w^j\overline{K}_w^m, E_w^iF_w^jJ_w :$ for all $i \geq 0$, $j \geq 0$, $l > 0$ and $m > 0\}$, and Y is the linear sub-space generated by $\{E_w^iF_w^j : i \geq 0, j \geq 0\}$. It is easy to see that $w\mathfrak{sl}_q(2) = Y \oplus W$;

$w\mathfrak{sl}_q(2)Ww\mathfrak{sl}_q(2) \subseteq W$, thus, W is an ideal; and, Y is a sub-algebra of $w\mathfrak{sl}_q(2)$. Note that the identity of W is J_w. Moreover, W is a Hopf algebra with the unit J_w, the comultiplication Δ_w^W satisfying

$$\Delta_w^W(E_w) = J_w \otimes E_w + E_w \otimes K_w, \tag{96}$$

$$\Delta_w^W(F_w) = F_w \otimes J_w + \overline{K}_w \otimes F_w, \tag{97}$$

$$\Delta_w^W(K_w) = K_w \otimes K_w, \quad \Delta_w^W(\overline{K}_w) = \overline{K}_w \otimes \overline{K}_w \tag{98}$$

and the same counit, multiplication and antipode as in $w\mathfrak{sl}_q(2)$. Let ρ be the algebra morphism from $\mathfrak{sl}_q(2)$ to W satisfying $\rho(E) = E_w$, $\rho(F) = F_w$, $\rho(K) = K_w$ and $\rho(K^{-1}) = \overline{K}_w$. Then ρ is, in fact, a Hopf algebra isomorphism since $\{E_w^i F_w^j K_w^l, E_w^i F_w^j \overline{K}_w^m, E_w^i F_w^j J_w$: for all $i \geq 0$, $j \geq 0$, $l > 0$ and $m > 0\}$ is a basis of W by Theorem 16. □

Let us assume here that q is a root of unity of order d in the field k where d is an odd integer and $d > 1$.

Set $I = (E_w^d, F_w^d, K_w^d - J_w)$ the two-sided ideal of U_q^w generated by $E_w^d, F_w^d, K_w^d - J_w$. Define the algebra $\overline{U}_q^w = U_q^w/I$.

REMARK. Note that $\overline{K}_w^d = J_w$ in $\overline{U}_q^w = U_q^w/I$ since $K_w^d = J_w$.

It is easy to prove that I is also a coideal of U_q and $T_w(I) \subseteq I$. Then I is a weak Hopf ideal. It follows that $\overline{U}_q^w$ has a unique weak Hopf algebra structure such that the natural morphism is a weak Hopf algebra morphism, so the comultiplication , the counit and the weak antipode of $\overline{U}_q^w$ are determined by the same formulas with U_q^w. We will show that $\overline{U}_q^w$ is a quasi-braided weak Hopf algebra. As a generalization of a braided bialgebra and R-matrix we have the following definitions [3].

Definition 31. Let in a k-linear space H there are k-linear maps $\mu : H \otimes H \to H, \eta : k \to H, \Delta : H \to H \otimes H, \varepsilon : H \to k$ such that (H, μ, η) is a k-algebra and (H, Δ, ε) is a k-coalgebra. We call H *an almost bialgebra*, if Δ is a k-algebra morphism, i.e. $\Delta(xy) = \Delta(x)\Delta(y)$ for every $x, y \in H$.

Definition 32. An almost bialgebra $H = (H, \mu, \eta, \Delta, \varepsilon)$ is called *quasi-braided*, if there exists an element R of the algebra $H \otimes H$ satisfying

$$\Delta^{op}(x)R = R\Delta(x) \tag{99}$$

for all $x \in H$ and

$$(\Delta \otimes \mathrm{id}_H)(R) = R_{13}R_{23}, \tag{100}$$

$$(\mathrm{id}_H \otimes \Delta)(R) = R_{13}R_{12}. \tag{101}$$

Such R is called *a quasi-R-matrix.*

By Theorem 30, we have $\overline{U}_q^w = U_q^w/I = Y/I \oplus W/I \cong Y/(E_w^d, F_w^d) \oplus \widetilde{U}_q$ where $\widetilde{U}_q = \mathfrak{sl}_q(2)/(E_w^d, F_w^d, K^d - 1)$ is a finite Hopf algebra. We know in [11] that the sub-algebra $\widetilde{B}_q$ of $\widetilde{U}_q$ generated by $\{E_w^m K_w^n : 0 \leq m, n \leq d-1\}$ is a finite dimensional Hopf sub-algebra and $\widetilde{U}_q$ is a braided Hopf algebra as a quotient of the quantum double of $\widetilde{B}_q$. The R-matrix of $\widetilde{U}_q$ is

$$\widetilde{R} = \frac{1}{d} \sum_{0 \leq i,j,k \leq d-1} \frac{(q - q^{-1})^k}{[k]!} q^{k(k-1)/2 + 2k(i-j) - 2ij} E_w^k K_w^i \otimes F_w^k K_w^j.$$

Since $\mathfrak{sl}_q(2) \overset{\rho}{\cong} W$ was Hopf algebras and $(E_w^d, F_w^d, K^d - 1) \overset{\rho}{\cong} I$, we get $\widetilde{U}_q \cong W/I$ as Hopf algebras under the induced morphism of ρ. Then W/I is a braided Hopf algebra with a R-matrix

$$R^w = \frac{1}{d} \sum_{0 \leq k \leq d-1; 1 \leq i,j \leq d} \frac{(q - q^{-1})^k}{[k]!} q^{k(k-1)/2 + 2k(i-j) - 2ij} E_w^k K_w^i \otimes F_w^k K_w^j.$$

Because the identity of W/I is J_w, there exists the inverse $\hat{R}^w$ of R^w such that $\hat{R}^w R^w = R^w \hat{R}^w = J_w$. Then we have

$$R^w \hat{R}^w R^w = R^w, \tag{102}$$

$$\hat{R}^w R^w \hat{R}^w = \hat{R}^w, \tag{103}$$

which shows that this R-matrix is regular in $\overline{U}_q$. It obeys the following relations

$$\Delta_w^{op}(x) R^w = R^w \Delta_w(x) \tag{104}$$

for any $x \in W/I$ and

$$(\Delta_w \otimes \mathrm{id}\,)(R^w) = R_{13}^w R_{23}^w \tag{105}$$

$$(\mathrm{id}\, \otimes \Delta_w)(R^w) = R_{13}^w R_{12}^w \tag{106}$$

which are also satisfied in $\overline{U}_q$. Therefore R^w is a von Neumann's regular quasi-R-matrix of $\overline{U}_q$. So, we get the following

Theorem 33. *$\overline{U}_q$ is a quasi-braided weak Hopf algebra with*

$$R^w = \frac{1}{d} \sum_{0 \le k \le d-1; 1 \le i,j \le d} \frac{(q-q^{-1})^k}{[k]!} q^{k(k-1)/2+2k(i-j)-2ij} E_w^k K_w^i \otimes F_w^k K_w^j$$

as its quasi-R-matrix, which is regular.

The quasi-R-matrix from J-weak Hopf algebra $v\mathfrak{sl}_q(2)$ has more complicated structure and will be considered elsewhere.

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О РЕГУЛЯРНЫХ РЕШЕНИЯХ КВАНТОВОГО УРАВНЕНИЯ ЯНГА-БАКСТЕРА И СЛАБЫХ АЛГЕБРАХ ХОПФА

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Изучаются обобщения алгебры Хопфа $\mathfrak{sl}_q(2)$ путем ослабления обратимости генератора K, т. е. заменой обратимости $KK^{-1} = 1$ на регулярность $K\overline{K}K = K$. Веедено две алгебры Хопфа: слабая алгебра Хопфа $w\mathfrak{sl}_q(2)$ и J-слабая алгебра Хопфа $v\mathfrak{sl}_q(2)$ которые детально исследованы. Показано, что моноид групповых элементов для $w\mathfrak{sl}_q(2)$ и $v\mathfrak{sl}_q(2)$ является регулярным. Построена quasi-braided слабая алгебра Хопфа $\overline{U}_q^w$ и показано, что соответствующая квази-R-матрица является регулярной $R^w\hat{R}^wR^w = R^w$.

КЛЮЧЕВЫЕ СЛОВА: алгебра Хопфа, регулярность, уравнение Янга-Бакстера, нетерово кольцо, групповой элемент, квази-R-матрица